A Peculiar Similarity Between The Hyperbolic and Trigonometric Sine Functions

Sebastián R. Papanikolaou Costa

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Abstract

While investigating the function the series expansion of $\frac{\sin x}{x}$, a striking similarity arose when comparing it to the series expansion for $\frac{\sinh x}{x}$. In this write up, this peculiar similarity will be stated for expository purposes. Follow up questions are posed for further study.

1 The Trigonometric Sine

The sine function is one of the most basic elementary functions taught in precalculus courses, here shall be presented a few key identities that shall prove useful.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

which leads use to our first identity, one that has actually been studied for a long time already,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} \tag{1}$$

2 The Hyperbolic Sine

Usually, the hyperbolic functions are considered a bit later, around the first to second levels of college calculus. These functions are actually defined in terms of sums of the exponential function, like so,

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

The hyperbolic functions are analytic, meaning that they have Taylor series expansions that converge in a given radius. The Maclaurin expansion for the hyperbolic sine is,

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Dividing by x we arrive at our second important identity,

$$\frac{\sinh x}{x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} \tag{2}$$

3 The Big Question

We can see that the expansion of $\frac{\sin x}{x}$ would be the exact same series as the series expansion of $\frac{\sinh x}{x}$, if the former wasn't an alternating series. For now, let us define an analytical function ϑ as follows,

Definition 3.1. *Let ϑ be a function of x defined as $\vartheta : \mathbb{R} \longrightarrow \mathbb{R}$ with $x \in \mathbb{R}$. Furthermore, let ϑ be convergent. Then we write ϑ as,*

$$\vartheta(x) = \sum_{n=0}^{\infty} \frac{\left|(-1)^n x^{2n}\right|}{(2n+1)!}. \tag{3}$$

The question is, what is the pointwise difference between $\vartheta(x)$ and $\frac{\sinh x}{x}$ More precisely, we wonder whether:

$$\frac{\sinh x}{x} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{\left|(-1)^n x^{2n}\right|}{(2n+1)!} = \vartheta(x)$$

is true or not.

The question is asked because the first n terms of both series agree with each other:

$$\frac{\sinh x}{x} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{\left|(-1)^n x^{2n}\right|}{(2n+1)!} = \vartheta(x).$$

4 Conclusions

Another question arises: what is the relation between $\vartheta(x)$ and $\frac{\sin x}{x}$?

The question in Section 3 was found while studying the series expansion techniques Euler used to arrive at his solution to the *Basel* problem in another write up that was made. This write up will be passed around for different mathematicians to look over and comment.

References

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[3] Weisstein, Eric W. "Hyperbolic Sine." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/HyperbolicSine.html