LINEARNA ALGEBRA

Međuispit (21. 11. 2019.)

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1. (a)

$$X^2 = \frac{1}{16}\begin{bmatrix} 1 & \sqrt{3} \\ 1 & 4\alpha \end{bmatrix}\begin{bmatrix} 1 & \sqrt{3} \\ 1 & 4\alpha \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 1+\sqrt{3} & \sqrt{3}(1+4\alpha) \\ 1+4\alpha & \sqrt{3}+16\alpha^2 \end{bmatrix}$$

$\Rightarrow \frac{1}{16}(1+\sqrt{3}) = \frac{1}{4} \Rightarrow$ Ne postoji takav α.

(b) $A^2 = A \quad / \det$

$\Rightarrow \det(A^2) = \det A$

$\Rightarrow (\det A)^2 = \det A$

Binet-Cauchy

$\Rightarrow \det A \cdot (\det A - 1) = 0 \Rightarrow \det A \in \{0, 1\}$

(c) $I - A$ idempotentna $\Leftrightarrow (I-A)^2 = I - A \Leftrightarrow \cancel{I^2} - \cancel{A} - A + A^2 = \cancel{I} - \cancel{A}$

$\Leftrightarrow -A + A^2 = 0 \Leftrightarrow A^2 = A \Leftrightarrow A$ idempotentna

2. (a) $(AB)^{-1} = B^{-1}A^{-1}$

Izvod:

$$(AB)(B^{-1}A^{-1}) = A\underbrace{(BB^{-1})}_{I}A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}B = I$$

(b)

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 = 1 \neq 0$$

$\det B = 3 \cdot 3 \cdot 3 = 27 \neq 0$, $\det C = 1 \cdot 1 \cdot 1 = 1 \neq 0 \Rightarrow A, B, C$ regularne

$$(A^{-1}B^{-1}C)^{-1} = C^{-1}BA = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 9 \\ -1 & 0 & 1 \\ 3 & 3 & 6 \end{bmatrix}$$

3. U ovisnosti o parametru $\alpha \in \mathbb{R}$ određujemo rang matrice čiji su stupci zadani vektori:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha & \alpha^2 \\ \alpha & 1 & \alpha^2 & \alpha^2 \end{bmatrix} \begin{matrix} |\cdot(-1) \\ + \\ |\cdot(-\alpha) \; + \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \alpha-1 & \alpha-1 & \alpha^2-1 \\ 0 & 1-\alpha & \alpha^2-\alpha & \alpha^2-\alpha \end{bmatrix} \begin{matrix} |\cdot 1 \\ + \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \alpha-1 & \alpha-1 & \alpha^2-1 \\ 0 & 0 & \alpha^2-1 & 2\alpha^2-\alpha-1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \alpha-1 & \alpha-1 & (\alpha-1)(\alpha+1) \\ 0 & 0 & (\alpha-1)(\alpha+1) & (2\alpha+1)(\alpha-1) \end{bmatrix}$$

1° $\alpha = 1$

Dobivamo matricu

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow r = 1 \Rightarrow$$ jedan linearno nezavisan vektor među zadanima

2° $\alpha \neq 1$

Drugi i treći redak možemo podijeliti s $\alpha - 1$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \alpha+1 \\ 0 & 0 & \alpha+1 & 2\alpha+1 \end{bmatrix}_{1\cdot(-1)\,+} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & \alpha \\ 0 & 0 & \alpha+1 & \alpha \end{bmatrix}_{1\cdot(-1)\,+} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -\alpha+1 & \alpha \\ 0 & 0 & 1 & \alpha \end{bmatrix}$$

$\Rightarrow r = 3 \Rightarrow$ tri linearno nezavisna vektora među zadanima

4. (a)
$$\left[\begin{array}{cccc|c} 2 & 3 & 1 & 2 & 3 \\ 4 & 6 & 3 & 4 & 5 \\ 6 & 9 & 5 & 6 & 7 \\ 8 & 12 & 7 & 8 & 9 \end{array}\right] \begin{array}{l} \\ 1\cdot(-2)\ + \\ 1\cdot(-3)\ + \\ 1\cdot(-4)\ + \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 2 & 3 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 & -3 \end{array}\right] \begin{array}{l} + \\ 1\cdot(-1) \\ 1\cdot(-2)\ + \\ 1\cdot(-3)\ + \end{array} \sim \left[\begin{array}{cccc|c} 2 & 3 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} 2x_1 + 3x_2 + 2x_4 = 4 \\ x_3 = -1 \end{cases} \qquad x_2 = \alpha,\ x_4 = \beta,\quad \alpha, \beta \in \mathbb{R}$$

$$\Rightarrow x_1 = 2 - \frac{3}{2}\alpha - \beta$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - \frac{3}{2}\alpha - \beta \\ \alpha \\ -1 \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}$$

(b)
$$x_h = \alpha \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \alpha, \beta \in \mathbb{R}$$

(c) Na primjer, za $(\alpha, \beta) = (0,0)$ i $(\alpha, \beta) = (2,0)$ dobivamo

$$x_p = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad x_p' = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}.$$

5. $(\vec{a}+\vec{b})\cdot(\vec{a}\times\vec{c})+(\vec{a}+\vec{c})\cdot(\vec{b}\times\vec{c})+(\vec{b}+\vec{c})\cdot(\vec{a}\times\vec{b})=$

$$= + \begin{cases} \underbrace{\vec{a}\cdot(\vec{a}\times\vec{c})}_{=0} + \vec{b}\cdot(\vec{a}\times\vec{c}) \\ \vec{a}\cdot(\vec{b}\times\vec{c}) + \underbrace{\vec{c}\cdot(\vec{b}\times\vec{c})}_{=0} \\ \underbrace{\vec{b}\cdot(\vec{a}\times\vec{b})}_{=0} + \vec{c}\cdot(\vec{a}\times\vec{b}) \end{cases}$$

$$= \underbrace{\vec{b}\cdot(\vec{a}\times\vec{c})} + \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$= -\vec{a}\cdot\cancel{(\vec{b}\times\vec{c})} + \vec{a}\cdot\cancel{(\vec{b}\times\vec{c})} + \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$= \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$= |\vec{c}|\cdot|\vec{a}\times\vec{b}|\cos\underbrace{\measuredangle(\vec{a}\times\vec{b},\vec{c})}_{90^\circ-60^\circ=30^\circ \text{ ili } 180^\circ-(90^\circ-60^\circ)=150^\circ}$$

$$= |\vec{c}|\cdot|\vec{a}|\cdot|\vec{b}|\sin\underbrace{\measuredangle(\vec{a},\vec{b})}_{30^\circ}\cdot(\pm\cos 30^\circ)$$

$$= 1\cdot 3\cdot 2\cdot\frac{1}{2}\cdot\left(\pm\frac{\sqrt{3}}{2}\right)$$

$$= \pm\frac{3\sqrt{3}}{2}$$