Statistical Inference of Discretely Observed Compound Poisson Processes and Related Jump Processes

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Abstract

Contents

1 Introduction

Definition 1.1 (Counting Process)**.** A *counting process* is a stochastic process $\{N(t) : t \geq 0\}$ with values that are non-negative, integer and non-decreasing i.e. $\forall s, t \geq 0 : s \leq t$:

1. $N(t) \geq 0$,
2. $N(t) \in \mathbb{N}$,
3. $N(s) \leq N(t)$.

Definition 1.2 (Poisson Process)**.** A *Poisson process with intensity* λ is a counting process $\{N(t) : t \geq 0\}$ with the following properties:

1. $N(0) = 0$,
2. It has independent increments i.e. $\forall n \in \mathbb{N} : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, $N(t_n) - N(t_{n-1}), N(t_{n-1}) - N(t_{n-2}), \ldots, N(t_1)$ are independent,
3. The number of occurrences in any interval of length t is a Poisson random variable with parameter λt i.e. $\forall s, t : s \leq t$, $N(t) - N(s) \sim$ Poisson$(\lambda(t - s))$.

Lemma 1.1. *A Poisson process with intensity λ has exponentially distributed inter-arrival times with rate λ.*

Definition 1.3 (Compound Poisson Process)**.** Let $N(t) : t \geq 0$ be a d-dimensional Poisson process with intensity λ.

Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d random variables taking values in $\mathbb{R}^d$ with common distribution F.

Also assume that the Y_i's are independent of the Poisson process $\{N(t) : t \geq 0\}$.

Then, a *Compound Poisson process (CPP)* is a stochastic process $\{X(t) : t \geq 0\}$ such that

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

where, by convention, we take $X(t) = 0$ if $N(t) = 0$.

Suppose we take discrete observations of a CPP i.e. we consider $X(\Delta), X(2\Delta), \ldots$ where $X(t) : t \geq 0$ is a CPP. We want to estimate F. Note that the jump size $X(n\Delta) - X((n-1)\Delta)$ is equivalent in distribution to a Poisson random sum of intensity Δ:

$$X(n\Delta) - X((n-1)\Delta) = \sum_{i=1}^{N(n\Delta)} Y_i - \sum_{i=1}^{N((n-1)\Delta)} Y_i$$
$$= \sum_{i=1}^{N(n\Delta)-N((n-1)\Delta)} Y_i$$
$$=^d \sum_{i=1}^{N} Y_i$$

where $N \sim \text{Poisson}(\Delta)$

2 Spectral Approach

Now we have formulated the problem, we visit some methods for estimating the unknown density f. Since adding a Poisson number of Y's is referred to as compounding, much of the literature refers to the problem of recovering density f of Y's from observations of X as decompounding.

The approach of decompounding was famously proposed by Buchmann and Grübel to estimate the density f for discrete and continuous cases of the distribution F of the Y's.

Van Es built on this idea for fixed sampling rate $\Delta = 1$ using the Lévy - Khintchine formula. We explain the idea behind this method and show its strength through various examples.

2.1 Van Es

2.1.1 Construction of Density Estimator via suitable inversion of characteristic functions

We first note the following property:

Proposition 2.1. *For Poisson random sum X, the characteristic function of X, denoted by ϕ_X, is given by $\phi_X(t) = \mathbb{E}e^{itX} = e^{-\lambda+\lambda\phi_f(t)}$*

Proof.

$$
\begin{aligned}
\phi_X(t) &= \mathbb{E}e^{itX} \\
&= \mathbb{E}\left[\exp\left(it\sum_{i=1}^{N(\lambda)} Y_i\right)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{N(\lambda)} \exp(itY_i)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{N(\lambda)} \exp(itY_i)\middle| N(\lambda)\right]\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{N(\lambda)} \mathbb{E}\left[\exp(itY_1) \mid N(\lambda)\right]\right] && \text{(by i.i.d assumption of the } Y_i\text{'s)} \\
&= \mathbb{E}\left[\prod_{i=1}^{N(\lambda)} \phi_f(t)\right] && (Y_1 \text{ and } N(\lambda) \text{ are independent)} \\
&= \mathbb{E}\left[\exp(N(\lambda)\ln\phi_f(t))\right] \\
&= \exp(\lambda(e^{\ln\phi_f(t)} - 1)) && \text{(MGF of a Poisson random variable)} \\
&= e^{-\lambda+\lambda\phi_f(t)}
\end{aligned}
$$

□

We can rewrite $\phi_X(t)$ as:

$$
\begin{aligned}
\phi_X(t) &= e^{-\lambda}(e^{\lambda\phi_f(t)} - 1 + 1) \\
&= e^{-\lambda} + e^{-\lambda}(e^{\lambda\phi_f(t)} - 1) \\
&= e^{-\lambda} + e^{-\lambda}\frac{e^{\lambda}-1}{e^{\lambda}-1}(e^{\lambda\phi_f(t)} - 1) \\
&= e^{-\lambda} + \frac{1-e^{-\lambda}}{e^{\lambda}-1}(e^{\lambda\phi_f(t)} - 1) \qquad (1)
\end{aligned}
$$

Let g be the density of $X \mid N(\lambda) > 0$.
Let $\phi_g(t) = \mathbb{E}\left[e^{itX} \mid N(\lambda) > 0\right] = \frac{\mathbb{E}\left[e^{itX}\mathbb{1}(N(\lambda)>0)\right]}{\mathbb{P}(N(\lambda)>0)}$.
Then

$$
\begin{aligned}
\phi_X(t) &= \mathbb{E}\left[e^{itX}\mathbb{1}(N(\lambda) = 0)\right] + \mathbb{E}\left[e^{itX}\mathbb{1}(N(\lambda) > 0)\right] \\
&= \mathbb{P}(N(\lambda) = 0) + \mathbb{P}(N(\lambda) > 0)\phi_g(t) \\
&= e^{-\lambda} + (1 - e^{-\lambda})\phi_g(t)
\end{aligned}
$$

Therefore, using (1), we get that

$$\phi_g(t) = \frac{1}{e^{\lambda} - 1}(e^{\lambda\phi_f(t)} - 1) \tag{2}$$

Thus, we can see from this that if we were to obtain an estimator for $\phi_g(t)$, then by suitable inversion of the formula in (2), we would obtain an estimator for $\phi_f(t)$.

In order to rewrite (2) in terms of $\phi_f(t)$, we must be able to invert the complex exponential function since $\phi_f(t)$ takes complex values. However, such function is not invertible since it is not bijective: in particular it is not injective as $e^{w+2\pi i} = e^w \ \forall w \in \mathbb{C}$.

Therefore, we use the following lemmas concerning the distinguished logarithm:

Lemma 2.1. *If $h_1 : \mathbb{R} \to \mathbb{C}$ and $h_2 : \mathbb{R} \to \mathbb{C}$ are continuous functions such that $h_1(0) = h_2(0) = 0$ and $e^{h_1} = e^{h_2}$, then $h_1 = h_2$.*

Proof. See Appendix. □

Lemma 2.2. *If $\phi : \mathbb{R} \to \mathbb{C}$ is a continuous function such that $\phi(0) = 1$ and $\phi_g(t) \neq 0 \ \forall t \in \mathbb{R}$ then there exists a unique continuous function $h : \mathbb{R} \to \mathbb{C}$ with $h(0) = 0$ and $\phi(t) = e^{h(t)}$ for $t \in \mathbb{R}$.*

Proof. See Appendix. □

Therefore, for such a function ϕ as described in the Lemma, we say that the unique function h is the distinguished logarithm and we denote $h(t) = \text{Log}(\phi(t))$. Note also that for ϕ and ψ satisfying the assumptions of the Lemma, we have $\text{Log}(\phi(t)\psi(t)) = \text{Log}(\phi(t)) + \text{Log}(\psi(t))$ as expected.

Therefore, noting that $\phi(t) = e^{\lambda(\phi_f(t)-1)}$ is a continuous function satisfying $\phi(0) = 1$ and $\phi(t) \neq 0 \ \forall t \in R$, we get that

$$\begin{aligned}
\lambda(\phi_f(t) - 1) &= \text{Log}\left(e^{\lambda(\phi_f(t)-1)}\right) && \text{(Lemma 2.1)} \\
&= \text{Log}\left(e^{-\lambda}\left[(e^{\lambda} - 1)\phi_g(t) + 1\right]\right) \\
&= -\lambda + \text{Log}\left((e^{\lambda} - 1)\phi_g(t) + 1\right)
\end{aligned}$$

Therefore,

$$\phi_f(t) = \frac{1}{\lambda}\text{Log}\left((e^{\lambda} - 1)\phi_g(t) + 1\right) \tag{3}$$