A method for estimating the rigid-body rotation of a sphere from an optic-flow field

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January 7, 2006

This set of notes outlines a method for estimating the rigid-body rotation of a sphere using the optic-flow field captured by a camera. We assume that the sphere, with radius R, is rotating about its center which is located at the position $(0, 0, d)^T$. The angular velocity of the sphere is given by $\Omega = (\omega_x, \omega_y, \omega_z)^T$. A pinhole camera model with the focal point located at the origin is assumed. The camera has focal length f and is assumed to be pointing straight down the z axis. Thus the center of projection $(0, 0)^T$ in the image plane is located at $(0, 0, f)^T$ in world coordinates and the image plane is parallel the xy-plane. The image plane coordinates are written $(u, v)^T$ whereas the world coordinates are written $(x, y, z)^T$.

The general strategy is to express the optic-flow field of the camera in terms of the image plane coordinates $(u, v)^T$, the camera focal length f, the distance to the sphere d, the radius of the sphere R, and the angular velocity of the sphere. The image plane coordinates $(u, v)^T$ and the parameters f, d, and R are all known quantities. Thus the only unknowns are the three components of the angular velocity vector which we can solve for provided we have enough data. This strategy leads to an over-determined system of linear equations which can be solved in least squares sense via singular value decomposition.

We will begin by deriving an expression relating points on the surface of the sphere with their projections on the image plane. The relationship between points in the image plane and points in the world coordinate system is given by:

$$u = \frac{xf}{z} \qquad \text{and} \qquad v = \frac{yf}{z}. \tag{1}$$

This relationship can be inverted to express the x, y position of a point in world coordinates in terms of its projection on the image plane and its distance along the z axis:

$$x = \frac{uz}{f} \qquad \text{and} \qquad y = \frac{vz}{f}. \tag{2}$$

The equation for the surface of a sphere with radius R whose center is located at the position $(0, 0, d)^T$ is given by

$$x^2 + y^2 + (z - d)^2 - R^2 = 0. \tag{3}$$

Inserting the expressions for x and y from (2) into equation (3) results in a quadratic equation for z in terms of f, u, v, d, and R:

$$\left(1 + \frac{u^2}{f^2} + \frac{v^2}{f^2}\right) z^2 - 2dz + d^2 - R^2 = 0. \tag{4}$$

We are interested only in solutions which are real and on the near side of the sphere, i.e., where $z < d$. A necessary and sufficient condition for the existence of real solutions is that

$$R^2 \left(1 + \frac{u^2}{f^2} + \frac{v^2}{f^2}\right) - d^2 \left(\frac{u^2}{f^2} + \frac{v^2}{f^2}\right) > 0. \tag{5}$$

The near side solution for z is then given by

$$z = \frac{d - \sqrt{d^2 - \left(1 + \frac{u^2}{f^2} + \frac{v^2}{f^2}\right)(d^2 - R^2)}}{\left(1 + \frac{u^2}{f^2} + \frac{v^2}{f^2}\right)}. \tag{6}$$

Inserting the equation (6) in to the equations relating world and image coordinates (2) yields expressions for x and y in terms of f, u, v, d, and R:

$$x = \frac{u}{f}\left[\frac{d - \sqrt{d^2 - \left(1 + \frac{u^2}{f^2} + \frac{v^2}{f^2}\right)(d^2 - R^2)}}{\left(1 + \frac{u^2}{f^2} + \frac{v^2}{f^2}\right)}\right] \tag{7}$$

and

$$y = \frac{v}{f}\left[\frac{d - \sqrt{d^2 - \left(1 + \frac{u^2}{f^2} + \frac{v^2}{f^2}\right)(d^2 - R^2)}}{\left(1 + \frac{u^2}{f^2} + \frac{v^2}{f^2}\right)}\right]. \tag{8}$$

Thus we have accomplished our first goal of finding an explicit relationship between points on the surface of the sphere and points in the image plane.

The next step is to find expressions for the image plane velocities $\dot{u}$ and $\dot{v}$ for points on the surface of the sphere in terms of the angular velocity of the sphere. Differentiating the equations in (1) yields:

$$\dot{u} = f\left(\frac{\dot{x}z - x\dot{z}}{z^2}\right) \qquad \text{and} \qquad \dot{v} = f\left(\frac{\dot{y}z - y\dot{z}}{z^2}\right). \tag{9}$$

We want to replace the $\dot{x}$, $\dot{y}$ and $\dot{z}$ terms in (9) by expressions involving x, y, z, and the components of the angular velocity vector $\Omega = (\omega_x, \omega_y, \omega_z)^T$. The velocity of a point on the sphere is given by $\Omega \times \mathbf{r}$ where $\mathbf{r} = (x, y, z - d)^T$ thus:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ x & y & z - d \end{vmatrix} = \begin{pmatrix} \omega_y(z - d) - \omega_z y \\ \omega_z x - \omega_x(z - d) \\ \omega_x y - \omega_y x \end{pmatrix} \tag{10}$$

Inserting (10) into (9) yields:

$$\dot{u} = f\left\{\frac{[\omega_y(z-d) - \omega_z y]\, z - x\,(\omega_x y - \omega_y x)}{z^2}\right\} \tag{11}$$

and

$$\dot{v} = f\left\{\frac{[\omega_z x - \omega_x(z-d)]\, z - y\,(\omega_x y - \omega_y x)}{z^2}\right\}. \tag{12}$$

We may rewrite equations (11) and (12) in matrix form as follows:

$$\frac{1}{f}\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{pmatrix} \frac{(z-d)}{z} + \frac{x^2}{z^2} & -\frac{xy}{z^2} & -\frac{y}{z} \\ -\frac{(z-d)}{z} - \frac{y^2}{z^2} & \frac{xy}{z^2} & \frac{x}{z}\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}. \tag{13}$$

Recall that for a given point in the image plane $(u, v)^T$ the corresponding point on the surface of the sphere $(x, y, z)^T$ may be expressed in terms of the known parameters f, u, v, d, and R using equations (6), (7), (8). Thus the x, y, and z in equation (13) are known quantities. Although this system of equations for ω_x, ω_y, and ω_z is under-determined we may generate more equations by choosing further points in the image plane in order to arrive at a system with sufficient rank to solve.

The velocity vector at each point in the image plane generates a pair of linear equations. Suppose that we have n such points, (u_i, v_i) where $i = 1, \ldots, n$. We then have an over-determined system of $2n$ linear equations for our 3 unknowns as follows:

$$\begin{pmatrix}\dot{u}_1\\ \dot{v}_1\\ \vdots\\ \dot{u}_n\\ \dot{v}_n\end{pmatrix} = \begin{pmatrix} \frac{(z_1-d)}{z_1} + \frac{x_1^2}{z_1^2} & -\frac{x_1 y_1}{z_1^2} & -\frac{y_1}{z_1} \\ -\frac{(z_1-d)}{z_1} - \frac{y_1^2}{z_1^2} & \frac{x_1 y_1}{z_1^2} & \frac{x_1}{z_1} \\ \vdots & \vdots & \vdots \\ \frac{(z_n-d)}{z_n} + \frac{x_n^2}{z_n^2} & -\frac{x_n y_n}{z_n^2} & -\frac{y_n}{z_n} \\ -\frac{(z_n-d)}{z_n} - \frac{y_n^2}{z_n^2} & \frac{x_n y_n}{z_n^2} & \frac{x_n}{z_n}\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} \tag{14}$$

where the x_i, y_i and z_i are the solutions to equations (6), (7), (8) for u_i and v_i. This system of equation may readily be solved for the unknown ω_x, ω_y, and ω_z in the least-squares sense using singular value decomposition.

In practice a slightly more sophisticated camera model might be required or at least one that includes some scaling factors for relating world and image plane coordinates. If it is only scaling factors then this method should work essentially unaltered. However, if a nonlinear camera model is required, then this method could serve as a method of generating the initial guess for an iterative method used to solve the nonlinear system. The system of equations for a nonlinear camera model can derived in an analogous manner to the method presented here. $\square$