① a) $\alpha \in \mathbb{R}$ td. $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \alpha \end{bmatrix}$ ortogonalna

$A^{-1} = A^t / \cdot A$

$I = A \cdot A^t \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \alpha \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \alpha \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} + \frac{1}{4} & \frac{\sqrt{3}}{4} - \frac{\alpha}{2} \\ \frac{\sqrt{3}}{4} - \frac{\alpha}{2} & \frac{1}{4} + \alpha^2 \end{bmatrix} \Rightarrow \begin{matrix} \frac{1}{4} + \alpha^2 = 1 \Rightarrow \alpha = \pm \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{4} - \frac{\alpha}{2} = 0 \Rightarrow \boxed{\alpha = \frac{\sqrt{3}}{2}} \end{matrix}$$

b) Determinanta ortogonalne matrice?

$A^{-1} = A^t \Rightarrow \det A^{-1} = \det A^t \Rightarrow \frac{1}{\det A} = \det A / \cdot \det A$

$\Rightarrow 1 = (\det A)^2 \Rightarrow \boxed{\det A = \pm 1}$

c) A-ortogonalna $\Rightarrow A^{-1} = A^t$

B-simetrična $\Rightarrow B^t = B$

dokazati $A^{-1}BA$ simetrična

Treba dokazati da je $(A^{-1}BA)^t = A^{-1}BA$

$(A^{-1}BA)^t = A^t B^t (A^{-1})^t = A^{-1} B (A^t)^t = A^{-1}BA$

QED

② a) Indukcija : • za $n=2$ $\det A = \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$

• pretpostavimo da tvrdnja vrijedi za sve determinante reda n.

• Neka je A reda $n+1$ s dva jednaka retka (i-ti, j-ti). Razvijamo determinantu po bilo kojem od preostalih redaka, npr. k-tom ($k \neq i$, $k \neq j$)

$\det A = \sum_{l=1}^{n} (-1)^{k+l} a_{kl} M_{kl}$ gdje je M_{kl} determinanta matrice reda n čija su dva retka jednaka, a ona je po pretpostavci 0 pa je $\det A = 0$. QED

b) Treba pokazati da vrijedi $\begin{vmatrix} a_1' + a_1'' \\ a_2 \\ \vdots \\ a_n \end{vmatrix} = \begin{vmatrix} a_1' \\ a_2 \\ \vdots \\ a_n \end{vmatrix} + \begin{vmatrix} a_1'' \\ a_2 \\ \\ a_n \end{vmatrix}$,

odnosno svaki element prvog retka prikazan je u obliku sume dvaju elemenata

$\det A = \sum_{j=1}^{n} (a_{1j}' + a_{1j}'') A_{1j} = \sum_{j=1}^{n} a_{1j}' A_{1j} + \sum_{j=1}^{n} a_{1j}'' A_{1j} = \det A' + \det A''$ QED

c) Bez smanjenja općenitosti možemo pretpostanti da se tvrdnja odnosi na prvi i drugi redak matrice:

$$\begin{vmatrix} a_1 \\ \lambda a_1 + a_2 \\ \vdots \end{vmatrix} \underset{\text{prethodna tvrdnja}}{=} \begin{vmatrix} a_1 \\ \lambda a_1 \\ \vdots \end{vmatrix} + \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix} = \lambda \begin{vmatrix} a_1 \\ a_1 \\ \vdots \end{vmatrix} + \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix} = 0 + \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix} = \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix}$$

QED

③ $$\begin{cases} \alpha x + y + z = -1 \\ \alpha x + \alpha y + z = 1 \\ x + \alpha y + \alpha z = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} \alpha & 1 & 1 & -1 \\ \alpha & \alpha & 1 & 1 \\ 1 & \alpha & \alpha & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & \alpha & \alpha & 1 \\ \alpha & \alpha & 1 & 1 \\ \alpha & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & \alpha & \alpha & 1 \\ 0 & \alpha-\alpha^2 & 1-\alpha^2 & 1-\alpha \\ 0 & 1-\alpha^2 & 1-\alpha^2 & -1-\alpha \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & \alpha & \alpha & 1 \\ 0 & \alpha(1-\alpha) & (1-\alpha)(1+\alpha) & 1-\alpha \\ 0 & (1-\alpha)(1+\alpha) & (1-\alpha)(1+\alpha) & -(1+\alpha) \end{array}\right]$$

- $1-\alpha=0 \Rightarrow \boxed{\alpha=1}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$ sustav nema rješenja

- $\boxed{\alpha=0}$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right]$ jedinstveno rješenje $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

- $1+\alpha=0 \Rightarrow \boxed{\alpha=-1}$ $\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & \textcircled{-2} & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ beskonačno mnogo rješenja

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \lambda \in \mathbb{R}$$

- $\boxed{\alpha \neq -1, 0, 1}$ drugi redak dijelimo s $(1-\alpha)$, a treći s $(1+\alpha)$

$$\sim \left[\begin{array}{ccc|c} 1 & \alpha & \alpha & 1 \\ 0 & \alpha & 1+\alpha & 1 \\ 0 & 1-\alpha & 1-\alpha & -1 \end{array}\right] \curvearrowright + \sim \left[\begin{array}{ccc|c} 1 & \alpha & \alpha & 1 \\ 0 & \textcircled{1} & 2 & 0 \\ 0 & 1-\alpha & 1-\alpha & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\alpha & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \alpha-1 & -1 \end{array}\right] /:\alpha-1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\alpha & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \textcircled{1} & \frac{1}{1-\alpha} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{1-\alpha} \\ 0 & 1 & 0 & \frac{-2}{1-\alpha} \\ 0 & 0 & 1 & \frac{1}{1-\alpha} \end{array}\right]$$

jedinstveno rješenje oblika

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\alpha} \\ \frac{-2}{1-\alpha} \\ \frac{1}{1-\alpha} \end{bmatrix}$$

④ $a_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad a_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad a_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad a_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

najveći mogući broj linearno nezavisnih vektora?

Vektore možemo posložiti u stupce matrice, tad će najveći broj linearno nezavisnih vektora biti jednak rangu te matrice

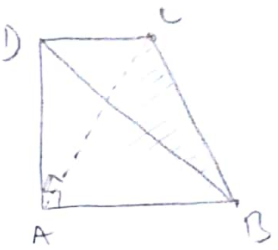
$$A = \begin{bmatrix} \boxed{1} & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{rang}(A) = 3$$

Najviše su 3 linearno nezavisna vektora.

⑤ $V=2$

$\vec{AD} \perp \vec{AB}$, $\vec{AD} \perp \vec{AC}$



$A(1,1,2)$, $B(2,3,3)$, $C(2,3,5)$

$D=?$ $D(x,y,z)$

$\vec{AB}=(1,2,1)$, $\vec{AC}=(1,2,3)$, $\vec{AD}=(x-1,\ y-1,\ z-2)$

1. način: Odredimo sve vektore $\vec{c}$ okomite na $\vec{AB}$ i $\vec{AC}$

$$\vec{AB}\times\vec{AC}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 4\vec{i}-2\vec{j} \Rightarrow \vec{c}=\lambda(4\vec{i}-2\vec{j}),\ \lambda\in\mathbb{R}$$

$$V=\frac{\text{površina baze}\cdot\text{visina}}{3} \Rightarrow \frac{|\vec{AB}\times\vec{AC}|\cdot|\vec{c}|}{6}=2\ /\cdot 6$$

$$\Rightarrow \sqrt{4^2+(-2)^2}\cdot\sqrt{\lambda^2(4^2+(-2)^2}=12 \Rightarrow 20|\lambda|=12 \Rightarrow \lambda=\pm\frac{3}{5}$$

$$c_1=\frac{12}{5}\vec{i}-\frac{6}{5}\vec{j} \Rightarrow \vec{AD}=\vec{c_1} \Rightarrow$$

$$c_2=-\frac{12}{5}\vec{i}+\frac{6}{5}\vec{j} \Rightarrow \vec{AD}=\vec{c_2} \Rightarrow$$

$$\boxed{\begin{matrix} D_1\left(\frac{17}{5},\ \frac{-1}{5},\ 2\right) \\ D_2\left(\frac{-7}{5},\ \frac{11}{5},\ 2\right) \end{matrix}}$$

II. način: Neka je $\vec{AD}=(d_1,d_2,d_3)$

$$\left.\begin{matrix} \vec{AB}\cdot\vec{AD}=0 \Rightarrow & d_1+2d_2+d_3=0 \\ \vec{AC}\cdot\vec{AD}=0 \Rightarrow & d_1+2d_2+3d_3=0 \end{matrix}\right\} \Rightarrow \begin{matrix} d_3=0 \\ d_1=-2d_2 \end{matrix} \quad \vec{AD}=(-2d_2,d_2,0)$$

$$V=\frac{|\vec{AB}\times\vec{AC}|\cdot|\vec{AD}|}{6} \Rightarrow \sqrt{20}\cdot\sqrt{4d_2^2+d_2^2}=12 \Rightarrow 5|d_2|=6 \begin{matrix} \nearrow d_2=\frac{6}{5} \\ \searrow d_2=\frac{-6}{5} \end{matrix}$$

$$\Rightarrow \vec{AD}=\left(-\frac{12}{5},\frac{6}{5},0\right) \Rightarrow$$

ili

$$\vec{AD}=\left(\frac{12}{5},\frac{-6}{5},0\right) \Rightarrow$$

$$\boxed{\begin{matrix} D_1\left(\frac{-7}{5},\ \frac{11}{5},\ 2\right) \\ D_2\left(\frac{17}{5},\ \frac{-1}{5},\ 2\right) \end{matrix}}$$