Appendix H

Inverse-Gaussian Distribution

The normalized inverse-Gaussian distribution is a probability distribution on the unit simplex, and can be viewed as an analog of the finite-dimensional Dirichlet distribution. It arises from normalizing a vector of independent inverse-Gaussian variables, which replace the gamma variables that would yield the Dirichlet distribution.

Recall that the *inverse-Gaussian distribution* $\mathrm{IGau}(\alpha, \gamma)$ with shape parameter $\alpha > 0$ and scale parameter $\gamma > 0$ is the probability distribution on $(0, \infty)$ with density

$$y \mapsto \frac{\alpha e^{\alpha\gamma}}{\sqrt{2\pi}} \frac{1}{y^{3/2}} e^{-\frac{1}{2}(\alpha^2/y + \gamma^2 y)}.$$

The $\mathrm{IGau}(0, \gamma)$ distribution is understood to be the distribution degenerate at 0. (The distribution takes its name from the distribution of the time until a Brownian motion $B = (B_t \colon t \geq 0)$ with linear drift reaches a given level: the hitting time $\inf\{t > 0 \colon B_t + t = \alpha\}$ possesses the $\mathrm{IGau}(\alpha, 1)$-distribution. This hitting time can be considered to be "inverse" to the (Gaussian) distribution of this process itself.)

Definition H.1 (Normalized inverse-Gaussian distribution) Given $Y_i \overset{\text{ind}}{\sim} \mathrm{IGau}(\alpha_i, 1)$ and $Y = \sum_{i=1}^k Y_i$, the vector $(Y_1, \ldots, Y_k)/Y$ is said to possess the *normalized inverse-Gaussian distribution* $\mathrm{NIGau}(k; \alpha)$ with parameters $k \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \geq 0$.

As the second parameter γ of an inverse-Gaussian distribution is a scale parameter, and one degree of freedom is lost by normalization, taking this parameter in the distribution of the variables Y_i in the preceding definition to be 1 is not a loss of generality.

The following propositions give elementary properties of inverse-Gaussian distributions; these resemble properties of the Dirichlet distribution.

Proposition H.2 (Aggregation)

(i) *If $Y_i \overset{ind}{\sim} \mathrm{IGau}(\alpha_i, \gamma)$, then $\sum_i Y_i \sim \mathrm{IGau}(\sum_i \alpha_i, \gamma)$.*
(ii) *If $(W_1, \ldots, W_k) \sim \mathrm{NIGau}(k; \alpha_1, \ldots, \alpha_k)$ and $Z_j = \sum_{i \in I_j} W_i$ for a given partition $I_1, \ldots, I_m$ of $\{1, \ldots, k\}$, then $(Z_1, \ldots, Z_m) \sim \mathrm{NIGau}(m; \beta_1, \ldots, \beta_m)$, where $\beta_j = \sum_{i \in I_j} \alpha_i$, for $j = 1, \ldots, m$.*

Proposition H.3 (Density) *If $\alpha_i > 0$ for $i = 1, \dots, k$, then the vector* $(W_1, \dots, W_k) \sim \mathrm{NIGau}(k; \alpha_1, \dots, \alpha_k)$ *possesses a probability density function, given by*

$$w = (w_1, \dots, w_k) \mapsto \frac{e^{\sum_{i=1}^k \alpha_i} \prod_{i=1}^k \alpha_i}{2^{k/2-1}\pi^{k/2}} \prod_{i=1}^k \frac{1}{w_i^{3/2}} \frac{K_{-k/2}\Big(\big(\sum_{i=1}^k \alpha_i^2/w_i\big)^{1/2}\Big)}{\big(\sum_{i=1}^k \alpha_i^2/w_i\big)^{k/4}}, \qquad w \in \mathbb{S}_k,$$

where K_m is the modified Bessel function of the third kind *of order m (see Watson (1995)). In particular, the marginal density of W_1 is*

$$w \mapsto \frac{e^{\alpha_1+\alpha_2}\alpha_1\alpha_2}{\pi} \frac{1}{w^{3/2}(1-w)^{3/2}} \frac{K_{-1}\Big(\big(\alpha_1^2/w + \alpha_2^2/(1-w)\big)^{1/2}\Big)}{\big(\alpha_1^2/w + \alpha_2^2/(1-w)\big)^{1/2}}, \qquad w \in (0,1).$$

Proposition H.4 (Laplace transform) *The Laplace transform of $Y \sim \mathrm{IGau}(\alpha, \gamma)$ is given by* $\mathrm{E}(e^{-\lambda Y}) = \exp\big[-\alpha(\sqrt{2\lambda + \gamma^2} - \gamma)\big]$, *for $\lambda \geq 0$.*

Proposition H.5 (Mean) *If $(W_1, \dots, W_k) \sim \mathrm{NIGau}(k; \alpha_1, \dots, \alpha_k)$, then,* $\mathrm{E}(W_i) = \alpha_i / \sum_{j=1}^k \alpha_j$.

Proof Write $W_i = Y_i/Y$, where $Y_j \overset{\text{ind}}{\sim} \mathrm{IGau}(\alpha_i, 1)$ and $Y = \sum_{j=1}^k Y_j$. Also let $Y_{-i} = \sum_{j \neq i} Y_j$ and $|\alpha| = \sum_{j=1}^k \alpha_j$. Then

$$\begin{aligned}
\mathrm{E}(W_i) &= \int_0^\infty \mathrm{E}(Y_i e^{-uY})\, du = \int_0^\infty \mathrm{E}(Y_i e^{-uY_i})\mathrm{E}(e^{-uY_{-i}})\, du \\
&= -\int_0^\infty \frac{d}{du}\mathrm{E}(e^{-uY_i})\, \mathrm{E}(e^{-uY_{-i}})\, du.
\end{aligned}$$

Now $Y_{-i} \sim \mathrm{IGau}(|\alpha| - \alpha_i, \gamma)$, by Proposition H.2. Therefore, by Proposition H.4 the preceding display is equal to

$$-\int_0^\infty \frac{d}{du} e^{-\alpha_i(\sqrt{2u+1}-1)}\, e^{-(|\alpha|-\alpha_i)(\sqrt{2u+1}-1)}\, du = \alpha_i \int_0^\infty (2u+1)^{-1/2} e^{-|\alpha|(\sqrt{2u+1}-1)} du.$$

We conclude the proof by the change of variables $y = \sqrt{2u+1} - 1$. ☐

The "generalized" inverse-Gaussian distribution replaces the power $y^{-3/2}$ in the density of an inverse-Gaussian distribution by a general one.

Definition H.6 (Generalized inverse-Gaussian distribution) The *generalized inverse-Gaussian distribution* $\mathrm{GIGau}(a, b, p)$ with parameters $\alpha > 0$, $\gamma > 0$ and $p \in \mathbb{R}$ is the probability distribution on $(0, \infty)$ with density

$$y \mapsto \frac{(\alpha/\gamma)^p}{2K_p(\alpha\gamma)} y^{p-1} e^{-\frac{1}{2}(\alpha^2/y + \gamma^2 y)},$$

where K_p is the modified Bessel function of the third kind.