Chapter 4

MODELING

In this chapter, we present the details of the two parts of the study. In Section 4.1, the first part, the details of the mathematical model for single-member district, which corresponds to a model for single-member district election system provided. For this part, we also present the notation used in the mathematical model. In Section 4.2, mathematical model for single-member district is generalized. The first mathematical model can work on a system in which only one representative can be elected in each electoral district. However, the mathematical model in Section 4.2 allows that more than one representative can be elected in each electoral district. The second model is more complex than the first one in terms of the total number of variables and constraints. The formulations are originated from a mathematical model in Nemoto and Hotta (2003) and modified for our purposes. The required definitions had been provided in Chapter 3. However, the definitions are also given in this chapter for the sake of completeness.

4.1 Mathematical Formulation for the Single-Member District

G=(I,A) denotes a contiguity graph with a set of nodes I and set of arcs between the nodes A. Here the political units and adjacency of the units are represented by the nodes and arcs in the graph respectively. After this representation, the authors developed a new network which is originated from the first one; $T = (\bar{I}, \bar{A})$. According to this network, each arc connecting node i and node j in G is replaced by the pair of arcs (i, j) and (j, i); n copies of this graph are formed, and node i in the h^{th} copy of the graph is denoted by w_i^h; $\forall i \in I$, $\forall h \in H$. Note that H denotes the set of districts; in other words, the copies of graph. In addition, $|H|$, which is equal to n, source-nodes and $|I|$ sink-nodes a.k.a tail-nodes are introduced. Each source-node s^h, $\forall h \in H$ is connected to all the nodes of the h^{th} copy of the graph with an arc (s^h, w_i^h) $\forall i \in I$, while for each sink-node t_i, $\forall i \in I$ there exists an arc (w_i^h, t_i), $\forall h \in H$. After introducing the additional nodes and arcs, $\bar{I}$ covers source-nodes (s^h), sink-nodes (t_i) and the nodes in the copies of the graph

(w_i^h); $\forall i \in I, \forall h \in H$. $\bar{A}$ covers the arcs between the source-nodes and the nodes of the copies of the graph (s^h, w_i^h), the arcs between the nodes w_i^h and w_j^h , and lastly the arcs between the nodes of the copies of the graph and sink-nodes (w_i^h, t_i); $\forall i, j \in I, \forall h \in H$. The objective function of this model is modified. The sets and parameters are provided in Table 4.1. The required decision variables are introduced in Table 4.2.

Table 4.1: Sets and parameters for the first formulation

H	set of the districts or copies of graph
I	set of the political units
P	set of the parties
C	set of units that cannot take part of the same district for the compactness
$\bar{A}$	set of the arcs between the pair of nodes (s^h, w_i^h), (w_i^h, w_j^h), and (w_i^h, t_i); $\forall i, j \in I, \forall h \in H, i \neq j$
$\delta^-(w_i^h)$	the set of arcs entering node w_i^h; $\forall i \in I, \forall h \in H$
$\delta^+(w_i^h)$	the set of arcs leaving node w_i^h; $\forall i \in I, \forall h \in H$
v_{ki}	the number votes that party k has in unit i; $\forall k \in P$, $\forall i \in I$
s_i	number of voters in unit i; $\forall i \in I$
ρ	the party chosen
$\bar{s}$	the average number of voters
β	the allowable percentage deviation from the average number voters
M	A sufficiently big value
F	the volume of the flow from each source

Table 4.2: Decision variables for the first formulation

x_{ih}	binary decision variable indicating if unit i is assigned to district h $\forall i \in I, \forall h \in H$
$f(a)$	the volume of the flow on arc a; $\forall a \in \bar{A}$
y_{ih}	binary decision variable indicating if the h^{th} copy of G the flow enters through node i; $\forall i \in I, \forall h \in H$
c_{kh}	binary decision variable indicating the party k wins in district h; $\forall k \in P, \forall h \in H$
t_h	the number of voters in district h; $\forall h \in H$
o_{kh}	auxiliary variable which shows the total votes of each party k in each district h $\forall k \in P, \forall h \in H$
z_{kph}	binary decision variable for the comparing of total votes of party k and p in district h; $\forall k, p \in P,\ \forall h \in H, k \neq p$

$$max \sum_{h \in H} c_{\rho h} \tag{4.1}$$

$$s.t.$$

$$\sum_{h \in H} x_{ih} = 1,\ \forall i \in I \tag{4.2}$$

$$\sum_{i \in I} x_{ih} v_{ki} = o_{kh},\ \forall k \in P,\ \forall h \in H \tag{4.3}$$

$$o_{ph} - o_{kh} + M z_{kph}\ \geq 0,\ \forall k, p \in P,\ \forall h \in H, k \neq p \tag{4.4}$$

$$\sum_{p \in P, p \neq k} z_{kph} - |P| + 2\ \leq c_{kh},\ \forall h \in H,\ \forall k \in P \tag{4.5}$$

$$\sum_{k \in P} c_{kh} = 1,\ \forall h \in H \tag{4.6}$$

$$\sum_{i \in I} x_{ih} s_i = t_h,\ \forall h \in H \tag{4.7}$$

$$t_h\ \leq \bar{s}(1 + \beta),\ \forall h \in H \tag{4.8}$$

$$t_h\ \geq \bar{s}(1 - \beta),\ \forall h \in H \tag{4.9}$$

$$\sum_{i \in I} y_{ih} = 1,\ \forall h \in H \tag{4.10}$$

$$f(s^h, w_i^h) = F y_{ih},\ \forall i \in I,\ \forall h \in H \tag{4.11}$$

$$\sum_{a \in \delta^-(w_i^h)} f(a) = \sum_{a \in \delta^+(w_i^h)} f(a),\ \forall i \in I,\ \forall h \in H \tag{4.12}$$

$$\sum_{a \in \delta^-(w_i^h)} f(a)\ \leq F x_{ih},\ \forall i \in I,\ \forall h \in H \tag{4.13}$$

$$x_{ih}\ \leq f(w_i^h, t_i),\ \forall i \in I,\ \forall h \in H \tag{4.14}$$

$$x_{ih} + x_{jh}\ \leq 1,\ \forall h \in H,\ \forall (i, j) \in C \tag{4.15}$$

$$f(a)\ \geq 0; o_{kh} \geq 0,\ \forall a \in \bar{A}; \forall h \in H,\ \forall k \in P \tag{4.16}$$

$$x_{ih}, y_{ih}, c_{kh}, z_{kph} \in \{0, 1\},\ \forall h \in H,\ \forall i \in I,\ \forall k, p \in P, k \neq p \tag{4.17}$$

In the objective function, the number of representatives of a particular party is maximized. Each unit must be assigned to a district in the constraints 4.2. The number of representatives of each party from each district is calculated by using the constraints from 4.3 to 4.6. The total number of votes of each party in each district is calculated in 4.3. The constraints with the number 4.4 are utilized to find the number representatives of each party by considering the total number of votes of each party. In order to find the winner party in each district, at first we compare the total votes of a pair of party and keep the result information that can be 0 or 1, then we determine which party dominates the others using the information from the constraints 4.4. This determination part is in 4.5. The mentioned information is kept in the variable o_{ph} $\forall p \in P, \forall h \in H$. In 4.6, the constraints enforce that only one candidate or party can be elected in each district. Calculation of the population of each district is in the constraints 4.7. The constraints of 4.8 and 4.9 are developed for the population equality criterion. The rest of the mathematical model is originated from the aforementioned article. Each source node must be assigned to one of the districts 4.10. The flow amount from each source node to the nodes we have in the original problem must be equal to a predetermined value if the corresponding arc is used 4.11. The total amount of flow on entering arcs must be equal to the total amount of flow on leaving arcs for each node 4.12. In 4.13, if node i is not assigned to unit h, then the total flow on entering arcs to this unit must be zero. If the flow between node w_i^h and tail node t_i is zero, then the node (unit) i cannot be assigned to the district h according to the constraints 4.14. Some predetermined units cannot be assigned to a same district in order to attain more compact results in the constraints 4.15. This constraint can be deleted according to the circumstance. The constraints of 4.16 are for nonnegativity. The last constraints are for defining the domains of the decision variables.

4.2 Mathematical Formulation for the Multi-Member District

The formulation above considers only the single-member district system. Multi-member case is also developed to have a more generic formulation. The single-member district problems can be solved using the formulation given in this section. The difference between two formulations is that the contraints from 4.4 to 4.6 in the first formulation are modified to the constraints that are from 4.22 to 4.26 in the second formulation. Calculating the number of representatives of each party in each district is handled by this group of constraints in both models. Second approach is more sophisticated than the first one. D'Hondt method is utilized for the allocation of the number of representations of each party according to its vote proportion in each district. The reader should check the mentioned allocation method to comprehend the constraints from 4.22 to 4.26 in the second formulation. The additional sets and parameters are provided in Table 4.3. The additional decision variables are introduced in Table 4.4.

Table 4.3: Additional sets and parameters for the second formulation

J	the set of denominators(number of representatives)
M_1	A sufficiently big value for the difference of total votes
M_2	A sufficiently big value for number of representatives and parties