(1)

1: $x_1 + x_4 = x_3$

2: $x_2 + x_5 = x_1$

3: $x_3 + x_6 = x_2$

4: $x_4 + x_5 + x_6 = 0$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{x} = 0$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{1} & \cancel{1} \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_3 + x_5 + x_6 \\ x_3 + x_6 \\ x_3 \\ -x_5 - x_6 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(2.) (a) Dokazujemo indukcijom po redu matrice n.

- $n=2$. $\det A = \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$
- Pretp. da tvrdnja vrijedi za sve matrice reda n.
- Neka je $A \in M_{n+1}$, sa dva jednaka retka (i-ti i j-ti).
  Razvijamo determinantu po k-tom retku, $k \neq i,j$
  $\det A = \sum_{l=1}^{n+1} (-1)^{k+l} a_{kl} M_{kl}$. M_{kl} je determinanta matrice reda n čija su 2 retka jednaka, a ona je po pretpostavci jednaka 0.

(b) Dokazujemo da vrijedi

$$\underbrace{\begin{vmatrix} a_1' + a_1'' \\ a_2 \\ \vdots \\ a_n \end{vmatrix}}_{\det A} = \underbrace{\begin{vmatrix} a_1' \\ a_2 \\ \vdots \\ a_n \end{vmatrix}}_{\det A'} + \underbrace{\begin{vmatrix} a_1'' \\ a_2 \\ \vdots \\ a_n \end{vmatrix}}_{\det A''}$$

$$\det A = \sum_{j=1}^{n} (a_{1j}' + a_{1j}'') M_{1j} = \sum_{j=1}^{n} a_{1j}' M_{1j} + \sum_{j=1}^{n} a_{1j}'' M_{1j} = \det A' + \det A''$$

(c) BSOMP 1. i 2. redak matrice

$$\begin{vmatrix} a_1 \\ \lambda a_1 + a_2 \\ \vdots \end{vmatrix} \overset{(b)}{=} \begin{vmatrix} a_1 \\ \lambda a_1 \\ \vdots \end{vmatrix} + \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix} = \lambda \begin{vmatrix} a_1 \\ a_1 \\ \vdots \end{vmatrix} + \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix} \overset{(a)}{=} \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix}$$

(3.) $v = \alpha a + \beta b + \gamma c$

$v \cdot a = 3 \Leftrightarrow (\alpha a + \beta b + \gamma c) \cdot a = 3 \Leftrightarrow \alpha a \cdot a + \beta b \cdot a + \gamma c \cdot a = 3 \Leftrightarrow$

$\Leftrightarrow \alpha \|a\|^2 + \beta \|b\| \|a\| \cos \measuredangle (b, a) + \gamma \cdot \|c\| \|a\| \cos \measuredangle (c a) = 3$

$\Leftrightarrow \alpha \cdot 1^2 + \beta \cdot 1 \cdot 2 \cdot \cos \frac{\pi}{3} + \gamma \cdot 1 \cdot 1 \cdot \cos \frac{\pi}{2} = 3$

$\Leftrightarrow \alpha + \beta = 3$

$v \cdot b = 12 \Leftrightarrow \alpha a \cdot b + \beta b \cdot b + \gamma c \cdot b = 12 \Leftrightarrow$

$\Leftrightarrow \alpha \cdot 1 \cdot 2 \cdot \cos \frac{\pi}{3} + \beta \cdot 4 + \gamma \cdot 1 \cdot 2 \cdot \cos \frac{\pi}{3} = 12$

$\Leftrightarrow \alpha + 4\beta + \gamma = 12$

$v \cdot c = 5 \Leftrightarrow \alpha a \cdot c + \beta b \cdot c + \gamma c \cdot c = 5$

$\Leftrightarrow \alpha \cdot 1 \cdot 1 \cdot \cos \frac{\pi}{2} + \beta \cdot 2 \cdot 1 \cdot \cos \frac{\pi}{3} + \gamma \cdot 1 = 5$

$\Leftrightarrow \beta + \gamma = 5$

$$\left.\begin{array}{l} \left.\begin{array}{l} \alpha + \beta = 3 \\ \alpha + 4\beta + \gamma = 12 \end{array}\right\} \Rightarrow 3\beta + \gamma = 9 \\ \beta + \gamma = 5 \end{array}\right\} \Rightarrow 2\beta = 4$$

$\beta = 2, \gamma = 3, \alpha = 1$

(4.) Pretp. da sustavi $Ax = b$ i $Cx = b$ imaju isti skup rješenja za svaki vektor b.

$A = [a_1 \ a_2 \ \dots \ a_n]$

$Ax = a_i$ ima rješenje $x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} i \Rightarrow Cx = a_i \Rightarrow A$ i C imaju isti i-ti stupac $\forall i$

$\Rightarrow A = C$

(5.) Zadano je preslikavanje $P: M_n(\mathbb{R}) \to M_n(\mathbb{R})$

$$P(A) = \frac{A+A^T}{2}.$$

(a) Dokažite da je P linearni operator.

(b) Dokažite da se jezgra operatora P sastoji od antisimetričnih matrica.

(c) Kolika je dimenzija $Ker(P)$?

(d) Kolika je dimenzija $Im(P)$? Opišite prostor $Im(P)$.

Rješenje

(a) $A, B \in M_n(\mathbb{R}),\ \alpha, \beta \in \mathbb{R}$

$$P(\alpha A + \beta B) = \frac{1}{2}\left[(\alpha A + \beta B) + (\alpha A + \beta B)^T\right] = \frac{1}{2}\left[\alpha A + \beta B + \alpha A^T + \beta B^T\right] =$$

$$= \alpha \cdot \frac{1}{2}(A + A^T) + \beta \cdot \frac{1}{2}(B + B^T) = \alpha P(A) + \beta P(B).$$

(b) $A \in Ker(P) \Leftrightarrow P(A) = 0 \Leftrightarrow \frac{A+A^T}{2} = 0 \Leftrightarrow A^T = -A$

(c) Promotrimo skup

$$S = \left\{ \begin{bmatrix} 0 & 1 & 0 & \\ -1 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ -1 & 0 & 0 & \\ & & & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & & \\ & \ddots & \\ & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{bmatrix} \right\} = \{F_{12}, F_{13}, \ldots, F_{n-1,n}\}$$

S je očito linearno nezavisan.

Neka je A antisimetrična matrica, $A = (a_{ij})$.

$$A^T = -A \Rightarrow a_{ii} = 0 \ \text{ i } \ a_{ij} = -a_{ji}$$

$$\Rightarrow A = a_{12}F_{12} + a_{13}F_{13} + \cdots + a_{n-1,n}F_{n-1,n} \in L(S)$$

$\Rightarrow S$ je baza prostora antisimetričnih matrica.

$$\Rightarrow d(P) = \frac{1}{2}(n^2 - n)$$

(d) $r(P) + d(P) = n^2 \Rightarrow r(P) = \frac{1}{2}(n^2 + n)$

$B \in Im(P) \Leftrightarrow B = \frac{1}{2}(A + A^T)$ za neku matricu $A \Rightarrow B^T = \frac{1}{2}(A^T + A) = B \Rightarrow B$ je simetrična

Obratno, ako je B sim. matrica onda je $B = \frac{1}{2}(B + B^T) \in Im(P)$.

Slika je jednaka prostoru simetričnih matrica.