

# Symmetric computations

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<https://surakuma.github.io/courses/daamtc.html>

# Symmetric Rank-K (SYRK) update

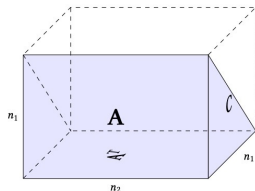
$C = A \cdot A^T$ , where  $A$  is an  $n_1 \times n_2$  matrix and  $C$  is an  $n_1 \times n_1$  symmetric matrix.

- It multiplies a matrix with its transpose
- It requires roughly half of the computation of general matrix multiplication due to symmetry
- $C_{ij} = \sum_{k=0}^{n_2-1} A_{ik} A_{jk}$  for  $0 \leq j \leq i \leq n_1$

SYRK pseudo code:

```
for  $i = 0:n_1 - 1$ 
  for  $j = 0:i$ 
    for  $k = 0:n_2 - 1$ 
       $C[i][j] += A[i][k] * A[j][k]$ 
```

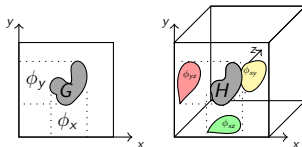
SYRK iteration space:



Here we assume that each  $C[i][j]$  is initialized to 0 in the beginning.

# Loomis-Whitney inequality

- For the 2d object  $G$ ,  $\text{Area}(G) \leq \phi_x \phi_y$
- For the 3d object  $H$ ,  $\text{Volume}(H) \leq \sqrt{\phi_{xy} \phi_{yz} \phi_{xz}}$



## Lemma

Let  $V \in \mathbb{Z}^3$  and  $\phi_{ij}(V)$  be the projection of  $V$  on the  $i - j$  plane, i.e.,  
 $\phi_{ij}(V) = \{(i, j) : \exists k, (i, j, k) \in V\}$ . Similarly  $\phi_{jk}(V)$  and  $\phi_{ik}(V)$  are defined.  
Then

$$|V| \leq |\phi_{ij}(V)|^{\frac{1}{2}} |\phi_{jk}(V)|^{\frac{1}{2}} |\phi_{ik}(V)|^{\frac{1}{2}}.$$

# Extension of Loomis-Whitney inequality

Theorem (Lemma 3, Ballard et al., SPAA 2023.)

Let  $V = \{(i, j, k) \in \mathbb{Z}^3 : j < i\}$  and  $\phi_{ij}(V)$  be the projection of  $V$  on the  $i - j$  plane, i.e.,  $\phi_{ij}(V) = \{(i, j) : \exists k, (i, j, k) \in V\}$ . Similarly  $\phi_{jk}(V)$  and  $\phi_{ik}(V)$  are defined. Then

$$2|V| \leq |\phi_{jk}(V) \cup \phi_{ik}(V)| (2|\phi_{ij}(V)|)^{\frac{1}{2}}.$$

Proof: Let  $\tilde{V} = \{(i, j, k) \in \mathbb{Z}^3 : (j, i, k) \in V\}$ .

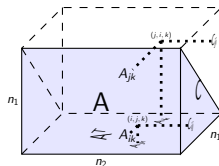
$$|V| = |\tilde{V}| \text{ and } |V \cup \tilde{V}| = 2|V|$$

$$\phi_{ij}(\tilde{V}) = \{(i, j) : (j, i) \in \phi_{ij}(V)\} \text{ and}$$

$$|\phi_{ij}(V) \cup \phi_{ij}(\tilde{V})| = 2|\phi_{ij}(V)|$$

$$\phi_{jk}(\tilde{V}) = \phi_{ik}(V), \phi_{ik}(\tilde{V}) = \phi_{jk}(V) \text{ and}$$

$$\phi_{ik}(V \cup \tilde{V}) = \phi_{jk}(V \cup \tilde{V}) = \phi_{jk}(V) \cup \phi_{ik}(V)$$



Applying the previous lemma (Loomis-Whitney inequality) on  $V \cup \tilde{V}$  yields the mentioned inequality.

# Parallel memory-independent communication lower bounds for SYRK

We focus on the computation of the entries below the diagonal of  $C$ .

```
for  $i = 0:n_1 - 1$ 
  for  $j = 0:i - 1$ 
    for  $k = 0:n_2 - 1$ 
       $C[i][j] += A[i][k] * A[j][k]$ 
```

- The computation is load balanced – each processor performs  $\frac{n_1(n_1-1)n_2}{2P}$  loop iterations
- One copy of data is in the system – there exists a processor whose input data at the start plus output data at the end must be at most  $\frac{n_1(n_1-1)/2 + n_1n_2}{P}$  words
  - Will analyze data transfers for this processor
  - $F$  be the set of indices  $(i, j, k)$  associated with loop iterations performed on this processor

# Optimization problem to compute communication lower bound

$$\begin{aligned} & \text{Minimize } |\phi_{ik}(F) \cup \phi_{jk}(F)| + |\phi_{ij}(F)| \quad \text{s.t.} \\ & (2|\phi_{ij}(F)|)^{\frac{1}{2}} |\phi_{ik}(F) \cup \phi_{jk}(F)| \geq 2|F| = \frac{n_1(n_1 - 1)n_2}{P} \end{aligned}$$

$|\phi_{ik}(F) \cup \phi_{jk}(F)|$  : number of accessed entries of  $A$

$|\phi_{ij}(F)|$  : number of accessed entries of  $C$

- Using Lagrange multipliers, optimal value obtained when

$$|\phi_{ik}(F) \cup \phi_{jk}(F)| = 2|\phi_{ij}(F)| = \left( \frac{n_1(n_1-1)n_2}{P} \right)^{\frac{2}{3}}$$

- Lower bound =  $|\phi_{ik}(F) \cup \phi_{jk}(F)| + |\phi_{ij}(F)| - \text{data owned by the processor}$   
$$= \frac{3}{2} \left( \frac{n_1(n_1-1)n_2}{P} \right)^{\frac{2}{3}} - \frac{n_1(n_1-1)/2 + n_1n_2}{P}$$

# Symmetric Rank-2k (SYR2K) update

$C = A \cdot B^T + B \cdot A^T$ , where  $A$  and  $B$  are  $n_1 \times n_2$  matrices.  $C$  is an  $n_1 \times n_1$  symmetric matrix.

SYR2K pseudo code:

for  $i = 0:n_1 - 1$

for  $j = 0:i$

for  $k = 0:n_2 - 1$

$C[i][j] += A[i][k] * B[j][k] + A[j][k] * B[i][k]$

## Assignment 3 – deadline Oct 5

**Question:** Obtain memory-independent communication lower bound for SYR2K. Assume that all the operations of each loop iteration are performed on the same processor, i.e,  $A[i][k] * B[j][k]$  and  $A[j][k] * B[i][k]$  are computed on one processor.