AN INTERESTING LIMIT AND THE BASEL PROBLEM

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Abstract. Leonhard Euler's first celebrated achievement was his solution to what was then known as the *Basel* problem, in 1735. The Basel problem was first posed by mathematician Pietro Mengoli and was so famous that even Jakob Bernoulli wrote about it. The Base problem asks the question: what does the sum of infinite integer reciprocals sum to? i.e. $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Using a heuristic approach, Euler found this sum to be precisely $\frac{\pi^2}{6}$. Ever since, there have been been countless proofs for this famous result. In this short write up, I will investigate some connections between the result of $\frac{\pi^2}{6}$ and the complex exponential function e^{ix}. In the spirit of Euler's approach I will be using the expansion of $\frac{\sin x}{x}$ for this study.

1. The Basel Problem

Here we define the Basel Problem and its result. The computation is irrelevant for our purposes. It is provided for sake of its result.

Definition 1.1 (Basel Problem)**.** *For* $n \in [1, \infty) \subset \mathbf{Z}$,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \ldots = \frac{\pi^2}{6}$$

.

2. Relevant Identities

The following identities are shown without proof, since they're standard in the study of complex analysis and analytic functions. We know that an *analytic* function is one whose Taylor series expansion is well defined and converges for its given radius of convergence. More formally, by [1] an analytic function is a complex function $f(z)$ that has a derivative at every point of some region in the complex plane. This also implies the existence of complex series and Taylor expansions for functions of the complex variable z and are thus well defined if these functions satisfy the *Cauchy-Riemann Criterion*. We know that the trigonometric functions are analytic as well as the exponential function and it's inverse, the natural logarithm. For our study on the result of Basel problem, we will consider the expansions of the trigonometric functions as well as the expansion and properties of the complex exponential. The following equations will prove useful for our purpose.

2.1. Expansion of $\sin x$ and $\cos x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \tag{1}$$

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$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{2}$$

2.2. **Complex Exponential.**

$$e^{xi} = \cos x + i \sin x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right), \tag{3}$$

$$\therefore e^{xi} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \tag{4}$$

From (3) we have the following identities:

$$e^{\pi i} = \cos \pi + i \sin \pi = -1, \tag{5}$$

and by extension,

$$e^{\pi i} + 1 = 0,$$

$$e^{2\pi i} = 1 \tag{6}$$

3. Results

In the spirit of Euler's technique, we start by dividing $\sin x$ by x to get the following,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots,$$

however, note that we can rewrite (6) as,

$$\begin{aligned} e^{2\pi i} &= (\cos \pi + i \sin \pi)(\cos \pi + i \sin \pi) \\ &= \cos^2 \pi \\ &= \left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \ldots\right)^2 \\ &= \left(\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}\right)^2 \\ &= \left(\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}\right) \\ &= (-1)(-1) \\ &= 1. \end{aligned}$$

We can now justify writing,

$$
\begin{aligned}
\frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \\
&= \left(\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!}\right)^2 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots .
\end{aligned}
$$

Collecting leftover terms into an infinite series sum itself, we get

$$
\begin{aligned}
\frac{\sin x}{x} &= \left(\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!}\right)^2 + \left(-\frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right) \\
&= \left(\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!}\right)^2 + \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n+2}}{(2n+3)!}.
\end{aligned}
$$

Note the following limit,

$$
\lim_{x \longrightarrow \pi} \frac{\sin x}{x} = \frac{\sin \pi}{\pi} = 0, \tag{7}
$$

which is an important result, because it implies the following,

$$
\begin{aligned}
\lim_{x \longrightarrow \pi} \frac{\sin x}{x} &= \left(\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!}\right)^2 + \lim_{x \longrightarrow \pi}\left[\sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n+2}}{(2n+3)!}\right] \\
&= \left(\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!}\right)^2 + \sum_{n=0}^{\infty}(-1)^{n+1} \frac{\pi^{2n+2}}{(2n+3)!} \\
&= 0.
\end{aligned}
$$

If we extract the limits (due to linearity), we arrive at the following,

$$
\lim_{x \longrightarrow \pi} \frac{\sin x}{x} = \lim_{x \longrightarrow \pi}\left[\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right)^2 + \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n+2}}{(2n+3)!}\right] = 0. \tag{8}
$$

The result of the Basel problem is hidden in this equation, with a little algebraic manipulation we see,

$$
\begin{aligned}
\lim_{x \longrightarrow \pi} \frac{\sin x}{x} &= \lim_{x \longrightarrow \pi}\left[\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right)^2 + \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n+2}}{(2n+3)!}\right] \\
&= -\frac{\pi^2}{3!} + \lim_{x \longrightarrow \pi}\left[\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right)^2 + \sum_{n=1}^{\infty}(-1)^{n+1} \frac{x^{2n+2}}{(2n+3)!}\right] \\
&= 0.
\end{aligned}
$$

Applying (3) we then get,

$$\begin{aligned}\lim_{x\longrightarrow\pi}\frac{\sin x}{x} &= 1-\frac{\pi^2}{3!}+\lim_{x\longrightarrow\pi}\left[\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+2}}{(2n+3)!}\right]\\ &= 1-\frac{\pi^2}{3!}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^{2n+2}}{(2n+3)!}.\end{aligned}$$

We now arrive at the end of our computation,

$$\lim_{x\longrightarrow\pi}\frac{\sin x}{x}=1-\frac{\pi^2}{3!}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^{2n+2}}{(2n+3)!}=0. \tag{9}$$

It would do well to stop any further algebraic manipulation to avoid any circular manipulation, and also to discuss *convergence* and other observations.

4. Observations and Further Study

The infinite series in (9) does indeed converge, this can be observed by applying the ratio test for convergence [2] as follows,

$$a_n=(-1)^{n+1}\frac{\pi^{2n+2}}{(2n+3)!},\, a_{n+1}=(-1)^{n+2}\frac{\pi^{2n+4}}{(2n+5)!},$$

therefore,

$$\begin{aligned}\lim_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\longrightarrow\infty}\left|\frac{(-1)^{n+2}\frac{\pi^{2n+4}}{(2n+5)!}}{(-1)^{n+1}\frac{\pi^{2n+2}}{(2n+3)!}}\right|\\ &= \lim_{n\longrightarrow\infty}\left|(-1)^{n+2}\frac{\pi^{2n+4}}{(2n+5)!}(-1)^{n+1}\frac{(2n+3)!}{\pi^{2n+2}}\right|\\ &= \pi^2\lim_{n\longrightarrow\infty}\frac{(2n+3)!}{(2n+5)!}\\ &= \pi^2\lim_{n\longrightarrow\infty}\frac{(2n+3)!}{(2n+5)(2n+3)!}\\ &= \pi^2\lim_{n\longrightarrow\infty}\frac{1}{2n+5}\\ &= (\pi)(0)=0,\end{aligned}$$

and since $0<1$ we know the series converges. That much we can be sure of. These results make some broad assumptions and they must be taken into account. We must keep in mind of that this depends on the fact that $\sin x$ and $\frac{\sin x}{x}$ are both analytic, thus allowing loose association with the complex exponential.

An interesting result that was found during this study is actually implied by Euler's identity, observe the following,

$$
\begin{aligned}
e^{\pi i} &= -1 \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} \\
\implies & \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\pi^{2n-2}}{(2n)!} = \frac{1}{\pi^2}.
\end{aligned}
$$

References

[1] Thomas, George B., *Calculus and Analytic Geometry*, Reading, Mass: Addison-Wesley Pub. Co, Mass, 1968.

[2] Clapham C, Nicholson J. *The Concise Oxford Dictionary of Mathematics*. 5 ed. ed. Oxford University Press; 2014.