Projection and its algorithm

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December 2020

Introduction

In this paper, we will encounter an algorithm to take the projection of any 3D polyhedron onto a plane passing through a specific point inside it. We will be engaged with linear algebra, linear space transformation, 3D geometry involving equation of planes and lines, vector algebra and 3D spherical coordinate system.

Let's say that we have a cuboid placed in a 3D coordinate system and we are observing it from a given set of coordinates and focusing on a specific point. Then how will this cuboid appear to us? The algorithm written below gives a solution to this very problem.

Preparing variables for the projection

Let us consider a cuboid having coordinates of vertices as follows:

$$A = (x_1, y_1, z_1)$$

$$B = (x_2, y_2, z_2)$$

$$C = (x_3, y_3, z_3)$$

$$D = (x_4, y_4, z_4)$$

$$E = (x_5, y_5, z_5)$$

$$F = (x_6, y_6, z_6)$$

$$G = (x_7, y_7, z_7)$$

$$H = (x_8, y_8, z_8)$$

In order to perform a projection operation on this cuboid, we would need to know the observers's position (let's call it O) and the location where the observer is focusing on (let's call it F_0).

$$Let, O \equiv (u, v, w)$$

$$F_0 \equiv (s, t, p)$$

Let us consider a vector originating from O and pointing towards F_0 and call it as $\vec{n}$. Therefore, we can write:

$$\vec{n} =< (s-u), (t-v), (p-w) >$$

For the sake convenience, let $(s-u) = l, (t-v) = n$ and $(p-w) = m$.

Therefore,

$$\vec{n} =< l, n, m >$$

Now consider this $\vec{n}$ as the normal vector for a plane that must also pass through point F_0. We will name this plane as P_1.

Then, the equation of the plane would be:

$$P_1 \equiv l \cdot (x-s) + n \cdot (y-t) + m \cdot (z-p) = 0$$

Now let's imagine a bunch of line segments connecting O to all the vertices of the cuboid. We will get eight different equations for each line. Let the 8 equation of lines be represented by: L_1(line joining O and A), L_2, L_3, $\ldots$,L_8.

The next step is to find the intersection point of each of these lines with the plane P_1. Let us say that the solution of the line L_1 and the plane P_1 is A'.

$$A' \equiv (x_{r1}, y_{r1}, z_{r1})$$

Like this we will get 8 set of coordinates and all of them would be lying on the same plane P_1 namely $A', B', C', \ldots, H'$. Let there be a set S having all of these 8 set of coordinates. These coordinates represent the 2D image/projection of this 3D cuboid on the plane P_1.

$$S = \{A', B', C', D', E', F', G', H'\}$$

Now in order to treat this plane P_1 as our new x-y 2D cartesian plane we have to define z and y axes in this plane which later on get transformed on the actual z and y axes of our 3D cartesian coordinate system. First of all, let us shift the origin to the point F_0 which changes $A'\,(x_{r1}, y_{r1}, z_{r1})$ to $A''\,(x_{r1}-s, y_{r1}-t, z_{r1}-p)$,$B'$ to B'' and so on. Now call the set of these shifted points to be S'.

$$S' = \{A'', B'', C'', D'', E'', F'', G'', H''\}$$

After that, we will use spherical coordinate system along with linear space transformation in such a manner so that all points in set S' could be transformed on plane:

$$P_2 \equiv x = 0$$

Brief introduction to spherical coordinate system

In spherical we represent a point P' as (r, θ, ϕ), where r is the radial distance of P from the origin, θ is the angle between the z-axis and the position vector of

P', and ϕ is the angle measured from x-axis to the projection of position vector of on x-y plane provided that:

$$0 \leq r < \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi < 2\pi$$

Let $\hat{a_r}$, $\hat{a_\theta}$ and $\hat{a_\phi}$ be three mutually orthogonal basis vectors along the position vector of P', the increasing direction of θ and the increasing direction of ϕ respectively. Therefore, any vector $\vec{A}$ could be written as:

$$\vec{A} =< A_r, A_\theta, A_\phi >= A_r\hat{a_r} + A_\theta\hat{a_\theta} + A_\phi\hat{a_\phi}$$

Also for any point $X \equiv (x, y, z)$ in 3D space, we can define:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arctan \frac{\sqrt{x^2 + y^2}}{z}$$

$$\phi = \arctan \frac{y}{x}$$

or we can also say that,

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Using the above information, we can conclude that -

$$\hat{a_r} = (\sin\theta\cos\phi)\hat{i} + (\sin\theta\sin\phi)\hat{j} + (\cos\theta)\hat{k}$$

$$\hat{a_\theta} = (\cos\theta\cos\phi)\hat{i} + (\cos\theta\sin\phi)\hat{j} - (\sin\theta)\hat{k}$$

$$\hat{a_\phi} = -(\sin\phi)\hat{i} + (\cos\phi)\hat{j}$$

Applying transformations to get the projection

We have $\vec{n}$ as the normal vector for plane P_1. Now let us find the r,θ and ϕ for this $\vec{n}$ considering our shifted origin and these conditions in our mind:

$$0 \leq r < \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi < 2\pi$$

Now defining,

$$r = \sqrt{l^2 + n^2 + m^2}$$

if $m > 0$, then:

$$\theta = \arctan \frac{\sqrt{l^2 + n^2}}{m}$$

and if $m < 0$, then:

$$\theta = \pi + \arctan \frac{\sqrt{l^2 + n^2}}{m}$$

if $l > 0$ and $n > 0$, then:

$$\phi = \arctan \frac{n}{l}$$

also if $l > 0$ and $n < 0$, then:

$$\phi = 2\pi + \arctan \frac{n}{l}$$

and if $l \leq 0$, then:

$$\phi = \pi + \arctan \frac{n}{l}$$

Now, after origin shifting, this plane P_1 has $\hat{a_r}$ (or $\frac{\vec{n}}{r}$) as its normal vector, $\hat{a_\theta}$ and $\hat{a_\phi}$ as its newly defined y and x axes.Now let us rename it as P_3.

$$P_3 \equiv (\sin\theta\cos\phi)x + (\sin\theta\sin\phi)y + (\cos\theta)z = 0$$

Defining a matrix which performs the transformation of $\hat{\imath}$,$\hat{\jmath}$ and $\hat{k}$ to $\hat{a_r}$,$\hat{a_\theta}$ and $\hat{a_\phi}$, respectively :

$$M = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$$

This matrix M will translate our 3D cartesian coordinates into the spherical coordinate system. But we have to find all the points in the set S', which already lies in the plane P_3,to get translated on the plane P_2 in such a way that $\hat{a_\theta}$ lines up with y-axis of our 3D cartesian system, $\hat{a_\phi}$ lines with z-axis and $\hat{a_r}$ lines up with x-axis.

For that purpose, we have to consider the inverse of matrix M. Now since, the matrix M has determinant equal to 1 and has eigenvalue of 1. Therefore, we can say that its inverse will be equal to its transpose.

$$M^{-1} = M^T = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}$$

On performing matrix vector transformation on each point of set S', we will get set of points having their z-coordinate as the x-coordinate of the finally projected image and y-coordinate as the y-coordinate of the finally projected image of its corresponding point in set S which itself corresponds to a vertex the given cuboid. Basically, this will give us a set of 2D cartesian coordinates which is the projection of the cuboid on the plane P_1.

One more thing to observe here is that since it is a 2D projection of 3D object, the x-coordinate of each vertex of this projected image of cuboid must be zero which can be easily seen by solving the equation given below for the transformation of A'' through matrix M^{-1}.

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} x_{r1} - s \\ y_{r1} - t \\ z_{r1} - p \end{bmatrix}$$

By performing this exact transformation on each point of set S' and plotting the resultant coordinates in 2D cartesian coordinate system, we have completely found the required projection.

There is a link to python program given below to illustrate this algorithm.

This is my link: https://github.com/sumit-6/Projection-and-its-algorithm.