Lesson 6: The Z-Transform and Convolution

Introduction

Required Reading:
Ch. 9.

Two signal processing tools — the *z transform* and *convolution* — are introduced in this lesson. These operations play important roles in the analysis of discrete-time signals (which is what we do in the computer). We shall see that they are related — the convolution of two time-domain signals (which is what we do when we filter a signal) is equivalent to multiplication of their corresponding z-transforms. This is one example of how these representations can greatly simplify computation.

After studying this lesson, you should be able to understand what the z-transform and convolution are, and how to implement them. You should understand the differences between them and the other transforms: Fourier series, Fourier transform, and discrete Fourier transform. You will enrich your knowledge of filter transfer functions with its time domain representation: its *impulse response*.

Domains

First let's recall the transforms we have learned so far:

1. Fourier series: transforms a finite and periodic, continuous signal in the time domain into an infinite, discrete spectrum in the frequency domain.

2. Fourier transform: transforms an infinite, continuous signal in the time domain into an infinite, continuous spectrum in the frequency domain.

3. Discrete Fourier transform: transforms a finite, discrete signal in the time domain into a finite, discrete spectrum in the frequency domain.

Now you will learn about another transform, called the *z-transform*, which converts an infinite, discrete signal in the time domain into a finite, continuous spectrum in the frequency domain. The z-transform fills the last combination among "finite vs. infinite; continuous vs. discrete". Table 6.1 summarizes all four transforms. As you can see, continuous versus discrete in the time domain transforms to infinite versus finite in the frequency domain, while finite versus infinite in the time domain transforms to discrete versus continuous in the frequency domain.

The z-transform

The *z-transform* of a discrete time signal f_k, $k = 0, \pm 1, \pm 2, \ldots, \pm\infty$ is defined as the power series

$$\mathcal{F}(z) \equiv \sum_{k=-\infty}^{\infty} f_k z^{-k} \tag{6-1}$$

Table 6.1: Summary of frequency transforms.

Transform	Time Domain	Frequency Domain
Fourier Series	Finite, Continuous	Infinite, Discrete
Fourier Transform	Infinite, Continuous	Infinite, Continuous
Discrete Fourier Transform	Finite, Discrete	Finite, Discrete
Z-Transforms	Infinite, Discrete	Finite, Continuous

where z is a continuous complex variable. It transforms the time domain infinite sequence into its complex plane representation $\mathcal{F}(z)$. Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges. The *region of convergence* (ROC) of $\mathcal{F}(z)$ is the set of all values of z for which $mathcalF(z)$ has a finite value. The inverse procedure, which obtains f_k from $\mathcal{F}(z)$ is called the *inverse z-transform*:

$$f_k = \frac{1}{j2\pi} \oint_C \mathcal{F}(z) z^{k-1} dz \tag{6-2}$$

where C denotes a closed contour in the ROC that includes $z = 0$, taken in a counterclockwise direction.

The z-transform is a general version of the *Discrete Time Fourier Transform* or DTFT. To obtain the DTFT, we restrict z to lie on the unit circle, that is $z = e^{j\omega}$. Next, we rewrite the z-transform and its inverse transform from the z-domain to the frequency domain. Substituting $z = e^{j\omega}$ into (6-1) and denoting $\mathcal{F}(e^{j\omega})$ as $F(\omega)$, we get,

$$F(\omega) = \mathcal{F}(e^{j\omega}) = \sum_{-\infty}^{\infty} f_k e^{-jk\omega} \tag{6-3}$$

Similarly, substitute $z = e^{j\omega}$ into (6-2) and recall that $e^{j\omega}$ needs to contain only frequencies up to the Nyquist cutoff (we restrict continuous frequency ω to lie between $-\pi$ and π radians/sample), so ω ranges from $-\pi$ and π and the complex exponential $e^{j\omega}$ ranges exactly once over the unit circle. The result is

$$\begin{aligned} f_k &= \frac{1}{j2\pi} \oint_C \mathcal{F}(e^{j\omega}) e^{j(k-1)\omega} de^{j\omega} \\ &= \frac{1}{j2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j(k-1)\omega} j e^{j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{jk\omega} d\omega \end{aligned} \tag{6-4}$$

because $de^{j\omega} = je^{j\omega}d\omega$. The signals we are dealing with in this section are discrete and infinitely long. From examination of (6-1) and (6-2) or (6-3) and (6-4), we know that this discrete and infinite signal is transformed into a continuous and finite spectrum.

Notice that the DTFT (6-3) and its inverse transform (6-1) are a special case of the z-transform for z on the unit circle ($z = re^{j\omega}$ and radius $r = 1$). For the general z-transform, z is a complex variable in the z plane, where $z = re^{j\omega}$ and r can have any value.

If the z-transform of signal f_k is denoted by

$$\mathcal{F}(z) \equiv \mathbf{Z}[f_k] \tag{6-5}$$

the relationship between f_k and $\mathcal{F}(z)$ can be indicated by the *transform pair*

$$f_k \stackrel{\mathbf{Z}}{\longleftrightarrow} \mathcal{F}(z) \tag{6-6}$$

Self-Test Exercises

1. Determine the z-transform and ROC for the sequence $f = \{1, 2, 5, 7, 0, 1\}, k = 0, 1, 2, 3, 4, 5$ (*Popup Answer:*
$$\mathcal{F}(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5} \tag{6-7}$$
The convergence region is the entire z plane, except for $z = 0$.)

2. Determine the z-transform and ROC of the sequence $f = \{1, 2, 5, 7, 0, 1\}, k = -2, -1, 0, 1, 2, 3$ (*Popup Answer:*
$$\mathcal{F}(z) = 1z^2 + 2z + 5 + 7z^{-1} + z^{-3} \tag{6-8}$$
The convergence region is entire z plane, except for $z = 0$ and $z = \infty$)

Example: z-transform of an impulse

The *unit impulse* or *unit sample* signal is a δ function,

$$\delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{6-9}$$

It has value of zero for every sample except $k = 0$, for which it has a value of one. Figure 4.1 in text book chapter 9 is a plot of this signal. Substituting the signal into (6-1), we have

$$\begin{aligned} \mathcal{F}(z) &= \sum_{k=-\infty}^{\infty} \delta_k z^{-k} \\ &= 1z^{-0} = 1 \end{aligned} \tag{6-10}$$

that is

$$\delta_k \stackrel{\mathbf{Z}}{\longleftrightarrow} 1 \tag{6-11}$$

Since the frequency content of the signal is the magnitude of its z-transform on the unit circle in the z-plane,

$$|F(\omega)| = |\mathcal{F}(e^{j\omega})| = 1 \tag{6-12}$$

This tells us that the frequency content is the same for all frequencies: an impulse has a *flat spectrum*.

What about time shifted impulses,

$$\delta_{k-n} = \begin{cases} 1 & k=n \\ 0 & k \neq n \end{cases}, \quad n>0 \tag{6-13}$$

$$\delta_{k+n} = \begin{cases} 1 & k=-n \\ 0 & k \neq -n \end{cases}, \quad n>0 \tag{6-14}$$

In these cases the nonzero value is not at time zero, but at time n or $-n$. We can compute the z-transform as before,

$$\begin{aligned} \mathcal{F}(z) &= \sum_{k=-\infty}^{\infty} \delta_{k-n} z^{-k} \\ &= 1z^{-n} = z^{-n} = \frac{1}{z^n} \end{aligned} \tag{6-15}$$

for δ_{k-n}. The convergence region is the entire z-plane, except $z=0$. We have learned in lesson 4 that $z=0$ is $\mathcal{F}(z)$'s pole. The z-transform for this shifted unit impulse has one value, z^{-n}, for any $z \neq 0$.

$$\delta_{k-n} \stackrel{\mathbf{z}}{\longleftrightarrow} \frac{1}{z^n}, \quad n>0 \tag{6-16}$$

Its frequency content is also one (remember that we compute the spectrum for values of z on the unit circle),

$$|F(\omega)| = |\mathcal{F}(e^{j\omega})| = |e^{-jn}| = 1 \tag{6-17}$$

This is not surprising at all, because it is, after all, just a time-shifted version of δ_k. I leave δ_{k+n} as a self-test exercise.

Self-Test Exercises

1. Sketch equation (6-12). (*Popup Answer: In a magnitude $|F(\omega)|$ vs. frequency plot, it is a horizontal line of value one.*)

2. Compute the z-transform and determine the ROC for the signal δ_{k+n}. (*Popup Answer:*

$$\begin{aligned} \mathcal{F}(z) &= \sum_{k=-\infty}^{\infty} \delta_{k+n} z^{k} \\ &= 1z^{n} = z^{n} \end{aligned} \tag{6-18}$$

The convergence region is entire z plane, except $z=\infty$.)

Example: z-transform of exponential signal

An exponential signal is defined as

$$f_k = \begin{cases} \alpha^k & k \geq 0 \\ 0 & k<0 \end{cases} \tag{6-19}$$

where α can be any real or complex value. The signal consists of an infinite number of samples. The z-transform of this signal is

$$\begin{aligned} \mathcal{F}(z) &= \sum_{k=-\infty}^{\infty} f_k z^{-k} \\ &= \sum_{k=0}^{\infty} \alpha^k z^{-k} \\ &= \sum_{k=0}^{\infty} (\alpha z^{-1})^k \end{aligned} \tag{6-20}$$

This is an infinite geometric series. Recall from lesson 5 that

$$1 + a + a^2 + a^3 + \ldots = \frac{1}{1-a}, \quad \text{if } |a| < 1 \tag{6-21}$$

(when the series in that lesson is infinite [$N \to \infty$] and $|a| < 1$, $a^N \to 0$). Consequently, for $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$, $\mathcal{F}(z)$ converges to

$$\mathcal{F}(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha| \tag{6-22}$$

In the z-plane, $|z| > |\alpha|$ refers to any z that is outside of the radius $|\alpha|$ circle. We see that in this case, the z-transform provides a compact alternative representation of the signal f_k.

Let's check out some special cases:

When α is a real number, $\alpha = 1/2$: The discrete signal in this case is

$$f_k = \begin{cases} \frac{1}{2}^k & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{6-23}$$

or

$$f_k = \left\{1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots, \left(\frac{1}{2}\right)^k, \ldots\right\}, \quad k \geq 0 \tag{6-24}$$

Replacing α with $1/2$ in (6-22), its z-transform is expressed as

$$\mathcal{F}(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \tag{6-25}$$

Figures 6.1 and 6.2 show graphs of the signal f_k and its corresponding region of convergence, which is outside of the radius 1/2 circle.

As you should be familiar with now, the frequency content of this signal is the magnitude of its z-transform on the unit circle $z = e^{j\omega}$ in the z-plane, which is

$$|F(\omega)| = |\mathcal{F}(e^{j\omega})| = \left|\frac{1}{1 - \frac{1}{2}e^{-j\omega}}\right| \tag{6-26}$$

Figure 6.3 is the plot of $|F(\omega)|$ versus frequency ω. The amplitude decreases along increasing frequency. Its peak is at zero frequency, which is called *DC* (which literally means "direct current," implying the constant — actually mean — component of the signal).

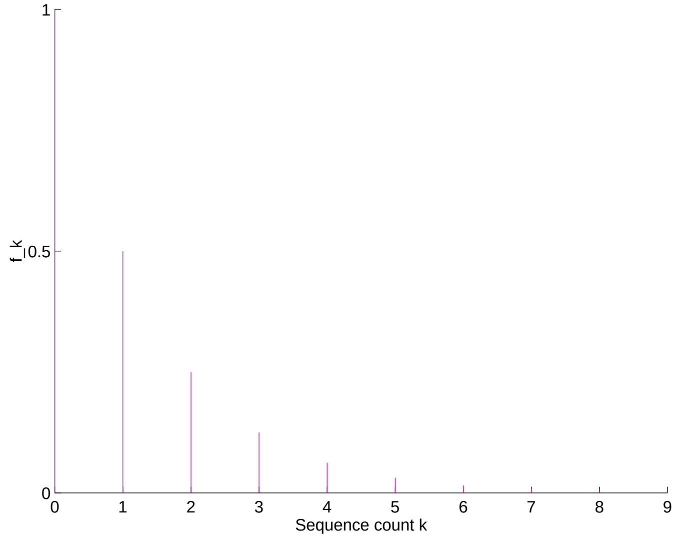

Figure 6.1: The exponential signal $f_k = (1/2)^k, k = 0, 1, 2, \ldots$.

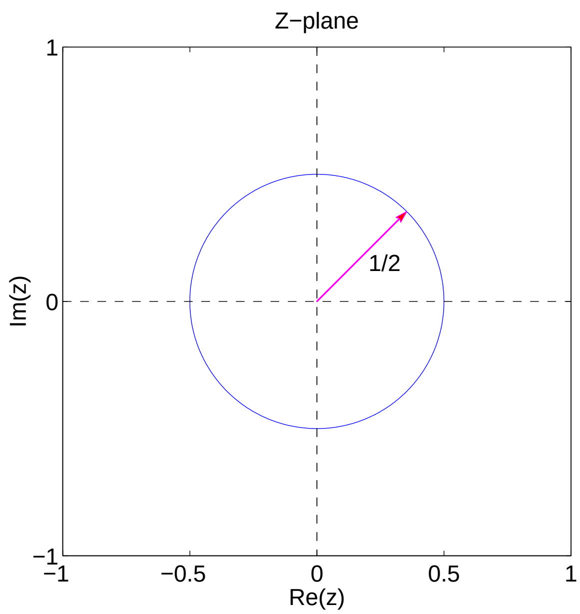

Figure 6.2: The convergence region (outside of the $r = 1/2$ circle) for the z-transform of the exponential signal $f_k = (1/2)^k, k = 0, 1, 2, \ldots$.

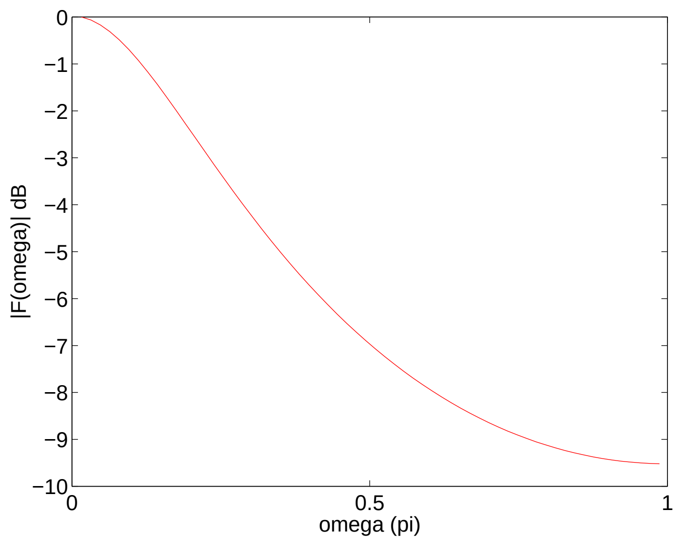

Figure 6.3: Frequency content of the signal shown in figure 6.1.

When α is a real number, $\alpha = 1$: For $\alpha = 1$, the sequence becomes

$$f_k = u_0 = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{6-27}$$

or

$$f_k = \{1, 1, 1, \ldots\}, \quad k \geq 0 \tag{6-28}$$

This is a discrete time, infinite duration *unit step* signal (see text book chapter 9, figure 4.2). Notice the difference between the unit step signal and the unit impulse signal. The latter only has one nonzero value at one particular time, the former has value one for all time after some particular time. By analogy with δ_k , a unit step occurring at time k is called u_k. Substituting $\alpha = 1$ into (6-22) we have

$$\mathcal{F}(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{6-29}$$

We can see that the pole is at $z = 1$, where the z-transform has an infinite value. Actually, we already know that the convergence region is $|z| > |\alpha|$ for a general α, so the region of convergence is outside of the unit circle $|z| = 1$, that is $|z| > 1$.

Let's evaluate the frequency content of the unit step signal. If we evaluate $F(\omega)$ on the unit circle (except at $z = 1$), we obtain

$$\begin{aligned} F(\omega) &= \mathcal{F}(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \\ &= \frac{e^{j\omega/2}}{e^{j\omega/2} - e^{-j\omega/2}} \\ &= \frac{e^{j\omega/2}}{2j \sin \omega/2} \\ &= \frac{e^{j(\omega/2 - \pi/2)}}{2 \sin \omega/2}, \quad \omega \neq 2\pi k, k = 0, 1, \ldots \end{aligned} \tag{6-30}$$

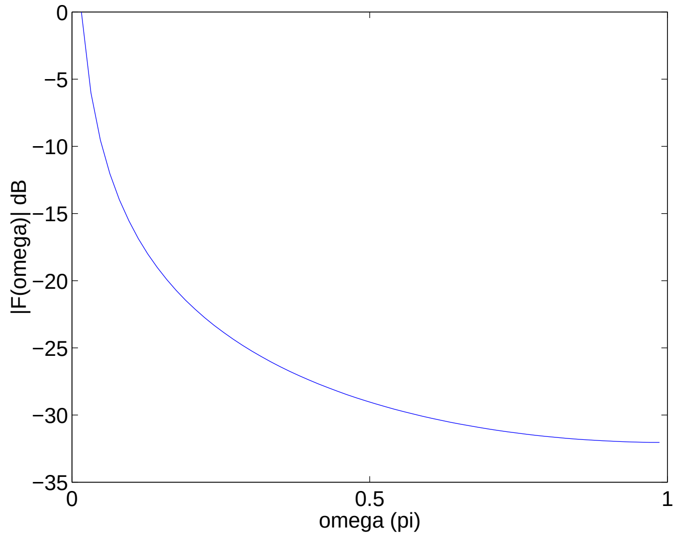

Figure 6.4: Frequency content of the unit step signal.

because $-j = e^{-j\pi/2}$ (see the self-test exercises). Hence, the presence of a pole at $z = 1$ (that is, at $\omega = 0$) creates a problem only when we want to compute $|F(\omega)|$ at $\omega = 0$, because $|F(\omega)| \to \infty$ as $\omega \to 0$. For any other value of ω, $|F(\omega)|$ is finite.

Figure 6.4 shows a plot of $|F(\omega)|$ vs. ω. Since the signal is a unit step, we might expect the signal to have zero frequency components at all frequencies except at $\omega = 0$, but that is not the case. The reason is that the signal is not a constant for all $-\infty < k < \infty$. Instead, it is turned on at $k = 0$. This abrupt jump creates all the frequency components existing in the range $0 < \omega \leq \pi$. Generally, all signals which start at a finite time will have non nonzero frequency components everywhere in the frequency axis from zero up to the Nyquist frequency. All such signals can be considered to be the product of some infinite signal with a unit step; we will see the effect of this on their spectrum when we explore convolution.

When α is a complex number, $\alpha = Re^{j\theta}$: When $\alpha = Re^{j\theta}$, equations (6-19) and (6-22) become

$$f_k = \begin{cases} R^k e^{jk\theta} & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{6-31}$$

$$\mathcal{F}(z) = \frac{1}{1 - Re^{j\theta} z^{-1}}, \quad |z| > |R| \tag{6-32}$$

Obviously $z = Re^{j\theta}$ is the pole. Equation (6-32) can be broken into real and imaginary parts:

$$\begin{aligned}
\mathcal{F}(z) &= \frac{1}{1 - Re^{j\theta}z^{-1}} \\
&= \frac{1}{1 - R(\cos\theta + j\sin\theta)z^{-1}} \\
&= \frac{1}{1 - R\cos\theta z^{-1} - jR\sin\theta z^{-1}} \\
&= \frac{1 - R\cos\theta z^{-1} + jR\sin\theta z^{-1}}{[1 - R\cos\theta z^{-1}]^2 - [jR\sin\theta z^{-1}]^2} \\
&= \frac{1 - R\cos\theta z^{-1} + jR\sin\theta z^{-1}}{1 - 2R\cos\theta z^{-1} + R^2 z^{-2}} \\
&= \frac{1 - R\cos\theta z^{-1}}{1 - 2R\cos\theta z^{-1} + R^2 z^{-2}} + j\frac{R\sin\theta z^{-1}}{1 - 2R\cos\theta z^{-1} + R^2 z^{-2}} \qquad (6\text{-}33)
\end{aligned}$$

We break the signal into two parts, too:

$$\Re f_k = R^k \cos(k\theta), \quad k \geq 0 \qquad (6\text{-}34)$$

$$\Im f_k = R^k \sin(k\theta), \quad k \geq 0 \qquad (6\text{-}35)$$

From real part we get

$$R^k \cos(k\theta) \overset{\mathbf{z}}{\longleftrightarrow} \frac{1 - R\cos\theta z^{-1}}{1 - 2R\cos\theta z^{-1} + R^2 z^{-2}} \qquad (6\text{-}36)$$

and from imaginary parts we have

$$R^k \sin(k\theta) \overset{\mathbf{z}}{\longleftrightarrow} \frac{R\sin\theta z^{-1}}{1 - 2R\cos\theta z^{-1} + R^2 z^{-2}} \qquad (6\text{-}37)$$

When $R < 1$, $R^k \cos k\theta$ and $R^k \sin k\theta$ are damped cosine and sine waves. An example of a damped cosine wave is shown in text book chapter 9 figure 5.2.

Self-Test Exercises

1. What is the derivative of u_k (the unit step at time step k)? (*Popup answer:* δ_k.)

2. Show that $e^{j\omega/2} - e^{-j\omega/2} = 2j\sin\omega/2$. (*Popup answer: Euler's formula states that* $e^{j\theta} = \cos\theta + j\sin\theta$. *For negative angles,* $e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta$. *The difference of these two is* $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$*; substitute* $\theta = \omega/2$ *to finish up.*)

3. Prove that $e^{-j\pi/2} = -j$. (*Popup answer:* $e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = 0 - j \times 1 = -j$.)

Convolution

Let's first define what convolution is. For two infinite, discrete signals x_k and h_k, the *convolution* y_t of them at time t is defined as

$$y_t = \sum_{k=-\infty}^{\infty} x_k h_{t-k} \qquad (6\text{-}38)$$

The notation for convolution is "$*$", so this can be written as

$$Y = X * H \tag{6-39}$$

where Y, X, and H are the signals with samples at time t being y_t, x_t, and h_t.

Let's consider the case when both x and h both start at zero. This is equivalent to saying that both have values of zero before that sample. So, $x_k = 0$ when $k < 0$ and $h_{t-k} = 0$ when $t - k < 0$. Equation (6-38) becomes

$$y_t = \sum_{k=0}^{t} x_k h_{t-k} \tag{6-40}$$

Actually, this is a more realistic situation than k's summation from $-\infty$ to ∞.

We expand the summation in (6-40) as

$$y_t = x_0 h_t + x_1 h_{t-1} + x_2 h_{t-2} + \ldots + x_k h_{t-k} + \ldots + x_{t-2} h_2 + x_{t-1} h_1 + x_t h_0 \tag{6-41}$$

Using this formulation, you may show that the convolution $X * H$ has the following properties:

1. Commutative: $X * H = H * X$
2. Distributive : $X * (H1 + H2) = X * H1 + X * H2$
3. Associative : $(X * H) * G = X * (H * G)$

Obviously, $x * \vec{0} = \vec{0} * x = 0$ (where $\vec{0}$ is a vector of all zeros). Then how about $\vec{1} * \vec{1}$ (where $\vec{1}$ is a vector of all ones, in this case $\vec{1} = u_0$; see the self-test exercise)?

Example of Convolution

Determine the convolution $e^t * e^t$, $t = 0, 1, 2, \ldots,$. Using (6-40),

$$\begin{aligned} e^t * e^t &= \sum_{k=0}^{t} e^k e^{t-k} \\ &= \sum_{k=0}^{t} e^t = e^t \sum_{k=0}^{t} 1 = te^t \end{aligned} \tag{6-42}$$

Self-Test Exercises

1. Determine if $u_0 * H \neq H$ is true, where $h_t = t$, $t = 0, 1, 2, \ldots$. (*Popup Answer: Yes, it is true:*

$$\begin{aligned} u_0 * H &= \sum_{k=0}^{t} 1(t-k) \\ &= \sum_{k=0}^{t} t - \sum_{k=0}^{t} k \\ &= t^2 - \frac{t(t+1)}{2} = \frac{t(t-1)}{2} \neq t \end{aligned}$$

).

2. Compute $u_0 * u_0$. (*Popup Answer:*

$$u_0 * u_0 = \sum_{k=0}^{t} 1 = t \tag{6-43}$$

So $u_0 * u_0 \neq u_0$.)

Implementing Convolution

From observation of equation (6-41), we know that for a fixed time t, y_t can be computed by the term-by-term multiplication of the sequence

$$\{x_0, x_1, x_2, \ldots, x_k, \ldots, x_{t-2}, x_{t-1}, x_t\} \tag{6-44}$$

and the time-reversed sequence

$$\{h_t, h_{t-1}, h_{t-2}, \ldots, h_{t-k}, \ldots, h_2, h_1, h_0\} \tag{6-45}$$

We just multiply the corresponding terms (for example, $x_k h_{t-k}$), then add the products. This produces the convolution for one time point t. Remember, however, that the output y_t is also a sequence, $t = 0, 1, 2, \ldots$. We need to repeat this process for at t to get the full sequence, as shown in algorithm 6.1.

Algorithm 6.1 Discrete convolution.

Require: x_t is a finite, discrete signal, $t = 0, 1, 2, \ldots$
Require: h_t is a finite, discrete signal, $t = 0, 1, 2, \ldots$
Ensure: y_t is the convolution $X * H$, $t = 0, 1, 2, \ldots$
  for $t = 0, 1, 2, \ldots$ **do**
    Reverse h_k to produce $h'_k = h_{t-k}$, $k = 0, 1, 2, \ldots, t$
    $s_k = x_k h'_k$
    $y_t = \sum_{k=0}^{t} s_k$
  end for

A more-or-less direct implementation of this algorithm in C is:

```
/* Convolution of two vectors

   Input: vectors x and h of lengths nx and nh (nh < nx)
   Output: vector y of length nx + nh - 1 (storage already allocated)
 */
void convolve(int x, unsigned nx, int h, unsigned nh, int y)
{
   for (unsigned t=0; t<nx+nh-1; t++) {
      y[t] = 0;
      for (unsigned k=max(0,t-nh+1); k<=min(nx-1,t); k++)
         y[t] += x[k] * h[t-k];
   }
}
```

Figure to be faxed.

Figure 6.5: Example of convolution function execution for $n_x = 5$ and $n_h = 2$.

The above implementation is for finite-length signals, rather than the infinite ones we've been discussing. It assumes that the input h is shorter than x ($n_h < n_x$). This h input is often called the convolution *kernel* because, as we shall see, h represents the action of our signal processing system while x is the actual input signal. Notice that, for small $t < n_h - 1$, not all of h is used. This is equivalent to multiplying the unused elements of h against the zero values of x_k, $k < 0$. Similarly, for large $t > nx - 1$, part of h is also unused — multiplied against the zero values of x_k, $k > n_x - 1$. Figure 6.5 illustrates function execution for a signal of length 5 and a kernel of length 2. This is one way to deal with the *boundary conditions* associated with the convolution: what to do with the ends of the signal X. In general, there are three ways of dealing with these boundary conditions:

1. "Pad" X with zeros past its ends. This is what was done in the code above.
2. "Reflect" X by copying element k to index $-k$. This is sometimes done in image processing operations.
3. "Truncate" the convolution at the ends of X. This means that t would cover the range $n_h - 1 \leq t \leq n_x$ and Y would be shorter than X.

MATLAB has the built-in convolution command `conv`, which takes two vectors as inputs and outputs the convolution result with length equal to one less than the sum of the two input vector lengths. You can use "`help conv`" to get more information (how does `conv` deal with boundary conditions? [*Popup answer: It zero pads*]).

Let's look at the convolution of $X * H = e^t * e^t$ again. In this case, X and H are the same function. In figure 6.6 (top), X is shown as a blue curve and the time-reversed H is shown as a red curve. The convolution result is in the bottom graph.

Self-Test Exercises

1. Use MATLAB to compute the convolution $e^{-t} * e^{-t}$ and plot the result. (*Popup Hint: use* `conv` *and* `plot`).

Properties of the Z-Transform

The z-transform is a very powerful signal processing tool because it has some very important properties. I list some of these properties in table 6.2, where the time-domain signals x_k and y_k have z-transforms of $\mathcal{X}(z)$ and $\mathcal{Y}(z)$.

Knowing these properties can be very convenient. For example, the z-transform of a signal shifted (delayed) by n samples, x_{k-n}, is $z^{-n}\mathcal{X}(z)$; this is our familiar z operator. You can also see that the convolution property of the z-transform means that convolution in the time domain is

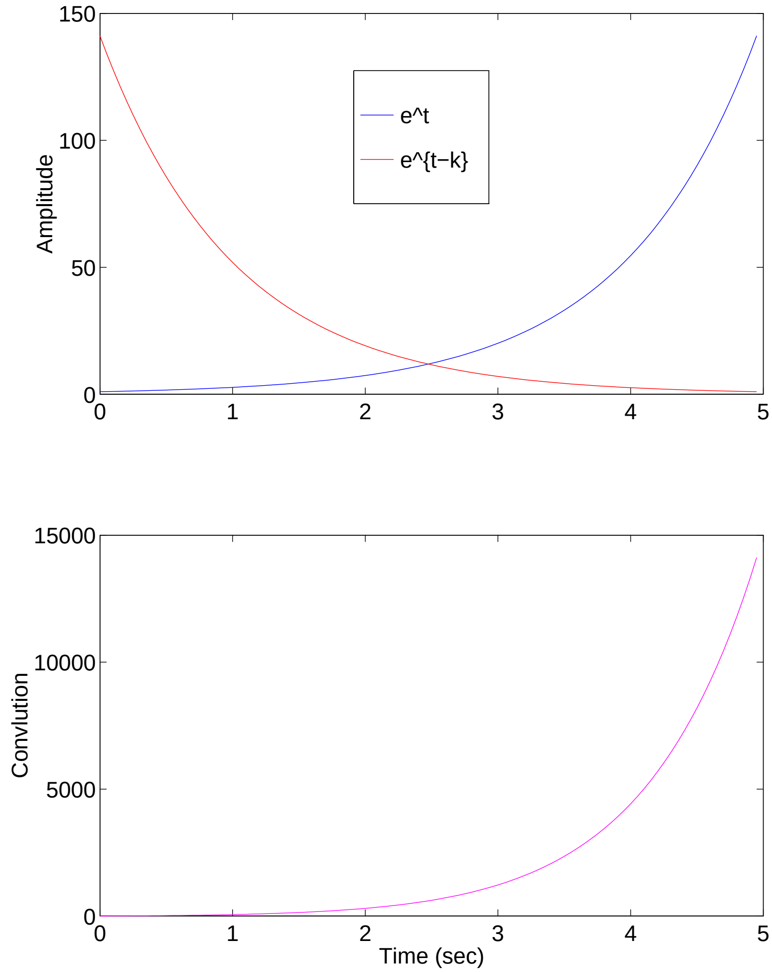

Figure 6.6: Convolution of $e^t * e^t$. Top blue line is e^t, top red line is time reversed version of e^t, the bottom plot is the result of convolution.

Table 6.2: Some properties of the z-transform.

Property	Time Domain, $Z^{-1}\{\cdot\}$	z-Domain, $Z\{\cdot\}$
Linearity	$a_1 x_k + a_2 y_k$	$a_1 \mathcal{X}(z) + a_2 \mathcal{Y}(z)$
Time shift	x_{k-n}	$z^{-n}\mathcal{X}(z)$
Scaling in the z-domain	$a^k x_k$	$\mathcal{X}(a^{-1}z)$
Time reversal	x_{-k}	$\mathcal{X}(z^{-1})$
Differentiation in the z-domain	$k x_k$	$-z\frac{\mathrm{d}\mathcal{X}(z)}{\mathrm{d}z}$
Convolution	$x_k * y_k$	$\mathcal{X}(z)\mathcal{Y}(z)$

multiplication in the z-domain. So, if we have the z-transform of two signals, it is much easier to perform convolution. Later on, you will find out that this is very important in filtering. Let's prove this property:

From (6-40) a convolution of x_t and y_t is defined as:

$$W = X * Y = \sum_{k=-\infty}^{\infty} x_k y_{t-k} \tag{6-46}$$

The z-transform of w_t is

$$\begin{aligned} \mathcal{W}(z) &= \sum_{t=-\infty}^{\infty} w_t z^{-t} \\ &= \sum_{t=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_k y_{t-k} \right) z^{-t} \end{aligned} \tag{6-47}$$

Interchanging the order of the summations (which is equivalent to factoring out the x_k and distributing the z^{-t} over the inner summation),

$$\mathcal{W}(z) = \sum_{k=-\infty}^{\infty} x_k \left(\sum_{t=-\infty}^{\infty} y_{t-k} z^{-t} \right) \tag{6-48}$$

Applying the time shift property of the z-transform, we obtain

$$\begin{aligned} \mathcal{W}(z) &= \sum_{k=-\infty}^{\infty} x_k \mathcal{Y}(z) z^{-k} \\ &= \mathcal{Y}(z) \sum_{k=-\infty}^{\infty} x_k z^{-k} = \mathcal{Y}(z)\mathcal{X}(z) \end{aligned} \tag{6-49}$$

I'll present a couple examples of using these properties.

Example: Time Shifting

Remember that I discussed that the z-transform of the unit impulse δ_k

$$\delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{6-50}$$

is 1 and that the z-transform of the shifted unit impulse δ_{k-n} is z^{-n}? Using the time-shifting property of the z-transform, this is easy to determine:

$$\delta_k \overset{\mathbf{Z}}{\longleftrightarrow} 1 \tag{6-51}$$

then

$$\delta_{k-n} \overset{\mathbf{Z}}{\longleftrightarrow} 1 * z^{-n} = z^{-n} \tag{6-52}$$

Example: Convolution

Given the signals

$$x_k = \{1, -3, 2, 1\}, \quad k = 0, 1, 2, 3 \tag{6-53}$$

and

$$y_k = \begin{cases} 1 & k = 0, 1 \\ 0 & k = 2, 3 \end{cases} \tag{6-54}$$

use the z-transform to compute their convolution $W = X * Y$.

According to (6-1),

$$\begin{aligned} \mathcal{X}(z) &= 1 - 3z^{-1} + 2z^{-2} + z^{-3} && (6\text{-}55) \\ \mathcal{Y}(z) &= 1 + z^{-1} && (6\text{-}56) \end{aligned}$$

Then using the convolution property of the z-transform, we have

$$\begin{aligned} \mathcal{W}(z) = \mathcal{X}(z)\mathcal{Y}(z) &= (1 - 3z^{-1} + 2z^{-2} + z^{-3})(1 + z^{-1}) \\ &= 1 - 2z^{-1} - z^{-2} + 3z^{-3} + z^{-4} && (6\text{-}57) \end{aligned}$$

Now we can easily get the inverse z-transform $w_k = x_k * y_k$ from the result $\mathcal{W}(z)$. Again from the z-transform definition (6-1),

$$w_k = x_k * y_k = \{1, -2, -1, 3, 1\} \tag{6-58}$$

We can also compute the convolution directly, according to the convolution definition (6-40). There are only 4 nonzero terms in x_k and 2 in y_k. For a fixed time t, the convolution is given by

$$\begin{aligned} w_t &= \sum_{k=0}^{t} x_k y_{t-k} \\ &= x_0 y_t + x_1 y_{t-1} + x_2 y_{t-2} + x_3 y_{t-3} && (6\text{-}59) \end{aligned}$$

Changing t gives the sequence of the convolution output as a function of time. In the following computation, I use $y_{t-k} = 0$, if $t - k < 0$ and $x_k = 0$ and $y_k = 0$ if $k > 3$:

$$\begin{aligned} w_0 &= x_0 y_0 = 1 \\ w_1 &= x_0 y_1 + x_1 y_0 = 1 - 3 = -2 \\ w_2 &= x_0 y_2 + x_1 y_1 + x_2 y_0 = 0 - 3 + 2 = -1 \\ w_3 &= x_0 y_3 + x_1 y_2 + x_2 y_1 + x_3 y_0 = 0 + 0 + 2 + 1 = 3 \\ w_4 &= x_1 y_3 + x_2 y_2 + x_3 y_1 = 0 + 0 + 1 = 1 \\ w_k &= 0, \quad k > 4 \end{aligned}$$

Therefore, this also gives the result

$$w_k = x_k * y_k = \{1, -2, -1, 3, 1\} \tag{6-60}$$

Self-Test Exercises

1. Prove the scaling property of the z-transform; that is, if

$$x_k \overset{\mathbf{z}}{\longleftrightarrow} \mathcal{X}(z)$$

then

$$a^k x_k \overset{\mathbf{z}}{\longleftrightarrow} \mathcal{X}(a^{-1}z)$$

(*Popup Answer : From (6-1)*,

$$\begin{aligned} Z\left\{a^k x_k\right\} = \sum_{k=-\infty}^{\infty} a^k x_k z^{-k} &= \sum_{k=-\infty}^{\infty} x_k (a^{-1}z)^{-k} \\ &= \mathcal{X}(a^{-1}z) \end{aligned} \tag{6-61}$$

).

Impulse Response and the Transfer Function

Recall the a filter's input/output relationship is summarized by the transfer function I discussed in lessons 3 and 4. Let's denote the input signal as x_t and output as y_t in the time domain. You learned that the the transfer function in the z domain is $\mathcal{H}(z)$. What is its time domain representation? We will answer this shortly; first I would like to give a name to the time domain transfer function, h_t: the filter's *impulse response*. The filter's input/output relationship can be written using h_t as:

$$Y = X * H \tag{6-62}$$

This says that the output of filter results from the convolution between input signal x_t and filter's impulse response h_t. From earlier in this lesson, you now know that convolution of two signals in the time domain is equivalent to multiplying their z-transforms in the z domain. So, we obtain the input/output relationship via the transfer function in the z domain as

$$\mathcal{Y}(z) = \mathcal{X}(z)\mathcal{H}(z) \tag{6-63}$$

where $\mathcal{X}(z)$ and $\mathcal{Y}(z)$ are the z-transforms of x_t and y_t and $\mathcal{H}(z)$ is the z-transform of h_t. Actually, this $\mathcal{H}(z)$ is just the transfer function we talked about in lessons 3 and 4. In other words, a filter's transfer function is the z-transform of its impulse response!

As a simple example of how to use the z-transform to determine a filter's transfer function from its defining equation, consider the feedforward filter in text book chapter 4, equation (5.1), shown again here:

$$y_t = a_0 x_t + a_1 x_{t-\tau} \tag{6-64}$$

Applying the z-transform to both sides,

$$\mathcal{Y}(z) = a_0\mathcal{X}(z) + a_1 z^{-\tau}\mathcal{X}(z) \tag{6-65}$$

because of the time shift property of the z-transform. This can be rearranged to be

$$\mathcal{Y}(z) = (a_0 + a_1 z^{-\tau})\mathcal{X}(z) \tag{6-66}$$

So the transfer function is

$$\mathcal{H}(z) = a_0 + a_1 z^{-\tau} \tag{6-67}$$

and therefore, we have

$$\mathcal{Y}(z) = \mathcal{H}(z)\mathcal{X}(z) \tag{6-68}$$

The $\mathcal{H}(z)$ is exactly same as derived in text book chapter 4, equation (5.7).

Text book chapter 9, figure 6.1 illustrates input and output via the transfer function. Applying the inverse z-transform, we can get the time domain representation of $\mathcal{H}(z)$, or the impulse response h_t. In a similar manner, we can get the output y_t from $\mathcal{Y}(z)$. In fact, sometime, using (6-63), we found an easy way to compute a filter's response to a signal if we have already know the signal's z-transform and the filter's transfer function.

Let's see why h_t is called the impulse response. Remember the unit impulse, which is a signal that has the value one at $t = 0$ and zero otherwise, and which can be expressed as

$$\delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{6-69}$$

The impulse response of a filter is its output when a unit impulse is applied. Now we compute these two function's convolution,

$$y_t = \sum_{k=0}^{t} \delta_k h_{t-k} = h_t \tag{6-70}$$

This tell us that the filter response y_t to a unit impulse is h_t. This is why h_t is called the impulse response.

Applying the z-transform to both side of the above equation, we get

$$\mathcal{Y}(z) = \mathcal{H}(z) \tag{6-71}$$

Therefore, the *transfer function* can be viewed as the z-transform of the filter's *impulse response*, or its impulse response in the z domain.

If we restrict z to lie on the unit circle, $z =^{j\omega}$, from (6-63) we obtain

$$Y(\omega) = H(\omega)X(\omega) \tag{6-72}$$

where $X(\omega)$ is the signal's frequency content, $Y(\omega)$ is the frequency content of the filter output and $H(\omega)$ is the filter's frequency response I discussed in lessons 3 and 4.

Example: Computing Transfer Function and Impulse Response

Compute the transfer function and impulse response of the system described by the feedforward filter,

$$y_t = 2x_t + \frac{1}{2}y_{t-1}. \tag{6-73}$$

By computing the z-transform of the both side of above equation and using the fact that

$$y_{t-1} \overset{\mathbf{Z}}{\longleftrightarrow} z^{-1}\mathcal{Y}(z) \tag{6-74}$$

we obtain,

$$\mathcal{Y}(z) = 2\mathcal{X}(z) + \frac{1}{2}z^{-1}\mathcal{Y}(z). \tag{6-75}$$

Hence the transfer function is

$$\mathcal{H}(z) = \frac{\mathcal{Y}(z)}{\mathcal{X}(z)} = \frac{2}{1 - \frac{1}{2}z^{-1}} \tag{6-76}$$

$\mathcal{H}(z)$ has a pole at $z = 1/2$ and zero at $z = 0$. Now we compute its impulse response. Previously we had an example of an exponential signal (6-19) and its z-transform (6-22), so we have

$$a^k \overset{\mathbf{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad k \geq 0, |z| > a \tag{6-77}$$

Using this result, we obtain the inverse transform

$$h_t = 2\left(\frac{1}{2}\right)^k, \quad k \geq 0 \tag{6-78}$$

This is the corresponding impulse response.

Inverse Z-transform by Partial Fraction Expansion

There are many ways to determine the inverse z-transform besides directly using definition (6-2). Here, I briefly introduce a simple method using *partial fraction expansion* for an example to avoid too much math. Suppose $\mathcal{Y}(z)$ is the z-transform of the signal y_k. If $\mathcal{Y}(z)$ is a rational function with N distinct poles p_1, p_2, . . ., p_N, we seek an expansion of the form

$$\mathcal{Y}(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \ldots + \frac{A_N}{1 - p_N z^{-1}} \tag{6-79}$$

and determine the coefficients $A_1, A_2, \ldots, A_N$. The final step in the inversion of $\mathcal{Y}(z)$ can be obtained by inverting each term in (6-79) and taking the corresponding linear combination. From the previous section, we know that

$$a^k \overset{\mathbf{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad , k \geq 0, |z| > a \tag{6-80}$$

so each of the $A_k/(1 - p_1 z^{-1})$ has the inverse z-transform $A_k p^k$, $k \geq 0$. The final result should be

$$y_k = A_1 p_1^k + A_2 p_2^k + \ldots + A_N p_N^k, \quad , k \geq 0 \tag{6-81}$$

In the case in which all poles are distinct but some of them are complex, and the signal y_k is real, then if p_j is a pole, its complex conjugate p_j^* is also a pole, and the corresponding coefficients in the partial fraction expansion are also complex conjugates. Therefore, the contribution of two complex conjugate poles is of the form

$$(y_k)_n = A_n p_n^k + A_n^* (p_n^*)^k, \quad k \geq 0 \tag{6-82}$$

Following the text book, I have skipped some complications that can arise in the partial fraction method.

Example: Partial Fraction Expansion

Given the z-transform of y_t,

$$\mathcal{Y}(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \tag{6-83}$$

determine y_t.

We first need to perform a partial fraction expansion and determine the coefficients. Multiplying both numerator and denominator by z^2,

$$\mathcal{Y}(z) = \frac{z^2}{z^2 - 1.5z + 0.5} \tag{6-84}$$

Solving for the roots of $z^2 - 1.5z + 0.5 = 0$ to determine the poles, we obtain $p_1 = 1$ and $p_2 = 0.5$, so the expansion is in the form of (6-79)

$$\begin{aligned} \frac{\mathcal{Y}(z)}{z} &= \frac{z}{(z-1)(z-0.5)} \\ &= \frac{A_1}{(z-1)} + \frac{A_2}{(z-0.5)} \end{aligned} \tag{6-85}$$

A simple way to determine A_1 and A_2 is to multiply the equation by the denominator term $(z-1)(z-0.5)$, which should be equal to the numerator of the original equation. We get

$$z = A_1(z - 0.5) + A_2(z - 1) \tag{6-86}$$

For $z = p_1 = 1$,

$$1 = A_1(1 - 0.5) \longrightarrow A_1 = 2 \tag{6-87}$$

For $z = p_2 = 0.5$,

$$0.5 = A_2(0.5 - 1) \longrightarrow A_2 = -1 \tag{6-88}$$

so the result of the partial fraction expansion is

$$\frac{\mathcal{Y}(z)}{z} = \frac{2}{(z-1)} - \frac{1}{(z-0.5)} \tag{6-89}$$

or

$$\mathcal{Y}(z) = \frac{2}{(1 - z^{-1})} - \frac{1}{(1 - 0.5z^{-1})} \tag{6-90}$$

Using (6-80) when $|z| > 1$, we obtain

$$y_k = 2(1)^k - (0.5)^k = 2 - 0.5^k, \quad k \geq 0, \tag{6-91}$$

Assignment 6

1. Compute the z-transform of

$$f_k = \begin{cases} (-1)^k & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{6-92}$$

and determine its frequency content.

2. Determine the z-transform of the signal

$$x_k = \begin{cases} \cos \omega_0 k & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{6-93}$$

3. Text book chapter 9 problem 9.3.

4. The impulse response of a system is $h_t = \{1, 2, 1, -1\}$, $t = -1, 0, 1, 2$. Determine the response of the system to the input signal $x_t = \{1, 2, 3, 1\}$, $t = 0, 1, 2, 3$.

5. Using text book chapter 9 equation (8.9), determine the filter output y_t and plot it using MATLAB to generate text book chapter 9, figure 8.1.