PRISMS PhaseField
Mechanics (Infinitesimal Strain)

Consider a strain energy expression of the form:

$$\Pi(\varepsilon) = \int_{\Omega} \frac{1}{2} \varepsilon : C : \varepsilon \; dV \tag{1}$$

where ε is the infinitesimal strain tensor, $C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ is the fourth order elasticity tensor and (λ, mu) are the Lame parameters.

1 Variational treatment

Considering variations on the displacement u of the from $u + \epsilon w$, we have

$$\delta\Pi = \frac{d}{d\epsilon} \int_{\Omega} \frac{1}{2} \varepsilon_{\epsilon} : C : \varepsilon_{\epsilon} \; dV \bigg|_{\epsilon=0} \tag{2}$$

$$= -\int_{\Omega} \boldsymbol{\nabla} w : C : \varepsilon \; dV + \int_{\partial\Omega} w \cdot (C : \varepsilon \cdot n) \; dS \tag{3}$$

$$= -\int_{\Omega} \boldsymbol{\nabla} w : \sigma \; dV + \int_{\partial\Omega} w \cdot (\sigma \cdot n) \; dS \tag{4}$$

$$= -\int_{\Omega} \boldsymbol{\nabla} w : \sigma \; dV + \int_{\partial\Omega} w \cdot t \; dS \tag{5}$$

where $\sigma = C : \varepsilon$ is the stress tensor and $t = \sigma \cdot n$ is the surface traction.

Now the minimization of the variation, $\delta\Pi = 0$, gives the weak formulation of the governing equation of mechanics:

$$\int_{\Omega} \boldsymbol{\nabla} w : \sigma \; dV - \int_{\partial\Omega} w \cdot t \; dS = 0 \tag{6}$$