$2a - b + 2c,\ 2b - 2c + 6d,\ 2c - 3d,\ 2d$

Prvi međuispit iz Linearne algebre

4. travnja 2013.

1. **[3 boda]** (a) Zadani su vektori $(1,1,0)$, $(2,1,4)$ i $(3,3,a)$ s koeficijentima iz polja $\mathbb{Z}_5$. Odredite sve $a \in \mathbb{Z}_5 = \{0,1,2,3,4\}$ za koje su ta tri vektora linearno nezavisna.

   (b) Zadana je matrica $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$ s koeficijentima iz polja $\mathbb{Z}_5$. Riješite matričnu jednadžbu $\mathbf{AX} = \mathbf{B}$, ako je $\mathbf{B} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix}$.

2. **[3 boda]** Neka je P_3 vektorski prostor svih polinoma $p = p(t)$ stupnja najviše 3. Pokažite da podskup

$$X = \{p(t) \in P_3 : p''(t) - p'(t) + 2p(t) = 0,\ \forall t \in \mathbb{R}\}$$

   čini vektorski podprostor prostora polinoma P_3. Nađite mu neku bazu i odredite dimenziju.

3. **[3 boda]** Neka je X unitarni vektorski prostor nad poljem realnih brojeva, te neka su $\mathbf{e}_1, \ldots, \mathbf{e}_k$ vektori koji su svi međusobno ortogonalni i različiti od nul-vektora.

   (a) Dokažite da su ti vektori linearno nezavisni.

   (b) Neka je $\mathbf{x}$ bilo koji zadani vektor u X. Odredite koeficijente $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ tako da broj

$$\|\mathbf{x} - (\lambda_1 \mathbf{e}_1 + \cdots + \lambda_k \mathbf{e}_k)\|$$

   bude minimalan.

4. **[5 bodova]** Zadan je vektorski prostor $\mathbb{R}^5$ s uobičajenim skalarnim produktom. Neka je $X = L(\mathbf{a}, \mathbf{b})$ njegov podprostor, pri čemu je $\mathbf{a} = (1,1,1,0,0)$ i $\mathbf{b} = (1,1,-1,0,0)$.

   (a) Odredite ortogonalnu projekciju vektora $\mathbf{c} = (1,1,1,1,1)$ na X. a · b nisu ortogon.

   (b) Odredite neku bazu u ortogonalnom komplementu $X^\perp$.

5. **[3 boda]** Zadani su podprostori

$$M = \{\mathbf{x} \in \mathbb{R}^3 : x_1 + x_2 - x_3 = 0\},$$
$$N = L((1,0,0,),(1,-1,0))$$

   od $\mathbb{R}^3$. Odredite neke baze i pripadajuće dimenzije prostora $M \cap N$ i $M + N$.

6. **[5 bodova]** Zadane su funkcije $f(x) = x$ i $g_\mu(x) = x^2 + \mu$ u Lebesgueovom prostoru $L^2(-1,1)$.

   (a) Odredite uvjet na parametar $\mu \in \mathbb{R}$ tako da funkcije f i g_μ budu međusobno okomite u tom prostoru.

   (b) Neka je $h(x) = |x|$, gdje je $x \in (-1,1)$. Provjerite da je $h \in L^2(-1,1)$ i izračunajte $\|h\|$.

   (c) Neka je parametar μ iz (a) dijela zadatka. Odredite funkciju $e = e(x)$ u prostoru $L(f, g_\mu)$ tako da bude najbliža funkciji h s obzirom na normu u prostoru $L^2(-1,1)$.

Okrenite!

b5)

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} \quad \mathbb{Z}_5 \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} \qquad \det A = 8 - 6 = 2$$

$$AX = B \quad / \cdot A^{-1}$$

$$X = A^{-1} B$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}^T = 2^{-1} \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} =$$

0 1 2 3 4

$$2 \cdot 2^{-1} = 1$$

$$2 \cdot \underline{\underline{3}} = 1$$

$$= 3 \cdot \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 2 \\ 14 & 8 & 6 \end{bmatrix}$$

$$= / \mathbb{Z}_5 / =$$

$$= \begin{bmatrix} 0 & 4 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

(2.) $\mathcal{P}_3$

$$X = \{ p(t) \in P_3 : p''(t) - p'(t) + 2p(t) = 0, \ \forall t \in \mathbb{R} \}$$

$p(t) = a + bt + ct^2 + dt^3$

$p'(t) = b + 2ct + 3dt^2$

$p''(t) = 2c + 6dt$

$2c + 6dt - b - 2ct - 3dt^2 + 2a + 2bt + 2ct^2 + 2dt^3$

$= (2a - b + 2c) + (2b - 2c + 6d)t + (2c - 3d)t^2 + (2d)t^3$

$\vec{p} = (2a - b + 2c, 2b - 2c + 6d, 2c - 3d, 2d)$

$p_1, p_2 \in X$

DOKAZ: 1) $p_1 + p_2 = (2a_1 - b_1 + 2c_1, 2b_1 - 2c_1 + 6d_1, 2c_1 - 3d_1, 2d_1) + (2a_2 - b_2 + 2c_2, 2b_2 - 2c_2 + 6d_2, 2c_2 - 3d_2, 2d_2)$

$= (2(a_1 + a_2) - (b_1 + b_2) + 2(c_1 + c_2), 2(b_1 + b_2) - 2(c_1 + c_2) + 6(d_1 + d_2), 2(c_1 + c_2) - 3(d_1 + d_2), 2(d_1 + d_2))$

$= \Big/ \begin{matrix} a_1 + a_2 = A \\ b_1 + b_2 = B \\ c_1 + c_2 = C \\ d_1 + d_2 = D \end{matrix} \Big/ = (2A - B + 2C, 2B - 2C + 6D, 2C - 3D, 2D) \in X$ ✓

2) $\alpha p = (2\alpha a - \alpha b + 2\alpha c, 2\alpha b - 2\alpha c + 6\alpha d, 2\alpha c - 3\alpha d, 2\alpha d) =$

$\alpha \in \mathbb{R}$

$p \in X$

$= \Big/ \begin{matrix} \alpha a = A \\ \alpha b = B \\ \alpha c = C \\ \alpha d = D \end{matrix} \Big/ = (2A - B + 2C, 2B - 2C + 6D, 2C - 3D, 2D) \in X$ ✓

iz (1) i (2) $\Rightarrow$ X je vekt. potpr. na $\mathcal{P}_3$ //

BAZA + DIM

$$2a - b + 2c + 0 = 0$$
$$0 + 2b - 2c + 6d = 0$$
$$\left.\begin{aligned} 2c - 3d &= 0 \\ 2d &= 0 \end{aligned}\right\} \Rightarrow c, d = 0$$

$$\Downarrow$$

$$\left.\begin{aligned} 2a - b &= 0 \\ 2b &= 0 \end{aligned}\right. \Rightarrow \begin{aligned} a &= 0 \\ b &= 0 \end{aligned}$$

$X \rightarrow$ samo nul-vektor

$\dim X = 0$

baza $\rightarrow$ nema

2 3 a)

$$\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_k e_k = 0 \quad / (\,|e_1)$$

$$(\lambda_1 e_1|e_1) + (\lambda_2 e_2|e_1) + \dots = 0$$

1 $$\lambda_1 (e_1|e_1) + \lambda_2 \underbrace{(e_2|e_1)}_{0} + \dots = 0$$

1

$$\lambda_1 \|e_1\|^2 = 0$$

1 $$\Downarrow$$

$$\lambda_1 = 0$$

postupak se ponavlja za $e_2, \dots, e_k$

slijedi da su $\lambda_1, \dots, \lambda_k$ redom $= 0$.

Time je dokazana lin. nezavisnost

b) $\|x - (\lambda_1 e_1 + \dots + \lambda_k e_k)\|$

Broj je minimalan za $\lambda_1 e_1 + \dots + \lambda_k e_k = x$ jer

$\|x - x\| = \|0\| = 0$

$$\lambda_1 = \frac{(x|e_1)}{\|e_1\|^2} e_1$$

$\vdots$

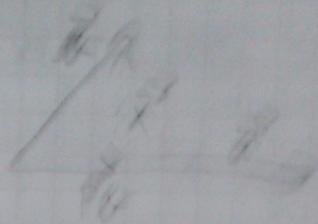
$$\lambda_i = \frac{(x|e_i)}{\|e_i\|^2} e_i$$

$\vdots$

$$\lambda_k = \frac{(x|e_k)}{\|e_k\|^2} e_k$$

(4) $\vec{a} = (1,1,1,0,0)$

$\vec{b} = (1,1,-1,0,0)$



$P_b = \frac{(b|a)}{\|a\|^2} a$

$= \frac{1+1-1}{3}(1,1,1,0,0)$

$= (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$

$\vec{b} = \vec{a}_{\|} + \vec{x}$

$x = b - P_b$

$= (1,1,-1,0,0) - (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$

$= (\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}, 0, 0)$

$c = (1,1,1,1,1)$

a) $\frac{(c|a)}{\|a\|^2} a + \frac{(c|x)}{\|x\|^2} x = \frac{3}{3}(1,1,1,0,0) + \underbrace{\frac{\frac{2}{3}+\frac{2}{3}-\frac{4}{3}}{\|x\|^2} x}_{0}$ $\quad \frac{2}{3}+\frac{2}{3}-\frac{4}{3} = 0$

$= (1,1,1,0,0)$

b) $X^{\perp}$

$\forall y \in X^{\perp} \quad (y|a) = 0 \qquad y_1 + y_2 + y_3 + 0 \cdot y_4 + 0 \cdot y_5 = 0$

$y = (y_1, y_2, y_3, y_4, y_5) \quad (y|b) = 0 \qquad y_1 + y_2 - y_3 + 0 \cdot y_4 + 0 \cdot y_5 = 0$

$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} y_1 = -y_2 \\ y_3 = 0 \end{array}$

$y = (-y_2, y_2, 0, y_3, y_4) = y_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{e}, \vec{k}, \vec{g}$ čine bazu jer:

1) razapinju $X^{\perp}$

2) su linearno nezavisni

$$\lambda_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \iff \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \\ \lambda_3 &= 0 \end{aligned}$$

5) $x_1+x_2-x_3=0$

$L\left((1,0,0),\ (1,-1,0)\right)$

<u>M</u>

$$(x_1,x_2,x_3)=(x_2-x_3,x_2,x_3)=x_2\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{m_1}+x_3\underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{m_2}$$

m_1, m_2 čine bazu za M jer:

1) razapinju M

2) su lin. nezav. $\longrightarrow \quad x_2\begin{bmatrix}1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}=0 \iff x_2=0,\ x_3=0$

$M\cap N$

$$\lambda_1\begin{bmatrix}1\\1\\0\end{bmatrix}+\lambda_2\begin{bmatrix}-1\\0\\1\end{bmatrix}=\lambda_3\begin{bmatrix}1\\0\\0\end{bmatrix}-\lambda_4\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$(-1)\begin{bmatrix}1&-1&-1&-1&\vdots&0\\1&0&0&1&\vdots&0\\0&1&0&0&\vdots&0\end{bmatrix}\sim\begin{bmatrix}1&-1&-1&-1&\vdots&0\\0&1&1&2&\vdots&0\\0&1&0&0&\vdots&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&1&\vdots&0\\0&1&1&2&\vdots&0\\0&0&-1&-2&\vdots&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&0&1&\vdots&0\\0&1&0&0&\vdots&0\\0&0&1&2&\vdots&0\end{bmatrix}\Rightarrow\begin{array}{l}\lambda_1=-\lambda_4\\\lambda_2=0\\\lambda_3=-2\lambda_4\end{array}\Rightarrow \sigma\in M\cap N$$

$$\sigma=-2\lambda_4\begin{bmatrix}1\\0\\0\end{bmatrix}+\lambda_4\begin{bmatrix}1\\-1\\0\end{bmatrix}=\lambda_4\begin{bmatrix}-2+1\\0-1\\0\end{bmatrix}=\lambda_4\underbrace{\begin{bmatrix}-1\\-1\\0\end{bmatrix}}_{\vec{e}}$$

$\vec{e}$ je baza za $M\cap N$ jer:

1) razapinje $M\cap N$

2) je linearn. nezav. (trivijalno)

$\hookrightarrow \dim M\cap N=1$

(6) $f(x)=x$

$g_\mu(x)=x^2+\mu$

a) $\int_{-1}^{1} x\cdot(x^2+\mu)\,dx=0$

$$\int_{-1}^{1}(x^3+\mu x)\,dx=\left(\frac{1}{4}x^4+\frac{1}{2}\mu x^2\right)\Big|_{-1}^{1}=0$$

$$\frac{1}{4}\cdot 1+\frac{1}{2}\mu-\left(\frac{1}{4}(-1)^2+\frac{1}{2}\mu(-1)^2\right)=\frac{1}{4}+\frac{1}{2}\mu-\frac{1}{4}-\frac{1}{2}\mu=0$$

$f\perp g$ za svaki $\mu\in\mathbb{R}$

b) $h(x)=|x|$

$$\int_{-1}^{1}h^2(x)\,dx=\int_{-1}^{0}(-x)^2\,dx+\int_{0}^{1}x^2\,dx=\frac{1}{3}x^3\Big|_{-1}^{0}+\frac{1}{3}x^3\Big|_{0}^{1}=$$

$$=\cancel{\frac{1}{3}\cdot 0}+\frac{1}{3}\cdot 1+\frac{1}{3}\cdot 1-\cancel{\frac{1}{3}\cdot 0}=\frac{2}{3}<\infty$$

$$\|h\|=\sqrt{\int_{-1}^{1}h^2(x)\,dx}=\sqrt{\frac{2}{3}}$$

$2a = b + 2c,\ 3b = 2c + 6d,\ 2c - 3d \geq 1$

(17) a)

1) $A(\alpha x + \beta x') = (\alpha x_1 + \alpha 3x_2 - 2\alpha x_3 + \beta x_1' + 3\beta x_2' - 2\beta x_3',$

$\alpha x_3 + \beta x_3',$

$\alpha x_1 + 3\alpha x_2 + \beta x_1' + 3\beta x_2',$

$2\alpha x_1 + 6\alpha x_2 - 5\alpha x_3 + 2x_1'\beta + 6x_2'\beta - 5x_3'\beta)$

$\alpha(x_1 + 3x_2 - 2x_3,\ x_3,\ x_1 + 3x_2,\ 2x_1 + 6x_2 - 5x_3) + \beta(x_1' + 3x_2' - 2x_3',\ x_3',$
$x_1' + 3x_2',\ 2x_1' + 6x_2' - 5x_3')$

$= \alpha A(x) + \beta A(x')$ ✓

b) $x_1' = x_1 + 3x_2 - 2x_3$

$x_2' = x_3$

$x_3' = x_1 + 3x_2$

$x_4' = 2x_1 + 6x_2 - 5x_3$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 6 & -5 \end{bmatrix}$$

c) $$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 = -3x_2$

$x_3 = 0$

$$(-3x_2, x_2, 0) = x_2 \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{a}}$$

$\vec{a}$ je baza jezgre jer: 1) razapinje jezgru
2) $\vec{a}$ je linearno nezavisan (trivijalno)