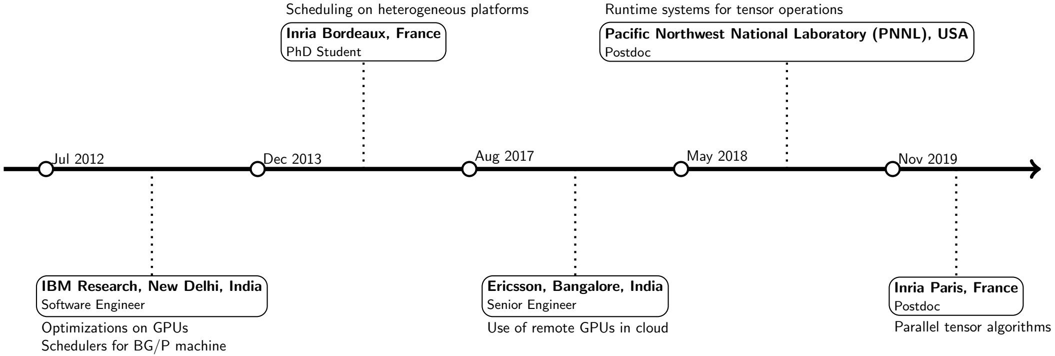
Scalable Tensor Algorithms for Modern Computing Systems

Suraj KUMAR

CNRS LIP/LaBRI Applicant

March 15, 2022

Resume in timeline



Jul 2012

IBM Research, New Delhi, India
Software Engineer

Optimizations on GPUs
Schedulers for BG/P machine

Dec 2013

Scheduling on heterogeneous platforms

Inria Bordeaux, France
PhD Student

Aug 2017

Ericsson, Bangalore, India
Senior Engineer

Use of remote GPUs in cloud

May 2018

Runtime systems for tensor operations

Pacific Northwest National Laboratory (PNNL), USA
Postdoc

Nov 2019

Inria Paris, France
Postdoc

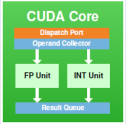
Parallel tensor algorithms

Advisors/Close collaborators

- PhD supervisors at Inria Bordeaux, France: Olivier Beaumont, Lionel Eyraud-Dubois, Samuel Thibault, Emmanuel Agullo
- Postdoc supervisors/Close collaborators: Marcin Zalewski (PNNL, USA), Sriram Krishnamoorthy (PNNL, USA), Laura Grigori (Inria Paris, France), Grey Ballard (Wake Forest University, USA)

Past research experience

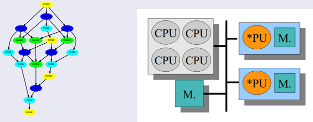
GPU algorithms



IBM Research, India, 2013

🟦 Stencil computations

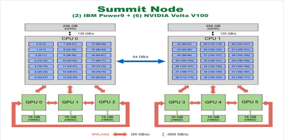
Scheduling on heterogeneous platforms



PhD thesis, Inria Bordeaux, France, 2017

🟦 Dense linear algebra computations

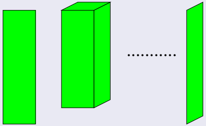
Fast molecular simulations



PNNL, USA, 2019

🟩🟦🟦 Tensor computations, TAMM library

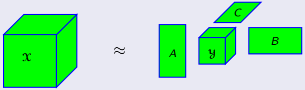
Parallel tensor approximations



Inria Paris, France, 2019

🟩🟦🟪 Tensor-train decomposition

Communication optimal algorithms



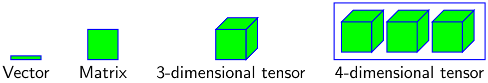
Inria Paris, France, 2019

🟩🟦🟦 Multiple Tensor-times-matrix computations (to obtain $\mathcal{Y}$)

Expertise: Tensor computations, Parallel algorithms, Communication costs, Low-rank approximations

Tensors and their uses

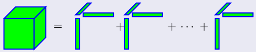
- **Neuroscience**: Neuron × Time × Trial
- **Media**: User x Movie x Time
- **Ecommerce**: User x Product x Time
- **Social-Network**: Person x Person x Time x Type



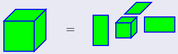
- High dimensional tensors: Neural network, Molecular simulation, Quantum computing

- People work with low dimensional structure (decomposition) of the tensors

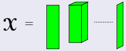
Canonical decomposition



Tucker decomposition



Tensor-train decomposition



Importance of communication

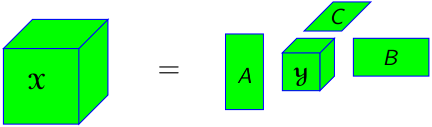
- Gaps between computation and communication costs growing exponentially

	time-per-operation	Network-bandwidth	Network-latency
Annual improvements	59 %	26 %	15 %

Source: Getting up to speed: The future of supercomputing

Goal : Scalable and communication optimal algorithms for tensor computations

Higher-order SVD (HOSVD) to compute Tucker decomposition



Algorithm 1 3-dimensional HOSVD Algorithm($\mathcal{X}$)

```
1: Obtain factor matrices $A, B$ and $C$ from the matrix representations of the input tensor $\mathcal{X}$
2: $\mathcal{Y} = \mathcal{X} \times_1 A^\mathsf{T} \times_2 B^\mathsf{T} \times_3 C^\mathsf{T}$
3: Return $\mathcal{Y}$, $A$, $B$, $C$
```

- $\mathcal{X}$, $\mathcal{Y}$: 3-dimensional input and output tensors (or arrays) & A, B, C: matrices
- $\times_i$: tensor contraction along the ith dimension (similar to matrix multiplication)
- Multiple Tensor-Times-Matrix (Multi-TTM) computation: $\mathcal{Y} = \mathcal{X} \times_1 A^\mathsf{T} \times_2 B^\mathsf{T} \times_3 C^\mathsf{T}$

Communication lower bounds and Communication optimal algorithms

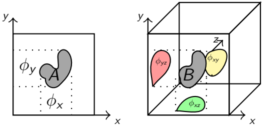
1. For Matrix Matrix Multiplications
2. For Multi-TTM Computations

Settings

- P number of processors
- Each processor performs (asymptotically) equal amount of operations
- One copy of data is in the system
  - $1/P$th amount of inputs (before the computation) and output (after the computation) on each processor
- Focus on bandwidth cost (volume of data transfers)

- Submitted this work to SPAA 2022 conference
- Joint work with L. Grigori, G. Ballard, K. Rouse, H. Al Daas

Approach to obtain communication lower bounds



- Loomis-Whitney inequalitiy: for $d-1$ dimensional projections
  - For the 2d object A, $\phi_x\phi_y \geq Area(A)$
  - For the 3d object B, $(\phi_{xy}\phi_{yz}\phi_{xz})^{\frac{1}{2}} \geq Volume(B)$
- Hölder-Brascamp-Lieb (HBL) inequality – generalization for arbitrary dimensional projections
  - Provide exponent for each projection

Constraints for parallel load balanced matrix matrix multiplication

- $C = AB$ with $A \in \mathbb{R}^{n_1 \times n_2}$, $B \in \mathbb{R}^{n_2 \times n_3}$, and $C \in \mathbb{R}^{n_1 \times n_3}$

$$\text{for } i = 1{:}n_1, \text{ for } k = 1{:}n_2, \text{ for } j = 1{:}n_3$$
$$C[i][j] += A[i][k] * B[k][j]$$

- ϕ_A, ϕ_B, ϕ_C: projections of computations on arrays A, B, C
- From Loomis-Whitney/HBL inequality: $\phi_A^{\frac{1}{2}}\phi_B^{\frac{1}{2}}\phi_C^{\frac{1}{2}} \geq$ number of multiplications per processor $= \frac{n_1n_2n_3}{P}$
- Our contributions: $\frac{n_1n_2}{P} \leq \phi_A \leq n_1n_2$, $\frac{n_2n_3}{P} \leq \phi_B \leq n_2n_3$, $\frac{n_1n_3}{P} \leq \phi_C \leq n_1n_3$

Optimization problem and Communication lower bounds

$$\textit{Minimize } \phi_A + \phi_B + \phi_C \quad \text{s.t.}$$

$$\phi_A^{\frac{1}{2}}\phi_B^{\frac{1}{2}}\phi_C^{\frac{1}{2}} \geq \frac{n_1 n_2 n_3}{P}$$

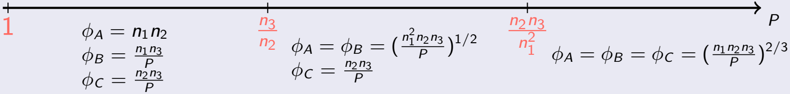
$$\frac{n_1 n_2}{P} \leq \phi_A \leq n_1 n_2$$

$$\frac{n_2 n_3}{P} \leq \phi_B \leq n_2 n_3$$

$$\frac{n_1 n_3}{P} \leq \phi_C \leq n_1 n_3$$

Amount of array accesses $= \phi_A + \phi_B + \phi_C$

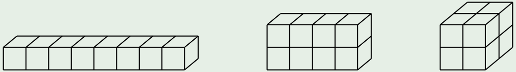
- Estimate the solution and prove optimality using all Karush–Kuhn–Tucker conditions are satisfied
- For $n_1 \leq n_2 \leq n_3$,



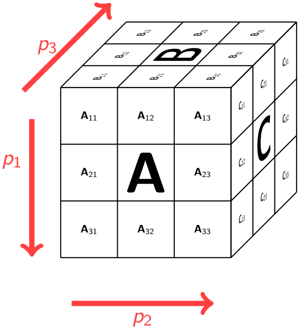
- Communication lower bound $= \phi_A + \phi_B + \phi_C -$ data owned by the processor $= \phi_A + \phi_B + \phi_C - \frac{n_1 n_2 + n_2 n_3 + n_1 n_3}{P}$

Design of communication optimal algorithms for $C = AB$

Arrangements of 8 processors



- P is organized into $p_1 \times p_2 \times p_3$ logical grid
- Select p_1, p_2 and p_3 based on the communication lower bounds
- Gather A on the set of processors along each slice of p_3
- Gather B on the set of processors along each slice of p_1
- Perform local computation
- Perform reduce operation along p_2 to obtain C



3-dimensional Multi-TTM ($\mathcal{Y} = \mathcal{X} \times_1 \mathbf{A}^{(1)^\mathsf{T}} \times_2 \mathbf{A}^{(2)^\mathsf{T}} \times_3 \mathbf{A}^{(3)^\mathsf{T}}$)

- TTM-in-Sequence approach (used in Tucker-MPI)
  - $\mathcal{Y} = \left(\left(\mathcal{X} \times_1 \mathbf{A}^{(1)^\mathsf{T}}\right) \times_2 \mathbf{A}^{(2)^\mathsf{T}}\right) \times_3 \mathbf{A}^{(3)^\mathsf{T}}$

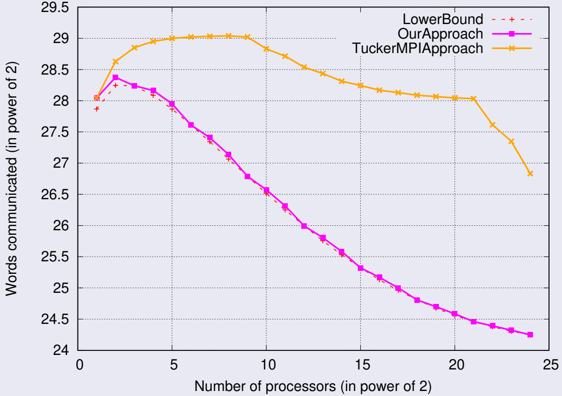
All-at-Once definition with $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, $\mathcal{Y} \in \mathbb{R}^{r_1 \times r_2 \times r_3}$, $\mathbf{A}^{(k)} \in \mathbb{R}^{n_k \times r_k}$ (our contribution)

```
for $n_1' = 1{:}n_1$, for $n_2' = 1{:}n_2$, for $n_3' = 1{:}n_3$
  for $r_1' = 1{:}r_1$, for $r_2' = 1{:}r_2$, for $r_3' = 1{:}r_3$
    $\mathcal{Y}(r_1', r_2', r_3') = \mathcal{Y}(r_1', r_2', r_3')$
        $+ \left(\mathcal{X}(n_1', n_2', n_3') * \mathbf{A}^{(1)}(n_1', r_1') * \mathbf{A}^{(2)}(n_2', r_2') * \mathbf{A}^{(3)}(n_3', r_3')\right)$
```

- Applied our framework to compute lower bounds
- Designed similar communical optimal algorithms (though 6-dimensional)

Simulated performance comparison

$n_1 = n_2 = n_3 = 2^{20}, r_1 = r_2 = r_3 = 2^8$



- Typical scenarios in data compression problems
- Our approach communicates much less than the state-of-the-art approach (TuckerMPI)

Project: Scalable Tensor Algorithms for Modern Computing Systems

1. Design of Scalable Communication Optimal Algorithms for Tensors (Main Focus)
2. Extension of Existing Approaches/Algorithms (Short/Mid Term Research Plans)
3. Exploratory Topics (Mid/Long Term Research Plans)

Scalable algorithms for popular tensor operations

- Determine the communication lower bounds for tensor decompositions
- Analyse the popular decomposition algorithms and communications performed by them
- Propose new scalable communication optimal algorithms
    - If possible design tiles/tasks based algorithms
- Implement the proposed algorithms
- Same for manipulation operations of popular tensor representations
- Create a tensor library

Strassen's concepts to tensors

Matrix multiplication of $n \times n$ square matrices

- Complexity of traditional matrix multiplication is $\mathcal{O}(n^3)$
- Strassen's matrix multiplication
  - Expressed matrix multiplication as a tensor computation
  - Canonical rank of the tensor determines the complexity of the computation
  - Complexity is $\mathcal{O}(n^{2.81})$
- Plan to extend Strassen's concepts to tensor contractions

Contraction of a 3-dimensional tensor with a matrix

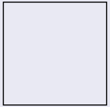
```
for $i_1 = 1 : n$ do
    for $i_2 = 1 : n$ do
        for $i_3 = 1 : n$ do
            for $j_2 = 1 : n$ do
                $\mathcal{G}(i_1, i_2, j_2) = \mathcal{G}(i_1, i_2, j_2) + \mathcal{A}(i_1, i_2, i_3) * B(i_3, j_2)$
            end for
        end for
    end for
end for
```

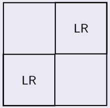
- Total $\mathcal{O}(n^4)$ operations
- Apply Strassen's algorithm for each i_1, total $\mathcal{O}(n^{3.81})$ operations
- Expressing as a canonical decomposition of $8 \times 8 \times 4$ tensor can further reduce the number of operations (first try)

Hierarchical matrix concepts to tensors

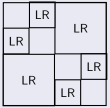
Hierarchical matrices

- Data sparse approximation of non-sparse matrices


Original Matrix



Step 1

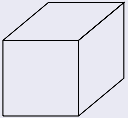


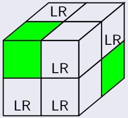
Step 2

LR: low rank block

Tensors

- $f(i,j,k) = \frac{1}{|i-j|+|j-k|+|k-i|}$
- Value is small if difference of any pair is large
- Formalize and evaluate this approach for tensors


Original Tensor



Step 1

High dimensional tensor representations

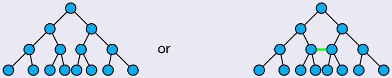
Notation: Tensors are denoted by solid shapes and number of lines denote the dimensions of the tensors. Connecting two lines implies summation (or contraction) over the connected dimensions.

- Adding tensors and applying an operator in Tensor-train representation

- Requires a truncation process which iterates over cores one by one – not suitable to work in parallel

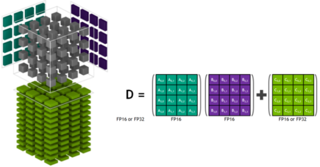
New tensor representations – suitable for parallelization

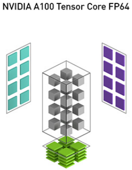
- Look at new representations in tree format



  - Data will be stored at the leaf nodes

Architecture aware algorithms and Training of tensorized neural networks





- Recent Nvidia GPUs have tensor cores to accelerate AI computations
- Design parallel algorithms which take architecture details into account
- Design parallel methods to train tensorized neural networks

Integration in the LIP/LaBRI laboratory

My research plans

- **Main focus**: scalable and communication optimal tensor algorithms
- **Short term plans**: Strassen's concepts to tensors, hierarchical matrix concepts to tensors
- **Long term plans**: new tensor representations, parallel training of tensorized neural networks

ROMA team (LIP laboratory)

- *Bora Ucar*: design of tensor compression and manipulation algorithms
- *Gregoire Pichon*: low-rank based methods
- *Anne Benoit, Loris Marchal, Yves Robert and Frederic Vivien*: scalability and scheduling aspects in the long term

SATANAS team (LaBRI laboratory)

- *Olivier Beaumont* and *Lionel Eyraud-Dubois*: parallel training of tensorized neural networks
- *Mathieu Faverge*: low-rank based methods
- *Abdou Guermouche, Samuel Thibault*: exploitation of maximum potential of HPC systems in the long term

Bringing additional skills

- High dimensional dense tensor computations, use of tensors in molecular simulations (for both teams)
- Communication lower bounds for linear algebra computations (for both teams)
- Scalable approaches for large HPC systems and GPU computations (only for the ROMA team)

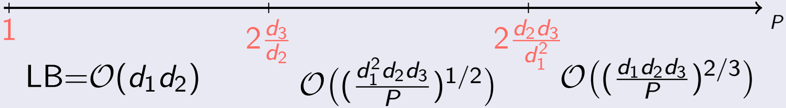
Thank You!

Backup Slides

Existing lower bounds for matrix matrix multiplication

- $C = AB$, where $A \in \mathbb{R}^{n_1 \times n_2}$, $B \in \mathbb{R}^{n_2 \times n_3}$, and $C \in \mathbb{R}^{n_1 \times n_3}$
- Let $d_1 = \min(n_1, n_2, n_3) \leq d_2 = median(n_1, n_2, n_3) \leq d_3 = \max(n_1, n_2, n_3)$

Existing communication lower bounds (CARMA [IPDPS 2013])

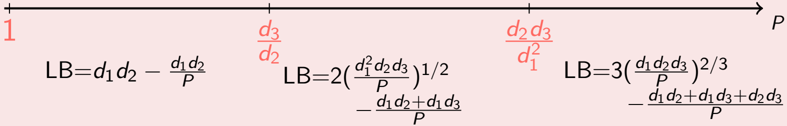


P

1 $\quad 2\frac{d_3}{d_2} \quad 2\frac{d_2 d_3}{d_1^2}$

$\text{LB}=\mathcal{O}(d_1 d_2) \quad \mathcal{O}((\frac{d_1^2 d_2 d_3}{P})^{1/2}) \quad \mathcal{O}((\frac{d_1 d_2 d_3}{P})^{2/3})$

Our communication lower bounds



P

1 $\quad \frac{d_3}{d_2} \quad \frac{d_2 d_3}{d_1^2}$

$\text{LB}=d_1 d_2 - \frac{d_1 d_2}{P}$

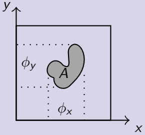
$\text{LB}=2(\frac{d_1^2 d_2 d_3}{P})^{1/2} - \frac{d_1 d_2 + d_1 d_3}{P}$

$\text{LB}=3(\frac{d_1 d_2 d_3}{P})^{2/3} - \frac{d_1 d_2 + d_1 d_3 + d_2 d_3}{P}$

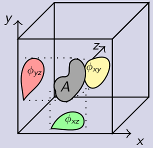
Loomis-Whitney & Hölder-Brascamp-Lieb inequalities

Size of $d-1$ dimensional projections (Loomis-Whitney inequalitiy)

- 2-dimensional object A and its 1-dimensional projections ϕ_x, ϕ_y
- $\phi_x \phi_y \geq Area(A)$



- 3-dimensional object A and its 2-dimensional projections: ϕ_{xy}, ϕ_{yz}, ϕ_{xz}
- $(\phi_{xy}\phi_{yz}\phi_{xz})^{\frac{1}{3-1}} \geq Volume(A)$



Hölder-Brascamp-Lieb (HBL) inequality – Generalization of Loomis-Whitney inequality

$$\boldsymbol{\Delta} = \begin{array}{c} \\ i \\ j \\ k \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

for $i = 1{:}n_1$, for $k = 1{:}n_2$, for $j = 1{:}n_3$

$$C[i][j] += A[i][k] * B[k][j]$$

- Find $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathsf{T}}$ such that $\boldsymbol{\Delta}.\mathbf{x} \geq \mathbf{1}$, $\mathbf{1}$ is vector of all ones
- ϕ_A, ϕ_B, ϕ_C: projections of computations on arrays A, B, C
- HBL inequality: $\phi_A^{x_1}\phi_B^{x_2}\phi_C^{x_3} \geq$ Amount of computations
- To make inequality tight select $\mathbf{x}$ such that $\mathbf{1}^{\mathsf{T}}\mathbf{x}$ is minimum $=> x_1 = x_2 = x_3 = \frac{1}{2}$

Optimization problem and Communication lower bounds

- ϕ_A, ϕ_B, ϕ_C indicate the amount of array accesses

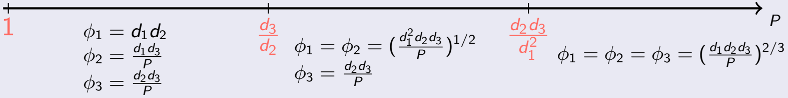
$$\textit{Minimize } \phi_A + \phi_B + \phi_C \text{ s.t.}$$
$$\phi_A^{\frac{1}{2}} \phi_B^{\frac{1}{2}} \phi_C^{\frac{1}{2}} \geq \frac{n_1 n_2 n_3}{P}$$
$$\frac{n_1 n_2}{P} \leq \phi_A \leq n_1 n_2$$
$$\frac{n_2 n_3}{P} \leq \phi_B \leq n_2 n_3$$
$$\frac{n_1 n_3}{P} \leq \phi_C \leq n_1 n_3$$

Generalized version ($d_1 \leq d_2 \leq d_3$)

$$\textit{Minimize } \phi_1 + \phi_2 + \phi_3 \text{ s.t.}$$
$$\phi_1^{\frac{1}{2}} \phi_2^{\frac{1}{2}} \phi_3^{\frac{1}{2}} \geq \frac{d_1 d_2 d_3}{P}$$
$$\frac{d_1 d_2}{P} \leq \phi_1 \leq d_1 d_2$$
$$\frac{d_1 d_3}{P} \leq \phi_2 \leq d_1 d_3$$
$$\frac{d_2 d_3}{P} \leq \phi_3 \leq d_2 d_3$$

Amount of accesses $= \phi_1 + \phi_2 + \phi_3$

- Estimate the solution and prove optimality using all Karush–Kuhn–Tucker conditions are satisfied



1

$\phi_1 = d_1 d_2$
$\phi_2 = \frac{d_1 d_3}{P}$
$\phi_3 = \frac{d_2 d_3}{P}$

$\frac{d_3}{d_2}$

$\phi_1 = \phi_2 = (\frac{d_1^2 d_2 d_3}{P})^{1/2}$
$\phi_3 = \frac{d_2 d_3}{P}$

$\frac{d_2 d_3}{d_1^2}$

$\phi_1 = \phi_2 = \phi_3 = (\frac{d_1 d_2 d_3}{P})^{2/3}$

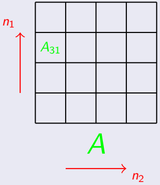
P

- Communication lower bound $= \phi_1 + \phi_2 + \phi_3 -$ data owned by the processor $= \phi_1 + \phi_2 + \phi_3 - \frac{d_1 d_2 + d_2 d_3 + d_1 d_3}{P}$

Design of Communication Optimal Algorithms

Data Distribution (P is organized into a $p_1 \times p_2 \times p_3$ grid)

- p_1, p_2, and p_3 evenly distribute n_1, n_2, and n_3
- Each processor has $\frac{1}{P}$th amount of input and output variables
- $A_{31} = A(2\frac{n_1}{p_1} + 1 : 3\frac{n_1}{p_1}, 1 : \frac{n_2}{p_2})$ is evenly distributed among $(3, 1, *)$ processors
- $B_{12} = B(1 : \frac{n_2}{p_2}, \frac{n_3}{p_3} + 1 : 2\frac{n_3}{p_3})$ is evenly distributed among $(*, 1, 2)$ processors



Algorithm 2 $C = AB$ Matrix Multiplication Algorithm

```
$(p'_1, p'_2, p'_3)$ is my processor id
//All-gather input matrices $A$ and $B$
$A_{p'_1 p'_2}$ = All-Gather($A$, $(p'_1, p'_2, *)$)
$B_{p'_2 p'_3}$ = All-Gather($B$, $(*, p'_2, p'_3)$)
$T$ = Local-Matrix-Multiplication($A_{p'_1 p'_2}$, $B_{p'_2 p'_3}$) // Local matrix multiplication in a temporary
Reduce-Scatter($C_{p'_1 p'_3}$, T, $(p'_1, *, p'_3)$) // Reduce-scatter the output
```

3-dimensional Multi-TTM ($\mathcal{Y} = \mathcal{X} \times_1 \mathbf{A}^{(1)^\mathsf{T}} \times_2 \mathbf{A}^{(2)^\mathsf{T}} \times_3 \mathbf{A}^{(3)^\mathsf{T}}$)

All-at-Once 3-dimensional Multi-TTM Computation

```
for $n_1' = 1{:}n_1$, for $n_2' = 1{:}n_2$, for $n_3' = 1{:}n_3$
    for $r_1' = 1{:}r_1$, for $r_2' = 1{:}r_2$, for $r_3' = 1{:}r_3$
        $\mathcal{Y}(r_1', r_2', r_3') = \mathcal{Y}(r_1', r_2', r_3')$
            $+ \left(\mathcal{X}(n_1', n_2', n_3') * \mathbf{A}^{(1)}(n_1', r_1') * \mathbf{A}^{(2)}(n_2', r_2') * \mathbf{A}^{(3)}(n_3', r_3')\right)$
```

$$\boldsymbol{\Delta} = \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{1}_3 & \mathbf{0}_3 \\ \mathbf{I}_{3\times 3} & \mathbf{0}_3 & \mathbf{1}_3 \end{bmatrix}$$

- Total number of $4-array$ operations $= n_1 r_1 n_2 r_2 n_3 r_3$
- $\boldsymbol{\Delta}$ is not full rank
    - Consider each vector $\mathbf{x}$ such that $\boldsymbol{\Delta} \cdot \mathbf{x} = \mathbf{1}$, $\mathbf{x}$ is of the form $[a\ a\ a\ 1\text{-}a\ 1\text{-}a]^\mathsf{T}$ and $0 \leq a \leq 1$
- $\phi_{\mathcal{X}}, \phi_{\mathcal{Y}}$: tensor projections & ϕ_1, ϕ_2, ϕ_3: matrix projections
- From HBL, $\phi_{\mathcal{X}}^{1-a} \phi_{\mathcal{Y}}^{1-a} \phi_1^a \phi_2^a \phi_3^a \geq$ Amount of computations

Solving Optimization Problem to Compute Lower Bounds

- Select a processor which performs $\frac{n_1 r_1 n_2 r_2 n_3 r_3}{P}$ amount of $4 - array$ operations
- After applying lower and upper bounds for each projection, we need to solve the following optimization problem

$$\textit{Minimize } \phi_{\mathcal{X}} + \phi_{\mathcal{Y}} + \phi_1 + \phi_2 + \phi_3 \quad \text{s.t.}$$

$$\phi_{\mathcal{X}}^{1-a}\phi_{\mathcal{Y}}^{1-a}\phi_1^a\phi_2^a\phi_3^a \geq \frac{n_1 r_1 n_2 r_2 n_3 r_3}{P}$$

$$\frac{n_1 n_2 n_3}{P} \leq \phi_{\mathcal{X}} \leq n_1 n_2 n_3$$

$$\frac{r_1 r_2 r_3}{P} \leq \phi_{\mathcal{Y}} \leq r_1 r_2 r_3$$

$$\frac{n_1 r_1}{P} \leq \phi_1 \leq n_1 r_1$$

$$\frac{n_2 r_2}{P} \leq \phi_2 \leq n_2 r_2$$

$$\frac{n_3 r_3}{P} \leq \phi_3 \leq n_3 r_3$$

Divide the problem into two parts

Matrix part

$$\textit{Minimize } \phi_1 + \phi_2 + \phi_3 \quad \text{s.t.}$$

$$\phi_1\phi_2\phi_3 \geq \frac{n_1r_1n_2r_2n_3r_3}{P}$$

$$\frac{n_1r_1}{P} \leq \phi_1 \leq n_1r_1$$

$$\frac{n_2r_2}{P} \leq \phi_2 \leq n_2r_2$$

$$\frac{n_3r_3}{P} \leq \phi_3 \leq n_3r_3$$

Tensor part

$$\textit{Minimize } \phi_{\mathcal{X}} + \phi_{\mathcal{Y}} \quad \text{s.t.}$$

$$\phi_{\mathcal{X}}\phi_{\mathcal{Y}} \geq \frac{n_1r_1n_2r_2n_3r_3}{P}$$

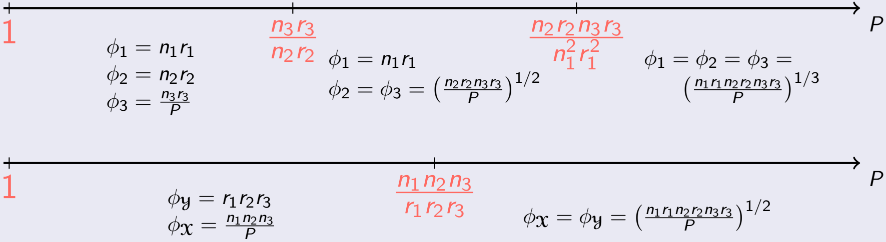
$$\frac{n_1n_2n_3}{P} \leq \phi_{\mathcal{X}} \leq n_1n_2n_3$$

$$\frac{r_1r_2r_3}{P} \leq \phi_{\mathcal{Y}} \leq r_1r_2r_3$$

Amount of Accesses and Lower bounds

- We assume $n_1 r_1 \leq n_2 r_2 \leq n_3 r_3$ and $r_1 r_2 r_3 \leq n_1 n_2 n_3$
- Estimate solutions for both parts using Lagrange multipliers (optimality can be proven using KKT conditions)

Amount of accesses $= \phi_x + \phi_y + \phi_1 + \phi_2 + \phi_3$



$\phi_1 = n_1 r_1$
$\phi_2 = n_2 r_2$
$\phi_3 = \frac{n_3 r_3}{P}$

$\phi_1 = n_1 r_1$
$\phi_2 = \phi_3 = \left(\frac{n_2 r_2 n_3 r_3}{P}\right)^{1/2}$

$\phi_1 = \phi_2 = \phi_3 = \left(\frac{n_1 r_1 n_2 r_2 n_3 r_3}{P}\right)^{1/3}$

$\phi_y = r_1 r_2 r_3$
$\phi_x = \frac{n_1 n_2 n_3}{P}$

$\phi_x = \phi_y = \left(\frac{n_1 r_1 n_2 r_2 n_3 r_3}{P}\right)^{1/2}$

Communication lower bound $= \phi_x + \phi_y + \phi_1 + \phi_2 + \phi_3 - \frac{n_1 n_2 n_3 + r_1 r_2 r_3 + n_1 r_1 + n_2 r_2 + n_3 r_3}{P}$