Order Encoding

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August 1, 2021

Abstract

In this short paper I will attempt to present a scheme for giving an "order" to an unordered sets of binary strings. We will use linear algebraic techniques to show that strings of $1's$ and $0's$ will correspond to natural numbers, thus allowing for an alternate ordering scheme from more traditional ones.

1 Order

In mathematics, the concept of an order can mean a lot of things. We have derivatives of increasing and decreasing orders in calculus as well as the ordering in a strictly monotonically increasing (or decreasing) sequence. We will concern ourselves with the latter.

Consider the set B_2 of binary strings of length 2:

$$B_2 = \{00, 01, 10, 11\}.$$

We know that by the technique of binary counting, each length-2 string corresponds to a non-negative natural number by considering the positions of the string to be powers of 2, from right to left, where a position with a 1 denotes the use of the power of 2 in that position and a 0 denotes the absence of the power of 2 in that position. To calculate the number that corresponds to the binary string, we simply add the powers of 2 that correspond to the 1's and 0's positions in the string, like so:

$$011010 \longrightarrow 0 + 2^4 + 2^3 + 0 + 2^1 + 0 = 26,$$

$$01 \longrightarrow 0 + 2^0 = 1,$$

$$0111 \longrightarrow 0 + 2^2 + 2^1 + 2^0 = 7.$$

Using this simple technique, we can establish an order to binary strings. Mathematically, this can be shown but first we must impose the constraint that all strings considered are of the same length.

2 The Order Encoding Scheme

Let's take a wild leap of faith here and take binary strings of length n to be vectors of a dimension n, mathematically we will need the vectors to be over the field of natural numbers $\mathbf{N}$. Furthermore, our vectors will contain only 1's and 0's.

Definition 1 (Order Encoding Function)**.** *Let Υ be a subspace of $\mathbf{N}^n$, or $\Upsilon \subset N^n$. Furthermore for all $\upsilon_i \in \Upsilon$, we have $\upsilon_{i,j} = 0 \vee \upsilon_{i,j} = 1$ for $1 \leq j \leq n$ and $1 \leq i \leq |\Upsilon|$. Then there exists a function T such that $T : \Upsilon \longrightarrow \mathbf{N}$ and we define it as follows:*

$$T = \left(\begin{bmatrix} 2^{n-1} & 0 & \cdots \\ \vdots & \ddots & \\ 0 & & 2^0 \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \vdots \\ \upsilon_n \end{bmatrix}_i \right) \bullet \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \vdots \\ \upsilon_n \end{bmatrix}_i \tag{1}$$

T is called the "Order Enoder" or "Order Enconding Function" and will map a binary string of length n to its correspong natural number representation.

Remark 1. *Note that we can "tighten" our restrictions on Υ by stating that $|\upsilon_i|^2 \leq n$.*

A brief example is given for sake of completion:

Example 1. *Consider $\Upsilon = \{(0,0),(0,1),(1,0),(1,1)\}$. Applying T an all elements of Υ, we get:*

$$\upsilon_1 = (0,0) \longrightarrow T = \left(\begin{bmatrix} 2^1 & 0 \\ 0 & 2^0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \bullet \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

$$\upsilon_2 = (0,1) \longrightarrow T = \left(\begin{bmatrix} 2^1 & 0 \\ 0 & 2^0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \bullet \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

$$\upsilon_3 = (1,0) \longrightarrow T = \left(\begin{bmatrix} 2^1 & 0 \\ 0 & 2^0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2$$

$$\upsilon_4 = (1,1) \longrightarrow T = \left(\begin{bmatrix} 2^1 & 0 \\ 0 & 2^0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \bullet \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3$$

3 Note

Please be advised that these results are preliminary and no proof has been worked on, however an induction process would be used.