LINEARNA ALGEBRA

Zimski ispitni rok (13.2.2020.)

RJEŠENJA ZADATAKA

1. (a) $\det A = 2\cdot 4\cdot 2 = 16 \neq 0 \Rightarrow A$ regularna

$\det B = 1\cdot 1\cdot 1 = 1 \neq 0 \Rightarrow B$ regularna

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} /:2 \\ /:4 \\ \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} /\cdot(-1) \\ \\ + \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 2 & -\frac{1}{2} & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ /:2 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{1}{2} \end{array}\right]$$

$$\underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}}_{A^{-1}}$$

$\Rightarrow \det(A^{-1}+I) = \left(\frac{1}{2}+1\right)\left(\frac{1}{4}+1\right)\left(\frac{1}{2}+1\right) = \frac{45}{16} \neq 0$

$\Rightarrow A^{-1}+I$ regularna

(b) $(AX)^{-1} + X^{-1} = B$

$X^{-1}A^{-1} + X^{-1} = B$

$X^{-1}(A^{-1}+I) = B \quad /^{-1}$

$(A^{-1}+I)\cdot / \quad (A^{-1}+I)^{-1}X = B^{-1}$

$X = (A^{-1}+I)B^{-1}$

$$= \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{5}{4} & 0 \\ -\frac{1}{4} & 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{5}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{2} \end{bmatrix}$$

2. (T1) Ne može vrijediti.

Budući da je $r(B)<4$, B je singularna matrica pa $\det B = 0$.

Po Binet-Cauchyjevom teoremu,

$$\det(AB) = \det A \cdot \det B = 0$$

pa je i AB singularna matrica, tj. $r(AB)<4$.

(T2) Može vrijediti.

Na primjer,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A+B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow r(A+B) = 4$$

(T3) Može vrijediti (štoviše, uvijek vrijedi)

$r(A)=4$ pa je A regularna matrica i sustav $A\vec{x}=\vec{0}$ uvijek ima jedinstveno (trivijalno) rješenje

$$A^{-1}\cdot | \quad A\vec{x} = \vec{0}$$

$$\Rightarrow \vec{x} = A^{-1}\vec{0} = \vec{0}$$

(T4) Ne može vrijediti

Dimenzija prostora rješenja sustava $B\vec{x}=\vec{0}$ jednaka je $4-r(B)=1$ te taj sustav uvijek ima beskonačno mnogo rješenja.

3. (a) 1. način

Tražena točka je nožište okomice iz ishodišta na pravac p.

Vektor smjera pravca p je kolinearan vektorskom produktu vektora normala ravnina čiji je presjek p:

$$\vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{vmatrix} = -4\vec{i} - 2\vec{j} - 2\vec{k}$$

Jednadžba ravnine koja prolazi ishodištem i okomita je na p (tj. ima vektor normale $\vec{s}$) glasi:

$$-4(x-0)-2(y-0)-2(z-0)=0$$

$$\pi \ \ldots \ 2x+y+z=0$$

Tražena točka je presjek p i π, tj. rješenje sustava:

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -6 \\ 1 & -1 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} + \\ \cdot(-1) \\ \cdot(-2) \\ + \end{array} \sim \left[\begin{array}{ccc|c} 0 & 2 & -2 & -6 \\ 1 & -1 & -1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \begin{array}{l} /:2 \\ \\ /:3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 0 & 1 & -1 & -3 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} + \\ \cdot(-1) \\ + \\ \cdot 1 \end{array} \sim \left[\begin{array}{ccc|c} 0 & 0 & -2 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow z = \frac{3}{2} \\ \Rightarrow x = 0 \\ \Rightarrow y = -z = -\frac{3}{2} \end{array}$$

Tražena točka je $\left(0, -\frac{3}{2}, \frac{3}{2}\right)$.

2. način

Odredimo najprije parametarske jednadžbe od p:

$$\begin{cases} x+y-3z+6=0 \\ x-y-z=0 \end{cases} \begin{array}{l} + \\ \cdot 1 \end{array} \Rightarrow 2x-4z+6=0 \Rightarrow x = 2z-3$$

$$\Rightarrow y = x - z = z - 3$$

$$\Rightarrow p \ldots \begin{cases} x = 2t-3 \\ y = t-3 \\ z = t \end{cases}$$

Za točku $T(2t-3, t-3, t) \in p$, njen kvadrat udaljenosti od ishodišta je jednak

$$|OT|^2 = (2t-3-0)^2 + (t-3-0)^2 + (t-0)^2$$

$$= 6(t^2 - 3t + 3)$$

Dobivena kvadratna funkcija svoj minimum postiže za $t = -\frac{-3}{2 \cdot 1} = \frac{3}{2}$ pa je tražena točka $T(0, -\frac{3}{2}, \frac{3}{2})$.

(b) Neka je $A(x_A, y_A, z_A)$ tražena točka. Točka dobivena u (a) podzadatku je polovište dužine $\overline{OA}$ pa imamo

$$\frac{x_A + 0}{2} = 0 \quad \Rightarrow x_A = 0$$

$$\frac{y_A + 0}{2} = -\frac{3}{2} \quad \Rightarrow y_A = -3$$

$$\frac{z_A + 0}{2} = \frac{3}{2} \quad \Rightarrow z_A = 3$$

Tražena točka je $A(0, -3, 3)$.

4. (a) $$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$$

$$\Rightarrow A(\vec{a}_1 - 2\vec{a}_2 + 2\vec{a}_3) = -5\vec{a}_1 - 3\vec{a}_2 + 2\vec{a}_3$$

(b) Matrica prijelaza iz baze $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ u bazu $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$:

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \substack{+ \\ 1\cdot(-1)} \;\; \substack{+ \\ 1\cdot(-1)} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \substack{+ \\ 1\cdot(-1)}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \underbrace{\begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}}_{T^{-1}}$$

Zato matrični prikaz od A u bazi $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ glasi

$$A' = T^{-1}AT = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ -3 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ -3 & -3 & -4 \\ 2 & 3 & 4 \end{bmatrix}$$

5. Matrični prikaz od A u bazi $\{\vec{a}_1, \vec{a}_2\}$

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}.$$

Odredimo svojstvene vrijednosti dobivene matrice:

$$\kappa_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 2 \\ 2 & \lambda-6 \end{vmatrix}$$

$$= \lambda^2 - 9\lambda + 18 - 4 = (\lambda-2)(\lambda-7) \Rightarrow \lambda_1 = 2, \ \lambda_2 = 7$$

Pripadni svojstveni vektori:

1° $(2I - A)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] \begin{matrix} \cdot 2 \\ + \end{matrix} \sim \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 2x_2$$

$$\Rightarrow \vec{x} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \alpha \in \mathbb{R}\setminus\{0\}$$

2° $(7I - A)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \begin{matrix} + \\ \cdot(-2) \end{matrix} \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 1 & 0 \end{array}\right] \Rightarrow x_2 = -2x_1$$

$$\Rightarrow \vec{x} = \alpha \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \alpha \in \mathbb{R}\setminus\{0\}$$

Dakle, A se može dijagonalizirati u bazi $\{2\vec{a}_1 + \vec{a}_2, \ \vec{a}_1 - 2\vec{a}_2\}$.

6. (a) Neka su $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ skalari takvi da

$$\alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \dots + \alpha_k \vec{e}_k = \vec{0}.$$

Za svaki $i \in \{1, 2, \dots, k\}$, skalarnim množenjem gornje jednakosti sa $\vec{e}_i$ dobivamo

$$0 = \left(\sum_{j=1}^{k} \alpha_j \vec{e}_j \,\middle|\, \vec{e}_i \right) = \sum_{j=1}^{k} \alpha_j \underbrace{(e_j | e_i)}_{\substack{=0 \\ \text{za } j \neq i}} = \alpha_i (e_i | e_i).$$

Budući da je $\vec{e}_i \neq \vec{0}$, slijedi $(\vec{e}_i | \vec{e}_i) = \|\vec{e}_i\|^2 > 0$ pa $\alpha_i = 0$.

Dakle $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$ pa su vektori $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k$ po definiciji linearno nezavisni.

(b) Budući da je $\{\vec{e}_1, \dots, \vec{e}_n\}$ baza za X, za svaki vektor $\vec{x} \in X$ postoje jedinstveni skalari $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ takvi da

$$\vec{x} = \sum_{j=1}^{n} \alpha_j \vec{e}_j .$$

Za svaki $i \in \{1, 2, \dots, n\}$, skalarnim množenjem gornje jednakosti sa $\vec{e}_i$ dobivamo

$$(\vec{x} | \vec{e}_i) = \left(\sum_{j=1}^{n} \alpha_j \vec{e}_j \,\middle|\, \vec{e}_i \right) = \sum_{j=1}^{n} \alpha_j \underbrace{(\vec{e}_j | \vec{e}_i)}_{\substack{=0 \\ \text{za } j \neq i}} = \alpha_i (\vec{e}_i | \vec{e}_i)$$

$$\Rightarrow \quad \alpha_i = \frac{(\vec{x} | \vec{e}_i)}{(\vec{e}_i | \vec{e}_i)} = \frac{(\vec{x} | \vec{e}_i)}{\|\vec{e}_i\|^2}, \qquad i = 1, 2, \dots, n$$

$$\Rightarrow \quad \vec{x} = \sum_{j=1}^{n} \frac{(\vec{x} | \vec{e}_j)}{\|\vec{e}_j\|^2} \vec{e}_j$$

(c) Neka je $\vec{x} \in X$ proizvoljan. Tada postoje jedinstveni skalari $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ takvi da

$$\vec{x} = \sum_{j=1}^{n} \alpha_j \vec{e}_j .$$

Računamo

$$\|\vec{x}\|^2 = (\vec{x} \mid \vec{x})$$

$$= \left(\sum_{j=1}^{n} \alpha_j \vec{e}_j \,\middle|\, \sum_{k=1}^{n} \alpha_k \vec{e}_k \right)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \alpha_k (\vec{e}_j \mid \vec{e}_k)$$

$$= \left[\begin{array}{l} \text{Budući da je } \{\vec{e}_1, \dots, \vec{e}_n\} \text{ ortonormiran skup,} \\ \text{za sve } j, k \in \{1, \dots, n\} \text{ vrijedi} \\ \qquad (\vec{e}_j \mid \vec{e}_k) = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases} \end{array} \right]$$

$$= \sum_{j=1}^{n} \alpha_j \alpha_j \cdot 1 = \sum_{j=1}^{n} \alpha_j^2 .$$

No, prema (b) dijelu je

$$\alpha_j = \frac{(\vec{x} \mid \vec{e}_j)}{\underbrace{\|\vec{e}_j\|^2}_{=1}} = (\vec{x} \mid \vec{e}_j), \qquad j = 1, 2, \dots, n,$$

pa slijedi

$$\|\vec{x}\|^2 = \sum_{j=1}^{n} (\vec{x} \mid \vec{e}_j)^2 .$$