Variable Order Fractional Fokker-Planck Equations derived from Continuous Time Random Walks

Peter Straka[a]

[a]*School of Mathematics & Statistics, UNSW Sydney, Sydney NSW 2052, Australia*

Abstract

Continuous Time Random Walk models (CTRW) of anomalous diffusion are studied, where the anomalous exponent $\beta(x) \in (0,1)$ varies in space. This type of situation occurs e.g. in biophysics, where the density of the intracellular matrix varies throughout a cell. Scaling limits of CTRWs are known to have probability distributions which solve fractional Fokker-Planck type equations (FFPE). This correspondence between stochastic processes and FFPE solutions has many useful extensions e.g. to nonlinear particle interactions and reactions, but has not yet been sufficiently developed for FFPEs of the "variable order" type with non-constant $\beta(x)$.

In this article, variable order FFPEs (VOFFPE) are derived from scaling limits of CTRWs. The key mathematical tool is the 1-1 correspondence of a CTRW scaling limit to a bivariate Langevin process, which tracks the cumulative sum of jumps in one component and the cumulative sum of waiting times in the other. The spatially varying anomalous exponent is modelled by spatially varying $\beta(x)$-stable Lévy noise in the waiting time component. The VOFFPE displays a spatially heterogeneous temporal scaling behaviour, with generalized diffusivity and drift coefficients whose units are $\text{length}^2/\text{time}^{\beta(x)}$ resp. $\text{length}/\text{time}^{\beta(x)}$. A global change of the time scale results in a spatially varying change in diffusivity and drift.

A consequence of the mathematical derivation of a VOFFPE from CTRW limits in this article is that a solution of a VOFFPE can be approximated via Monte Carlo simulations. Based on such simulations, we are able to confirm that the VOFFPE is consistent under a change of the global time scale.

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1. Introduction

Subdiffusive processes are characterized by a sublinearly growing mean squared displacement proportional to t^β, $0 < \beta < 1$, and have been reported in many experimental systems [1, 2, 3, 4, 5]. For a majority of these systems, long rests of a walker are thought to be the main mechanism causing subdiffusion, and Continuous Time Random Walks (CTRWs) with heavy-tailed waiting times effectively capture this phenomenon [6, 7, 1]. CTRWs have become a widely used model, particularly because scaling limits of their probability distributions can be modelled by fractional differential equations and Fokker-Planck equations [8, 9, 10, 11] which can be extended tractably to model particle reactions [12, 13] and nonlinear interaction effects [14].

In the majority of the literature on CTRWs and FFPEs (fractional Fokker-Planck equations), the fractional parameter $0 < \beta < 1$ is treated as a global constant. The situation where β varies in space, however, is of interest in physics, since the strength of a trapping effect may vary throughout

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Email address: `p.straka@unsw.edu.au` (Peter Straka)

a disordered medium [15, 16, 17, 18, 14]. The experiments in [15] find that for the diffusion of a bead in F-actin networks, the fractional parameter depends solely on the ratio of bead size to mesh width, and hence a physical system with continuously varying fractional parameter can be designed.

There are various works in the statistical physics and computational literature that consider "variable order" fractional subdiffusion equations (see e.g. [19, 20, 21, 22] and the references therein). We highlight three such variable variable order FFPE (VOFFPE) below:

[19] Equation (11), the original derivation of a VOFFPE from CTRWs:

$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x^2} \left[K(x) D_t^{1-\beta(x)} P(x,t) \right] \tag{1.1}$$

[20] Equation (8):

$$D_t^{\beta(x)} P(x,t) = K \frac{\partial^2 P(x,t)}{\partial x^2} \tag{1.2}$$

[21] Equation (1.2):

$$\frac{\partial P(x,t)}{\partial t} = K_0 D_t^{1-\beta(x)} \frac{\partial^2 P(x,t)}{\partial x^2} + f(x,t) \tag{1.3}$$

In (1.2), the fractional temporal derivative is of Caputo type, whereas in (1.3), it is of Riemann-Liouville type. We note the following issues in the applicability of these three prototypes:

- In the above works, the consistency of physical dimensions is not addressed: An application of the fractional derivative of order β is dimensionally equivalent to a division by time$^\beta$, and it appears that one side of the equation has integer powers of time whereas the other side has a fractional power of time. A non-dimensional interpretation of the differential equation seems to be necessary, but it is unclear how time scale parameters enter into the equation, or how dynamics are related at two different time scales.

- Since the operators $\partial/\partial x$ and $\partial^2/\partial x^2$ do not commute with the variable order fractional derivative $D_t^{1-\beta(x)}$, the three equations have intrinsically different solutions $P(x,t)$. Only Equation (1.1) has been derived from a CTRW scaling limit, which means that Equations (1.2) and (1.3) are disadvantageous, because i) it is not immediately clear if solutions are positive, ii) probability densities cannot be computed via Monte Carlo simulations, and iii) extensions for chemical reactions or other particle interaction are ad hoc in the sense that they are not based on a particle model.

- The three works above do not address the issue of how to model space- and time-dependent drift and diffusivity, though this is useful for the modelling of e.g. chemotaxis and chemokinesis [14].

In this article, we resolve the above issues: From CTRW limit dynamics, we derive the VOFFPE (3.12), as a logical extension of (1.1). We show that our VOFFPE is dimensionally consistent, and elucidate consequences of the spatially inhomogeneous temporal scaling. Our framework is very general, and rests on only one unrestrictive assumption: The *Semi-Markov property*, which states that the next waiting time of a CTRW is independent of the current time, and depends only on the current position in space [23, 24][1]. This framework allows for a space- and time-dependent diffusivity

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[1]But the next jump may depend on both location and current time.

and drift, for which we detail conditions guaranteeing existence and uniqueness of a solution to the VOFFPE (3.12).

There is a 1-1 correspondence between Semi-Markov processes and *Markov additive processes* (MAP) [25]. These are Markov processes in space-time, with the property that the future trajectory is independent of the past, given the present *space* component *only* (as opposed to given the present value of *both* components). We fully exploit this 1-1 correspondence and represent CTRWs and their scaling limits as Langevin processes in space-time $\mathbb{R} \times [0, \infty)$. In the Langevin setting, spatially varying fractional parameters for the waiting time distribution can be introduced relatively easily, which is a useful tool in the derivation of the VOFFPE.

A recent result in [26] then provides the generalized FFPE (Fractional Fokker-Planck Equation) (3.3) which governs a CTRW scaling limit X_t in the sense that its unique solution $P(x, t)$ is the probability distribution of X_t. The generalized FFPE (3.3) contains a space-dependent memory kernel $V(x, s)$ which admits an interesting interpretation based on renewal theory. Moreover, if the dynamics are locally fractional with parameter $\beta(x)$, the VOFFPE (3.12) results from the generalized FFPE (3.3).

Section 2 describes the underlying dynamics in space-time, where the first component tracks the location of a particle and the second component the physical time that has passed. Scaling limits are identified as bivariate Langevin processes which uniquely correspond to CTRW limits. In Section 3 we utilize a result from the mathematics literature that links CTRW limits with fractional Fokker-Planck equations. In Section 4 we show that the variable-order FFPE (3.12) is consistent, by calculating matching solutions at different time scales via a Monte-Carlo simulation of CTRW trajectories.

2. The space-time setting

2.1. CTRWs are space-time Markov Chains

The trajectory of a CTRW can be reduced to a series of points in space-time, without loss of information: If $t_1, t_2, \ldots$ denote the jump times of a CTRW and $x_1, x_2, \ldots$ the locations of the end points of the jumps, then the location of the CTRW at time t is

$$X_t^{(c)} = x_k \text{ for } k \geq 0 \text{ such that } t_k \leq t < t_{k+1}. \tag{2.1}$$

We write c for a CTRW scaling parameter, as will be detailed below. Moreover, write t_0 for the starting time and x_0 for the starting location. Then the pair sequence $(t_0, x_0), (t_1, x_1), (t_2, x_2), \ldots$ must define a Markov chain in space-time $\mathbb{R} \times [0, \infty)$, by the Semi-Markov assumption that the $k+1$-th waiting time $t_{k+1} - t_k$ and the $k+1$-th jump $x_{k+1} - x_k$ are both independent of the past $(t_0, x_0), \ldots, (t_{k-1}, x_{k-1})$ given the current (t_k, x_k).

The distribution of the Markov chain (t_k, x_k) is thus uniquely determined by the initial location (x_0, t_0) in space-time, and a probability kernel on space-time

$$K^{(c)}(y, w | x, s) \tag{2.2}$$

denoting the probability distribution of the next jump $y = x_{k+1} - x_k$ after the next waiting time $w = t_{k+1} - t_k$, conditional on $(x_k, t_k) = (x, s)$. As an example, let $(t_0, x_0) = (0, 0)$ and

$$K^{(c)}(y, w | x, s) = \left[\frac{1}{2} \delta_{+\chi}(y) + \frac{1}{2} \delta_{-\chi}(y) \right] \beta(x)(1 + w/\tau)^{-1-\beta(x)} \, dw, \quad w > 0, \quad \beta(x) \in (0, 1). \tag{2.3}$$

Then (2.3) defines a CTRW with Pareto waiting times at scale τ and jumps $\pm\chi$ with equal probability $1/2$. The tail parameter β varies as a function $\beta(x)$ in space.

We will view $c > 0$ as a "master" scaling parameter, in the sense that as $c \uparrow \infty$, both $\chi \downarrow 0$ and $\tau \downarrow 0$. Thus for every c we have defined a Markov chain on $\mathbb{R} \times [0, \infty)$, which in turn defines a CTRW at scale χ in space and τ in time via (2.1). We are interested in the limiting CTRW process as $c \to \infty$ (if there is one). Theorem 2.1 specifies the exact conditions on the rate at which $\chi \downarrow 0$ and $\tau \downarrow 0$, so that a limiting CTRW process exists. See Subsection 2.4 for more details.

2.2. The limiting space-time Langevin Process

We seek to identify and characterize a class of possible limit processes (Y_u, Z_u) of the above Markov chains, in the scaling limit as $c \to \infty$. As it turns out, this class of limiting stochastic processes are solutions to Langevin equations in space-time (or solutions to stochastic differential equations with Lévy noise, or "jump-diffusions"). We characterize this class of Langevin processes in this section, and detail a construction of these processes via Poisson Random Measures. Then in Section 2.3, we utilize results from the theory of convergence of stochastic processes to give exact conditions of convergence to these Langevin processes.

Poisson Random Measure. The standard construction of Lévy noise is via Poisson Random Measures [27]. We describe an approach which is amenable to a spatially varying coefficient $\beta(x)$. We first define the mean measure M to be the Lebesgue measure on $(0, \infty) \times [0, \infty)$; the first component $(0, \infty)$ corresponds to the size of the jumps $w > 0$ of the temporal component Z_u, whereas $[0, \infty)$ corresponds to an auxiliary (or "operational") time, during which these jumps accumulate. Lebesgue measure is nothing but the surface measure, defined uniquely by the requirement that a rectangle $B = (a, b) \times (c, d) \subset (0, \infty) \times [0, \infty)$ have the measure $M(B) = (b - a)(d - c)$. A Poisson Random Measure with intensity measure M then is a random measure $N(dw, ds) = \sum_{r \in \mathcal{R}} \delta_r(dw, ds)$ where δ_r denotes a Dirac point measure at the point $r \in (0, \infty) \times [0, \infty)$ and $\mathcal{R} \subset (0, \infty) \times [0, \infty)$ is a random discrete set of points such that the following two criteria hold:

- For any bounded rectangle B, the number of points of $\mathcal{R}$ that lie in B is Poisson distributed with mean $M(B)$.

- The point counts in disjoint rectangles are independent.

Lévy measure. The points in $\mathcal{R}$ are transformed in their first component according to the *coefficient function F*. For instance, if we let

$$F(w) = (\Gamma(1 - \beta)w)^{-1/\beta}, \quad 0 < \beta < 1 \tag{2.4}$$

then it is not difficult to confirm that the set of points $F(\mathcal{R}) := \{(F(w), s) : (w, s) \in \mathcal{R}\}$ define a Poisson Random Measure on $(0, \infty) \times [0, \infty)$ with mean measure

$$M_F((a, b) \times (c, d)) = \int_a^b \nu(w) \, dw \times (d - c), \tag{2.5}$$

where

$$\nu(w) = \frac{\beta}{\Gamma(1 - \beta)} \, w^{-1-\beta}, \quad w > 0, \quad 0 < \beta < 1. \tag{2.6}$$

The density ν satisfies[2]

$$\int_{w>0} (1 \wedge w)\nu(w) \, dw < \infty, \tag{2.7}$$

and is hence called a *Lévy measure*; the points in $F(\mathcal{R})$ are referred to as *Lévy noise*. Finally, the Lévy process Z_u is constructed from Lévy noise via

$$Z_u = \int\limits_{0 \le s \le u} \int\limits_{w > 0} F(w) N(dw, ds) := \sum_{(w,s) \in \mathcal{R}: \, s \le u} F(w). \tag{2.8}$$

Note that condition (2.7) guarantees the convergence of the sum in (2.8). Increasing Lévy processes are called subordinators in the mathematics literature. If F is chosen as in (2.4), then the above subordinator has stable marginals, i.e. its Laplace transforms are [28]

$$\langle \exp(-\lambda Z_u) \rangle = \exp(-u\lambda^\beta), \quad u \ge 0. \tag{2.9}$$

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[2] $a \wedge b := \min\{a, b\}$

Tempered β-stable subordinator. A tempered β-stable subordinator has a Lévy measure of the form

$$\nu(w) = \frac{\beta}{\Gamma(1-\beta)}\, w^{-1-\beta}\, e^{-\theta w}\, dw, \quad w > 0, \quad 0 < \beta < 1, \quad \theta \geq 0. \tag{2.10}$$

Here θ is called the tempering parameter. A positive θ decreases the occurrence of very large jumps w of Z_u, and the mean of Z_u becomes finite. If instead of (2.4) we let

$$F(w) = \inf\{z > 0 : \overline{\nu}(z) \leq w\} \tag{2.11}$$

where $\overline{\nu}(z) = \int_z^\infty \nu(w')\, dw'$ and ν is as in (2.10), then (2.8) yields a tempered stable subordinator Z_u.

Varying noise. We now assume that the coefficients β and θ of the tempered β-stable subordinator are functions of x, the location in space. To this end, we make (2.10) space-dependent via

$$\nu(w|x) = \frac{\beta(x)}{\Gamma(1-\beta(x))}\, w^{-1-\beta(x)}\, e^{-\theta(x)w}\, dw, \quad w > 0, \quad 0 < \beta(x) < 1, \quad \theta(x) \geq 0, \tag{2.12}$$

where we assume that $\beta(x) \in (0,1)$ and $\theta(x) \geq 0$ are functions of x. We also make the coefficient function (2.11) space-dependent via

$$F(w|x) = \inf\{z > 0 : \overline{\nu}(z|x) \leq w\}, \tag{2.13}$$

where $\overline{\nu}(z|x) = \int_z^\infty \nu(w'|x)\, dw'$. Then Z_u as defined in (2.8) becomes a tempered stable subordinator with variable parameters $\beta(x)$ and $\theta(x)$. We have now set up the necessary tools for the limiting Langevin dynamics (Y_u, Z_u) in space-time.

Space-time Langevin Process. Before we discuss the stochastic process scaling limit mechanism, we define the class of scaling limits (Y_u, Z_u) which is suitable for our CTRW setting. This class consists of all Langevin processes in $\mathbb{R} \times [0, \infty)$ of the form

$$dY_u = b(Y_u, Z_u)\, du + \sqrt{a(Y_u, Z_u)}\, dW_u \tag{2.14}$$

$$dZ_u = d(Y_u)\, du + \int_{w>0} F(w|Y_u)\, N(dw, du). \tag{2.15}$$

Here Y_u denotes location in space; Z_u the corresponding (physical) time; u denotes auxiliary time, resulting from the rescaled number of CTRW steps; W_u is an independent Brownian motion; b and a are space- and time-dependent spatial drift resp. diffusivity functions; $d \geq 0$ is a temporal drift coefficient function; and F is as in (2.13).

The distinction between physical time Z_u and auxiliary time u is helpful as it allows to maintain a coherent physical interpretation of the FFPE derived in Section 3. The coefficient functions $b(y,t)$ and $a(y,t)$ measure length (resp. length squared) *per unit auxiliary time*. Similarly, d measures physical time per unit auxiliary time. F is dimensionless, while $N(dw, du)$ counts the point process points per auxiliary time, and hence its two-dimensional integral over both physical time w and auxiliary time u is physical time valued. Equation (2.14) has already appeared in [29], where it was assumed that Z_u is a homogeneous β-stable subordinator (Lévy flight). The inhomogeneous form of Z_u in (2.15) is key to the generalization to CTRW scaling limits with spatially inhomogeneous waiting times.

Conditions on coefficient functions. If the coefficient functions a, b, d and F satisfy certain local Lipschitz and growth conditions [27, Chapter 6], then the bivariate Langevin equation (2.14) – (2.15) has a unique stochastic process solution. Specifying $F(w|x)$ to the form (2.13) which depends on $\beta(x)$ and $\theta(x)$, we can employ a result by [30], which states that the Lipschitz and growth conditions are satisfied if we assume that the functions a, b, d, β and θ are bounded and continuously differentiable with bounded derivative. See Section Appendix A.1 for details.

2.3. Limit theorem for the Langevin equation

Infinitesimal generator. The limiting Langevin stochastic processes (Y_u, Z_u) are Markov processes with the Feller property, which means that at any time u, their distribution depends continuously on their starting point (x, s) (see the textbooks [31, 27] for more details). The distributions of their sample trajectories are uniquely determined by their infinitesimal generator $\mathcal{A}$, which is the operator

$$\mathcal{A}f(x,s) = \lim_{u \downarrow 0} \langle f(Y_u, Z_u) \rangle_{(x,s)} / u, \quad f \in \text{dom}(\mathcal{A}). \tag{2.16}$$

Its domain $\text{dom}(\mathcal{A})$ is the space of all real-valued, continuous functions vanishing at ∞ for which the above limit exists; within this space, the twice continuously differentiable functions f are dense, and hence it suffices to consider such f. The angle brackets, $\langle \rangle_{(x,s)}$ denote the ensemble average over all trajectories starting at (x, s). The general form of $\mathcal{A}$ is given e.g. in [27, Section 6.7.1]. However, since Z_u has increasing sample paths and Y_u has continuous sample paths, $\mathcal{A}$ has the simpler form

$$
\begin{aligned}
\mathcal{A}f(x,s) = {} & b(x,s)\frac{\partial}{\partial x}f(x,s) + \frac{1}{2}a(x,s)\frac{\partial^2}{\partial x^2}f(x,s) \\
& + d(x)\frac{\partial}{\partial s}f(x,s) + \int_0^\infty [f(x, s+w) - f(x,s)]\,\nu(w|x)\,dw.
\end{aligned}
\tag{2.17}
$$

Note that by the Semi-Markov assumption, the distribution of waiting times depends on space only and not on the current time; hence $d(x)$ and $\nu(w|x)$ only depend on x and not on s.

Conditions for stochastic process convergence. First, we embed the Markov chain with kernel (2.2) from Section 2.1 into continuous time. Define a continuous time Markov chain $(Y_u^{(c)}, Z_u^{(c)})$ on $\mathbb{R} \times [0, \infty)$ as follows: at starting location (x, s), wait an amount of time which is exponentially distributed with mean $1/c$. (This time is in "auxiliary time" u, and is not a CTRW waiting time!) Then jump to the location $(x + y, s + w)$, where the distribution of (y, w) is determined by (2.2), and repeat. The resulting process has the infinitesimal generator (see e.g. [31, Proposition 17.2])

$$\mathcal{A}^{(c)}f(x,s) = c \iint [f(x+y, s+w) - f(x,s)]K^{(c)}(y,w|x,s)\,dy\,dw. \tag{2.18}$$

A key result for Feller processes is that stochastic process convergence is equivalent to the convergence of infinitesimal generators [31, Theorem 17.25]:

$$\lim_{c \to \infty} (Y_u^{(c)}, Z_u^{(c)}) = (Y_u, Z_u) \text{ if and only if } \lim_{c \to \infty} \mathcal{A}^{(c)}f(x,s) = \mathcal{A}f(x,s), \quad f \in \text{Dom}(\mathcal{A}). \tag{2.19}$$

The convergence on the left side of (2.19) is in the sense of stochastic processes with respect to the Skorokhod J_1 topology, see e.g. [32]. This mode of convergence implies *finite dimensional convergence*, i.e. the fact that for any collection of times $u_1 < \ldots < u_n$, the convergence in distribution of the random vectors

$$\left((Y_{u_1}^{(c)}, Z_{u_1}^{(c)}), \ldots, (Y_{u_n}^{(c)}, Z_{u_n}^{(c)})\right) \to \left((Y_{u_1}, Z_{u_1}), \ldots, (Y_{u_n}, Z_{u_n})\right) \tag{2.20}$$

holds.

The two theorems below connect (2.19) to the convergence of CTRWs:

Theorem 2.1. *The right side of* (2.19) *holds if the following conditions all hold:*

$$\lim_{\epsilon \downarrow 0} \lim_{c \to \infty} c \iint\limits_{|z|<\epsilon,\, 0<w<\epsilon} z K^{(c)}(z,w|x,s)\, dz\, dw = b(x,s) \tag{2.21}$$

$$\lim_{\epsilon \downarrow 0} \lim_{c \to \infty} c \iint\limits_{|z|<\epsilon,\, 0<w<\epsilon} z^2 K^{(c)}(z,w|x,s)\, dz\, dw = a(x,s) \tag{2.22}$$

$$\lim_{\epsilon \downarrow 0} \lim_{c \to \infty} c \iint\limits_{|z|<\epsilon,\, 0<w<\epsilon} w K^{(c)}(z,w|x,s)\, dz\, dw = d(x) \tag{2.23}$$

$$\lim_{c \to \infty} c \iint\limits_{|z|\geq\varepsilon \ or \ w\geq 0} g(z,w) K^{(c)}(z,w|x,s)\, dz\, dw = \int g(0,w)\nu(w|x)\, dw, \quad \varepsilon > 0, \tag{2.24}$$

where $g(z,w)$ is any bounded continuous function which vanishes in a neighbourhood of the origin.

A proof of Theorem 2.1 is given in Appendix A.2.

Theorem 2.2. *The left side of* (2.19) *implies that*

$$\lim_{c \to \infty} X_t^{(c)} = X_t, \tag{2.25}$$

where $X_t^{(c)}$ is as in (2.1), *and*

$$X_t = Y_{E(t)} \ where \ E(t) = \inf\{u : Z_u > t\}. \tag{2.26}$$

The limit in (2.25) *is in the sense of stochastic processes with respect to the Skorokhod J_1 topology* [32].

A proof of Theorem 2.2 is given in Appendix A.3. Putting together the two theorems and result (2.19), we see that conditions (2.21)–(2.24) are sufficient for the convergence (2.25).

Example 2.3. Consider the kernel

$$K^{(c)}(y,w|x,s) = \mathcal{N}\left(y|b(x,s+w)/c, a(x,s+w)/c\right)\psi^{(c)}(w|x) \tag{2.27}$$

where $\mathcal{N}(\cdot|\mu,\sigma^2)$ denotes a Gaussian probability density with mean μ and variance σ^2, and where $\psi^{(c)}(w|x)$ is related to the spatially varying tempered stable Lévy measure (2.12) as follows: Write

$$\overline{\psi}^{(c)}(w|x) = \int_w^\infty \psi^{(c)}(w'|x)\, dw' \quad \text{and} \quad \overline{\nu}(w|x) = \int_w^\infty \nu(w'|x)\, dw' \tag{2.28}$$

for the tail functions of $\psi^{(c)}(w|x)$ and $\nu(w|x)$, and let

$$\overline{\psi}^{(c)}(w|x) = 1 \wedge \left(c^{-1}\overline{\nu}\left(w - \frac{d(x)}{c}\,\bigg|\,x\right)\right), \quad w > 0 \tag{2.29}$$

where we use the convention that $\overline{\nu}(w|x) = \infty$ if $w \leq 0$. Then $\overline{\psi}^{(c)}(w|x)$ is decreasing and satisfies $\overline{\psi}^{(c)}(0|x) = 1$ and $\overline{\psi}^{(c)}(\infty|x) = 0$, so is hence the tail function of a probability density $\psi^{(c)}(w|x)$.

The CTRW corresponding to (2.27) has tempered heavy-tailed Pareto waiting times with spatially varying tail parameter $\beta(x)$ and tempering parameter $\theta(x)$. Its jumps are Gaussian with mean and variance equal to b and a, evaluated at the current location and the time of jump. Using (A.7), it is straightforward to check conditions (2.21)–(2.22). Finally, using

$$\mathcal{N}\left(y|b(x,s+w)/c, a(x,s+w)/c\right) \to \delta_{(0,0)}(y), \quad c \to \infty, \tag{2.30}$$

(2.23) follows from (A.8) and (2.24) from (A.5). More details are in Appendix A.4. Finally, we note that the construction of waiting time densities $\psi^{(c)}(w|x)$ via (2.29) from the tail function $\overline{\nu}(w|x)$ of a Lévy measure satisfies (2.23)–(2.24) for *any* Lévy measure, not just (2.12).

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2.4. Remarks on spatio-temporal scaling parameters

If the waiting time distribution $\psi^{(c)}(w|x)$ in (2.27) has a finite mean τ, a scaling $\tau \sim 1/c$ would yield a positive value for $d(x)$ in (2.23) and a vanishing Lévy measure in (2.24). If the tails of $\psi^{(c)}(w|x)$ fall off as $w^{-1-\beta}$ (not depending on x), for instance $\overline{\psi}^{(c)}(w|x) = (1 + w/\tau)^{-\beta}$, where $\beta \in (0, 1)$, then the temporal scale τ needs to satisfy the scaling $\tau \sim c^{-1/\beta}$ to yield a meaningful limit in both (2.23) and (2.24) [33, Section 3.7].

If $\nu(w|x)$ has a variable exponent $\beta(x)$ as in (2.12), it is no longer possible to satisfy the requirement (2.24) via a global scaling relation $\tau \sim c^{-1/\beta}$. For example, let the waiting times be as defined in (2.3). The reader may check easily that for any choice of constant $\beta \in (0, 1)$, the left side of (2.24) will diverge or vanish, depending on whether $\beta(x)$ in $\nu(w|x)$ is smaller or larger than β. Instead, the *local* scaling relation

$$\tau(x) \sim c^{-1/\beta(x)}, \quad c \to \infty \tag{2.31}$$

needs to hold for (2.23)–(2.24) to be satisfied. In classical derivations of the FFPE (e.g. [8]), this is problematic, because the fractional diffusion coefficient can no longer be defined as the (global) limit of χ^2/τ^β. The bivariate Langevin process $(Y_u^{(c)}, Z_u^{(c)})$, however, converges to a meaningful limit, and hence so does the CTRW $X_t^{(c)}$ (by Theorem 2.2). Moreover, as discussed below, the distribution of the CTRW limit X_t is the unique solution of the VOFFPE (3.12).

For finite variance jumps as in the examples (2.3) and (2.27), the scaling $\chi^2 \sim 1/c$ is necessary to give a meaningful limit in equations (2.21)–(2.22). For infinite variance jumps, where the tails of the jump probability density fall off as $x^{-\alpha-1}$ with $\alpha \in (0, 2)$, the scaling $\chi^\alpha \sim 1/c$ is meaningful, see e.g. [33, Section 3.8]. This article only considers finite variance jumps; in order to apply to infinite variance jumps, the right side of (2.24) needs to be replaced by $\iint g(z, w)K(z, w|x)\, dz\, dw$, where for each x, $(z, w) \mapsto K(z, w|x)$ is a Lévy density in $\mathbb{R} \times [0, \infty)$ (see e.g. Equation (3.2) in [34]).

3. Fokker-Planck Equations with Memory

In [26], FFPEs for CTRW limit processes X_t were derived from the underlying infinitesimal generator $\mathcal{A}$ of the bivariate process (Y_u, Z_u). Unlike the infinitesimal generator, the FFPE is expressed in the adjoint space with "forward" variables (y, t) (interpreted as the endpoint and ending time of a trajectory) as opposed to the "backward" variables (x, s) (i.e. the starting point and starting time; see e.g. [27, Section 3.5.3]). The adjoint of the infinitesimal generator $\mathcal{A}$ in (2.17) is

$$\mathcal{A}^* g(y, t) = -\frac{\partial}{\partial y}[b(y, t)g(y, t)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[a(y, t)g(y, t)] \tag{3.1}$$

$$-\frac{\partial}{\partial t}[d(y)g(y, t)] + \int\limits_{w > 0}[g(y, t - w) - g(y, t)]\nu(w|y)\, dw \tag{3.2}$$

$$= [\mathcal{L}^* + \mathcal{D}^*]g(y, t)$$

where $\mathcal{L}^*$ (the Fokker-Planck operator) is the spatial differential operator on the right-hand side of (3.1) and $\mathcal{D}^*$ is the temporal integro-differential operator in (3.2). Since the infinitesimal generator decomposes as $\mathcal{A} = \mathcal{L} + \mathcal{D}$, the probability distribution $P(y, t)$ of X_t satisfies the Fokker-Planck equation in [26, (5.2)]:

$$\frac{\partial P(y, t)}{\partial t} = \mathcal{L}^*\left[\frac{\partial}{\partial t}\int_0^t P(y, t - s)V(y, s)\, ds\right] + h(y, t). \tag{3.3}$$

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Here $h(y,t)$ is an initial condition (typically $h(y,t) = \delta(y)\delta(t)$) and $V(y,t)$ is a spatially dependent memory kernel defined via its Laplace transform[3]

$$\hat{V}(y,\lambda) = \int_0^\infty e^{-\lambda t}V(y,t)\,dt = \frac{1}{d(y)\lambda + \int_0^\infty (1-e^{-\lambda w})\nu(w|y)\,dw} = \frac{1}{\lambda}\frac{1}{d(y)+\hat{\bar{\nu}}(\lambda|y)}. \quad (3.4)$$

Renewal measure. As can be seen from the infinitesimal generator (2.17), if $x \in \mathbb{R}$ is held fixed, the temporal component Z_u of the Langevin process is a subordinator with infinitesimal generator

$$\mathcal{D}f(x,s) = d(x)\frac{\partial}{\partial s}f(x,s) + \int_0^\infty [f(x,t+w) - f(x,s)]\nu(w|x)\,dw. \quad (3.5)$$

By the Lévy-Khintchine theorem (see e.g. [28] for its form for subordinators), its Laplace transform is $\langle e^{-\lambda Z_u(x)} \rangle = e^{-u\Upsilon(\lambda|x)}$, where the so-called Lévy symbol $\Upsilon(\lambda|x)$ is

$$\Upsilon(\lambda|x) = d(x)\lambda + \int_0^\infty (1-e^{-\lambda w})\nu(w|x)\,dw. \quad (3.6)$$

Moreover, $1/\Upsilon(\lambda|x) = \hat{V}(x,\lambda)$ is the Laplace transform of the renewal measure density $V(x,s)$ of $Z_u(x)$, which measures the expected occupation time of $Z_u(x)$ (see e.g. [35, Eq.(15)], or [28]):

$$\int_a^b V(x,s)\,ds = \left\langle \int_0^\infty \mathbf{1}(a < Z_u(x) < b)\,du \right\rangle, \quad (3.7)$$

where $\langle \rangle$ denotes expectation or the ensemble average over all (Y_u, Z_u) trajectories. Note that the interval (a,b) is a *physical* time interval, and the (random) amount of time spent in it by Z_u is *auxiliary* time, akin to the number of steps that the walker takes in the time window (a,b). Hence $V(y,t)$ measures unit auxiliary time per unit physical time. The brackets in (3.3) are hence a quantitiy of unit "probability × auxiliary time / physical time", and the Fokker-Planck operator of unit "1 / auxiliary time", thus giving (3.3) a dimensionally consistent interpretation.

Remark 3.1. The temporal drift coefficient $d(x)$ in (3.6) has a more straightforward representation in a less general case: Assume that $\nu(w|y) \equiv \nu(w)$ and $d(x) \equiv d$ do not depend on x, and that drift $b(x,t)$ and diffusivity $a(x,t)$ are constant in t. Then by comparison of Laplace transforms, it follows that (3.3) is equivalent to

$$d\frac{\partial}{\partial t}P(y,t) + \int_0^t \left[\frac{\partial}{\partial t}P(y,t-s)\right]\bar{\nu}(s)\,ds = \mathcal{L}^*P(y,t) + H(y,t) \quad (3.8)$$

where $H(y,t) = dh(y,t) + \int h(y,t-s)\bar{\nu}(s)\,ds$. Three special cases are of interest:

- If $d=1$ and $\nu(w) \equiv 0$, then (3.8) reduces to the (non-fractional) Fokker-Planck equation.

- If $d=0$ and $\nu(w)$ is as in (2.6), then (3.8) becomes the governing equation for Hamiltonian chaos in the sense of [36], see also Eq.(24) in [9].

- If $d \geq 0$ and $\nu(w)$ is as in (2.6), then (3.8) is a special case of a "distributed order" FFPE [22], where the distribution over the orders is a mixture of the two orders 1 and β.

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[3]To see that this definition is equivalent to Proposition 5.1 in [26], note that $H(y,w) \equiv \bar{\nu}(w|y) = \int_w^\infty \nu(w|y)$, and apply integration by parts.

Inhomogeneous scaling. For spatially varying tempered subdiffusion, i.e. when $\nu(w|x)$ has the form (2.12), the Laplace transform (3.4) of $V(y,t)$ takes the form

$$\hat{V}(y,\lambda) = \frac{1}{d(y)\lambda + (\lambda + \theta(y))^{\beta(y)} - \theta(y)^{\beta(y)}}. \tag{3.9}$$

We are only able to invert the Laplace transform in the case $d \equiv 0$ and $\theta \equiv 0$, whence $V(y,t) = t^{\beta(y)-1}/\Gamma(\beta(y))$; for the remainder of this article, we restrict ourselves to this case. To make the spatio-temporal scaling in the dynamics explicit, we introduce the time units T_0 for physical time, A_0 for auxiliary time and L_0 for length, and recall that $V(y,t)$ measures unit auxiliary time per unit physical time:

$$V(y,t) = \frac{(t/T_0)^{\beta(y)-1}}{\Gamma(\beta(y))} \frac{A_0}{T_0} = \frac{A_0}{T_0^{\beta(y)}} \frac{t^{\beta(y)-1}}{\Gamma(\beta(y))}$$

and hence convolution with $V(y,t)$ becomes the Riemann-Liouville fractional derivative of spatially varying order $1 - \beta(y)$,

$$\frac{\partial}{\partial t} \int_0^t P(y,t-s)V(y,s)\,ds = {}_0D_t^{1-\beta(y)}P(y,t)\frac{A_0}{T_0^{\beta(y)}} \tag{3.10}$$

(of unit A_0/T_0). The Fokker-Planck operator is

$$\mathcal{L}^*g(y,t) = -\frac{\partial}{\partial y}[b(y,t)g(y,t)] + \frac{\partial^2}{\partial y^2}[a(y,t)g(y,t)], \tag{3.11}$$

and the FFPE (3.3) then reads

$$\frac{\partial P(y,t)}{\partial t} = \frac{\partial^2}{\partial y^2}\left[a_{\beta(y)}(y,t)\,{}_0D_t^{1-\beta(y)}P(y,t)\right] - \frac{\partial}{\partial y}\left[b_{\beta(y)}(y,t)\,{}_0D_t^{1-\beta(y)}P(y,t)\right] + h(y,t), \tag{3.12}$$

where

$$a_{\beta(y)}(y,t) = a(y,t)\frac{A_0}{T_0^{\beta(y)}}, \qquad\qquad b_{\beta(y)}(y,t) = b(y,t)\frac{A_0}{T_0^{\beta(y)}} \tag{3.13}$$

are the generalized diffusivity and drift, with units $L_0^2/T_0^{\beta(y)}$ and $L_0/T_0^{\beta(y)}$, respectively.

If the fractional exponent and the diffusivity are constant, i.e. $\beta(y) \equiv \beta \in (0,1)$ and $a(y,t) \equiv 1$ (with unit L_0^2/A_0), we retrieve the well-known FFPE (see e.g. [10, Eq.(3)]) with fractional diffusion constant $a_\beta = L_0^2/T_0^\beta$. (Note that in [10], the space- and time-dependent fractional drift $b(y,t)/T_0^\beta$ coefficient is expressed as a space- and time-dependent force $F(y,t)$, divided by a fractional friction parameter η_β.) We remind the reader that in our Langevin framework, $a(y,t)$ and $b(y,t)$ need not be constant, but can be any bounded differentiable functions with bounded derivative.

The auxiliary time scale parameter A_0 cancels in (3.13), since $a(y,t)$ is measured in L_0^2/A_0 and $b(y,t)$ is measured in L_0/A_0, for some unit of length L_0. Thus the choice of A_0 *has no effect on* (3.12). The interpretation is that the Fokker-Planck dynamics are independent of the "speed" of the Langevin process (Y_u, Z_u). This is to be expected; to see why, first observe that multiplying $\mathcal{A}^*$ by a constant $C > 0$ is equivalent to dividing A_0 by C and uniquely corresponds to the Langevin process (Y_{Cu}, Z_{Cu}). Now note that the graphs of the two processes, i.e. the sets

$$\{(u, Y_u, Z_u) : u \geq 0\} \quad \text{and} \quad \{(u, Y_{Cu}, Z_{Cu}) : u \geq 0\}$$

have different distributions, whereas the *images*

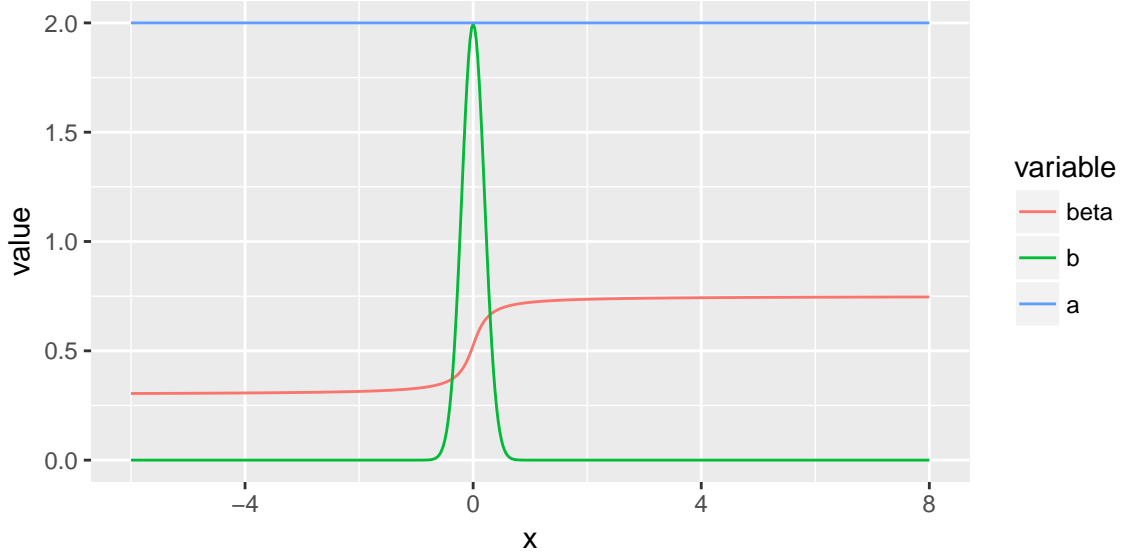$$\{(Y_u, Z_u) : u \geq 0\} \quad \text{and} \quad \{(Y_{Cu}, Z_{Cu}) : u \geq 0\},$$

Figure 1: Coefficients imitating the interface problem from [16].

i.e. the points traversed by (Y_u, Z_u) resp. (Y_{Cu}, Z_{Cu}), have identical distributions. Since the CTRW scaling limit is uniquely constructed from the image via (2.26), as opposed to from the graph, it must have the same distribution for both Langevin processes.

The choice of time scale is intrinsic in the generalized diffusivity and drift. If the unit of time T_0 is changed, e.g. increased by a factor of 10, then according to (3.13), $a_{\beta(y)}(y, t)$ and $b_{\beta(y)}(y, t)$ must be divided by $10^{\beta(y)}$. Then the CTRW limit X_{10t} at time $10t$ with the new coefficients must have the same distribution as the CTRW limit X_t at time t with the old coefficients. We check if this is the case in the next section.

4. Monte Carlo approximation of an FFPE solution

In [16], the authors study an interface problem, where for $x < 0$ the medium is more subdiffusive ($\beta = 0.3$) than for $x > 0$ (where $\beta = 0.75$). Even if a positive drift is applied at the interface, eventually all particles end up in the left half, even though the first moment $\langle x \rangle$ of the solution is always increasing. The reason for this "paradoxical" behaviour is as follows: the random walk on the infinite line is recurrent, i.e. each point is due to be visited an infinite number of times. Since the heavier-tailed waiting times in the left dominate the waiting times in the right, as $t \to \infty$ particles tend to be trapped in the left. Compared to the left, particles on the right are moving freely, and hence spread out further towards $+\infty$, hence the mean $\langle x \rangle$ is ever increasing.

We aim to reproduce this interface problem, and let the anomalous exponent be $\beta(y) = 0.525 + 0.45 * \arctan(y/0.2)/\pi$, which interpolates smoothly between $\lim_{y \to -\infty} \beta(y) = 0.3$ and $\lim_{y \to +\infty} \beta(y) = 0.75$, yet creates a sharp change at $y = 0$, see Figure 1. For the drift $b(y, t)$, we use the density of a Gaussian distribution with mean 0 and standard deviation 0.2, to approximate smoothly a delta function pulse at 0. The diffusivity $a(y, t) = 2$ is constant.

We check whether equation (3.12) is consistent if measured in two different time units, $T_0 = 1$ (second) and $T_0 = 10$ (seconds). If it is, then the probability densities of X_t at the times $1/8, 1, 8, 64$ for $T_0 = 1$ must be the same as the probability densities at the times $10/8, 10, 80, 640$ for $T_0 = 10$. Observe that changing $T_0 = 1$ to $T_0 = 10$ is equivalent to dividing both $a(x, t)$ and $b(x, t)$ by $10^{\beta(x)}$.

We have simulated 10,000 trajectories with the two coefficients scenarios above, at the scale parameter $c = 200$ (view the R-code at [37]). The value $c = 200$ was chosen so that the mean number of steps at the smallest time $t = 1/8$ was above 30. The trajectories were evaluated at the times mentioned above, and a kernel density estimator (with automated bandwidth selection) was

applied to the empirical distributions at each time, thus producing a Monte Carlo estimate of the probability distributions. The probability density estimates for T_0 are shown in the upper panel in Figure 2. The lower panel shows the difference between the densities in the two coefficient scenarios. This difference is seen to be falling within the expected range of Monte Carlo fluctuations, and is uniformly below 0.04 at all times. This fact supports the consistency of the VOFFPE over varying time scales (3.12).
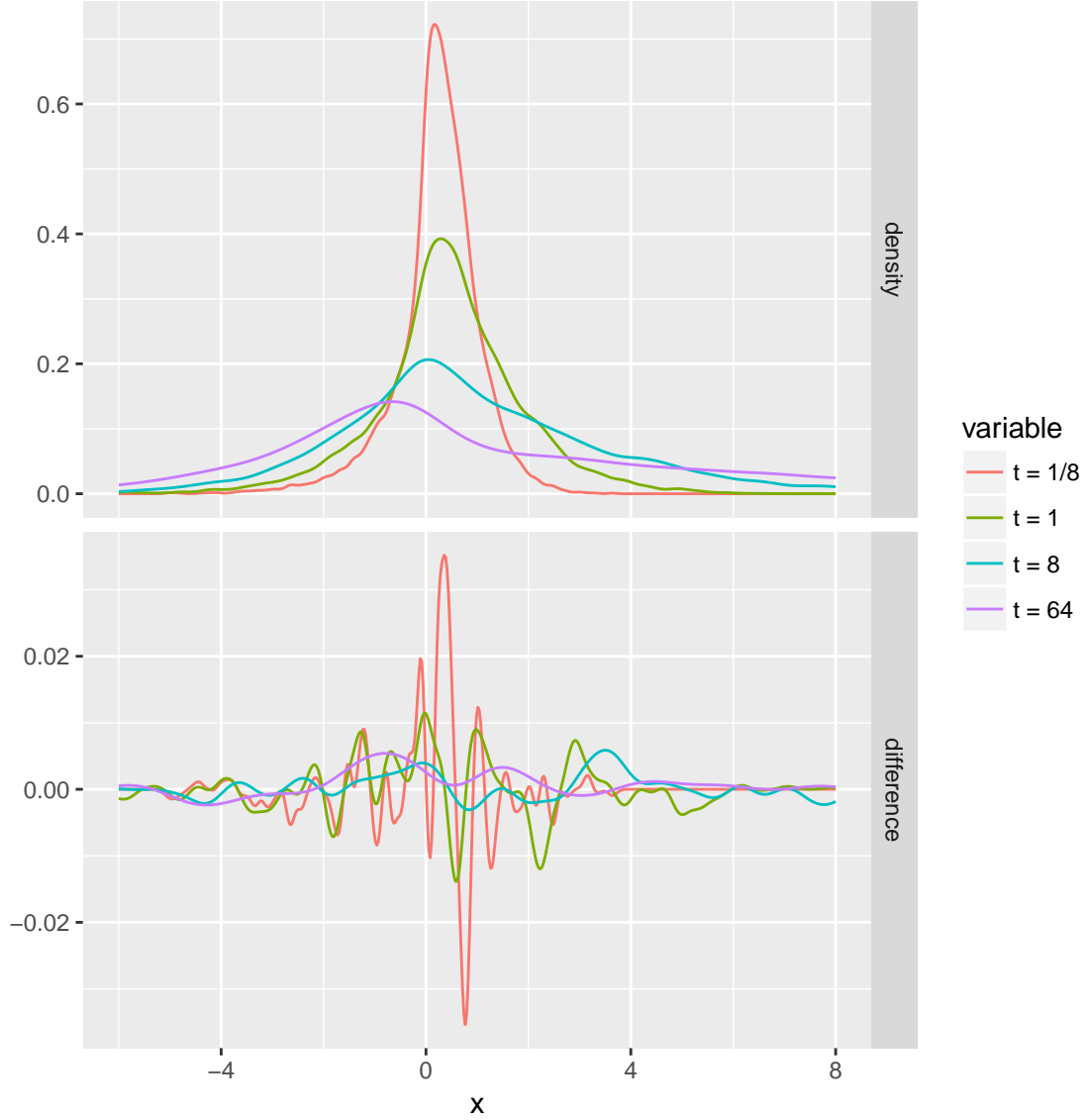

Figure 2: Comparison of the dynamics of variable fractional order, at two different time scales $T_0 = 1$ and $T_0 = 10$. The upper panel shows the probability densities for $T_0 = 1$ at different times. The lower panel shows the difference between probability densities for $T_0 = 10$ and for $T_0 = 1$.

5. Conclusion

In this article, we have employed the theory of stochastic differential equations (SDEs) with Lévy noise for the general representation of CTRW limit processes with spatially varying memory kernels. The theory allows for memory kernels of any variety, the only restriction on the type of memory kernel being the Lévy measure condition (2.7). The memory kernel can then be meaningfully varied in space, as long as certain technical assumptions on the SDE coefficient functions (local linear growth & local Lipschitz continuity) are maintained.

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Our main motivation was to give a rigorous treatment of CTRW scaling limits and FFPEs with spatially varying fractional order, and hence we have focussed on situations of this type. The discussion surrounding variable tempering and temporal drift parameters we have left for future work. Here, we have given a derivation which identifies the unique solutions of variable-order FFPE as probability distributions of CTRW scaling limits. Moreover, we have constructed a sequence of CTRW processes which converges to the solution of a variable-order FFPE. We have used this approach to approximate a solution to a variable-order FFPE via Monte Carlo simulation.

The fractional exponent β has traditionally been used to define a global fractional diffusion coefficient, of dimension Length^2 / Time$^\beta$. For spatially varying $\beta(x)$, such a global diffusion coefficient no longer exists. Instead, a factor $T_0^{-\beta(x)}$ arises in the drift and diffusion coefficients of the variable order FFPE, where T_0 corresponds to the applicable unit of time. We have given a dimensional analysis which shows that the variable order FFPE is dimensionally consistent. Finally, our simulations confirm that although the drift and diffusivity parameters seemingly depend on the chosen time scale T_0 and thus yield different governing FFPEs, the solutions of these different FFPEs are identical and hence that the theory is consistent.

Currently, there exist few numerical methods for the solution of FFPEs with memory kernels which vary in space, although such FFPEs may well be useful to the physics community, as discussed in the introduction. The algorithm from [38] may be extended to the spatially inhomogeneous setting, an idea which we will follow up on in future work.

Acknowledgements

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Appendix A. Appendix

Appendix A.1. Lipschitz and growth conditions on the coefficients

After truncation[4] at a number $l > 0$, the variable tempered stable Lévy measure (2.12) has the same form as the function g in Equation (3.1) of [30]. We note that we may ignore t in their notation (our u), since we only consider Langevin processes with dynamics that are homogeneous in u. We may also drop θ in their notation, as our Lévy noise occurs only in the one direction $(0, 1)$. Then identifying $n(x; \rho) \equiv \exp(\theta(x)w)$, $\alpha(x) \equiv \beta(x)$, $\beta(x) \equiv 0$, we have $g(x; d\rho) \equiv \nu(w|x)\, dw$. The condition $\mathcal{L}_x([0, \infty), R^d, S, (0, l])$ on n, α and β is satisfied if we assume that $\theta(x)$ and $\beta(x)$ are continuously differentiable with bounded derivative. If $\theta(x)$ and $\beta(x)$ are also bounded, then all four sub-conditions of Condition (II) are easily seen to be satisfied. Hence (s. top of page 111) part (ii) of Condition (I) is satisfied. Part (i) of Condition (I) is trivially fulfilled since $y(x; \rho) = \rho \equiv w$ is the identity function. Hence Theorem 2.1 applies, which means that $F(w|x)$ satisfies the Lipschitz and growth conditions from Theorem 1.1. Since we assume that the remaining coefficients are bounded and differentiable with bounded derivative, they also satisfy the Lipschitz and growth conditions, and hence the Langevin equations (2.14)–(2.15) have a unique solution (Y_u, Z_u).

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[4]Truncation of jumps has no effect on the existence and uniqueness of solutions to Stochastic Differential Equations [27].

Appendix A.2. Proof of Theorem 2.1

Assume that conditions (2.21)–(2.24) all hold. Split the domain of integration in (2.18) into a local part $L = \{(y, w) : |y| \leq \varepsilon,\ 0 < w \leq \varepsilon\}$ and its complement $L^{\complement}$. On L, we do a Taylor expansion and calculate

$$c \iint\limits_{L} [f(x + y, s + w) - f(x, w)] K^{(c)}(y, w | x, s)\, dy\, dw$$

$$= \iint\limits_{L} \left[y f_x(x, s) + w f_s(x, s) + \frac{1}{2} y^2 f_{xx}(x, s) + y w f_{xs}(x, s) + w^2 f_{ss}(x, s) + o(|y|^2 + |y||w| + |w|^2) \right]$$

$$c K^{(c)}(y, w | x, s)\, dy\, dw$$

$$= f_x(x, s) \iint\limits_{L} y\, c K^{(c)}(y, w | x, s)\, dy\, dw + f_s(x, s) \iint\limits_{L} w\, c K^{(c)}(y, w | x, s)\, dy\, dw$$

$$+ f_{xx}(x, s) \iint\limits_{L} y^2\, c K^{(c)}(y, w | x, s)\, dy\, dw + f_{xs}(x, s) \iint\limits_{L} y w\, c K^{(c)}(y, w | x, s)\, dy\, dw$$

$$+ f_{ss}(x, s) \iint\limits_{L} w^2\, c K^{(c)}(y, w | x, s)\, dy\, dw + o(1)$$

where $o(1)$ is a term that tends to 0 as $\varepsilon \downarrow 0$. For the yw-term, we find

$$\left| \iint\limits_{L} y w\, c K^{(c)}(y, w | x, s)\, dy\, dw \right| \leq \iint\limits_{L} |y| w\, c K^{(c)}(y, w | x, s)\, dy\, dw$$

$$\leq \varepsilon \iint\limits_{L} w\, c K^{(c)}(y, w | x, s)\, dy\, dw \to \varepsilon[d(x) + o(1)],$$

and the same bound holds for the w^2-term. We add the above and the integral over the complement $L^{\complement}$ and let $c \to \infty$, to get

$$f_x(x, s) b(x, s) + o(1) + f_s(x, s) d(x) + o(1) + f_{xx}(x, s) a(x, s) + o(1) + 2\varepsilon[d(x) + o(1)]$$

$$+ \iint\limits_{L^{\complement}} [f(x, s + w) - f(x, s)] \nu(w | x)\, dw.$$

As ε was arbitrary to begin with, we let $\varepsilon \downarrow 0$, and the above equals $\mathcal{A} f(x, s)$.

Appendix A.3. Proof of Theorem 2.2

Write (y, z) for a generic path in $\mathbb{R} \times [0, \infty)$, which is such that z is unbounded and non-decreasing. By Proposition 2.3 in [39], the following path mapping is continuous on the subset of all paths (y, z) which are such that z is strictly increasing: $(y, z) \mapsto (y_- \circ z_-^{-1})_+$, which composes the left-continuous version of y with the left-continuous version of the inverse

$$z^{-1}(t) = \inf\{u : z(u) > t\} \tag{A.1}$$

and changes the resulting composition back to right-continuous again. This path mapping maps the continuous time Markov chain paths $u \mapsto (Y_u^{(c)}, Z_u^{(c)})$ to the CTRW paths $t \mapsto X_t^{(c)}$ (Lemma 3.5 in [39]). Since the limiting process (Y_u, Z_u) has unbounded and strictly increasing Z_u, with probability 1 its samples paths lie in the set of continuity of the above path mapping. The continuous mapping theorem [40] applies, saying that $\lim_{c \to \infty}(Y_u^{(c)}, Z_u^{(c)}) = (Y_u, Z_u)$ with respect to J_1 implies $\lim_{c \to \infty} X_t^{(c)} = (Y_- \circ Z_-^{-1})_{t+}$ with respect to J_1. Note that Y_u has continuous paths, and so does Z_u^{-1} because Z_u is strictly increasing. But then $(Y_- \circ Z_-^{-1})_{t+} = Y \circ Z^{-1}(t) = Y_{E(t)}$.

Appendix A.4. Calculation for Example 2.3

First, we note that

$$\bar{\nu}(w|x) \sim w^{-\beta(x)}/\Gamma(1 - \beta(x)), \quad w \downarrow 0, \tag{A.2}$$

and thus the inverse function $\bar{\nu}^{-1}(c|x)$ of $z \mapsto \bar{\nu}(z|x)$ satisfies

$$\bar{\nu}(z|x) \leq c \Leftrightarrow z \geq \bar{\nu}^{-1}(c|x) \sim (\Gamma(1 - \beta(x))c)^{-1/\beta(x)}, \tag{A.3}$$

Hence from the definition (2.29) of $\overline{\psi}^{(c)}(w|x)$,

$$c\psi^{(c)}(w|x) = \begin{cases} \nu(w|x) & \text{if } w > C(x,c) := \bar{\nu}^{-1}(c|x) + d(x)/c, \\ 0 & \text{if } 0 < w \leq C(x,c). \end{cases} \tag{A.4}$$

Since $C(x,c) \downarrow 0$ as $c \to \infty$, for any $w > 0$

$$c\psi^{(c)}(w|x) \to \nu(w|x), \quad c\overline{\psi}^{(c)}(w|x) \to \bar{\nu}(w|x), \quad c \to \infty. \tag{A.5}$$

We also note that $\psi^{(c)}(w|x)$ is a probability density on the positive numbers. Due to (A.4), for any $w > 0$, and large enough c,

$$\psi^{(c)}(w|x) = \nu(w|x)/c \to 0, \quad c \to \infty. \tag{A.6}$$

Hence as $c \to \infty$, the probability mass of the density $\psi^{(c)}(w|x)$ must get concentrated near 0, i.e.

$$\int_0^\infty f(w)\psi^{(c)}(w|x) \to f(0) \tag{A.7}$$

for any bounded continuous f.

Finally, for any $\varepsilon > 0$,

$$\begin{aligned}
c \int_0^\varepsilon w\psi^{(c)}(w|x)\,dw &= c \int_{C(x,c)}^\varepsilon w\psi^{(c)}(w|x)\,dw \\
&= -c\left[w\overline{\psi}^{(c)}(w|x)\right]_{C(x,c)}^\varepsilon + c\int_{C(x,c)}^\varepsilon \overline{\psi}^{(c)}(w|x)\,dw \\
&= -c\varepsilon\overline{\psi}^{(c)}(\varepsilon|x) + cC(x,c) \times 1 + c\int_{C(x,c)}^\varepsilon \overline{\psi}^{(c)}(w|x)\,dw \\
&\to \varepsilon\bar{\nu}(\varepsilon|x) + d(x) + \int_0^\varepsilon \bar{\nu}(w|x)\,dw \\
&= d(x) + \mathcal{O}(\varepsilon^{1-\beta(x)}).
\end{aligned} \tag{A.8}$$