3 Compound Probability

3.1 Set Theory

A probability measure P is a function that maps subsets of the state space Ω to numbers in the interval $[0, 1]$. In order to study these functions, we need to know some basic set theory.

3.1.1 Basic Definitions

Definition 3.1. *A **set** is a collection of items, or elements, with no repeats. Usually we write a set A using curly brackets and commas to distinguish elements. For example, we write*

$$A = \{a_0, a_1, a_2\}$$

*to mean A is a set that has three distinct elements: a_0, a_1, and a_2. The size of the set A is denoted $|A|$ and is called the **cardinality** of A. In the above example, $|A| = 3$. The **empty set** is denoted $\varnothing$ and means*

$$\varnothing = \{\ \},$$

i.e. the set containing no elements.

Some essential set operations in probability are the intersection, union, and complement operators, denoted $\cap, \cup$, and ^{c}. They are defined below

Definition 3.2. ***Intersection**, **union**, and **complementation** are defined as follows. If A and B are subsets of our sample space Ω, then we write*

$$\begin{aligned}
&A \cap B = \{x \in \Omega : x \in A \text{ and } x \in B\} && (Intersection).\\
&A \cup B = \{x \in \Omega : x \in A \text{ or } x \in B\} && (Union).\\
&A^c = \{x \in \Omega : x \notin A\} && (Complementation).
\end{aligned}$$

Notation: The use of the colon ":" above is another way to say "such that". For example, the first line would read, "A intersect B is equal to the set of x in Ω such that x is in A and x is in B".

Another concept that we need to be familiar with is that of disjointness. For two sets to be disjoint, they must share no common elements, i.e. their intersection is empty.

Definition 3.3. *We say two sets A and B are **disjoint** if*

$$A \cap B = \varnothing$$

It turns out that if two sets A and B are disjoint, then we can write the probability of their union as

$$P(A \cup B) = P(A) + P(B)$$

3.1.2 Set Algebra

There is a neat analogy between set algebra and regular algebra. Roughly speaking, when manipulating expressions of sets and set operations, we can see that " $\cup$ " acts like " $+$ " and " $\cap$ " acts like " $\times$ ". Taking the complement of a set corresponds to taking the negative of a number. This analogy isn't perfect, however. If we considered the union of a set A and its complement A^c, the analogy would imply that $A \cup A^c = \varnothing$, since a number plus its negative is 0. However, it is easily verified that $A \cup A^c = \Omega$ (Every element of the sample space is either in A or not in A.)

Although the analogy isn't perfect, it can still be used as a rule of thumb for manipulating expressions like $A \cap (B \cup C)$. The number expression analogy to this set expression is $a \times (b + c)$. This would suggest we could write them as

$$\begin{aligned} a \times (b + c) &= a \times b + a \times c \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned}$$

The second set equality is true. Remember that what we just did was not a proof, but rather a non-rigorous rule of thumb to keep in mind. We still need to actually prove this expression.

Exercise 3.4. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. To show set equality, we can show that the sets are contained in each other. This is usually done in two steps.

Step 1: "$\subset$". First we will show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Select an arbitrary element in $A \cap (B \cup C)$, denoted ω. Then by definition of intersection, $\omega \in A$ and $\omega \in (B \cup C)$. By definition of union, $\omega \in (B \cup C)$ means that $\omega \in B$ or $\omega \in C$. If $\omega \in B$, then since ω is also in A, we must have $\omega \in A \cap B$. If $\omega \in C$, then since ω is also in A, we must have $\omega \in A \cap C$. Thus we must have either

$$\omega \in A \cap B \text{ or } \omega \in A \cap C$$

Hence, $\omega \in (A \cap B) \cup (A \cap C)$. Since ω was arbitrary, this shows that any element of $A \cap (B \cup C)$ is also an element of $(A \cap B) \cup (A \cap C)$. Thus we have shown

$$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$$

Step 2: " $\supset$ ". Next we will show that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Select an arbitrary element in $(A \cap B) \cup (A \cap C)$, denoted ω. Then $\omega \in (A \cap B)$ or $\omega \in (A \cap C)$. If $\omega \in A \cap B$, then $\omega \in B$. If $\omega \in A \cap C$, then $\omega \in C$. Thus ω is in either B or C, so $\omega \in B \cup C$. In either case, ω is also in A. Hence $\omega \in A \cap (B \cup C)$. Thus we have shown

$$(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$$

Since we have shown that these sets are included in each other, they must be equal. This completes the proof. $\square$

On the website, plug in each of the sets $(A \cap B) \cup (A \cap C)$ and $A \cap (B \cup C)$. Observe that the highlighted region doesn't change, since the sets are the same!

3.1.3 DeMorgan's Laws

In this section, we give two important set identities that are useful for manipulating expressions of sets. These rules are known as DeMorgan's Laws.

Theorem 3.5 (DeMorgan's Laws)**.** *Let A and B be subsets of our sample space Ω. Then*

(a) $(A \cup B)^c = A^c \cap B^c$

(b) $(A \cap B)^c = A^c \cup B^c$.

Proof.

(a) We will show that $(A \cup B)^c$ and $A^c \cap B^c$ are contained within each other.

Step 1: "$\subset$". Suppose $\omega \in (A \cup B)^c$. Then ω is not in the set $A \cup B$, i.e. in neither A nor B. Then $\omega \in A^c$ and $\omega \in B^c$, so $\omega \in A^c \cap B^c$. Hence $(A \cup B)^c \subset A^c \cap B^c$.

Step 2: "$\supset$". Suppose $\omega \in A^c \cap B^c$. Then ω is not in A and ω is not in B. So ω is in neither A nor B. This means ω is not in the set $(A \cup B)$, so $\omega \in (A \cup B)^c$. Hence $A^c \cap B^c \subset (A \cup B)^c$.

Since $A^c \cap B^c$ and $(A \cup B)^c$ are subsets of each other, they must be equal.

(b) Left as an exercise.

$\square$

If you're looking for more exercises, there is a link on the Set Theory page on the website that links to a page with many set identities. Try to prove some of these by showing that the sets are subsets of each other, or just plug them into the website to visualize them and see that their highlighted regions are the same.