LINEARNA ALGEBRA

Ljetni ispitni rok (10. 7. 2020.)

– RJEŠENJA ZADATAKA –

1. (a)

$$\begin{vmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{vmatrix} = \begin{vmatrix} -3 & 1 & 1 & 1 \\ 4 & -4 & 0 & 0 \\ 4 & 0 & -4 & 0 \\ 4 & 0 & 0 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{vmatrix} = 0 \cdot (-4)^3 = 0$$

(b)

$$\begin{vmatrix} a & 1 & 1 & \cdots & 1 \\ 1 & a & 1 & \cdots & 1 \\ 1 & 1 & a & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & a \end{vmatrix} = \begin{vmatrix} a & 1 & 1 & \cdots & 1 \\ 1-a & a-1 & 0 & \cdots & 0 \\ 1-a & 0 & a-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-a & 0 & 0 & \cdots & a-1 \end{vmatrix}$$

$$= \begin{vmatrix} a+(n-1) & 1 & 1 & \cdots & 1 \\ 0 & a-1 & 0 & \cdots & 0 \\ 0 & 0 & a-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a-1 \end{vmatrix} = \big(a+(n-1)\big)(a-1)^{n-1}$$

Dakle, determinanta je jednaka 0 za $a = -n+1$ i $a = 1$.

2. (a)
$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ -1 & 4 & -1 & 4 \\ 1 & -1 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -7 & 0 & -6 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow x_1 = -6 + 7x_2 \\ \Rightarrow x_3 = 2 - 3x_2 \end{array}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} -6+7t \\ t \\ 2-3t \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix}, \quad t \in \mathbb{R}$$

(b)
$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ -1 & 2a & -1 & 4 \\ 1 & -1 & a & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 2a-1 & 1 & 2 \\ 0 & 0 & a-2 & b+2 \end{array}\right]$$

Razlikujemo slučajeve:

1° $a = 2$

Imamo
$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & b+2 \end{array}\right]$$
→ za $b \neq -2$ sustav nema rješenja, dok za $b = -2$ sustav ima beskonačno mnogo rješenja prema (a) podzadatku

2° $a \neq 2$
$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 2a-1 & 1 & 2 \\ 0 & 0 & a-2 & b+2 \end{array}\right] \Big| : (a-2) \neq 0$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 2a-1 & 1 & 2 \\ 0 & 0 & 1 & \frac{b+2}{a-2} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & \frac{-2a-2b-8}{b-2} \\ 0 & 2a-1 & 0 & \frac{2a-b-6}{b-2} \\ 0 & 0 & 1 & \frac{b+2}{a-2} \end{array}\right]$$

Ponovno razlikujemo slučajeve:

2.1° Za $2a-1=0$, tj. $a=\frac{1}{2}$ imamo

- ako je $2a-b-6=0$, tj. $b=2a-6=-5$, onda sustav ima beskonačno mnogo rješenja:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -\frac{1}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \Rightarrow x_1 = x_2 - \frac{1}{7} \\ \\ \Rightarrow x_3 = 2 \end{array} \Rightarrow \vec{x} = \begin{bmatrix} -\frac{1}{7} \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

- ako je $2a-b-6 \neq 0$, tj. $b \neq -5$, sustav nema rješenja

2.2° Za $2a-1 \neq 0$, tj. $a \neq \frac{1}{2}$ vidimo da je rang matrice sustava 3 (tj. ta matrica je regularna) pa sustav ima jedinstveno rješenje.

Dakle, zadani sustav:

1) ima jedinstveno rješenje za $a \neq 2$ i $a \neq \frac{1}{2}$, $b \in \mathbb{R}$,

2) ima beskonačno mnogo rješenja za $a=2$, $b=-2$ te $a=\frac{1}{2}$, $b=-5$

3) nema rješenja za $a=2$, $b \neq -2$ te $a=\frac{1}{2}$, $b \neq -5$

3\. (a) Vektor normale tražene ravnine π mora biti okomit na vektore normale od π_1 i π_2 pa možemo uzeti

$$\vec{n}_{\pi} = \vec{n}_{\pi_1} \times \vec{n}_{\pi_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 1 & 2 & 1 \end{vmatrix} = -7\vec{i} + \vec{j} + 5\vec{k}.$$

Zato je opća jednadžba ravnine π oblika

$$-7x + y + 5z = D.$$

Budući da π prolazi ishodištem

$$D = -7 \cdot 0 + 0 + 5 \cdot 0 = 0$$

$$\Rightarrow \pi \ldots -7x + y + 5z = 0$$

(b)
$$\begin{cases} 2x - y + 3z = 1 \\ x + 2y + z = 0 \end{cases} \; \Big| \cdot (-2) \; +$$

$$\Rightarrow \begin{cases} -5y + z = 1 \\ x + 2y + z = 0 \end{cases} \; \Big| \cdot (-1) \; +$$

$$\Rightarrow \begin{cases} -5y + z = 1 & \Rightarrow z = 1 + 5y \\ x + 7y = -1 & \Rightarrow x = -1 - 7y \end{cases}$$

Ukoliko stavimo $y = t$, $t \in \mathbb{R}$, dobivamo da je presjek ravnina π_1 i π_2 pravac čije su parametarske jednadžbe

$$p \ldots \begin{cases} x = -1 - 7t \\ y = t \\ z = 1 + 5t \end{cases} \qquad t \in \mathbb{R},$$

ili u kanonskom obliku

$$p \ldots \frac{x+1}{-7} = \frac{y}{1} = \frac{z-1}{5}$$

4. $D: \mathcal{P}_3 \to \mathcal{P}_4$, $\quad (Dp)(t) = (t^2+t)\,p'(t)$

Za vektore e_1, e_2, e_3, e_4 kanonske baze za $\mathcal{P}_3$ vrijedi

$$(De_1)(t) = (t^2+t)\cdot 1' = (t^2+t)\cdot 0 = 0,$$

$$(De_2)(t) = (t^2+t)\cdot t' = (t^2+t)\cdot 1 = t^2+t,$$

$$(De_3)(t) = (t^2+t)\cdot (t^2)' = (t^2+t)\cdot 2t = 2t^3+2t^2,$$

$$(De_4)(t) = (t^2+t)\cdot (t^3)' = (t^2+t)\cdot 3t^2 = 3t^4+3t^3,$$

pa je matrični zapis od D u paru kanonskih baza

$$[D] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Za proizvoljan vektor $p(t) = at^3+bt^2+ct+d \in \mathcal{P}_3$ imamo

$$(Dp)(t) = (t^2+t)(3at^2+2bt+c)$$

$$= 3a(t^4+t^3)+2b(t^3+t^2)+c(t^2+t),$$

pa vidimo da skup $\{t^4+t^3,\ t^3+t^2,\ t^2+t\}$ razapinje $\operatorname{Im} D$. Uočimo i da su ti vektori linearno nezavisni:

$$\alpha(t^4+t^3)+\beta(t^3+t^2)+\gamma(t^2+t) = \alpha t^4 + (\alpha+\beta)t^3 + (\beta+\gamma)t^2 + \gamma t = 0$$

$$\begin{cases} \alpha = 0 \\ \alpha+\beta = 0 \quad \Rightarrow \beta = 0 \\ \beta+\gamma = 0 \quad \Rightarrow \gamma = 0 \\ \gamma = 0 \end{cases}$$

Dakle, ti vektori čine bazu za $\operatorname{Im} D$ i $r(D) = \dim(\operatorname{Im} D) = 3$.

Prema teoremu o rangu i defektu slijedi $d(D) = \dim \mathcal{P}_3 - r(D) = 4-3 = 1$.

Odredimo bazu za jezgru: za $p(t) = at^3 + bt^2 + ct + d \in \operatorname{Ker} D$ imamo

$$(Dp)(t) = 3at^4 + (3a+2b)t^3 + (2b+c)t^2 + ct = 0$$

$$\Leftrightarrow \begin{cases} 3a & = 0 \Rightarrow a = 0 \\ 3a + 2b & = 0 \Rightarrow b = 0 \\ 2b + c & = 0 \Rightarrow c = 0 \\ c & = 0 \end{cases} \qquad d \in \mathbb{R},$$

pa je $p(t) = d \cdot 1$. Dakle, $\{1\}$ je (jedna) baza za jezgru od D.

Da bismo odredili jedan polinom iz $\mathcal{P}_4$ koji nije element $\operatorname{Im} D$, možemo naći neki polinom koji je linearno nezavisan s onima iz dobivene baze za $\operatorname{Im} D$ (kao kad bismo tu bazu nadopunjavali do baze za $\mathcal{P}_4$). Uzmimo, na primjer, vektor kanonske baze za $\mathcal{P}_4$, t^4:

$$\alpha(t^4+t^3) + \beta(t^3+t^2) + \gamma(t^2+t) + \delta t^4 = (\alpha+\delta)t^4 + (\alpha+\beta)t^3 + (\beta+\gamma)t^2 + \gamma t = 0$$

$$\Rightarrow \begin{cases} \alpha + \delta & = 0 \Rightarrow \delta = 0 \\ \alpha + \beta & = 0 \Rightarrow \alpha = 0 \\ \beta + \gamma & = 0 \Rightarrow \beta = 0 \\ \gamma & = 0 \end{cases}$$

Dakle, t^4 je linearno nezavisan s vektorima baze za $\operatorname{Im} D$ pa vrijedi $t^4 \notin \operatorname{Im} D$ (t^4 ne može biti element prostora razapetog vektorima s kojima je linearno nezavisan).

5. (a) Neka su $x, y \in \operatorname{Ker} A$ i $\alpha, \beta \in \mathbb{R}$ proizvoljni. Po definiciji jezgre, $A(x) = A(y) = 0$.

Imamo

$$A(\alpha x + \beta y) = \alpha \underbrace{Ax}_{=0} + \beta \underbrace{Ay}_{=0} = 0,$$

pa slijedi $\alpha x + \beta y \in \operatorname{Ker} A$, tj. $\operatorname{Ker} A$ je potprostor od X.

(b) $\boxed{\Rightarrow}$ Neka je $A : X \to Y$ injekcija i $x \in \operatorname{Ker} A$. Budući da je A linearan operator, vrijedi $A(0) = 0$, tj. $A(0) = A(x)$, odakle zbog injektivnosti slijedi $x = 0$. Dakle, $\operatorname{Ker} A = \{0\}$.

$\boxed{\Leftarrow}$ Obratno, pretpostavimo $\operatorname{Ker} A = \{0\}$. Neka su $x, y \in X$ takvi da $A(x) = A(y)$. Imamo

$$\begin{aligned} A(x) = A(y) &\Rightarrow A(x) - A(y) = 0 \\ &\Rightarrow A(x-y) = 0 \\ &\Rightarrow x - y \in \operatorname{Ker} A \\ &\Rightarrow x - y = 0 \quad \Rightarrow x = y, \end{aligned}$$

pa po definiciji slijedi da je A injekcija.

(c) Prema teoremu o rangu i defektu slijedi

$$\dim(\operatorname{Im} A) = \dim X - \dim(\operatorname{Ker} A) = 5 - 3 = 2 < 3 = \dim Y.$$

Dakle, $\operatorname{Im} A \neq Y$ pa A nije surjekcija.

6.

$$A = \begin{bmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{bmatrix}$$

Odredimo svojstvene vrijednosti od A. Karakteristični polinom od A je:

$$\kappa_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda+4 & -2 & 2 \\ -2 & \lambda+7 & -4 \\ 2 & -4 & \lambda+7 \end{vmatrix} \begin{matrix} \\ + \\ \cdot 1 \end{matrix}$$

$$= \begin{vmatrix} \lambda+4 & -2 & 2 \\ 0 & \lambda+3 & \lambda+3 \\ 2 & -4 & \lambda+7 \end{vmatrix} = (\lambda+3) \begin{vmatrix} \lambda+4 & -2 & 2 \\ 0 & 1 & 1 \\ 2 & -4 & \lambda+7 \end{vmatrix}$$

$\cdot(-1)$ $+$

$$= (\lambda+3) \begin{vmatrix} \lambda+4 & -2 & 4 \\ 0 & 1 & 0 \\ 2 & -4 & \lambda+11 \end{vmatrix} \leftarrow \; = (\lambda+3) \begin{vmatrix} \lambda+4 & 4 \\ 2 & \lambda+11 \end{vmatrix}$$

$$= (\lambda+3)\left[(\lambda+4)(\lambda+11)-8\right] = (\lambda+3)(\lambda^2+15\lambda+36)$$

$$= (\lambda+3)(\lambda^2+3\lambda+12\lambda+36) = (\lambda+3)(\lambda+3)(\lambda+12) = (\lambda+3)^2(\lambda+12)$$

$\Rightarrow$ svojstvene vrijednosti su $\lambda_1 = -12$ i $\lambda_2 = -3$

Odredimo pripadne svojstvene potprostore:

1° $\lambda_1 = -12$

$$(-12I - A)\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -8 & -2 & 2 & 0 \\ -2 & -5 & -4 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right] \begin{matrix} +\,|\cdot 4 \\ +\,|\cdot 1 \\ \end{matrix} \sim \left[\begin{array}{ccc|c} 0 & -18 & -18 & 0 \\ 0 & -9 & -9 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right] \begin{matrix} |:(-18) \\ |:(-9) \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right] \begin{matrix} |\cdot(-1) \\ + \\ +\,|\cdot 4 \end{matrix} \sim \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \begin{matrix} \Rightarrow x_2 = -x_3 = -2x_1 \\ \\ \Rightarrow x_3 = 2x_1 \end{matrix}$$

$x_1 = t,\ t \in \mathbb{R}$

$$\Rightarrow \vec{x} = \begin{bmatrix} t \\ -2t \\ 2t \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}_{=\vec{v}_1}, \quad t \in \mathbb{R}$$

Normirajmo dobiveni vektor:

$$\vec{e}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{1^2+(-2)^2+2^2}} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$2^\circ \ \lambda_2 = -3$

$(-3I - A)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ -2 & 4 & -4 & 0 \\ 2 & -4 & 4 & 0 \end{array}\right] \begin{matrix} \cdot 2 \\ + \\ \cdot(-2)\ + \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 2x_2 - 2x_3 \\ x_2 = t \\ x_3 = u \end{array} \quad t, u \in \mathbb{R}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 2t - 2u \\ t \\ u \end{bmatrix} = t \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{= \vec{v}_2} + u \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_3}, \quad t, u \in \mathbb{R}$$

Ortonormirajmo skup $\{\vec{v}_2, \vec{v}_3\}$ Gram-Schmidtovim postupkom:

$$\vec{e}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{2^2 + 1^2 + 0^2}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{f}_3 = \vec{v}_3 - \langle \vec{v}_3 | \vec{e}_2 \rangle \vec{e}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{5}(-2 \cdot 2 + 0 \cdot 1 + 1 \cdot 0) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$$

$$\vec{e}_3 = \frac{1}{\|\vec{f}_3\|}\vec{f}_3 = \frac{1}{\frac{1}{5}\sqrt{(-2)^2 + 4^2 + 5^2}} \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} = \frac{\sqrt{5}}{3} \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$$

Budući da su vektori $\vec{e}_1$ i $\vec{e}_2$, tj. $\vec{e}_1$ i $\vec{e}_3$ već ortogonalni (to su svojstveni vektori pridruženi različitim svojstvenim vrijednostima simetrične matrice A), tražena ortonormirana baza je

$$\left\{ \frac{1}{3}(1, -2, 2), \ \frac{1}{\sqrt{5}}(2, 1, 0), \ \frac{\sqrt{5}}{3}(-2, 4, 5) \right\}.$$