Statistical Inference of Discretely Observed Compound Poisson Processes and Related Jump Processes

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Chapter 1

Introduction

Jump processes are a group of continuous-time stochastic processes whose values move in discrete amounts at random times. The compound Poisson process sits near the centre of this rich class of processes, and its versatility makes it an exceptional candidate for modelling phenomena in the world. However, its versatility comes at a price - statistical inference on them is difficult when we only observe values at discrete points in time because a significant amount of information is lost.

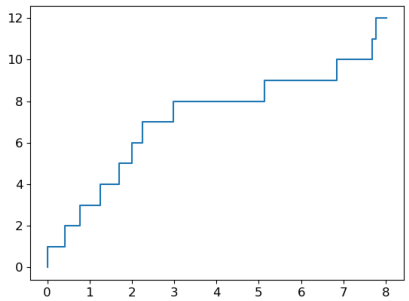
In this essay, we discuss and implement various non-parametric methods to perform inference on compound Poisson processes. In particular, we first employ a spectral approach using suitable kernel functions as shown in van Es et al. [6] and in Comte et al. [2]. We then visit non-parametric Bayesian inference. We simplify our problem to the parametric case in order to construct the density estimator laid out in Gugushvili et al. [4]. Finally, we generalise the Bayesian procedure using Dirichlet processes and illustrate its performance via numerical simulations.

Since this essay is computational in flavour, we focus mainly on carefully deriving the forms of the estimators and analyse their performance on various example simulations. Therefore, we omit many theoretical guarantees of these estimators, but such results exist and are abundant in the literature.

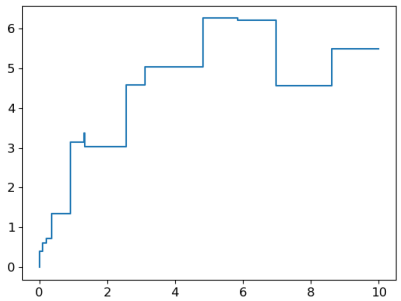
1.1 Compound Poisson Processes

Properties. Compound Poisson processes push the boundaries of Poisson processes, by allowing the jump sizes to follow a distribution rather than being of fixed unit size. We associate compound Poisson processes with the following three key properties:

1. The occurrence of the jumps follow a Poisson process,
2. The jumps sizes are independent, and follow a common distribution,
3. The inter-arrival times between the jumps and the sizes of the jumps are mutually independent random variables.



(a) Poisson Process



(b) Compound Poisson Process

Figure 1.1.1: Simulations of Jump Processes

Definition 1.1.1 (Poisson Process)**.** A Poisson process with intensity λ is a non-negative, non-decreasing, integer-valued stochastic process $(N_t)_{t\geq 0}$ starting at 0 with the following properties:

1. **Independent Increments:** For every $n \in \mathbb{N}$ and $t_1, \ldots, t_n$ such that $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, we have that
$$N_{t_n} - N_{t_{n-1}}, \ldots, N_{t_2} - N_{t_1}, N_{t_1}$$
are mutually independent,

2. **Stationary Increments:** The number of occurrences in any interval of length Δ is a Poisson random variable with parameter $\lambda\Delta$:
For every $\Delta > 0$ and $t \geq 0$ we have that
$$N_{(t+1)\Delta} - N_{t\Delta} \sim \mathcal{P}(\lambda\Delta).$$

As we can see in Figure 1.1.1a, the jump sizes are of unit length, whilst the compound Poisson process allows the jumps to vary in both magnitude and direction.

Definition 1.1.2 (Compound Poisson Process)**.** Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity λ. Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d random variables with common distribution F. Also assume that this sequence is independent of the Poisson process.

Then, a compound Poisson process with intensity λ and jump distribution F is a stochastic process $(X_t)_{t\geq 0}$ such that
$$X_t = \sum_{i=1}^{N_t} Y_i$$
We will say that Y_i are the jumps of the compound Poison process. By convention, we take $X_t = 0$ if $N_t = 0$.

1.2 The Statistical Inverse Problem

Suppose we are only able to observe the values of a CPP at times $\Delta, 2\Delta, \ldots, n\Delta$, thus giving us observations
$$\{X_{i\Delta} : i = 1, \ldots, n\}.$$

Assumptions. For our purposes, we assume that the intensity λ of the CPP is known. We also assume that the jump distribution F has continuous probability density function f and assigns zero mass at the origin $\{0\}$, i.e.

$$F(\{0\}) = 0.$$

This is because, if not, then the event of a jump at some time t could potentially be indistinguishable to the event of no jump at time t.

Goal. We want to recover the density f from our observations $\{X_{i\Delta} : i = 1, \dots, n\}$. We approach this problem using non-parametric estimation, as we want to put only few assumptions on density f. For example, we may only assume that f comes from the set of Lipschitz continuous functions. Such spaces are infinite-dimensional, and so parametric models are unsuitable for this task.

Transformation of Observations. Given our observations $\{X_{i\Delta} : i = 1, \dots, n\}$, consider increments $\{Z_i : i = 1, \dots, n\}$ given by

$$Z_i = X_{i\Delta} - X_{(i-1)\Delta}. \tag{1.1}$$

By the independent increments (Property 1) of the Poisson process, all Z_i are mutually independent.

Proposition 1.2.1. *For Z_i defined in* (1.1)*, we have that*

$$Z_i \overset{\mathcal{L}}{=} \mathbb{1}(N > 0) \sum_{j=1}^{N} Y_i \tag{1.2}$$

where $N \sim \mathcal{P}(\lambda\Delta)$ and N is independent of jumps Y_i.

Proof.

$$\begin{aligned} Z_i &= X_{i\Delta} - X_{(i-1)\Delta} \\ &= (Y_{N_{(i-1)\Delta}+1} + \cdots + Y_{N_{i\Delta}}) \mathbb{1}(N_{i\Delta} > N_{(i-1)\Delta}) \\ &\overset{\mathcal{L}}{=} (Y_1 + \cdots + Y_{N_{i\Delta} - N_{(i-1)\Delta}}) \mathbb{1}(N_{i\Delta} > N_{(i-1)\Delta}) \qquad (1.3) \\ &\overset{\mathcal{L}}{=} \mathbb{1}(N > 0) \sum_{j=1}^{N} Y_i \qquad (1.4) \end{aligned}$$

where $N \sim \mathcal{P}(\lambda\Delta)$. Line (1.3) follows by the i.i.d property of the jumps and line (1.4) follows by the stationary increments (Property 2) of a Poisson process. The independence of N and the jumps follows from the independence of the Poisson process and the jumps. □

Since all Z_i are mutually independent, it will be more ideal to deal with observations Z_i rather than $X_{i\Delta}$. Henceforth, we will refer to our observations as $\{Z_i : i = 1, \dots, n\}$ and call such Z_i as a Poisson random sum. The non-linearity of this inverse problem stems from the randomness of our Poisson random variable N in the Poisson random sum.

1.3 Properties of Poisson Random Sums

We have now converted our problem into that involving Poisson random sums. Therefore, we ought to investigate the properties of such random variables and exploit these properties for our inference. This section is important and these properties will be referred to throughout the essay.

Poisson Random Sums. Let $Y_1, Y_2, \dots$ be a sequence of i.i.d random variables with probability density function f. As before, we will call these jumps. Let

$$Z = \mathbb{1}(N > 0) \sum_{j=1}^{N} Y_i, \qquad \text{with } N \sim \mathcal{P}(\lambda\Delta) \tag{1.5}$$

be a Poissom random sum.

Proposition 1.3.1. *The characteristic function, ϕ_Z, of Z defined in* (1.5) *is given by*

$$\phi_Z(t) = \mathbb{E}e^{itZ} = e^{-\lambda\Delta + \lambda\Delta\phi_f(t)}$$

where ϕ_f denotes the characteristic function of a single jump.

Proof.

$$\begin{aligned}
\phi_Z(t) &= \mathbb{E}e^{itZ} = \mathbb{E}e^{it\mathbb{1}(N>0)\sum_{i=1}^{N} Y_i} \\
&= \mathbb{E}[\mathbb{1}(N=0)] + \mathbb{E}\left[\mathbb{1}(N>0)\prod_{i=1}^{N} e^{itY_i}\right] \\
&= e^{-\lambda\Delta} + \mathbb{E}\left[\mathbb{1}(N>0)\mathbb{E}\left[\prod_{i=1}^{N} e^{itY_i}\middle| N\right]\right] && \text{(law of total expectation)} \\
&= e^{-\lambda\Delta} + \mathbb{E}\left[\mathbb{1}(N>0)\prod_{i=1}^{N} \mathbb{E}\left[e^{itY_1}\middle| N\right]\right] && \text{(i.i.d property of jumps)} \\
&= e^{-\lambda\Delta} + \mathbb{E}\left[\mathbb{1}(N>0)\prod_{i=1}^{N} \phi_f(t)\right] && \text{(independence of } Y_1 \text{ and } N) \\
&= e^{-\lambda\Delta} + \mathbb{E}\left[\mathbb{1}(N>0)e^{N\ln\phi_f(t)}\right] \\
&= \mathbb{E}\left[e^{N\ln\phi_f(t)}\right] \\
&= \exp(\lambda\Delta(e^{\ln\phi_f(t)} - 1)) && \text{(MGF of } \mathcal{P}(\lambda\Delta)) \\
&= e^{-\lambda\Delta + \lambda\Delta\phi_f(t)}
\end{aligned}$$

□

For the next property, we require the following Lemma.

Lemma 1.3.1. *Let X and Y be independent random variables with density functions f_X, f_Y and characteristic functions ϕ_X, ϕ_Y respectively. Then the sum $Z = X + Y$ is a random variable with density function f_Z and characteristic function ϕ_Z, where*

$$f_Z = f_X * f_Y, \qquad \phi_Z = \phi_X\phi_Y$$

and $*$ denotes the convolution i.e.

$$(f * g)(t) = \int_{\mathbb{R}} f(\tau)g(t-\tau)d\tau.$$

Proposition 1.3.2. *The distribution $\mathbb{P}_Z$ of Z is absolutely continuous with respect to measure $\mu = \delta_{\{0\}} + \text{Leb}$ and has Radon-Nikodym derivative*

$$\frac{\mathrm{d}\mathbb{P}_Z}{\mathrm{d}\mu}(x) = e^{-\lambda\Delta}\mathbb{1}_{\{0\}}(x) + (1 - e^{-\lambda\Delta})\sum_{m=1}^{\infty} a_m(\lambda\Delta)f^{*m}(x)\mathbb{1}_{\mathbb{R}\setminus\{0\}}(x)$$

where

$$a_m(\lambda\Delta) = \frac{1}{e^{\lambda\Delta} - 1}\frac{(\lambda\Delta)^m}{m!}$$

and $f^{*m} = f\overbrace{*\cdots*}^{m}f$ *denotes the m-fold convolution of* f *with itself.*

Proof. Suppose we have $A \in \mathcal{B}$ such that $\mu(A) = 0$. Then $0 \notin A$ and A has Lebesgue measure 0. Therefore, under event A, we have that $N > 0$ i.e.

$$\left\{\mathbb{1}(N > 0)\sum_{j=1}^{N} Y_i \in A\right\} \subseteq \{N > 0\} \cap \left\{\sum_{j=1}^{N} Y_i \in A\right\}.$$

Using this, we get that

$$\begin{aligned}
\mathbb{P}_Z(A) &= \mathbb{P}\left(\mathbb{1}(N > 0)\sum_{j=1}^{N} Y_i \in A\right) \\
&\leq \mathbb{P}\left(N > 0, \sum_{j=1}^{N} Y_i \in A\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{n} Y_i \in A, N = n\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{n} Y_i \in A\right)\mathbb{P}(N = n) \\
&= 0
\end{aligned}$$

since $\sum_{j=1}^{n} Y_i$ has a density by Lemma 1.3.1. Therefore, $\mathbb{P}_Z$ is absolutely continuous with respect to μ. Furthermore, for $A \in \mathcal{B}$,

$$\begin{aligned}
\mathbb{P}_Z(A) &= \mathbb{P}\left(0 \in A, N = 0\right) + \sum_{m=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{m} Y_i \in A\right)\mathbb{P}(N = m) \\
&= \mathbb{P}(N = 0)\int_A d\delta_0 + \sum_{m=1}^{\infty} e^{-\lambda\Delta}\frac{(\lambda\Delta)^m}{m!}\int_A f^{*m}(x)dx \\
&= \int_A e^{-\lambda\Delta}\mathbb{1}_{\{0\}}(x) + \sum_{m=1}^{\infty} e^{-\lambda\Delta}\frac{(\lambda\Delta)^m}{m!}f^{*m}(x)\mathbb{1}_{\mathbb{R}\setminus\{0\}}(x)d\mu(x)
\end{aligned}$$

giving the result. □

In light of Proposition 1.3.2, we see that a zero-valued Poisson random sum Z (corresponding to $N = 0$) provides no additional information about the density f. In this case, conditional on $N > 0$, by slight modification of our proof, Z has probability density function given by

$$g(x) = \frac{e^{-\lambda\Delta}}{1 - e^{-\lambda\Delta}}\sum_{m=1}^{\infty}\frac{(\lambda\Delta)^m}{m!}f^{*m}(x) \tag{1.6}$$

Chapter 2

Kernel Density Estimation

Kernel density estimation is a non-parametric method for estimating the probability density function from a finite data sample. Its ability to generate smooth curves from a discrete set of observations without assuming any parametric model make it an ideal candidate for estimating continuous probability density functions.

Motivation Let X be a random variable with probability density p with respect to the Lebesgue measure on $\mathbb{R}$. The corresponding distribution function is

$$F(x) = \int_{-\infty}^{x} p(t)dt, \qquad \text{and we have} \qquad \frac{\mathrm{d}F}{\mathrm{d}x} = p.$$

Consider n i.i.d observations $X_1, \ldots, X_n$ with same distribution as X. The empirical distribution function is given by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x).$$

By the Strong Law of Large Numbers, since for fixed x, $I(X_i \leq x)$ are i.i.d, we have that

$$F_n(x) \to \mathbb{E}\left[I(X_1 \leq x)\right] = \mathbb{P}(X \leq x) = F(x) \text{ a.s. as } n \to \infty.$$

Therefore, $F_n(x)$ is a consistent estimator of $F(x)$ for every $x \in \mathbb{R}$. Also, since $p(x) = F'(x)$, for sufficiently small $h > 0$ we can write an approximation

$$p(x) \approx \frac{F(x+h) - F(x-h)}{2h}$$

Thus, replacing F by our empirical distribution function F_n to gives us an estimator $\hat{p}_n(x)$ of $p(x)$ where

$$\begin{aligned}
\hat{p}_n(x) &= \frac{F_n(x+h) - F_n(x-h)}{2h} \\
&= \frac{1}{2nh}\sum_{i=1}^{n} I(x-h < X_i \leq x+h) \\
&= \frac{1}{nh}\sum_{i=1}^{n} k_0\left(\frac{x - X_i}{h}\right)
\end{aligned}$$

where $k_0(u) = \frac{1}{2}I(-1 < u \leq 1)$. Note that $k_0(-u) = k_0(u)$ and $\int k_0(u)du = 1$. A simple generalisation gives us a kernel function and corresponding kernel density estimator.

Definition 2.0.1. (Kernel Density Estimator). Let $k : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function such that

$$\int_{\mathbb{R}} k(u)du = 1, \qquad \text{and} \qquad k(-u) = k(u).$$

Then we say that k is a kernel function. Let $h > 0$. Then the kernel density estimator of n i.i.d observations $X_1, \ldots, X_n$ is given by

$$\hat{p}_n(x) = \frac{1}{nh} \sum_{i=1}^{n} k\left(\frac{x - X_i}{h}\right) \tag{2.1}$$

2.1 Estimation of Convolution Powers

Returning back to our problem, suppose we have non-zero observations $\{Z_i : i = 1, \ldots, n\}$ of the compound Poisson process. By (1.6) following Proposition 1.3.2, we have that each Z_i has density g given by

$$g = \frac{e^{-\lambda\Delta}}{1 - e^{-\lambda\Delta}} \sum_{m=1}^{\infty} \frac{(\lambda\Delta)^m}{m!} f^{*m} = \frac{1}{e^{\lambda\Delta} - 1} \sum_{m=1}^{\infty} \frac{(\lambda\Delta)^m}{m!} f^{*m} \tag{2.2}$$

We can see that density g is a weighted sum of convolution power of density f. If we could rewrite this expression in terms of f, then an estimator of density g directly provides an estimator of f. We will show that this is possible for λ, Δ sufficiently small.

Proposition 2.1.1. *Provided that $\lambda\Delta < \log 2$, we have*

$$f = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \frac{(e^{\lambda\Delta} - 1)^m}{\lambda\Delta} g^{*m}$$

Proof. Let $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ denote the Fourier transform, defined by

$$\mathcal{F}[f](t) = \int_{\mathbb{R}} e^{itx} f(x)dx, \qquad t \in \mathbb{R}.$$

Note that this is the same as the characteristic function of a random variable with probability density function f. We first show, using the Fourier Inversion Theorem, that the Fourier transform defined above is injective.

Fourier Inversion Theorem. Let $f \in L^1(\mathbb{R})$ be a continuous function. Suppose also that $\mathcal{F}[f]$ is integrable. Then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \mathcal{F}[f](t)dt \tag{2.3}$$

We denote $\mathcal{F}[f] : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ by

$$\mathcal{F}^{-1}[f](t) = \frac{1}{2\pi} \int e^{-itx} f(x)dx$$

Then we can write the Fourier Inversion Theorem more compactly as

$$\mathcal{F}^{-1}\mathcal{F}[f] = \mathcal{F}\mathcal{F}^{-1}[f] = f.$$

To show that $\mathcal{F}$ is injective, suppose that f is in $L^1(\mathbb{R})$ such that $\mathcal{F}[f] = 0$. Then, we have that $\mathcal{F}[f]$ is in $L^1(\mathbb{R})$ since it is the zero function. Therefore, by the Fourier Inversion Theorem, we get that

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \mathcal{F}[f](t)dt = 0,$$

thus showing that $f \mapsto \phi_f$ is injective on the space of Lebesgue-integrable functions, which is where our densities live.

We next use the Convolution Theorem that states that $\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$ for integrable functions f, g. From (2.2), the linearity of an integral and the Convolution Theorem we get that

$$\begin{aligned}\mathcal{F}[g] &= \frac{1}{e^{\lambda\Delta} - 1} \sum_{m=1}^{\infty} \frac{(\lambda\Delta)^m}{m!} (\mathcal{F}[f])^m \\ &= \frac{\exp(\lambda\Delta\mathcal{F}[f]) - 1}{e^{\lambda\Delta} - 1}\end{aligned}$$

Rearranging, we get that

$$\exp(\lambda\Delta\mathcal{F}[f]) = 1 + (e^{\lambda\Delta} - 1)\mathcal{F}[g]$$

Note that $\left\|(e^{\lambda\Delta} - 1)\mathcal{F}[g]\right\|_\infty < \left\|e^{\lambda\Delta} - 1\right\|_\infty < 1$ for $\lambda\Delta < \log 2$. Therefore, the distinguished logarithm defined in the previous section reduces to the principal branch of the logarithm. Thus, we get that

$$\mathcal{F}[f] = \frac{\log(1 + (e^{\lambda\Delta} - 1)\mathcal{F}[g])}{\lambda\Delta} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \frac{(e^{\lambda\Delta} - 1)^m}{\lambda\Delta} \mathcal{F}[g]^m \tag{2.4}$$

by the Taylor expansion of the logarithm, which holds since

$$\lambda\Delta < \log 2 \implies \left\|(e^{\lambda\Delta} - 1)\mathcal{F}[h]\right\|_\infty < 1.$$

Applying Fourier Inversion Theorem, since f is continuous and a probability density function so $f \in L^1(\mathbb{R})$, gives the result. □

2.1.1 Construction

Consider the kernel density estimator, shown in Definition 2.0.1, given by

$$\hat{g}_n(x) = \frac{1}{nh} \sum_{i=1}^{n} k\left(\frac{x - Z_i}{h}\right), \qquad \text{for some fixed } h > 0. \tag{2.5}$$

We take the kernel function k to be the sinus cardinal function

$$k(x) = \frac{\sin(\pi x)}{\pi x}. \tag{2.6}$$

Proposition 2.1.2. *The sinus cardinal function given in* (2.6) *is a kernel function and has characteristic function*

$$\phi_k(t) = \int_{\mathbb{R}} k(x) e^{itx} dx = \mathbb{1}_{[-\pi,\pi]}(t)$$

Proof.

$$\frac{\sin(\pi x)}{\pi x} = \frac{1}{2\pi} \frac{e^{-i\pi x} - e^{i\pi x}}{-ix} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ixt} dt = \mathcal{F}^{-1}\left[\mathbb{1}_{[-\pi,\pi]}\right](x)$$

Applying the Fourier Inversion Theorem and noting that the characteristic function of the kernel is the same as the Fourier transform of the kernel gives the result. □

We also have the following result:

Lemma 2.1.1. *Let k be a kernel function and let bandwidth $h > 0$. Let $\hat{g}_n$ be the kernel density estimator given in* (2.5) *and let ϕ_k be the characteristic function of kernel k. Also, let ϕ_{emp} be the empirical characteristic function given by*

$$\phi_{emp}(t) = \sum_{j=1}^{n} e^{itZ_j}.$$

Then for all $t \in \mathbb{R}$,

$$\phi_{\hat{g}_n}(t) = \phi_{emp}(t)\phi_k(ht)$$

Proof.

$$\begin{aligned}
\phi_{\hat{g}_n}(t) &= \int_{-\infty}^{\infty} e^{itx}\hat{g}_n(x)dx \\
&= \int_{-\infty}^{\infty} e^{itx}\frac{1}{nh}\sum_{j=1}^{n} k\left(\frac{x - Z_j}{h}\right)dx \\
&= \frac{1}{n}\sum_{j=1}^{n} e^{itZ_j}\int_{-\infty}^{\infty} e^{ithy}k(y)dy \qquad \left(y = \frac{x - Z_j}{h}\right) \\
&= \phi_{\text{emp}}(t)\phi_k(ht)
\end{aligned}$$

□

Therefore, using the Fourier Inversion Theorem and Lemma 2.1.1 for our sinus cardinal kernel k, we see that the kernel density estimator can be rewritten as

$$\begin{aligned}
\hat{g}_n(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \phi_{\text{emp}}(t)\phi_k(ht)e^{-itx}dt \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \phi_{\text{emp}}(t)e^{-itx}dt.
\end{aligned}$$

Furthermore, using Lemma 1.3.1, which is equivalent to the Convolution Theorem, we note that $\phi_{g^{*m}} = (\phi_g)^m$. Since $\hat{g}_n$ is an estimator of g, it is sensible to use $\hat{g}_n^{*m}$ as an estimator of convolution power g^*. Note that

$$\begin{aligned}
\hat{g}_n^{*m}(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \phi_{\hat{g}_n^{*m}}(t)e^{-itx}dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\phi_{\hat{g}_n}(t)\right)^m e^{-itx}dt \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left(\phi_{\text{emp}}(t)\right)^m e^{-itx}dt
\end{aligned}$$

Therefore, using $\hat{g}_n^{*m}$ as our estimator of g^{*m}, we immediately obtain, provided $\lambda\Delta < \log 2$, an estimator for f given by

$$\hat{f}_n(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\frac{(e^{\lambda\Delta} - 1)^m}{\lambda\Delta}\hat{g}_n^{*m}(x)$$

For small $\lambda\Delta < \log 2$, $e^{\lambda\Delta} - 1$ will be close to 0. In particular, $\frac{(e^{\lambda\Delta}-1)^m}{m} \to 0$ as $m \to \infty$. Therefore, it is sensible to truncate the series up to some sufficiently large K to give

$$\hat{f}_n(x) = \sum_{m=1}^{K} \frac{(-1)^{m+1}}{m}\frac{(e^{\lambda\Delta} - 1)^m}{\lambda\Delta}\hat{g}_n^{*m}(x) \tag{2.7}$$

2.1.2 Simulation Results

To compute estimators $\hat{g}_n^{*m}$, we decompose it into two terms, approximate the integral using the trapezoid rule and then perform the Fast Fourier Transform (FFT) to obtain a range of values for the density function estimator.

We can express $\hat{g}_n^{*m} = \hat{g}_{n,1}^{*m} + \hat{g}_{n,2}^{*m}$ where

$$\hat{g}_{n,1}^{(m)}(x) = \frac{1}{2\pi} \int_0^{\frac{\pi}{h}} (\phi_{\text{emp}}(t))^m \, e^{-itx} dt$$

$$\begin{aligned} \hat{g}_{n,2}^{(m)}(x) &= \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{0} (\phi_{\text{emp}}(t))^m \, e^{-itx} dt \\ &= \frac{1}{2\pi} \int_0^{\frac{\pi}{h}} (\phi_{\text{emp}}(-t))^m \, e^{itx} dt \end{aligned}$$

We work with $\hat{g}_{n,1}^{(m)}$ - the case for $\hat{g}_{n,2}^{(m)}$ is very similar. We approximate $\hat{g}_{n,1}^{(m)}(x)$ using the trapezoid rule.

Trapezoid Rule. Let $\{t_j\}_{j=0}^{N-1}$ be a set of N equally spaced values partitioning $[a, b]$, with spacing $\eta = \frac{b-a}{N}$. Then, for integrable function h we get the following approximation

$$\int_a^b h(x)dx \approx \eta \sum_{j=0}^{N-1} h(t_j) \tag{2.8}$$

We take a grid $t_j = j\eta$ for $j = 0, 1, \ldots, N-1$ where N is some large power of 2 and $\eta = \frac{\pi}{(N-1)h}$.

$$\hat{g}_{n,1}^{(m)}(x) \approx \frac{1}{2\pi} \sum_{j=0}^{N-1} (\phi_{\text{emp}}(t_j))^m \, e^{-it_j x} \eta$$

We take $x_k = -\frac{N\delta}{2} + \delta k$ for $k = 0, 1, \ldots, N-1$ and δ is some constant to be defined later. Then

$$\hat{g}_{nh}^{(m)(1)}(x) \approx \frac{1}{2\pi} \sum_{j=0}^{N-1} (\phi_{\text{emp}}(t_j))^m \, e^{it_j \frac{N\delta}{2}} e^{-ijk\eta\delta} \eta$$

Similarly,

$$\hat{g}_{nh}^{(m)(2)}(x) \approx \frac{1}{2\pi} \sum_{j=0}^{N-1} (\phi_{\text{emp}}(-t_j))^m \, e^{-it_j \frac{N\delta}{2}} e^{ijk\eta\delta} \eta$$

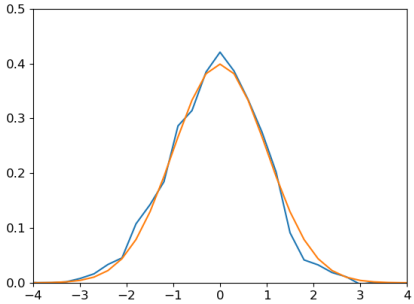
Fast Fourier Transform. Let $\{x_j\}_{j=0}^{N-1}$ be a sequence of complex numbers. The Fast Fourier Transform (FFT) computes the sequence $\{Y_k\}_{k=0}^{N-1}$ where

$$Y_k = \sum_{j=0}^{N-1} x_j e^{-ik\frac{2\pi j}{N}} \tag{2.9}$$

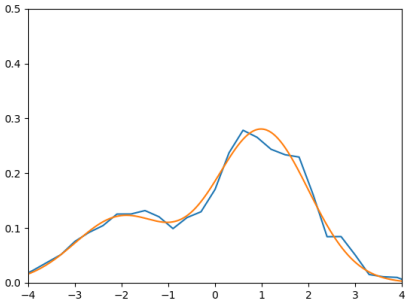
The inverse transform is given by

$$Y_k = \frac{1}{N} \sum_{j=0}^{N-1} x_j e^{ik\frac{2\pi j}{N}} \tag{2.10}$$

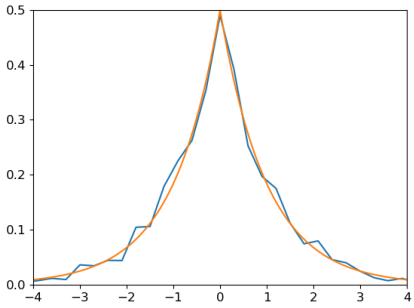
We choose $\delta = \frac{2h(N-1)}{N}$ so that $\eta\delta = \frac{2\pi}{N}$ and then apply FFT to these terms to obtain an estimate for $\hat{g}_{nh}^{(m)}$. Plugging in convolution estimators into (2.7) gives an estimator $\hat{f}_n$ of f.



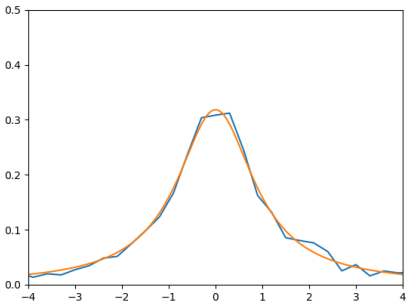
(a) Standard Gaussian



(b) Mixture of Gaussian



(c) Laplace



(d) Cauchy

Figure 2.1.1: Density Estimates via the Estimation of Convolution Powers

Examples. We simulated a compound Poisson process with intensity $\lambda = 0.3$ and we observed equally spaced points of separation $\Delta = 0.4$. We obtained a sample size of 5000 non-zero increments. For the kernel density estimators, we took $N = 4096$ number of points, bandwidth $h = 0.15$ and a truncation up to the 10th convolution power. The simulations show a satisfactory general fit, capturing the overall shape well.

In particular, the estimator is robust to distributions with strong peaks and heavy tails. We can see this from the estimates of the Laplace and Cauchy distributions. Furthermore, the estimator nicely depicts the bimodal nature of the mixture of Gaussians.

One major drawback of this estimator is the interdependence between the bandwidth h and the spacing δ of the points we evaluate the estimator over. By choosing $\delta = \frac{2h(N-1)}{N}$, we see that a larger bandwidth would result in a larger spacing, reducing the number of points of the estimate at regions of high probability mass. The resulting increased interpolation reduces the smoothness of the density estimates that we hoped to achieve through kernel density estimators.

2.2 Inversion of Characteristic Functions

Once again, suppose we have non-zero observations $\{Z_i : i = 1, \ldots, n\}$ of the compound Poisson process. Let

$$X = \mathbb{1}(N > 0) \sum_{j=1}^{N} Y_i$$

be a Poisson random sum with $N \sim \mathcal{P}(\lambda\Delta)$. Then

$$\begin{aligned}\phi_X(t) &= \mathbb{E}\left[e^{itX}\mathbb{1}(N=0)\right] + \mathbb{E}\left[e^{itX}\mathbb{1}(N>0)\right] \\ &= \mathbb{P}(N=0) + \mathbb{P}(N>0)\phi_{Z_i}(t) \\ &= e^{-\lambda\Delta} + (1-e^{-\lambda\Delta})\phi_{Z_i}(t)\end{aligned}$$

Using Proposition 1.3.1 and letting the probability density function of an observation Z_i be denoted by g (which exists by Proposition 1.3.2), we get that

$$e^{-\lambda\Delta+\lambda\Delta\phi_f(t)} = e^{-\lambda\Delta} + (1-e^{-\lambda\Delta})\phi_g(t)$$

Rearranging, we have

$$\phi_g(t) = \frac{1}{e^{\lambda\Delta}-1}(e^{\lambda\Delta\phi_f(t)}-1). \tag{2.11}$$

This suggests that if we could suitably invert the formula in (2.11) to get an expression in terms of ϕ_f, then an estimator of ϕ_g would induce an estimator for ϕ_f. Then, using Fourier Inversion Theorem, we would obtain an estimator for f.

2.2.1 Construction

The main issue to address is inverting relationship (2.11) to obtain an expression in terms of ϕ_f. However, ϕ_f takes complex values, and so we must find an inverse to the map $\exp : \mathbb{C} \to \mathbb{C}$. Such an inverse does not exist in our the sense, since it is not injective: $e^{w+2\pi i} = e^w \ \forall w \in \mathbb{C}$. Therefore, the following Lemmas are significant to obtain our desired expression.

Distinguished Logarithm.

Lemma 2.2.1. *Suppose $\phi : \mathbb{R} \to \mathbb{C}$ is a continuous function such that $\phi(0) = 1$ and $\phi_g(t) \neq 0$ for every $t \in \mathbb{R}$. Then there exists a unique continuous function $h : \mathbb{R} \to \mathbb{C}$ with $h(0) = 0$ and $\phi(t) = e^{h(t)}$ for $t \in \mathbb{R}$.*

Proof. See Theorem 7.6.2 in [1]. □

For a function ϕ satisfying the assumptions of Lemma 2.2.1, we say that the unique function h is the distinguished logarithm and we denote

$$h(t) = \mathrm{Log}(\phi)(t).$$

The following property is an easy consequence of Lemma 2.2.1:

Lemma 2.2.2. *For ϕ_1, ϕ_2 satisfying the assumptions of Lemma 2.2.1, we have for $\psi(t) = \phi_1(t)\phi_2(t)$,*

$$\mathrm{Log}(\psi) = \mathrm{Log}(\phi_1) + \mathrm{Log}(\phi_2).$$

Proof. We have that $\psi(0) = 1$ and $\psi(t) \neq 0$ for every $t \in \mathbb{R}$. Therefore, $\mathrm{Log}(\psi)$ exists by Lemma 2.2.1. Furthermore,

$$e^{\mathrm{Log}(\psi)(t)} = \psi(t) = \phi_1(t)\phi_2(t) = e^{\mathrm{Log}(\psi_1)(t)}e^{\mathrm{Log}(\psi_2)(t)} = e^{\mathrm{Log}(\psi_1)(t)+\mathrm{Log}(\psi_2)(t)}$$

The result then follows by the uniqueness of the distinguished logarithm. □

Therefore, noting that $\psi(t) \triangleq e^{\phi_f(t)-1}$ is a continuous function satisfying $\psi(0) = 1$ and $\psi(t) \neq 0$ for every $t \in \mathbb{R}$, we get that

$$\begin{aligned}
\phi_f(t) - 1 &= \mathrm{Log}\left(e^{\phi_f - 1}\right)(t) \\
&= \mathrm{Log}\left(e^{\frac{\lambda\Delta(\phi_f-1)}{\lambda\Delta}}\right)(t) \\
&= \mathrm{Log}\left(\left[e^{-\lambda\Delta}((e^{\lambda\Delta}-1)\phi_g + 1)\right]^{\frac{1}{\lambda\Delta}}\right)(t) && \text{(using (2.11))} \\
&= \frac{1}{\lambda\Delta}\mathrm{Log}\left((1-e^{-\lambda\Delta})\phi_g + e^{-\lambda\Delta})\right)(t) && \text{(Lemma 2.2.2)}
\end{aligned}$$

Therefore,

$$\phi_f(t) = 1 + \frac{1}{\lambda\Delta}\mathrm{Log}\left((1-e^{-\lambda\Delta})\phi_g + e^{-\lambda\Delta}\right)(t)$$

By Levy's Inversion formula, for integrable ϕ_f we have

$$f(x) = \frac{1}{2\pi\lambda\Delta}\int_{-\infty}^{\infty} e^{-itx}\left(\lambda\Delta + \mathrm{Log}\left((1-e^{-\lambda\Delta})\phi_g + e^{-\lambda\Delta}\right)(t)\right)dt \tag{2.12}$$

Using our theory of kernel density estimation, for some kernel k with characteristic function ϕ_k, bandwidth $h > 0$ and observations $Z_1, \ldots, Z_n$, we estimate density g by the kernel density estimator

$$\hat{g}_n(x) = \frac{1}{nh}\sum_{i=1}^{n} k\left(\frac{x - Z_i}{h}\right)$$

Technical Issues. Using Lemma 2.1.1, it is tempting to introduce an estimator $\hat{f}_n$ of f

$$\hat{f}_n(x) = \frac{1}{2\pi\lambda\Delta}\int_{-\infty}^{\infty} e^{-itx}\left(\lambda\Delta + \mathrm{Log}\left((1-e^{-\lambda\Delta})\phi_{\mathrm{emp}}\phi_k(h\,\cdot) + e^{-\lambda\Delta}\right)(t)\right)dt \tag{2.13}$$

but this brings two main issues:

1. In light of Lemma 2.2.1, we may have some measurable set A with non-zero Lebesgue measure such that $(1-e^{-\lambda\Delta})\phi_{\mathrm{emp}}(t)\phi_k(ht) + e^{-\lambda\Delta}$ is zero for $t \in A$. The distinguished logarithm is undefined under such sets and thus our estimator of f is undefined in this case.

2. There is no guarantee that the integral is finite. For example,

$$\phi_{\hat{g_n}}(t) = \frac{\exp(e^{it}) - e^{-\lambda\Delta}}{1 - e^{-\lambda\Delta}}$$

would give $\hat{f}_n(1)$ to be infinity.

In order to prove asymptotic properties, we must adjust our estimators by bounding $\hat{f}_n$ for each n using a suitable sequence $(M_n)_{n\geq 1}$. However, for our discussion, we note such limitations and provide simulations for examples where these two cases do not occur.

2.2.2 Simulation Results

We note that for $\lambda\Delta < \log 2$, the distinguished logarithm in (2.13) reduces to the principal branch of the logarithm. Therefore, we can directly use the built-in logarithm from scientific computing packages, and bounding $\hat{f}_n$ by a suitable sequence is not needed. Thus, we use (2.13) directly to compute our estimator on cases where $\lambda\Delta < \log 2$.

Kernel. We use the following kernel function k given by

$$k(t) = \frac{48t(t^2-1)\cos t - 144(2t^2-5)\sin t}{\pi t^7} \tag{2.14}$$

This expression is fairly non-trivial, but its characteristic function has a much simpler expression.

Proposition 2.2.1. *The function k defined in* (2.14) *is a kernel and has characteristic function*

$$\phi_k(t) = (1-t^2)^3 \mathbb{1}\{|t| < 1\}$$

In a similar fashion to Section 2.1.2, we rewrite (2.13) as $\hat{f}_n(x) = \hat{f}_{n,1}(x) + \hat{f}_{n,2}(x)$ where

$$\hat{f}_{n,1}(x) = \frac{1}{2\pi\lambda\Delta} \int_0^\infty e^{-itx} \log\Big((e^{\lambda\Delta}-1)\phi_{\text{emp}}(t)\phi_k(ht)+1\Big)dt \tag{2.15}$$

$$\begin{aligned}\hat{f}_{n,2}(x) &= \frac{1}{2\pi\lambda\Delta} \int_{-\infty}^0 e^{-itx} \log\Big((e^{\lambda\Delta}-1)\phi_{\text{emp}}(t)\phi_k(ht)+1\Big)dt \\ &= \frac{1}{2\pi\lambda\Delta} \int_0^\infty e^{itx} \log\Big((e^{\lambda\Delta}-1)\phi_{\text{emp}}(-t)\phi_k(ht)+1\Big)dt \end{aligned} \tag{2.16}$$

Line (2.16) follows since ϕ_k is symmetric. We again deal with $\hat{f}_{n,1}$ - the case of $\hat{f}_{n,2}$ is very similar.

Approximating (2.15) using the Trapezoid rule (2.8), we get, for spacing parameter $\eta > 0$ and $t_j = j\eta$, that

$$\hat{f}_{n,1}(x) \approx \frac{\eta}{2\pi\lambda\Delta} \sum_{k=0}^{N-1} e^{-it_j x} \log\Big((e^{\lambda\Delta}-1)\phi_{\text{emp}}(t_j)\phi_k(ht_j)+1\Big), \tag{2.17}$$

We evaluate our function $\hat{f}_{nh}^{(1)}$ at points $\{x_k\}_{k=0}^{N-1}$ given by

$$x_k = \frac{-N\delta}{2} + \delta k$$

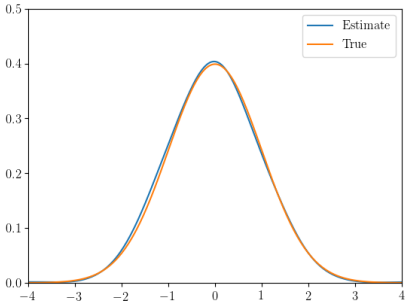
for some $\delta > 0$ to be chosen later. Thus we have

$$\hat{f}_{n,1}(x_k) \approx \frac{1}{2\pi\lambda} \sum_{j=0}^{N-1} e^{-ijk\eta\delta} e^{it_j \frac{N\delta}{2}} \psi(t_j)\eta, \tag{2.18}$$

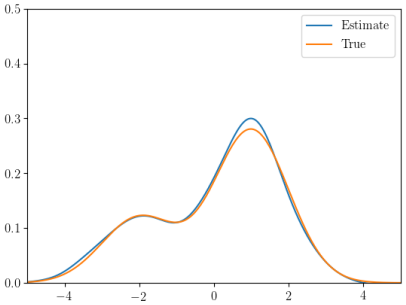
where $\psi(t) = \log\big((e^{\lambda\Delta}-1)\phi_{\text{emp}}(t)\phi_k(ht)+1\big)$. Similarly,

$$\hat{f}_{n,2}(x_k) \approx \frac{1}{2\pi\lambda} \sum_{j=0}^{N-1} e^{ijk\eta\delta} e^{-it_j \frac{N\delta}{2}} \psi(-t_j)\eta, \tag{2.19}$$

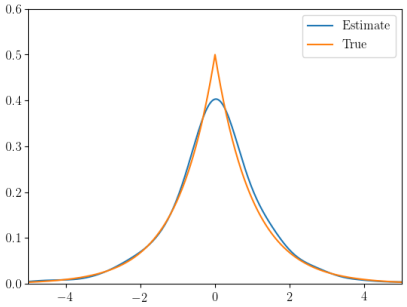
Taking N to be some large power of 2 and choosing η, δ such that $\eta\delta = \frac{2\pi}{N}$, we can then apply FFT to these expressions to obtain an approximation for estimator $\hat{f}_n$ of f. We choose η to be relatively smaller so that δ can be relatively larger and thus, points at which we evaluate our density estimator are relatively separate from one another.



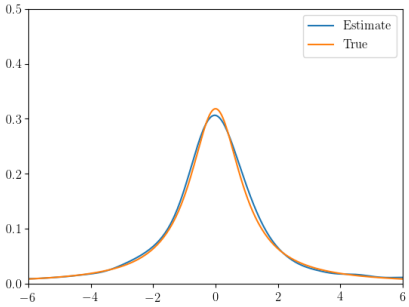
(a) Standard Gaussian



(b) Mixture of Gaussian



(c) Laplace

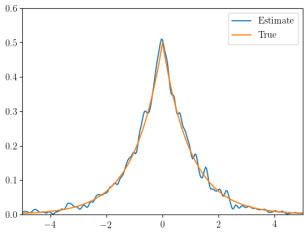


(d) Cauchy

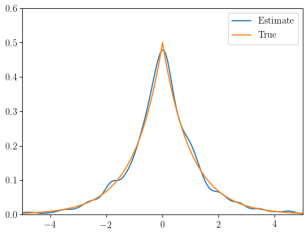
Figure 2.2.1: Density Estimates via Inversion of Characteristic Functions.

Examples. We took $\lambda = 0.3$, $\Delta = 0.4$, $N = 16384$, $\eta = 0.01$ and bandwidth $h = 0.14$. We can see immediately that the estimates are much closer to the true densities than those constructed in the previous section. This is largely due to the independence of the spacing δ and bandwidth h. Therefore, we get smooth estimates with little interpolation required.

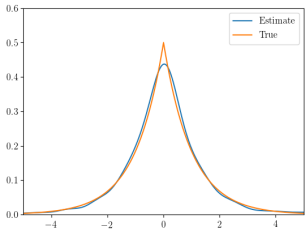
One area of concern is the inability to achieve a steep peak in the estimate of the Laplace distribution. This is mainly due to the bandwidth acting as a smoothing parameter. We can attempt to obtain a better estimate by varying the bandwidth over a range of values and choosing the one that gives the smallest error. We take bandwidth $h = 0.02, 0.06, 0.1$.



(a) Laplace with $h = 0.02$



(b) Laplace with $h = 0.06$



(c) Laplace with $h = 0.1$

As we can see, a smaller bandwidth captures the peak of the Laplace distribution in a quite satisfactory manner, but causes the tails of the distribution to be less smooth. A higher bandwidth results in the contrary, therefore, we must find a middle ground. In our example, a bandwidth $h = 0.06$ seems to provide the best overall fit.

Chapter 3

Bayesian Density Estimation

The kernel density estimators provide a satisfactory attempt in recovering jump probability density function f. We will now focus on the Bayesian approach to density estimation. Advantages of using a Bayesian approach to density estimation compared to kernel density estimators include the following:

1. We obtain a distribution over the space of probability densities, rather than just a point estimate. This allows us to readily obtain uncertainty quantification through credible sets.
2. We can assert prior beliefs quantitatively through the prior distribution. This is useful in practice when we have information about the parameters in question.
3. Under suitable conditions, as shown in [4], the posterior contracts around the 'true' density at a $\sqrt{n\Delta}$-rate, where n is the number of observations and Δ is the separation size. This shows the feasibility of the Bayesian approach.

3.1 Bayes Theorem on Function Spaces

The Bayesian approach treats the unknown quantities in question as random variables. Prior beliefs about the unknown quantities are represented by the prior distribution, and the posterior distribution captures our beliefs about the unknown quantities after they have been modified in light of the observed data. If all such distributions have probability densities, then we can simply write Bayes Theorem for parameter θ and observations $X_1, \ldots, X_n$ as

$$p(\theta|X_1, \ldots, X_n) = \frac{p(X_1, \ldots, X_n|\theta)p(\theta)}{\int p(X_1, \ldots, X_n|\theta)p(\theta)d\theta} \tag{3.1}$$

The issue that arises is that the probability density we would like to recover comes from an infinite-dimensional space. Therefore, we would need to define a distribution over such a space. In infinite-dimensional Banach spaces, there is no analogue with the Lebesgue measure so any distribution over such space does not have an equivalent probability density form. Therefore, for our purposes, we must formulate the theorem more abstractly to gain a rigorous understanding. We follow [5] to formulate the theorem on function spaces.

Background. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and let $\theta, \mathcal{D}$ be random elements with values in measurable spaces $(\Theta, \mathcal{B}_\Theta), (X, \mathcal{B}_X)$ respectively. For our purposes, Θ is some function space and that $X = \mathbb{R}^n$ is the space that our observations live in. Furthermore, let $\mathcal{F}_\theta = \sigma(\theta), \mathcal{F}_\mathcal{D} = \sigma(\mathcal{D})$ be the smallest σ-algebras in Ω such that $\theta, \mathcal{D}$ are measurable respectively. Denote

3.2 Parametric Estimation using a Data Augmentation Scheme

Since explicitly describing a prior distribution on a function space is no trivial task, we first begin by simplifying our problem to the parametric case. We then use a data augmentation scheme to make our likelihood tractable and implement a Metropolis-Hastings-within-Gibbs algorithm for sampling from the posterior.

Mixture of Gaussians. We assume that the density f is a mixture of Gaussians:

$$f(\cdot) = \sum_{j=1}^{J} \rho_j \psi(\cdot\,; \mu_j, 1/\tau) \tag{3.2}$$

where J is known and $\psi(\cdot\,; \mu, \sigma^2)$ denotes the normal density with mean μ and variance σ^2. For convenience, we will refer to the precision $\tau = \frac{1}{\sigma^2}$ instead of the variance. To make use of conjugate priors, we assume that the Gaussians have common precision τ.

Therefore, estimating density f is equivalent to estimating parameters τ and

$$\begin{aligned} \rho &= (\rho_1, \ldots, \rho_J)^T \\ \mu &= (\mu_1, \ldots, \mu_J)^T \end{aligned}$$

As such, we now deal with parametric estimation for our parameters (ρ, μ, τ).

3.2.1 Bypassing the Intractable Likelihood

We have seen in Proposition 1.3.2 that for non-zero observation Z, the likelihood of Z given f is given by

$$p(z|f) = \frac{e^{-\lambda\Delta}}{1 - e^{-\lambda\Delta}} \sum_{m=1}^{\infty} \frac{(\lambda\Delta)^m}{m!} f^{*m}(x)$$

We have seen in Section 2.2 that computing convolutions, or even estimates of convolutions, is expensive. Therefore, we would like to avoid this computation and introduce auxiliary variables to circumvent dealing with the likelihood.

Auxiliary Variables. Suppose for (possibly zero) observation Z, we knew how many terms it consists of in its Poisson sum, and how many terms arise from each of the components $1, \ldots, J$ in the mixture. In other words, for observation $Z_i = \mathbb{1}(N > 0) \sum_{j=1}^{N} Y_j$ suppose we knew

$$a_i = (a_{ij} : j = 1, \ldots, J) \tag{3.3}$$

where a_{ij} denotes the number of terms in the Poisson sum occuring from component j. Then

$$Z_i \overset{\mathcal{L}}{=} \sum_{j=1}^{J} \sum_{k=1}^{a_{ij}} Y_k^{(j)} \mathbb{1}(a_{ij} > 0), \qquad Y_k^{(j)} \overset{\text{ind}}{\sim} \mathcal{N}\left(\mu_j, \frac{1}{\tau_j}\right),\ j = 1, \ldots, J. \tag{3.4}$$

Since a (deterministic) sum of independent Gaussian random variables is a Gaussian random variable, we get that

$$Z_i |\ a_i, \mu, \tau \sim \mathcal{N}\left(a_i^T \mu, \tau^{-1} a_i^T \mathbf{1}\right) \text{ for } i = 1, \ldots, n.$$

Our likelihood now becomes tractable, and is given by the Gaussian probability density function

$$p(Z_i |\ a_i, \mu, \tau) = \psi(Z_i; a_i^T \mu, \tau^{-1} a_i^T \mathbf{1})$$

We will denote $a = (a_1, \ldots, a_n)$, where n the the number of observations (we allow for zero-valued observations) to be our auxiliary variable. We can now use version (3.1) of Bayes Theorem and construct a MCMC algorithm with invariant distribution $p(\mu, \rho, \tau, a|Z)$.

3.2.2 Construction of Hierarchical Model

Priors. We take the following priors for (ρ, μ, τ):

$$\begin{aligned} \rho &\sim \text{Dir}(\alpha, \ldots, \alpha) \\ \tau &\sim \mathcal{G}(\eta, \gamma) \\ \mu_j|\tau &\overset{\text{ind}}{\sim} \mathcal{N}(\xi_j, \kappa^{-1}\tau^{-1}) \end{aligned} \tag{3.5}$$

where $(\alpha, \eta, \gamma, \xi, \kappa)$ are hyperparameters.

Proposition 3.2.1. *For $a = (a_1, \ldots, a_n)$ defined in (3.3), we have that*

$$a_{ij}|\rho_j \overset{\text{ind}}{\sim} \mathcal{P}(\lambda\rho_j\Delta)$$

Proof. For a mixture, component j is chosen with probability ρ_j. Therefore, for m independent draws of components, the number of draws corresponding to each component has Multinomial$(m; \rho_1, \ldots, \rho_J)$ distribution. Let $n_i = \sum_{j=1}^{J} a_{ij}$. We know that $n_i \sim \mathcal{P}(\lambda\Delta)$ and that each jump term is independent. Then, by the probability mass function of a Multinomial distribution, we have

$$\begin{aligned} p(a_i|\rho) &= p(n_i|\rho)\binom{n_i}{a_{i1}, \ldots, a_{iJ}} \prod_{j=1}^{J} \rho_j^{a_{ij}} \\ &= e^{-\lambda\Delta}\frac{(\lambda\Delta)_i^n}{n_i!}\binom{n_i}{a_{i1}, \ldots, a_{iJ}} \prod_{j=1}^{J} \rho_j^{a_{ij}} \\ &= \prod_{j=1}^{J} e^{-\lambda\Delta\rho_j}\frac{(\lambda\Delta\rho_j)^{a_{ij}}}{a_{ij}!}. \end{aligned}$$

Since this is the distribution of J independent $\mathcal{P}(\lambda\Delta\rho_j)$ random variables, the result follows. □

Hierarchical Model. Let π denote the joint prior distribution on (ρ, μ, τ) specified above. Then, we can write our model as:

$$\begin{aligned} \rho &\sim \text{Dir}(\alpha, \ldots, \alpha) \\ \tau &\sim \mathcal{G}(\eta, \gamma) \\ \mu_j|\tau &\overset{\text{ind}}{\sim} \mathcal{N}(\xi_j, \kappa^{-1}\tau^{-1}) \\ a_{ij}|\ \rho &\overset{\text{ind}}{\sim} \mathcal{P}(\lambda\rho_j\Delta) \\ Z_i|a_i, \mu, \tau &\overset{\text{ind}}{\sim} \mathcal{N}(a_i^T\mu, \tau^{-1}a_i^T\mathbf{1}) \end{aligned}$$

This model gives us the following joint distribution decomposition:

$$p(\rho, \mu, \tau, a, Z) = p(\rho)p(\tau)p(\mu|\tau)p(a|\rho)p(Z|a, \mu, \tau).$$

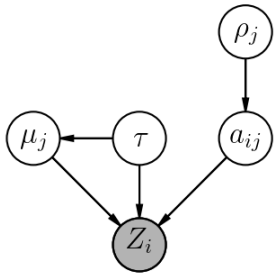


Figure 3.2.1: Probabilistic Graphical Model for the Hierarchical Model.

3.2.3 Contruction of MCMC Algorithm

We can use Gibbs sampling to sample from the joint distribution $p(\rho, \mu, \tau, a, Z)$ as follows:

Algorithm 1: Gibbs Sampler for Finite Mixture Hierarchical Model

Result: Samples from the posterior distribution $p(a, \mu, \tau, \rho|Z)$.
Initialise $\rho^{(0)}, \mu^{(0)}, \tau^{(0)}, a^{(0)}$;
for $t = 1, \ldots, N$ **do**
Update auxiliary variable a:
1. Sample $a^{(t)} \sim p(a|\rho^{(t-1)}, \mu^{(t-1)}, \tau^{(t-1)}, Z)$ via Metropolis-Hastings step

Update parameters ρ, τ, μ:
1. Sample $\rho^{(t)} \sim p(\rho|a^{(t)}, Z)$
2. Sample $\tau^{(t)} \sim p(\tau|a^{(t)}, Z)$
3. Sample $\mu^{(t)} \sim p(\mu|a^{(t)}, \tau^{(t)}, Z)$

end

Updating Auxiliary Variable. We would like to sample from $p(a|\rho, \mu, \tau, Z) \propto p(Z|a, \mu, \tau, \rho)p(a|\rho)$. We do this using a Metropolis-Hastings step. Note that by the Hierarchical model:

$$\begin{aligned} p(Z|a, \mu, \tau, \rho)p(a|\rho) &= \left(\prod_{i=1}^{n} p(Z_i|a_i, \mu, \tau, \rho)\right)\left(\prod_{i=1}^{n}\prod_{j=1}^{J} p(a_{ij}|\rho)\right) \\ &= \prod_{i=1}^{n}\left(\psi\left(Z_i; a_i^T\mu, a_i^T\tau^{-1}\right)\prod_{j=1}^{J} e^{-\lambda\rho_j\Delta}\frac{(\lambda\rho_j\Delta)^{a_{ij}}}{a_{ij}!}\right) \end{aligned}$$

Therefore, conditional on (ρ, μ, τ, Z), we have that each a_i is independent and so for each $i = 1, \ldots, n$, we perform a Metropolis Hastings step to sample from

$$\psi\left(Z_i; a_i^T\mu, a_i^T\tau^{-1}\right)\prod_{j=1}^{J} e^{-\lambda\rho_j\Delta}\frac{(\lambda\rho_j\Delta)^{a_{ij}}}{a_{ij}!}$$

We construct our proposal distribution as follows:

1. We draw $n_i^\circ \sim \mathcal{P}(\lambda\Delta)$.
2. We draw $a_i^\circ = (a_{i1}^\circ, \ldots, a_{iJ}^\circ) \sim \text{Multinomial}(n_i^\circ; \rho_1, \ldots, \rho_J)$.

Then, our proposal density function $q(a_i^\circ|\rho)$ is given by

$$\begin{aligned} q(a_i^\circ|\rho) &= p(n_i^\circ)p(a_i^\circ|n_i^\circ, \rho) \\ &= e^{-\lambda\Delta}\frac{(\lambda\Delta)^{n_i^\circ}}{n_i^\circ!}\binom{n_i^\circ}{a_{i1}^\circ, \ldots, a_{iJ}^\circ}\prod_{j=1}^{J}\rho_j^{a_{ij}^\circ} \\ &= \prod_{j=1}^{J} e^{-\lambda\rho_j\Delta}\frac{(\rho_j\lambda\Delta)^{a_{ij}^\circ}}{a_{ij}^\circ!} \end{aligned}$$

by the same calculation as in Proposition 3.2.1. Therefore, our acceptance probability A is

$$A = \frac{p(a_i^\circ|\rho,\mu,\tau,Z_i)q(a_i|\rho)}{p(a_i|\rho,\mu,\tau,Z_i)q(a_i^\circ|\rho)}$$

$$= \frac{\psi\left(Z_i; (a_i^\circ)^T\mu, \tau^{-1}(a_i^\circ)^T\mathbf{1}\right)}{\psi\left(Z_i; a_i^T\mu, \tau^{-1}a_i^T\mathbf{1}\right)}$$

and we accept a_i° with probability $1 \wedge A$.

Updating Mixture Weights. Let

$$s_j = \sum_{i=1}^{n} a_{ij} \tag{3.6}$$

be the number of jumps in component j. We use the following Lemma to update our mixture weights.

Lemma 3.2.1. *Conditional on a, we have that $\rho_1, \ldots, \rho_J$ are independent and*

$$\rho_j|a \overset{ind}{\sim} \mathcal{G}(\alpha + s_j, \lambda n\Delta)$$

Proof. We calculate $p(\rho|a)$ using Bayes Theorem:

$$\begin{aligned}
p(\rho|a) &\propto p(a|\rho)\pi(\rho) \\
&= \prod_{j=1}^{J} \pi(\rho_j) \left(\prod_i p(a_{ij}|\rho)\right) \\
&\propto \prod_{j=1}^{J} \rho_j^{\alpha-1} \left(\prod_i e^{-\rho_j\lambda\Delta}(\rho_j\lambda\Delta)^{a_{ij}}\right) \\
&= \prod_{j=1}^{J} \rho_j^{\alpha-1} e^{-\rho_j\lambda n\Delta}(\rho_j\lambda\Delta)^{s_j} \\
&\propto \prod_{j=1}^{J} \rho_j^{s_j+\alpha-1} e^{-\rho_j\lambda n\Delta}.
\end{aligned}$$

Since this is the probability density of independent $\mathcal{G}(s_j+\alpha, \lambda n\Delta)$ distributed random variables, the result follows. □

Updating Individual Component Parameters. Let

$$n_i = \sum_{j=1}^{J} a_{ij}$$

be the number of jumps in observation i. We use the following Lemma to update the parameters of each component.

Lemma 3.2.2. *Conditional on (Z, a), we have*

$$\begin{aligned}
\tau|Z,a &\sim \mathcal{G}(\eta + n/2, \gamma + (R - q^TP^{-1}q)/2) \\
\mu|\tau,Z,a &\sim \mathcal{N}(P^{-1}q, \tau^{-1}P^{-1})
\end{aligned}$$

where P is the symmetric $J \times J$ matrix given by

$$P = \kappa I_{J\times J} + \tilde{P}, \qquad \tilde{P}_{jk} = \sum_{i=1}^{n} n_i^{-1} a_{ij} a_{ik}$$

q is the J-dimensional vector with

$$q_j = \kappa \xi_j + \sum_{i=1}^{n} n_i^{-1} a_{ij} Z_i$$

$R > 0$ is given by

$$R = \kappa \sum_{j=1}^{J} \xi_j^2 + \sum_{i=1}^{n} n_i^{-1} Z_i^2$$

and $R - q^T P^{-1} q > 0$.

Note that adding $\kappa I_{J\times J}$ ensures the invertibility of P.

Proof. For (μ, τ) we get

$$\begin{aligned}
p(\mu,\tau|Z,a) &\propto p(Z|a,\mu,\tau)p(\mu,\tau) \\
&\propto p(Z|\mu,\tau,a)p(\mu|\tau)p(\tau) \\
&\propto \left(\prod_{i=1}^{n} \psi(Z_i; a_i^T\mu, n_i/\tau)\right)\left(\tau^{J/2}\exp\left\{-\frac{\tau\kappa}{2}\sum_{j=1}^{J}(\mu_j-\xi_j)^2\right\}\right)\left(\tau^{\eta-1}e^{-\gamma\tau}\right) \\
&\propto \tau^{n/2}\exp\left\{-\frac{\tau}{2}\sum_{i=1}^{n} n_i^{-1}(Z_i - a_i^T\mu)^2\right\}\left(\tau^{J/2}\exp\left\{-\frac{\tau\kappa}{2}\sum_{j=1}^{J}(\mu_j-\xi_j)^2\right\}\right)\left(\tau^{\eta-1}e^{-\gamma\tau}\right) \\
&\propto \tau^{\eta-1+(n+J)/2}\exp\left\{-\gamma\tau - \frac{D(\mu)}{2}\tau\right\}
\end{aligned}$$

where

$$\begin{aligned}
D(\mu) &= \kappa\sum_{j=1}^{J}(\mu_j-\xi_j)^2 + \sum_{i=1}^{n} n_i^{-1}(Z_i - a_i^T\mu)^2 \\
&= \mu^T P\mu - 2q^T\mu + R
\end{aligned}$$

by easy calculation. Note that, by completing the square,

$$\mu^T P\mu - 2q^T\mu + R = (\mu - P^{-1}q)^T P(\mu - P^{-1}q) - q^T P^{-1} q + R$$

Therefore,

$$p(\mu|\tau,a,Z) \propto \exp\left\{-\frac{\tau}{2}(\mu - P^{-1}q)^T P(\mu - P^{-1}q)\right\}$$

It follows by this that $\mu|\tau,z,a \sim \mathcal{N}(P^{-1}q, \tau^{-1}P^{-1})$.

Also,

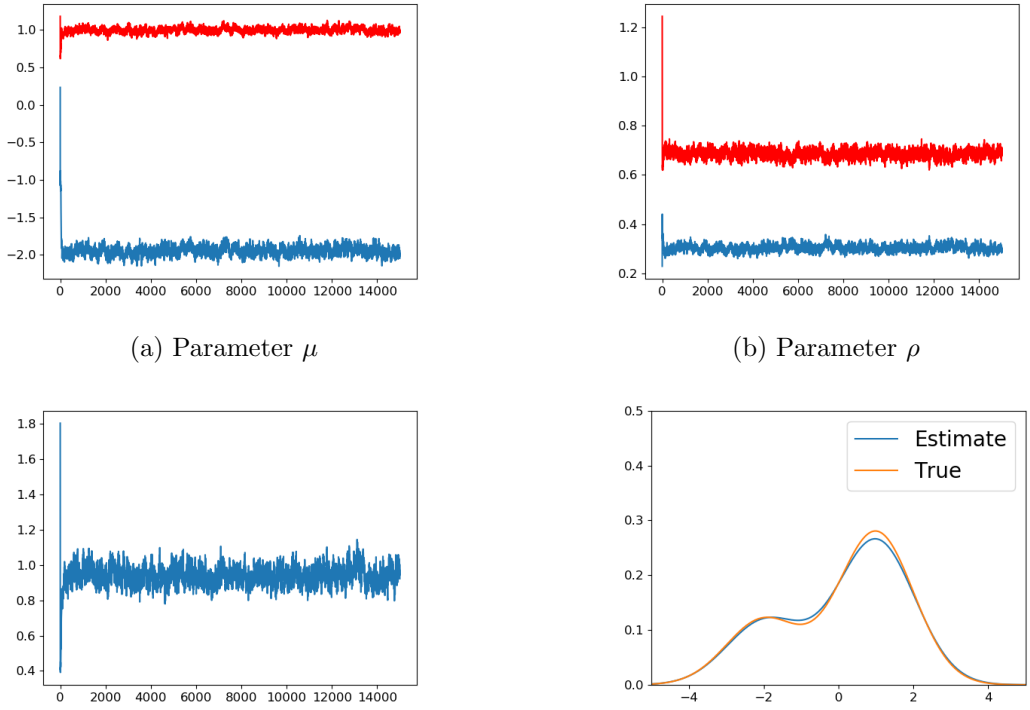
$$\begin{aligned}
\int \exp(-\frac{\tau}{2}D(\mu))d\mu &= \exp\left\{-\frac{\tau}{2}(R - q^T P^{-1}q)\right\}\int \exp\left(-\frac{\tau}{2}(\mu - P^{-1}q)^T P(\mu - P^{-1}q)\right)d\mu \\
&= \exp\left\{-\frac{\tau}{2}(R - q^T P^{-1}q)\right\}(2\pi)^{J/2}\sqrt{|\tau^{-1}P^{-1}|} \qquad (3.7)
\end{aligned}$$

Line (3.7) follows by the integral of a multivariate Gaussian distribution being equal to 1. Thus, we can write $p(\tau|Z,a)$ as

$$\begin{aligned} p(\tau|Z,a) &= \int p(\tau,\mu|z,a)d\mu \\ &= \int \tau^{\eta-1+(n+J)/2}\exp\left\{-\gamma\tau - \frac{D(\mu)}{2}\tau\right\}d\mu \\ &\propto \tau^{\eta-1+(n+J)/2}e^{-\gamma\tau}(2\pi)^{J/2}\sqrt{|\tau^{-1}P^{-1}|}\exp\left\{-\frac{\tau}{2}(R-q^TP^{-1}q)\right\} \qquad \text{using (3.7)} \\ &\propto \tau^{\eta+(n+J-J)/2-1}\exp\left\{-\tau\left(\gamma+\frac{1}{2}(R-q^TP^{-1}q)\right)\right\} \end{aligned}$$

giving that $\tau|Z,a \sim \mathcal{G}(\eta+n/2, \gamma+(R-q^TP^{-1}q)/2)$ □

3.2.4 Simulation Results



(a) Parameter μ

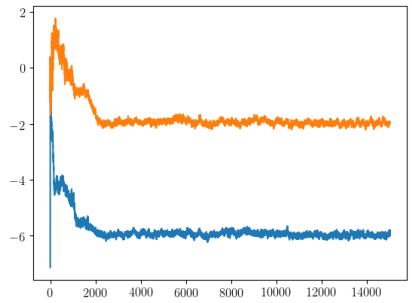
(b) Parameter ρ

(c) Parameter τ

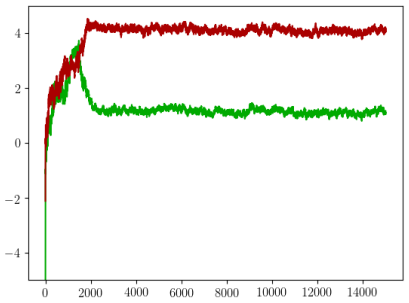
(d) Density Estimate

Figure 3.2.2: Run of MCMC algorithm for mixture of two Gaussians.

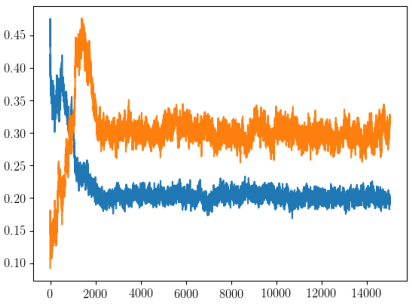
Example 1. We simulated a CPP with intensity $\lambda = 1$ and jump density function $f(x) = 0.3\psi(x;-2,1) + 0.7\psi(x;1,1)$. We obtained $n = 8000$ observations with spacing $\Delta = 1$ (giving roughly $5000 \approx (1-e^{-\Delta\lambda})n$ non-zero observations) and performed 15000 MCMC iterations. We obtained an average acceptance rate of 51% for the auxiliary variable Metropolis-Hastings proposals. The plots show that the chains move close to the true values of the mixture parameters in a satisfactory manner. A plot of posterior mean estimates of the parameters can be seen in Figure 3.2.2d.



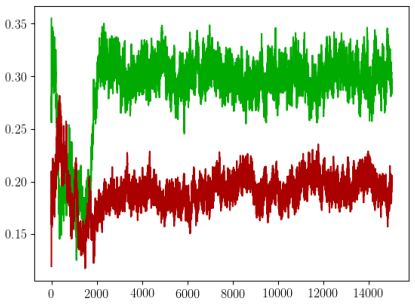
(a) Parameter μ_1, μ_2



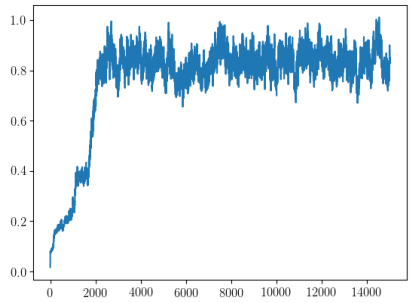
(b) Parameter μ_3, μ_4



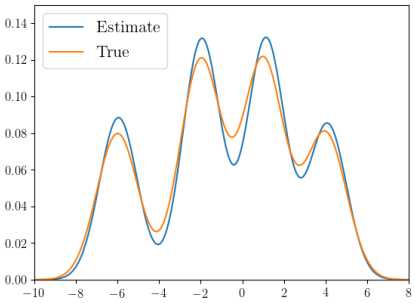
(c) Parameter ρ_1, ρ_2



(d) Parameter ρ_3, ρ_4



(e) Parameter τ



(f) Density Estimate

Figure 3.2.3: Run of MCMC algorithm for mixture of four Gaussians.

Example 2. We simulated a CPP with intensity $\lambda = 1$ and jump density function $f(x) = 0.2\psi(x; -6, 1) + 0.3\psi(x; -2, 1) + 0.3\psi(x; 1, 1) + 0.2\psi(x; 4, 1)$. As before, we obtained $n = 8000$ observations which consisted of roughly 5000 non-zero observations and performed 15000 MCMC iterations. We obtained an average acceptance rate of 29.7% for the auxiliary variable Metropolis-Hastings proposals.

The plots show good convergence around the true means and mixture weights, however, the chain for the precision parameter τ shows consistent underestimation and the sample variance is higher in the majority of parameters than in Example 1. A plot of posterior mean estimates of the parameters can be seen in Figure 3.2.2d. Overall, we can see that the increase in number of components causes a significant reduction in the performance of the algorithm. This is due to the increased difficulty in moving around a higher dimensional space for the Gibbs sampler.

3.3 Non-Parametric Estimation via Dirichlet Process Mixture Model

In the previous section, we assumed that J was known, allowing us to reduce the problem to one of parametric estimation. However, such an assumption reduces the parameter space of densities significantly. Ideally we would like to have very little assumed about the space in which the jump probability density function lives.

We relax this assumption on J being known by using a Dirichlet Process Mixture Model (DPMM). The Dirichlet process is a distribution over the space of probability distributions. Therefore, the Dirichlet process allows us to treat Bayes' Theorem in the most general case as in Section 3.1, by specifying a Dirichlet Process prior on the space of probability distributions for the jump distribution F.

3.3.1 Dirichlet Processes

Definition 3.3.1. (Random Measure). Let $(\Omega, \mathcal{F}, \mathcal{P})$ be some arbitrary probability space and $(\mathfrak{X}, \mathcal{X})$ be a measurable space. Then a map $P : \Omega \times \mathcal{X} \to \mathbb{R}$ is a random measure on $(\mathfrak{X}, \mathcal{X})$ if

1. For every $\omega \in \Omega$, the map $A \mapsto P(\omega, A)$ is a probability measure on $(\mathfrak{X}, \mathcal{X})$,
2. For every $A \in \mathcal{X}$, the map $\omega \mapsto P(\omega, A)$ is a random variable from $\Omega \to \mathbb{R}$.

Definition 3.3.2. (Dirichlet Process). A random measure P on $(\mathfrak{X}, \mathcal{X})$ is said to possess a Dirichlet process distribution $\mathrm{DP}(\alpha)$ with base measure α on the measurable space $(\mathfrak{X}, \mathcal{X})$ if, for every finite measurable partition $A_1, \ldots, A_k$ of $\mathfrak{X}$, we have that the joint distribution of random variables $P(A_1), \ldots, P(A_k)$ satisfy

$$(P(A_1), \ldots, P(A_k)) \sim \mathrm{Dir}\,(k+1; \alpha(A_1), \ldots, \alpha(A_k))$$

This definition does not provide much intuition about how a Dirichlet process could be used to deal with our problem of density estimation. Therefore, for our purposes, we simplify our discussion to a countable sample space.

Countable Dirichlet Process. A probability measure on a countable sample space S (equipped with the σ-algebra $\mathfrak{S}$ generated by all finite subsets) can be represented as an infinite-length probability vector $s = (s_1, s_2, \ldots)$ assigning probability weights to each element in the sample space. For example, a Poisson distribution on countable measurable space $(\mathbb{N}, \sigma(\mathbb{N}))$ can be represented as the probability vector

$$\left(e^{-\lambda}\frac{\lambda^k}{k!}\right)_{k=0}^{\infty}$$

Thus, the space of probability measures on a countable space corresponds to the unit simplex

$$S_\infty = \left\{ s = (s_1, s_2, \ldots) : s_j \geq 0, j \in \mathbb{N}, \sum_{j=1}^{\infty} s_j = 1 \right\}. \tag{3.8}$$

Consider the smallest σ-algebra $\mathfrak{S}_\infty$ on S_∞ that makes coordinate maps $s \mapsto s_i,\ i \in \mathbb{N}$ measurable. Consider some arbitrary probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a random measure $P : \Omega \times \mathfrak{S} \to \mathbb{R}$. Then, clearly for every $\omega \in \Omega$, P_ω is in the space S_∞, where $P_\omega(A) = P(\omega, A)$. Through this, we can see that P is a random element from $(\Omega, \mathcal{F})$ to $(S_\infty, \mathfrak{S}_\infty)$. It then makes sense to talk about

$$\mathcal{P}(P \in M) \qquad \text{for } M \in \mathfrak{S}_\infty$$

As such, the distribution of P is a probability measure on $(S_\infty, \mathfrak{S}_\infty)$. Therefore, by constructing a random element that takes values in our space S_∞, we generate a distribution on the space S_∞.

Proposition 3.3.1. *Let $(S_\infty, \mathfrak{S}_\infty)$ be the measurable space defined in (3.8) and let $(\Omega, \mathcal{F}, \mathcal{P})$ be some arbitrary probability space. Suppose that map $p : \Omega \to S_\infty$ is a random element. Then, every coordinate p_i is a random variable.*

Proof. This is a trivial result that is a consequence of our definition of $\mathfrak{S}_\infty$. Let $\phi_i : S_\infty \to \mathbb{R}$ denote the coordinate map $\phi_i(s) = s_i$. Then, we know that ϕ_i is measurable for every i. As a direct result, for $B \in \mathcal{B}(\mathbb{R})$, we have

$$\phi_i^{-1}(B) \in \mathfrak{S}_\infty, \qquad i \in \mathbb{N}.$$

Therefore,

$$p^{-1}(\phi_i^{-1}(B)) \in \mathcal{F}, \qquad i \in \mathbb{N}.$$

It follows that for every $i \in \mathbb{N}$,

$$p_i^{-1}(B) = \{\omega : \phi_i(p(\omega)) \in B\} = p^{-1}(\phi_i^{-1}(B)) \in \mathcal{F}$$

as required. □

In other words, Proposition 3.3.1 tells us that a distribution on $(S_\infty, \mathfrak{S}_\infty)$ gives a sequence of random variables $(p_1, p_2, \dots)$ such that $\sum_{j=1}^{\infty} p_j = 1$ almost surely. From this and Definition 3.3.2, we see that random element $p = (p_1, p_2, \dots) \sim \mathrm{DP}(\alpha)$ if for every $k \in \mathbb{N}$,

$$\left(p_1, \dots, p_k, 1 - \sum_{j=1}^{k} p_j\right) \sim \mathrm{Dir}\left(k+1; \alpha_1, \dots, \alpha_k, \sum_{j=k+1}^{\infty} \alpha_j\right)$$

where $\alpha = (\alpha_1, \alpha_2, \dots)$ is a (deterministic) sequence such that

$$\sum_{j=1}^{\infty} \alpha_j < \infty, \qquad \alpha_j \geq 0,\ j \in \mathbb{N}.$$

Construction through the Stick Breaking Process We perform the following algorithm to distribute the total probability mass 1, conceptually thought of as a stick of length 1, randomly to each coordinate $p_1, p_2, \dots$.

1. We first break the stick at the point given by the random variable V_1 where $0 \leq V_1 \leq 1$ and assign mass V_1 to p_1.
2. We think of the remaining mass $1 - V_1$ as a new stick and break it into two pieces of relative lengths V_2 and $1 - V_2$ according to the value of random variable V_2. We assign mass $V_2(1 - V_1)$ to the point p_2.
3. We repeat in this way so that point p_j has mass

$$p_j = V_j \prod_{l=1}^{j-1} (1 - V_l) \tag{3.9}$$

Proposition 3.3.2. *Let α be a probability distribution on $\mathbb{R}^d$ and let $M > 0$ be fixed. Suppose $\theta_1, \theta_2 \ldots \overset{\text{i.i.d}}{\sim} \alpha$ and $V_1, V_2, \ldots \overset{\text{i.i.d}}{\sim} \text{Beta}(1, M)$ are all mutually independent random variables. Then, for the random element $p = (p_1, p_2, \ldots)$ defined in (3.9), we have that*

$$\sum_{j=1}^{\infty} p_j \delta_{\theta_j} \sim \text{DP}(M\alpha).$$

where $(M\alpha)(A) := M\alpha(A)$ for every $A \in \mathcal{B}(\mathbb{R}^d)$ is a measure on $\mathbb{R}^d$ of total mass M. We call $M > 0$ the concentration parameter.

Proof. See Theorem 4.12 of [3]. □

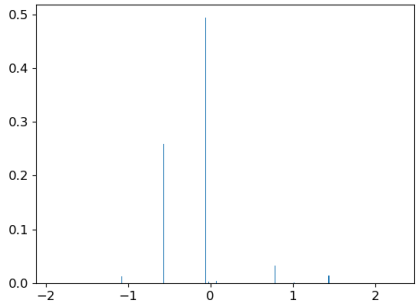


Figure 3.3.1: Realisation of Dirichlet process with $\alpha = \mathcal{N}(0, 1), M = 1$.

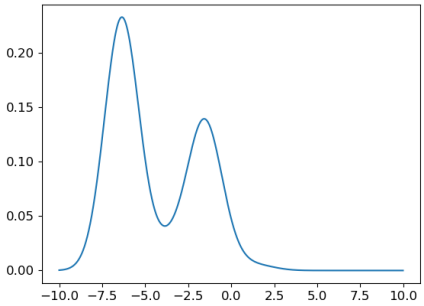


Figure 3.3.2: Realisation of Dirichlet process mixture.

From Proposition 3.3.2 and as we can see in Figure 3.3.1, it is clear that realisations of $\text{DP}(\alpha)$ are almost surely discrete. This does not seem suitable for our purposes as we would like realisations of a DP to be distributions with continuous probability density functions. To solve this, we convolve the distribution generated from a Dirichlet process with a kernel, creating a Dirichlet process mixture.

Dirichlet Process Mixtures Let α be a probability measure on $\mathbb{R}^d$. Let Θ be some parameter set and, for $\theta \in \Theta \subset \mathbb{R}^d$, let the map $x \mapsto \psi(x, \theta)$ be a probability density function. For example, we could take $\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_{>0}$ and Gaussian probability density function

$$\psi(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Let $F \sim \text{DP}(\alpha)$. Then we define the Dirichlet process mixture to be the probability density function $p_{F,\psi}$ given by

$$p_{F,\psi}(x) = \int \psi(x, \theta) dF(\theta).$$

Since $F \overset{\mathcal{L}}{=} \sum_{j=1}^{\infty} p_j \delta_{\theta_j}$ for some $p = (p_1, p_2, \ldots)$, $\theta = (\theta_1, \theta_2, \ldots)$ constructed in Proposition 3.3.2, we get that

$$p_{F,\psi}(x) \overset{\mathcal{L}}{=} \sum_{j=1}^{\infty} p_j \psi(x, \theta).$$

Therefore, for Gaussian probability density function ψ, we have now extended (3.2) into a mixture of infinite number of Gaussians with a Dirichlet process prior on the mixture weights. In Figure 3.3.2, we have taken $\alpha_\mu = \mathcal{N}(0, 10), M = 1$ for mean parameter μ and

$$\psi(x; \mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}.$$

We can see that a realisation of a Dirichlet process mixture is almost surely continuous, as desired. We our now back to estimation over an infinite number of parameters, therefore, such estimation is non-parametric. As we will see, even though we have an infinite number of parameters, the mixture weights will be significant in only a finite number of components.

3.3.2 Construction of the Hierarchical Model

In similar fashion to Section 3.2, we will assume that the density f comes from a mixture of Gaussians as in (3.2). The notable difference is that we assume J is unknown and we try to infer it from our observations.

The stick breaking construction allows us to write down our Hierarchical model easily. We assume the same priors for (μ, τ) as in (3.5). We also put a prior on the concentration parameter α. Then we have the model:

$$\begin{aligned}
\alpha &\sim \mathcal{G}(\nu, \epsilon) \\
\tau &\sim \mathcal{G}(\eta, \gamma) \\
\mu_j|\tau &\overset{\text{ind}}{\sim} \mathcal{N}(\xi_j, \kappa^{-1}\tau^{-1}) \\
\beta_j|\alpha &\overset{\text{ind}}{\sim} \text{Beta}(1, \alpha) \\
\rho_j|\beta &= \beta_j \textstyle\prod_{k=1}^{j-1}(1-\beta_k) \\
a_{ij}|\rho &\overset{\text{ind}}{\sim} \text{Po}(\lambda\rho_j\Delta) \\
Z_i|a, \mu, \tau &\overset{\text{ind}}{\sim} \mathcal{N}(a_i^T\mu, \tau^{-1}a_i^T\mathbf{1})
\end{aligned}$$

where $(\epsilon, \nu, \eta, \gamma, \xi, \kappa)$ are hyperparameters. We can illustrate this using the following probabilistic graphical model:

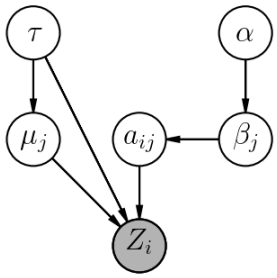


Figure 3.3.3: Probabilistic Graphical Model for the Hierarchical Model.

Truncating Mixture. Since we cannot generate infinite component mixtures in our simulations, we truncate our mixtures up to some sufficiently large number of components K. This can be justified intuitively, since with a finite number of observations, it seems quite likely that the number of mixture components that contribute non-negligible mass to the mixture will grow slower than the number of samples. This intuition can be formalized through the fact that the expected number of components that contribute non-negligible mass to the mixture approaches $\log n$, where n is the number of observations. Therefore, we take $K = \log n$.

3.3.3 Construction of MCMC Algorithm

We again use a Metropolis-Hastings-within-Gibbs algorithm to sample from the posterior distribution, but this time additionally performing a Metropolis-Hastings step to sample from conditional distribution $\beta|a, Z$.

Algorithm 2: Gibbs Sampler for DPMM Hierarchical Model

Result: Samples from the posterior distribution $p(a, \mu, \tau, \beta, \rho|Z)$.
Initialise $\alpha^{(0)}, \beta^{(0)}, \mu^{(0)}, \tau^{(0)}, a^{(0)}$;
for $k = 1, \ldots, K$ **do**
  Set $\rho_k^{(0)} \leftarrow \beta_k^{(0)} \prod_{j=1}^{k-1} (1 - \beta_j^{(0)})$
end
for $t = 1, \ldots, N$ **do**
  Update auxiliary variable a:
    1. Sample $a^{(t)} \sim p(a|\rho^{(t-1)}, \mu^{(t-1)}, \tau^{(t-1)}, Z)$ via Metropolis-Hastings step
  Update parameters α, β, τ, μ:
    1. Sample $\alpha^{(t)} \sim p(\alpha|\beta^{(t-1)})$
    2. Sample $\beta^{(t)} \sim p(\beta|a^{(t)}, \alpha^{(t)}, Z)$ via Metropolis-Hastings step
    3. Sample $\tau^{(t)} \sim p(\tau|a^{(t)}, Z)$
    4. Sample $\mu^{(t)} \sim p(\mu|a^{(t)}, \tau^{(t)}, Z)$
  Update parameter ρ:
    1. Set $\rho_k^{(t)} \leftarrow \beta_k^{(t)} \prod_{j=1}^{k-1} (1 - \beta_j^{(t)})$
end

Updating Concentration Parameter. We update auxiliary variable a using the same Metropolis-Hastings step as in Section 3.2.2. We also update parameters τ, μ using the same distributions in Lemma 3.2.2. To update α, we derive the conditional distribution $p(\alpha|\beta)$.

Lemma 3.3.1. *Conditional on β, we have*

$$\alpha|\beta \sim \mathcal{G}\left(\epsilon - K, \nu - \sum_{k=1}^{K} \log(1 - \beta_k)\right)$$

Proof. We have

$$\begin{aligned}
p(\alpha|\beta) &\propto p(\beta|\alpha)p(\alpha) \\
&= \left(\prod_{k=1}^{K} (1 - \beta_k)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)}\right) \alpha^{\epsilon - 1} e^{-\nu\alpha} \\
&= \alpha^{\epsilon - K - 1} \exp\left\{\sum_{k=1}^{K} (\alpha - 1) \log(1 - \beta_k) - \nu\alpha\right\} \\
&\propto \alpha^{\epsilon - K - 1} \exp\left\{-\alpha\left(\nu - \sum_{k=1}^{K} \log(1 - \beta_k)\right)\right\}
\end{aligned}$$

giving the required result. □

Updating Stick Breaking Weights. We update β using a Metropolis-Hastings step since its conditional distribution does not have a tractable form. For each $k = 1, \ldots, K$, we want to sample from $p(\beta_k|a, \beta_{-k}, \alpha) \propto p(a|\beta)p(\beta_k|\alpha)$, where

$$\beta_{-k} = (\beta_1, \ldots, \beta_{k-1}, \beta_{k+1}, \ldots, \beta_K).$$

Note that

$$
\begin{aligned}
p(a|\beta)p(\beta_k) &\propto p(a|\rho)p(\beta_k) \\
&\propto \left(\prod_{i=1}^{n}\prod_{j=1}^{K} p(a_{ij}|\rho)\right)(1-\beta_k)^{\alpha-1} \\
&\propto (1-\beta_k)^{\alpha-1}\prod_{j=1}^{K}\prod_{i=1}^{n} e^{-\lambda\Delta\rho_j}\rho_j^{a_{ij}} \\
&= (1-\beta_k)^{\alpha-1}\prod_{j=1}^{K} e^{-n\lambda\Delta\rho_j}\rho_j^{s_j} && (s_j \text{ defined in (3.6)}) \\
&= (1-\beta_k)^{\alpha-1}\prod_{j=1}^{K}\left(\exp\left\{-n\lambda\Delta\beta_j\prod_{l=1}^{j-1}(1-\beta_l)\right\}\beta_j^{s_j}\prod_{l=1}^{j-1}(1-\beta_l)^{s_j}\right) \\
&\propto (1-\beta_k)^{\alpha-1+\sum_{j=k+1}^{K}s_j}\beta_k^{s_k}\prod_{j=k}^{K}\exp\left\{-\lambda\Delta n\beta_j\prod_{l=1}^{j-1}(1-\beta_l)\right\}
\end{aligned}
$$

Therefore, we propose

$$\beta_k^{\circ} \sim \text{Beta}(s_k+1, \sum_{j=k+1}^{K} s_j + \alpha).$$

where the sum $\sum_{j=k+1}^{K} s_j$ is taken to be zero when $k = K$. Let

$$\beta^{\circ} = (\beta_1, \ldots, \beta_{k-1}, \beta_k^{\circ}, \beta_{k+1}, \ldots, \beta_K).$$

Our acceptance probability A is

$$
\begin{aligned}
A &= \frac{\prod_{j=k}^{K}\exp\left\{-\lambda\Delta n\beta_j^{\circ}\prod_{l=1}^{j-1}(1-\beta_l^{\circ})\right\}}{\prod_{j=k}^{K}\exp\left\{-\lambda\Delta n\beta_j\prod_{l=1}^{j-1}(1-\beta_l)\right\}} \\
&= \exp\left\{\lambda\Delta n\sum_{j=k}^{K}\left(\beta_j\prod_{l=1}^{j-1}(1-\beta_l) - \beta_j^{\circ}\prod_{l=1}^{j-1}(1-\beta_l^{\circ})\right)\right\}
\end{aligned}
$$

We accept β° with probability $1 \wedge A$. After we complete all Metropolis-Hastings steps, we update stick-breaking weights ρ accordingly with the new β.

3.3.4 Simulation Results

Label Switching Problem. Notice that

Chapter 4

Comparison of Estimators

Bibliography

[1] Kai Lai Chung. *A course in probability theory*. Academic press, 2001.

[2] Fabienne Comte, Cline Duval, and Valentine Genon-Catalot. Nonparametric density estimation in compound poisson process using convolution power estimators. *Metrika*, 77, 01 2014. URL https://doi.org/10.1007/s00184-013-0475-3.

[3] Subhashis Ghosal and Aad Van der Vaart. *Fundamentals of nonparametric Bayesian inference*, volume 44. Cambridge University Press, 2017.

[4] Shota Gugushvili, Frank van der Meulen, and Peter Spreij. A non-parametric bayesian approach to decompounding from high frequency data. *Statistical Inference for Stochastic Processes*, 21(1):53–79, Apr 2018. URL https://doi.org/10.1007/s11203-016-9153-1.

[5] Robert S. Liptser and Albert N. Shiryaev. *Absolute Continuity of Measures corresponding to the Itô Processes and Processes of the Diffusion Type*, pages 251–315. Springer Berlin Heidelberg, Berlin, Heidelberg, 2001. ISBN 978-3-662-13043-8. doi: 10.1007/978-3-662-13043-8_8. URL https://doi.org/10.1007/978-3-662-13043-8_8.

[6] Bert van Es, Shota Gugushvili, and Peter Spreij. A kernel type nonparametric density estimator for decompounding. *Bernoulli*, 13(3):672–694, 08 2007. doi: 10.3150/07-BEJ6091. URL https://doi.org/10.3150/07-BEJ6091.