3.2 Combinatorics

In many problems, to find the probability of an event, we will have to count the number of outcomes in Ω which satisfy the event, and divide by $|\Omega|$, i.e. the total number of outcomes in Ω. For example, to find the probability that a single die roll is even, we count the total number of even rolls, which is 3, and divide by the total number of rolls, 6. This gives a probability of $\frac{1}{2}$. But what if the event isn't as simple as "roll an even number"? For example, if we flipped 10 coins, we could be interested in the event, "flipped 3 heads total". Some valid outcomes include

$$\begin{gathered}(\mathtt{H}, \mathtt{H}, \mathtt{H}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}) \\ (\mathtt{H}, \mathtt{H}, \mathtt{T}, \mathtt{H}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}) \\ (\mathtt{H}, \mathtt{H}, \mathtt{T}, \mathtt{T}, \mathtt{H}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}) \\ (\mathtt{H}, \mathtt{T}, \mathtt{T}, \mathtt{H}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{H}, \mathtt{T}).\end{gathered}$$

Immediately some questions arise. Is there a way to systematically count these outcomes? How could we count the number of outcomes that have 3 heads in them without listing them all out? In this section, we will discover how to count the outcomes of such an event, and generalize the solution to be able to conquer even more complex problems.

3.2.1 Permutations

Suppose there are 3 students waiting in line to buy a spicy chicken sandwich. A question we could ask is, "How many ways can we order the students in this line?" Since there are so few students, let's just list out all possible orderings. We could have any of

$$\text{6 of these } \begin{cases} (1, 2, 3) \\ (1, 3, 2) \\ (2, 1, 3) \\ (2, 3, 1) \\ (3, 1, 2) \\ (3, 2, 1) \end{cases}$$

So there are 6 total possible orderings. If you look closely at the list above, you can see that there was a systematic way of listing them. We first wrote out all orderings starting with 1. Then came the orderings starting with 2, and then the ones that started with 3. In each of these groups of orderings starting with some particular student, there were two ways to complete the orderings. This is because once we fixed the first person in line, there were two ways to order the remaining two students. Let's now formalize this thought process.

Denote N_i to be the number of ways to order i students. Now we observe that the number of orderings can be written

$$N_3 = 3 \cdot N_2$$

since there are 3 ways to pick the first student, and N_2 ways to order the remaining two students. By similar reasoning,

$$N_2 = 2 \cdot N_1$$

Since the number of ways to order 1 person is just 1, we have $N_1 = 1$. Hence,

$$N_3 = 3 \cdot N_2 = 3 \cdot (2 \cdot N_1) = 3 \cdot 2 \cdot 1 = 6$$

which is the same as what we got when we just listed out all the orderings and counted them.

Now suppose we want to count the number of orderings for 10 students. 10 is a big enough number that we can no longer just list out all possible orderings and count them. Instead, we will make use of the recursive method we described above. The number of ways to order 10 students is

$$N_{10} = 10 \cdot N_9 = 10 \cdot (9 \cdot N_8) = \cdots = 10 \cdot 9 \cdot 8 \cdot 7 \cdot ... \cdot 2 \cdot 1 = 3,628,800.$$

That's a lot of orderings! It would have been nearly impossible for us to list out over 3 million orderings of 10 students, but we were still able to count these orderings using our neat trick. We have a special name for this operation.

Definition 3.6. *The number of* ***permutations****, or orderings, of n distinct objects is given by the* ***factorial*** *expression,*

$$n! = n \cdot (n-1) \cdot ... \cdot 2 \cdot 1$$

The factorial symbol is an exclamation point, which is used to indicate the excitement of counting.

3.2.2 Combinations

Now that we've established a quick method of counting the number of ways to order n distinct objects, let's figure out how to do our original problem. At the start of this section we asked how to count the number of ways we could flip 10 coins and have 3 of them be heads. The valid outcomes include

$$\begin{gathered}(\mathtt{H},\mathtt{H},\mathtt{H},\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T})\\(\mathtt{H},\mathtt{H},\mathtt{T},\mathtt{H},\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T})\\(\mathtt{H},\mathtt{H},\mathtt{T},\mathtt{T},\mathtt{H},\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T})\\\vdots\end{gathered}$$

But its not immediately clear how to count all of these, and it definitely isn't worth listing them all out. Instead let's apply the permutations trick we learned in Section 3.2.2.

Suppose we have 10 coins, 3 of which are heads up, the remaining 7 of which are tails up. Label the 3 heads as coins 1, 2, and 3. Label the 7 tails as coins 4, 5, 6, 7, 8, 9, and 10. There are 10! ways to order, or permute, these 10 (now distinct, since we labeled them) coins. However, many of these permutations correspond to the same string of `H`'s and `T`'s. For example, coins 7 and 8 are both tails, so we would be counting the two permutations

$$(1, 2, 3, 4, 5, 6, \mathbf{7}, \mathbf{8}, 9, 10)$$
$$(1, 2, 3, 4, 5, 6, \mathbf{8}, \mathbf{7}, 9, 10)$$

as different, when they both correspond to the outcome

$$(\mathtt{H}, \mathtt{H}, \mathtt{H}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T})$$

hence we are *over counting* by just taking the factorial of 10. In fact, for the string above, we could permute the last 7 coins in the string (all tails) in 7! ways, and we would still get the same string, since they are all tails. To any particular permutation of these last 7 coins, we could permute the first 3 coins in the string (all heads) in 3! ways and still end up with the string

$$(\mathtt{H}, \mathtt{H}, \mathtt{H}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T})$$

Hence there are $7! \cdot 3!$ ways in which we can order the 10 labeled coins to get the above outcome.

Now consider the following possible outcome.

$$(\mathtt{H}, \mathtt{H}, \mathtt{T}, \mathtt{H}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T}, \mathtt{T})$$

We can still permute the 3 heads up coins in 3! ways, and all of these ways would still correspond to the outcome above. Similarly, we can still permute the 7 tails up coins in 7! ways without changing the outcome. Again, we arrive at $7! \cdot 3!$ ways to order the labeled coins corresponding to the above outcome. We can now generalize from these simple examples.

To each string of `H`'s and `T`'s, we can rearrange the coins in $3! \cdot 7!$ ways without changing the actual grouping of `H`'s and `T`'s in the string. So if there are 10! total ways of ordering the labeled coins, we are counting each unique grouping of heads and tails $3! \cdot 7!$ times, when we should only be counting it once. Dividing the total number of permutations by the factor by which we over count each unique grouping of heads and tails, we find that the number of unique groupings of `H`'s and `T`'s is

$$\#\text{ of outcomes with 3 heads and 7 tails} = \frac{10!}{3!7!}$$

This leads us to the definition of the binomial coefficient.

Definition 3.7. *The* ***binomial coefficient*** *is defined*

$$\binom{n}{k} \doteq \frac{n!}{k!(n-k)!}$$

The binomial coefficient, denoted $\binom{n}{k}$, represents the number of ways to pick k objects from n objects where the ordering within the chosen k objects doesn't matter. In the previous example, $n = 10$ and $k = 3$. We could rephrase the question as, "How many ways can we pick 3 of our 10 coins to be heads?" The answer is then

$$\binom{n}{k} = \binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = 120$$

We read the expression $\binom{n}{k}$ as "n choose k". Let's now apply this counting trick to compute the probabilities of poker hands.

3.2.3 Poker

One application of counting includes computing probabilities of poker hands. A poker hand consists of 5 cards drawn from the deck. The order in which we receive these 5 cards is irrelevant. The number of possible hands is thus

$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = 2,598,960$$

since there are 52 cards to choose 5 cards from.

In poker, there are types of hands that are regarded as valuable in the following order, from most to least valuable.

1. Royal Flush: A, K, Q, J, 10 all in the same suit.
2. Straight Flush: Five cards in a sequence, all in the same suit.
3. Four of a Kind: Exactly what it sounds like.
4. Full House: 3 of a kind with a pair.
5. Flush: Any 5 cards of the same suit, but not in sequence.
6. Straight: Any 5 cards in sequence, but not all in the same suit.
7. Three of a Kind: Exactly what it sounds like.
8. Two Pair: Two pairs of cards.
9. One Pair: One pair of cards.

10. High Card: Anything else.

Let's compute the probability of drawing some of these hands.

Exercise 3.8. Compute the probabilities of the above hands.

Solution.

1. There are only 4 ways to get this hand. Either we get the royal cards in diamonds, clubs, hearts, or spades. We can think of this has choosing 1 suit from 4 possible suits. Since there are $\binom{52}{5}$ possible hands, the probability of this particular hand is

$$P(\text{Royal Flush}) = \frac{\binom{4}{1}}{\binom{52}{5}} \approx 1.5 \cdot 10^{-6}$$

2. Assuming hands like K, A, 2, 3, 4 don't count as consecutive, there are in total 10 valid consecutive sequences of 5 cards (each starts with any of A,2,...,10). We need to pick 1 of 10 starting values, and for each choice of a starting value, we can pick 1 of 4 suits to have them all in. This gives a total of $\binom{10}{1} \cdot \binom{4}{1} = 40$ straight flushes. However, we need to subtract out the probability of a royal flush, since one of the ten starting values we counted was 10 (10, J, Q, K, A is a royal flush, which, in poker, is distinct from getting a straight flush). Hence the probability of this hand is

$$P(\text{Straight Flush}) = \frac{\binom{10}{1}\binom{4}{1} - \binom{4}{1}}{\binom{52}{5}} \approx 1.5 \cdot 10^{-5}$$

3. There are 13 values and only one way to get 4 of a kind for any particular value. However, for each of these ways to get 4 of a kind, the fifth card in the hand can be any of the remaining 48 cards. Formulating this in terms of our binomial coefficient, there are $\binom{13}{1}$ ways to choose the value, $\binom{12}{1}$ ways to choose the fifth card's value, and $\binom{4}{1}$ ways to choose the suit of the fifth card. Hence the probability of such a hand is

$$P(\text{Four of a Kind}) = \frac{\binom{13}{1}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}} \approx 0.00024$$

4. For the full house, there are $\binom{13}{1}$ ways to pick the value of the triple, $\binom{4}{3}$ ways to choose which 3 of the 4 suits to include in the triple, $\binom{12}{1}$ ways to pick the value of the double, and $\binom{4}{2}$ ways to choose which 2 of the 4 suits to include in the double. Hence the probability of this hand is

$$P(\text{Full House}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} \approx 0.0014$$

5. through 10. are left as exercises. The answers can be checked on the Wikipedia page titled “Poker probability”.

$\square$