2.2 Expectation

Consider the outcome of a single die roll, and call it X. A reasonable question one might ask is "What is the average value of X?". We define this notion of "average" as a weighted sum of outcomes.

Since X can take on 6 values, each with probability $\frac{1}{6}$, the weighted average of these outcomes should be

$$\begin{aligned}
\text{Weighted Average} &= \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 \\
&= \frac{1}{6}\cdot(1+2+3+4+5+6) \\
&= \frac{21}{6} \\
&= 3.5
\end{aligned}$$

This may seem dubious to some. How can the average roll be a non-integer value? The confusion lies in the interpretation of the phrase *average roll*. A more correct interpretation would be the long term average of the die rolls. Suppose we rolled the die many times, and recorded each roll. Then we took the average of all those rolls. This average would be the fraction of 1's, times 1, plus the fraction of 2's, times 2, plus the fraction of 3's, times 3, and so on. But this is exactly the computation we have done above! In the long run, the fraction of each of these outcomes is nothing but their probability, in this case, $\frac{1}{6}$ for each of the 6 outcomes.

From this very specific die rolling example, we can abstract the notion of the *average value* of a random quantity. The concept of average value is an important one in statistics, so much so that it even gets a special bold faced name. Below is the mathematical definition for the **expectation**, or average value, of a random quantity X.

Definition 2.8. *The* ***expected value****, or* ***expectation*** *of X, denoted by $E(X)$, is defined to be*

$$E(X) = \sum_{x\in X(\Omega)} xP(X=x)$$

This expression may look intimidating, but it is actually conveying a very simple set of instructions, the same ones we followed to compute the average value of X.

The $\sum$ sign means to sum over, and the indices of the items we are summing are denoted below the $\sum$ sign. The $\in$ symbol is shorthand for "contained in", so the expression below the $\sum$ is telling us to sum over all items *contained in* the set $X(\Omega)$ (defined in the next paragraph). We can think of the expression to the right of the $\sum$ sign as the actual items we are summing, which in this case, is the weighted contribution of each item in our sample space.

The notation $X(\Omega)$ is used to deal with the fact that Ω may not be a set of numbers, so a weighted sum of elements in Ω isn't even well defined. For instance, in the case of a coin flip, how can we compute $\mathtt{H} \cdot \frac{1}{2} + \mathtt{T} \cdot \frac{1}{2}$? We would first need to assign *numerical values* to `H` and `T` in order to compute a meaningful expected value. For a coin flip we typically make the following assignments,

$$\begin{aligned} \mathtt{T} &\mapsto 0 \\ \mathtt{H} &\mapsto 1 \end{aligned}$$

So when computing an expectation, the indices that we would sum over are contained in the set

$$X(\Omega) = \{0, 1\}$$

Let's use this set of instructions to compute the expected value for a coin flip.

2.2.1 Expectation of a Coin Flip

Now let X denote the value of a coin flip with bias p. That is, with probability p we flip `H`, and in this case we say $X = 1$. Similarly, with probability $1 - p$ we flip `T`, and in this case we say $X = 0$. The expected value of the random quantity X is then

$$\begin{aligned} E(X) &= \sum_{x \in X(\Omega)} x P(X = x) \\ &= \sum_{x \in \{0,1\}} x P(X = x) \\ &= 0 \cdot P(X = 0) + 1 \cdot P(X = 1) \\ &= 0 \cdot P(\mathtt{T}) + 1 \cdot P(\mathtt{H}) \\ &= 0 \cdot (1 - p) + 1 \cdot p \\ &= p \end{aligned}$$

So the expected value of this experiment is p. If we were flipping a fair coin, then $p = \frac{1}{2}$, so the average value of X would be $\frac{1}{2}$.

Again, we can never get an outcome that would yield $X = \frac{1}{2}$, but this is not the interpretation of the expectation of X. Remember, the correct interpretation is to consider what would happen if we flipped the coin many times, obtained a sequence of 0's and 1's, and took the average of those values. For a fair coin, we would expect around half of the flips to give 0 and the other half to give 1, giving an average value of $\frac{1}{2}$.

Exercise 2.9. Show the following properties of expectation.

(a) If X and Y are two random variables, then

$$E(X+Y) = E(X) + E(Y)$$

(b) If X is a random variable and c is a constant, then

$$E(cX) = cE(X)$$

(c) If X and Y are independent random variables, then

$$E[XY] = E[X]E[Y]$$

Proof. For now, we will take (a) and (c) as a fact, since we don't know enough to prove them yet (and we haven't even defined independence of random variables!). (b) follows directly from the definition of expectation given above. □