Module 4:

(1)

RFS - Random finite sets

In real life tracking - objects disappear and appear

Our interest: present object's states.

anything outside our FOV, we don't care much

RFS: way to model $\{x_1, x_2, \dots x_{nk}\}$

where order of objects isn't preserved.

→ consider total set of (x) not individual elements

→ easy to add/remove elements

→ 1-1 relation b/w phys. reality & set.

Prediction - update sequence is same.

~~But for~~

Pros: unified framework for all aspects

- births/deaths, models (motion, measurements) etc

- extended obje. tracking
- powerful derivations
- Diff. metrics (RMSE can't be used for MOT)
- converges to Bayes optimal soln (theory)

RFS - What is it? Integrals? Distributions?

MOT algorithms etc.

RFS: Random var. with outcomes :→ sets with finite no. of unique elements ②

No. & elements are random.
(same for measurements)

RFS elements ⇒ $D = R^{nx}$

Euclidean space of size nx.

$\boxed{X \in F(D)}$

Examples $Z_k \in R^{n_z}$

X_k. $X = \phi$ $Z = \phi$.

$X = \{x^1\}$ $Z = \{z^1\}$

$X = \{x^1, x^2\}$ $Z = \{z^1, z^2\}$ $\boxed{z_1 \neq z_2}$

$x^1 \neq x^2$

Sets are equal if they contain same elements
invariant to order
never repeated elements

$a \cup b \triangleq \{x : x \in a \text{ or } x \in b\}$

$a \cap b \triangleq \{x ; x \in a \text{ and } x \in b\}$

Disjoint sets $a \cap b = \phi$

cardinality of set $a = |a|$ = no. of unique elements for a finite set

Multiobject PDF: (3)

M.O. PDF of RFS. to describe its distribution

$p_x(\{x^1, \dots, x^n\})$ = non-negative fn that integrats to one.

: capture cardinality & distribution over elements of set (given cardinality).

Must be indifferent to order of elements

$$p_x(\{x^1, x^2\}) = p_x(\{x^2, x^1\})$$

$x \sim N(0,1)$ $\quad X = \{x\}$ then (only 1 element)

$$p_x(X) = \begin{cases} N(v; 0,1) & \text{if } x = \{v\} \\ 0 & \text{if } |x| \neq 1 \end{cases}$$

multiobject PDF takes same value as $N(v; 0,1)$ if $X = \{v\}$ evaluated @ v.

If sets contain more than one element, then $p_x(X) = 0$

eg: 1

$p_x(\{1, -2\}) = 0$ $\quad$ [$\because$ 2 elements]

$p_x(\{-0.6\}) = N(-0.6; 0, 1) = 0.3$

eg 2:

$x^1 \sim \text{unif}(0,1)$ $\quad x^2 \sim \text{unif}(1,2)$

independent & $x = \{x^1, x^2\}$

$$p_x(X) = \begin{cases} p_1(v^1) p_2(v^2) + p_2(v^1) p_1(v^2) & \text{if } X = \{v_1, v_2\} \\ 0 & \text{if } x \neq 2. \end{cases}$$

where $p_1(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ $\quad p_2(x) = \begin{cases} 1 & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

x contains 2 elements

then $p_X(X) = \{ {}^nC_n$ all combinations of probability

$p_X(\{1.5, 0.5\}) = p_1(1.5)\, p_2(0.5) + p_1(0.5)\, p_2(1.5)$
$= (0)(0) + (1)(1) = 1$

$D = R$ (real numbers) scalars

for real valued Random variables

$$\Pr[x \in (v, v+\Delta v)] = \int_{v}^{v+\Delta v} p_X(s)\,ds \approx p_X(v)\,\Delta v \text{ if } (\Delta v = \text{small})$$

Density

if $\Delta v_1, \Delta v_2, \ldots, \Delta v_n$ = small

$p_X(\{v_1, v^2 \ldots v^n\}) \times \Delta v^1 \times \Delta v^2 \ldots \times \Delta v^n$
= (approx) probability that x contains one element in each of disjoint intervals
$(v_1, v_1 + \Delta v^1) \ldots (v_n, v_n + \Delta v^n)$

eg: 2 $\quad v^1 = 1.5,\ v^2 = 0.5,\ \Delta v^1 = \Delta v^2 = 0.2$

$p_X(\{v_1, v_2\}) \times \Delta v^1 \Delta v^2 = (1)(0.2)(0.2) = (0.2)^2$

Proof

$v_1 = 0.5,\ v_2 = 1.5$

$\therefore \Pr[x^1 \in (0.5, 0.7),\ x^2 \in (1.5, 1.7)]$
$= \Pr[x^1 \in (0.5, 0.7)]\, \Pr[x^2 \in (1.5, 1.7)]$
$= (1)(0.2)(1)(0.2) = (0.2)^2$

Multi-object PDFs vs. Ordered densities

$x = \{x^1, x^2 \dots x^n\}$ and $X = \Pi(x^1, x^2, x^3 \dots x^n)$

$\downarrow$ RFS

then $p_X([x^1, x^2 \dots x^n]) = \frac{1}{n!}(p_X(\{x_1, x_2 \dots x_n\}))$

as we can order $x^1 \dots x^n$ in $n!$ ways.

$n=2$, $p_X(\{x^1, x^n\}) = p_X([x^1, x^2]) + p_X([x^2, x^1])$

$= 2 p_X([x^1, x^2])$

$$p_X(X) = \begin{cases} \frac{1}{2}[p_1(v_1) p_2(v_2)] + \frac{1}{2}[p_1(v_2) p_2(v_1)] & X = \Pi(x^1, x^2) \\ 0 & \text{if } |X| \neq 2 \end{cases}$$

0.5 probability that $(x_1, x_2) = (v_2, v_1)$

0.5 probability $(x_1, x_2) = (v_1, v_2)$

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(4) Convolution formula:

Total RFS = union of individual RFS

eg: measurements = (clutter mea) $\cup$ (obj meas)

Given independent distribution of components. We need to find distribution of union

eg: fair coin twice, X = total no. of heads.

x_1 = first flip (no. of heads)

x_2 = second flip (no. of heads)

$\boxed{X = x_1 + x_2}$ $\quad P_X[x_i = j] = \frac{1}{2}$ for $i = 1, 2$ (time instance)

$j = 0, 1$

success / failure

$Pr[X=0] = Pr[X_1=0]\, Pr[X_2=0] = (0.5)^2 = 0.25$

$Pr[X=2] = Pr[X_1=1]\, Pr[X_2=1] = (0.5)^2 = 0.25$

$Pr[X=1] = Pr[X_1=1]\, Pr[X_2=0] + Pr[X_1=0]\, Pr[X_2=1] = (0.5)^2 + (0.5)^2 = (0.5)$

Rolling a dice

X_1 = no. of dots in 1st roll
X_2 = " in 2nd roll

$\boxed{X = X_1 + X_2}$

$$Pr[X=4] = P_{X_1}(3)P_{X_2}(1) + P_{X_1}(2)P_{X_2}(2) + P_{X_1}(1)P_{X_2}(3)$$
$$= \left(\frac{1}{36}\right)(3) = \frac{3}{36}$$

X_1, X_2 are independent, integer valued, random var.

if $\boxed{X = X_1 + X_2}$ $\boxed{Pr[X=V] = \sum_{S=-\infty}^{\infty} P_{X_1}(S)\, P_{X_2}(V-S)}$

Discrete $\boxed{Pr[X=V] = P_{X_1} * P_{X_2}(V)}$

Convolution $P_{X_1} \times P_{X_2}$ evaluated @ V.

Concept for Random variables can be extended to RFS (sets cannot be added, just $\cup$ & $\cap$)

$\boxed{X = X_1 \cup X_2}$ X_1, X_2 = ind. RFS

$$P_X(\{1,3\}) = P_{X_1}(\phi)\, P_{X_2}(\{1,3\}) + P_{X_1}(\{1,3\})\, P_{X_2}(\phi)$$

$\boxed{X_1 = X_2 = \{1.3\}}$?? ignored becauses in practical cases, 2 objects can't have same state

(Proof beyond scope)

also $Pr[X_1 = X_2 = \{\tilde{x}\}, \tilde{x} \in (1.2, 1.4)] = 0$

Set containing 2 element.

$X = X^1 \cup X^2$

one set is empty

$p_X(\{1.3, 2.7\}) = p_{X_1}(\{\phi\}) \, p_{X_2}(\{1.3, 2.7\}) +$

$p_{X_1}(\{1.3, 2.7\}) \, p_{X_2}(\phi) + p_{X_1}(\{1.3\}) \, p_{X_2}(\{2.7\})$

$+ p_{X_1}(\{2.7\}) \, p_{X_2}(\{1.3\})$

Convolution formula for union of 2 RFs (independent)

$X = X_1 \cup X_2$ $\quad p_X(X) = \sum_{X' \subseteq X} p_{X_1}(X') \, p_{X_2}(X \setminus X')$

for elements not in X_1

possible

for all elements in X

union of 'n' RFS

$\boxed{\sum_{X^1 \uplus X^2 \cdots \uplus X^n = X}}$ Denotes summation over all mutually disjoint (& possibly emty) sets) $X^1, X^2 \ldots X^n$ whose union is X

Disjoint sets $\boxed{X^1 \cap X^2 = \phi}$

eg. of summation. $\sum_{X_1 \uplus X_2 = \{1\}} f(X^1, X^2) = (f(\{1\}, \phi) + f(\phi, \{1\}))$

$\sum_{X^1 \uplus X^2 \uplus X^3 = \{4\}} f(X^1, X^2, X^3)$

$= f(\phi, \phi, \{4\}) + f(\phi, \{4\}, \phi) + f(\{4\}, \phi, \phi\})$

$\sum_{X^1 \uplus X^2 = \{3, 5\}} = 4$ cases.

$$\sum_{x^1 \uplus x^2 = x} f(x^1, x^2) = \sum_{x^1 \subseteq x} f(x^1, x \setminus x^1)$$ every term assigns elements.

$x = x^1 \cup x^2 \cup \dots x^n$

multi-object PDF $\quad p_x(x) = \sum_{x^1 \uplus x^2 \dots} \prod_{i=1}^{n} p_{x^i}(x^i)$

③ Set integrals

All pdf must $\int$ to 1

$$f : F(D) \to R.$$

$$\int f(x) \cdot \delta x = \sum_{i=0}^{\infty} \frac{1}{i!} \int f(\{x^1 \dots x^i\}) dx^1 . dx^2 . dx^i$$

$$= f(\phi) + \sum_{i=1}^{\infty} \frac{1}{i!} \int$$

for $p_x(x) = \begin{cases} N(x; 0, 1) & \text{if } x = \{x\} \\ 0 & \text{if } |x| \neq 1 \end{cases}$

Gaussian pdf

$$\int p_x(x) \cdot \delta x = \int p_x(\{x^1\}) . dx^1 = \int N(x; 0, 1) . dx^1 = 1$$

accounts for all possible combination (as elements are unordered)

if $i!$ not included, we'll integrate over same set $i!$ times

eg:- for $i=2$ case

$$\int_{v_1 > v_2} p_x(\{v^1, v^2\}) dv^1 dv^2 = \frac{1}{2} \int p_x(\{v^1, v^2\}) dv^1 dv^2$$

half area

descending order

Similarly for scalars of n-order, a descending order would give set integral = 1

But for vectors not trivial; hence better to use $\boxed{\frac{1}{i!}}$

Expected value

$f : F(D) \rightarrow R$

$$\underbrace{E[f(X)]}_{\text{Cannot be computed}} = \int f(X) \cdot P_X(X) \delta X = \sum_{i=0} \frac{1}{i!} \int f(\{x^1, x^2, \ldots\}) P_X(\{x^1, x^2 \ldots\}) dx^1 \ldots dx^i$$

Why? we cannot average (add sets)

eg: $\{0.3, 0.7\}$ & $\{1\} + \{2, 0\}$ is not defined

Cardinality distributions

RFS

$X \sim P_X(\cdot)$

$P_X(n) = Pr[|X| = n]$

set integral

Let kronecker delta fn be $\delta_i^0 = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$

$$Pr[|X| = n] = E[\delta_{n-|X|}]$$
$$= 0 \times Pr[\delta_{n-|X|} = 0] + 1 \times Pr[\delta_{n-|X|} = 1]$$
$$= \sum_{i=0}^{\infty} \frac{1}{i!} \int \delta_{n-i} P_X(\{x^1, x^2 \ldots x^i\}) dx^1 \ldots dx^i$$
$$= \frac{1}{n!} \int P_X(\{x^1, x^2, \ldots x^n\}) dx^1 \ldots dx^n$$

eg: for this case

$$Pr[|X| = n] = \begin{cases} \int N(x; 0, 1) & \text{if } |X| = 1 \\ 0 & \text{otherwise} \end{cases}$$

⑥ Belief mass function & prob. generating fns.
(bmf) (p.g.fl.s)

alternative description of RFS distribution
- helpful in derivation of expression, models

eg. FFT : convolution ⇔ pgfl.s ⇒ RFS

transform to pg.fl domain, do easy operations, and then inverse transform

mathemati rigorous, transparent,

X

diff. to understand @ start
less intuitive

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⑦ PPP ↔ one RFS distribution.

MO-pdf of PPP (X) is

$$P_X(X) = \exp\left(-\int \lambda(x')\,dx'\right) \prod_{x \in X} \lambda(x)$$

normalisation factor

all elements in set X

intensity function

$\bar{\lambda}$ = poisson rate = $\int \lambda(x)\,dx$.

$$P_X(\{x_1 \dots x_n\}) = \exp(-\bar{\lambda}) \prod_{i=1}^{n} \lambda(x_i)$$

MO-case

for SO-case, similar but $\frac{1}{n!}$ for normalising ordered duplicates

PPP model:
- clutter detections $D = R^{n_z}$
- appearing obj. in prediction step $D = R^{n_x}$
- med. from ext. objects $D = R^{n_z}$

Cardinality pmf of PPP

$$\Pr[|X| = n] = \frac{1}{n!}\int P_X\{x^1 \ldots x^n\} \cdot dx$$

$$= \frac{1}{n!}\int \exp(-\bar{\lambda}) \prod_{i=1}^{n} \lambda(x_i) \cdot dx$$

$$= \frac{\exp(-\bar{\lambda})}{n!}\int \prod_{i=1}^{n} \lambda(x_i)\, dx = \frac{\exp(-\bar{\lambda})}{n!} \prod_{i=1}^{n} \int \lambda(x_i)\, dx_i$$

$$\boxed{= \frac{\exp(-\bar{\lambda})}{n!} (\bar{\lambda}^n)} \Rightarrow P_o(n; \bar{\lambda})$$

$\downarrow$ mean $\bar{\lambda}$

Cardinality is Poisson distributed

Generating samples

sampling a PPP

1) Initialise $X = \phi$
2) Generate $n \sim P_o(\bar{\lambda})$
3. for $i = 1$ to n do
   Generate $x_i = \lambda(\cdot) / \bar{\lambda}$
   Set $X = X \cup \{x_i\}$
end for.

$$\lambda(x) = 4N\left(x; \begin{bmatrix} 3 \\ 3 \end{bmatrix}, I\right) + N\left(x; \begin{bmatrix} -3 \\ -3 \end{bmatrix}; I\right)$$

Weighted Gaussian sum

> 4 times more probable in region of $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$;

RFS ; PPP used interchangably

(8) Bernoulli process (RFS)
→ Multi-Bernoulli RFS

Bernoulli RFS (X) has MO-pdf

$$P_X(X) = \begin{cases} 1-r & \text{if } \cancel{X \neq \phi}\ X = \phi \\ r\,p_x(x) & \text{if } X = \{x\} \\ 0 & \text{if } |X| > 1 \end{cases}$$

where $0 \leq r \leq 1$ & $p_x(x) = $ pdf

$$\Pr[|X| = n] = \begin{cases} 1-r & \text{if } n = 0 \\ r & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

model
→ measurements from single obj $D = \mathbb{R}^{n_z}$
→ potential object $D = \mathbb{R}^{n_x}$
(both may / may not be present & uncertainty in location of object)

Sampling Bernoulli RFS

1. Initialise $X = \phi$
2. if rand $< r$ then
   $x = p_x(\cdot)$
   $X = \{x\}$ $\quad p_x = N(x; 0, I)$.
   endif.

if rand $> r$, $X = \phi$ (empty set).

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(9) Multi Bernoulli RFSs

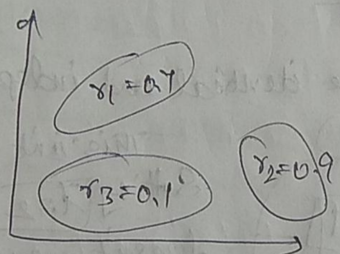
$X_1, X_2 \ldots X_N$ = independent Ber, RFSs with MO-pdfs $P_{X_1}(X_1)$, $P_{X_2}(X_2)$, ...

~~$P_X(X)$~~ and $X = \bigcup_{i=1}^{N} X^i$ multi-bernoulli (MB-RFS) with mo-pdf $\underbrace{P_X(X) = \sum_{\uplus_{i=1}^{N} X_i = X} \prod_{j=1}^{N} P_{X_j}(X_j)}_{\text{sum of all disjoint sets forming set } \{X\}}$ → using convolution formula.

MB RFSs model multiple potential ~~RFSs~~ objects
- according to posterior, $D = \mathbb{R}^{n_x}$
- appearing objects.

$P_{X_i}(X_i) =$ parametrised by r_i and $p_i(\cdot)$

eg:



(independent ~~funct~~ distributions)

N=3

$r_1 = 0.7$, $r_2 = 0.9$, $r_3 = 0.1$

$P_1(x)$, $P_2(x)$, $P_3(x)$ = gaussian

MB - RFS (X) represents that there are 3 potential objects

<u>Sampling MB-RFS</u>

1) Initialise $X = \phi$
2. for i = 1 to N do
   if rand < r_i then
   $x_i \sim p_i(\cdot)$ → spatial pdf.
   $X = X \cup \{x_i\}$
   end if
- endfor.

<u>MB vs PPP</u> MB ≈ PPP?

A bernoulli RFS with $r < 0.1$ ⟺ PPP

A MB RFS with $r_1, r_2 \ldots r_n < 0.1$ is approx PPP

Any PPP can be approx by MB but N is high

PPP ⇒ computationally efficient

Why?? if X is PPP, both mean and variance of $|X|$ is $\bar{\lambda}$

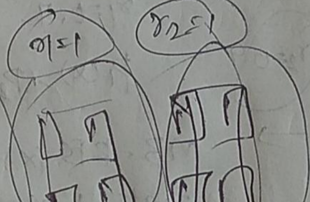
Cardinality is poisson distributed

if we know ~~A~~ there are 10 objects precisely, and we use PPP, we're forced to set mean, variance to 10, inspite of us knowing there are 10 objects

→ In such cases - ~~10 objects~~ $\boxed{MB}$

elements of MB-RFS can be identical and independent

$\boxed{r_1 = 1, r_2 = 1}$



This isn't trivial in PPP

Setting $\boxed{\lambda = 2}$ doesn't ensure 2 objects in PPP.

It can't differentiate 2 cars in same ~~position and 2 cars in 2 diff~~

~~Poisson~~ PPP - compact, efficient, can represent imprecise info

MB-PP - Precise info

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(10) Bayesian filtering recursions & models

Point objects.

Objective: $p(X_k | Z_{1:k}) \longrightarrow$ Posterior.

X_k ↓ ~~state~~ (dist. over set of objects)

$Z_{1:k}$ ↓ measurements

(no & state of objects), uncertainty in

Concept is same

Prediction $P(x_k|z_{k-1}) = \overbrace{\int P(x_k|x_{k-1}) P(x_{k-1}|z_{k-1}) \, dx_{k-1}}^{\text{chapman kolmogrov}}$

update Bayes rule.

$$P(x_k|z_k) = \frac{P(z_k|x_k) P(x_k|z_{k-1})}{\int \text{normalisation.}}$$

motion models need to capture prob of new objects (death_birth)

& how existing stats evolve

measurement

(11) Measurement models - object detections

n = time variant - RFS of estimates.

meas - objects as RFS now $\boxed{Z_k = O_k \cup C_k}$

~~$k=2$~~

objective $\boxed{g_k(O_k|x_k) = P(O_k|x_k)}$

$\boxed{\text{MO - pdf}}$ object detection

Assumptions. $\boxed{P^D(x)}$ $g_k(o|x)$. each obj measurement is independent of all other obj & meas (incl. clutters)

Point - object assumption

Cases: no state $x_k = \phi$ Consider ~~case~~ where there is atmost 1 object

$$g(O|\phi) = \begin{cases} 1 & \text{if } O = \phi \\ 0 & \text{otherwise.} \end{cases}$$

$O_k|x_k$ is are Bern. RFS with $\boxed{r=0}$

Case 2: $x = \{x\}$

$$g(o_k|\{x\}) = \begin{cases} 1 - P_D(x) & \text{if } o = \phi \\ P_D(x) g(o|x) & o = \{o\} \\ 0 & \text{if } |o| \geq 2 \end{cases}$$

Ber. RFS with $\boxed{r = P_D}$

and pdf = $\boxed{g(o|x)}$

Multiple objects

Set of obj. measurement from 1 object is Ber. RFS

As objects are indep. set of object measurements from multiple objects is multi Bern RFS.

$\boxed{x_k = \{x_k^1 \ldots x_k^{n_k}\}}$

set of object states

$O_k(x_k^i)$ = RFS repres obj. meas from x_k^i

$$\boxed{O_k = O_k(x_k^1) \cup O_k(x_k^2) \ldots O_k(x_k^{n_k})}$$

We assume same model $g_k(\cdot | \{x_k^i\})$ for all objects. can be different too no problem in changing it.

$\boxed{O^i = O_k(x_k^i)}$

$\boxed{O_k = O^1 \cup O^2 \ldots O^{n_k}}$

Convolution formula

$$g_k(O_k|\{x^1 \ldots x^{n_k}\}) = \sum_{O^1 \uplus O^2 \uplus O^{n_k}} \prod_{i=1}^{n_k} g_k(O^i|\{x^i\})$$

$O_k | X_k$ = multi Bern. RFS

Σ = combine all combinations of disjoint sets $O^1, \dots, O^{n_k}$

Example: $X_k = \{x^1, x^2\}$ (~~single~~ object measurement model)

$P^D = 0.85$ $\quad g_k(o|x) = N(o; x; 0.3I)$

2 objects.

$$g_k(O_k|X_k) = \sum_{O^1 \uplus O^2 = O_k} g_k(O^1|\{x^1\})\, g_k(O^2|\{x^2\})$$

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(12) Measurement model - complete model

Given X_k, $\boxed{Z_k = O_k \cup C_k}$, O_k - C_k are inde

$$\boxed{p(Z_k|X_k) = \sum_{C \uplus O = Z_k} P_{C_k}(C)\, g_k(O|X_k)}$$

association hypo

clutter distribution.

all combinations in Z_k, look @ Z_k and say these are object meas, these are clutter etc.

(clutter is Poisson RFS)

$$P_{C_k}(C) = \exp\Big(-\underbrace{\int \lambda_c(c')\, dc'}\Big) \prod_{c \in C} \lambda_c(c)$$

Poission RFS distribution

set C

$\boxed{\lambda_c = \text{poisson rate}}$

intensity fn

$Z_k|X_k$ = ~~union of~~ Poisson clutter C_k $\uplus$ multi Berno object $O_k|X_k$

$\Downarrow$

Poisson multi Bern RFS

~~Poisson~~ convolution formula 2 times

→ in $c \uplus o$ union

→ in $g(\cdot|x_k)$

can Simplify formula to be used just once.

$$P(z_k|x_k) = \sum_{c \uplus o^1 \uplus o^2 \ldots = z_k} P_{ck}(c) \prod_{i=1}^{n_k} g_k(o^i|\{x_k^i\})$$

$$\boxed{P_{ck}(c) = \exp(-\bar{\lambda}_c) \prod_{c \in C} \lambda_c(c)}$$

$$g_k(o|\{x\}) = \begin{cases} P^D(x) g_k(o|x) & \text{if } o = \{o\} \\ 1 - P^D(x) & \text{if } o = \phi \\ 0 & \text{if } |o| > 1 \end{cases}$$

can contain impossible combinations

eg: (10 measurements to 1 object)

if $g_k(o|\{x\})$ is adjusted, then its can be used for extended obj. tracking

Summing over $c \uplus o^1 \uplus o^2 \ldots = z_k$ or θ_k is analgous to summing over $\theta_k = [\theta_k^1, \theta_k^2, \ldots, \theta_k^{n_k}]$

$$o^i = \begin{cases} \phi & \text{if } \theta_k^i = 0 \\ \{z_k^{\theta_k^i}\} & \text{if } \theta_k^i > 0 \end{cases}$$

$$\boxed{c = z_k \setminus \bigcup_{i=1}^{n_k} o^i}$$

$X_k = \{x^1\}$; $\quad Z_k = \{z^1\}$ (one object, 1 measurement)

$$p(Z_k|X_k) = \sum_{C \uplus O^1 = Z_k} P_{CK}(C)\, g_k(O^1|\{x^1\})$$

$$\boxed{= P_{CK}(\{z^1\})\, g_k(\phi|\{x^1\}) + P_{CK}(\phi)\, g_k(\{z^1\}|\{x^1\})}$$

$$\boxed{= \exp(-\bar{\lambda}_c)\lambda_c(z^1)(1-P^D(x^1)) + \exp(-\bar{\lambda}_c)\, P^D(x^1)\, g_k(z^1|x^1)}$$

$$\cancel{= \exp(-\bar{\lambda}_c)}$$

Expressed in θ_k convention

EQUIVALENT

$\theta_k = [\theta_k^1]$

$$p(Z_k|X_k) = \sum_{\theta_k^1 = 0}^{1} \exp(-\bar{\lambda}_c)\lambda_c(z^1) \prod_{i:\theta_k^i = 0} (1-P^D(x^i)) \prod_{i:\theta_k^i > 0} \frac{P^D(x^i)\, g_k(z^{\theta_k}|x)}{\lambda_c(z^{\theta_k})}$$

General measurement model

(multiple obj. meas)

$$p(Z_k|X_k) = \sum_{\theta_k} \exp(-\bar{\lambda}_c) \prod_{j=1}^{m_k} \lambda_c(z_k^j) \prod_{i:\theta^i = 0} (1-P^D(x_k^i)) \prod_{i:\theta^i > 0} \frac{P^D(x^i)\, g_k(z|x)}{\lambda_c(z_k^{\theta_k^i})}$$

multi-object pdf vs. matrix version

$$\boxed{p(Z_k|X_k) = m!\, p(Z_k|X_k)}$$

Convolution formula simplifies process

earlier we used θ, m to condition Z and used law of total prob etc.

(13) Motion model surviving objects — stochastic distribution

obj. appear - disappear with time

Given, X_{k-1} $X_k = S_k \cup B_k$
(survive) (birth)

if object disappear w.r.t. time no. of elements in X_k reduces

$\boxed{\pi_k(S_k \mid X_{k-1})}$

($S_k \leftrightarrow O_k$ | $B_k \leftrightarrow C_k$)

motion - measurement model analogy.

single object motion model
→ survival prob $P^S(x)$
→ if it survives, it moves acc. to $\pi_k(s \mid x)$

each object moves independent of other

set of surviving objects when we have atmost 1 object

case 1: $X_{k-1} = \phi$ (no object.

$$\pi_k(S \mid \phi) = \begin{cases} 1 & \text{if } S = \phi \\ 0 & \text{otherwise} \end{cases}$$ → an object can survive if it exists

$S_k \mid X_{k-1}$ is Bern. RFS with $r = 0$.

case 2: $X_{k-1} = \{x\}$

$$\pi_k(S \mid \{x\}) = \begin{cases} 1 - P^S(x) & \text{if } S = \phi \\ P^S(x)\, \pi_k(s \mid x) & \text{if } S = \{s\} \\ 0 & \text{if } |S| > 1. \end{cases}$$

$S_k \mid X_{k-1}$ = Bern. RFS with $r = P^S(x)$
pdf $= \pi(\cdot \mid x)$

set of surviving objects from mult. obj = mult. Berno. RFS
RFS repr.

$S_k(X_{k-1}^i)$ = set of surviving objects from X_{k-1}^i

$$\boxed{S_k = S_k(X_{k-1}^1) \cup S_k(X_{k-1}^2) \dots S_k(X_{k-1}^{n_{k-1}})}$$

$$\boxed{S^i = S_k(X_{k-1}^i)}$$

$\Downarrow$

$$\boxed{S_k = S^1 \cup S^2 \cup \dots S^{n_{k-1}}}$$

$$\Pi_k(S_k \mid \{x^1, x^2 \dots x^{n_{k-1}}\}) = \sum_{S^1 \uplus S^2 \dots = S_k} \prod_{i=1}^{n_{k-1}} \Pi_k(S^i \mid \{x^i\})$$

multi-bernoulli RFS

$X = \{x_1, x_2\}$ $p_S = 0.7$ $\Pi_k(S \mid x) = N(S; x, 0.3I)$.

$$x_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad x_2 = \begin{bmatrix} .2 \\ -1 \end{bmatrix}$$

$$\Pi_k(S_k \mid \{x_1, x_2\}) = \sum_{S^1 \uplus S_2 = S_k} \Pi_k(S_k \mid \{x_1\}) \, \Pi_k(S_k \mid \{x_2\})$$

---

(4) Complete motion model.

$X_k = S_k \cup B_k$

Convolution formula

$$f(X_k \mid X_{k-1}) = \sum_{b \uplus S = X_k} P_{B_k}(b) \, \Pi_k(S \mid X_{k-1})$$

association hypothesis

Poisson PP

Some of state vector X_k have appeared now (B_k)

$$P_{b_K}(b) = \exp\left(-\underbrace{\int \lambda_b(b') \cdot db'}_{\text{Poisson rate}}\right) \prod_{b \in b} \lambda_b(b)$$

intensity fn

$X_K | X_{K-1}$ = Poisson MB. RFS

$$X_K = b_K \cup S_K(x^1_{K-1}) \cup \ldots S_K(x^{n_{K-1}}_{K-1})$$

$$\pi_K(X_K | \{x^1_{K-1}, x^2_{K-1} \ldots\}) = \sum_{b \uplus s^1 \uplus s^2 \ldots = X_K} P_{b_K}(b) \prod_{i=1}^{n_{K-1}} \pi_K(s^i | \{x^i_{K-1}\})$$

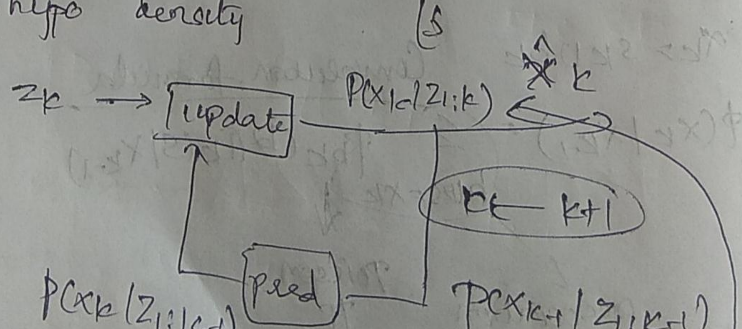
$$P_{b_K}(b) = \exp(-\bar{\lambda}_b) \prod_{b \in b} \lambda_b(b)$$

$$\pi_K(s | \{x\}) = \begin{cases} P^S(x)\,\pi(s|x) & \text{if } s = \{s\} \\ 1 - P^S(x) & \text{if } s = \phi \\ 0 & \text{if } |s| > 1 \end{cases}$$

---

(15) PHD filtering - introduction ← Simple efficient.

prob. hypo density



assumed density filtering

$$P(x_K | z_K) = P(x_{K-1} | z_{K-1})$$

must be same type.

$p(x_k|Z_k)$ $p(x_{k+1}|Z_{k+1}) =$ Poisson MO pdf
not good @ capturing detailed info (eg:- cardinality)

Idea: how to approximate $p(x_k|Z_{k-1})$
To make $p(x_k|Z_k)$ as Poisson RFS

RFS 'X $\sim p(\cdot)$ as a poisson RFS. $\lambda(x) = D(x)$ (we set poisson intensity)

$D(x) =$ prob. hypo. density (PHD) of $X \sim p(X)$

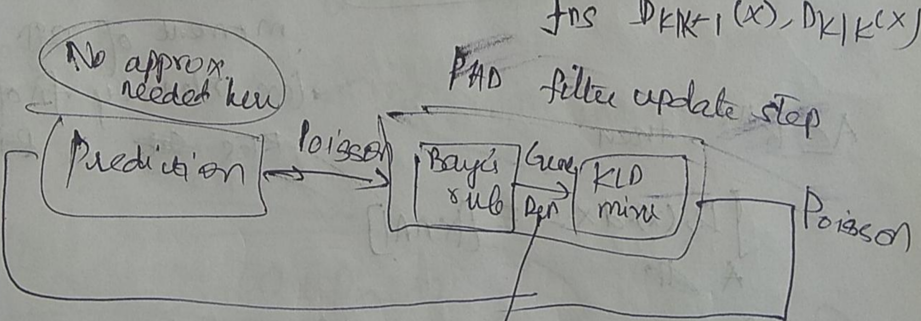
$\boxed{D(x) = fn(x)}$ any RFS

optimal in Kullback - Leibler divergence

Recursive compute $D_{k|k-1}(x)$ PHD of Predicted
$D_{k|k}(x)$ [PHD of updated]

approximate

$p(x_k|Z_{k-1})$, $p(x_k|Z_k)$ as Poisson MO-pdf with int. fns $D_{k|k-1}(x)$, $D_{k|k}(x)$



PHD are fns of single object state

approximation

PHDs parametrise MO-pdf

$p(X_k|Z_k) = \exp\left(\int - D_{k|k}(x')\right) \prod_{x \in X} D_{k|k}(x)$

we approximate $p(X_k|Z_k)$ as PPD with $D_{k|k}$ as intensity fn

elements in ~~posterior~~ PPP are independent and identically distributed (given cardinality) often rough approx of posterior

(1b) PHD and its properties

How to approximate Bern. or MBern. RFS as PPP?

$$\boxed{\begin{array}{c}\text{PHD of vector}(x)\\ \text{RFS } x\end{array}} \; D_X(x) = \int P_X(X) \sum_{x' \in X} \delta(x - x') \, \delta X$$

all elements of set.

Dirac delta fn.

= Expected value of summation

$$\boxed{D_X(x) = \int P_X(\{x\} \cup X) \, \delta X}$$ = First order stat. moment of RFS (or intensity fn of RFS ~~set~~ $\hat{x}$ ~~RFS~~

$A \subseteq \mathbb{R}^{n_x}$, then

$$\int_{A} D_X(x) \cdot dx = E[|X \cap A|]$$

$D_X(x) dx$ = expected no. of objects in dx

By setting A = entire space $\int D(x) \cdot dx = E[|X|]$

$$= \int_A \int P_X(X) \sum_{x' \in X} \delta(x - x') \cdot d\delta X \, dx$$

as $\int_A$ depends only on x, change order of $\int$

$$\Rightarrow \int P_X(X) \sum_{x' \in X} \underbrace{\underbrace{\int_A \delta(x-x')\,dx}_{=1 \text{ if } x' \in A}}_{|X \cap A|} \delta X$$

PHD of Bernoulli RFS

$$P_X(X) = \begin{cases} 1-r & \text{if } X = \phi \\ r\,p_X(x) & \text{if } X = \{x\} \\ 0 & \text{if } |X| \geq 2 \end{cases}$$

$\therefore$ PHD of X is

$$D_X(x) = \int P_X(\{x\} \cup X)\, \delta X$$
$$= P_X(\{x\} \cup \phi) + 0 \quad (\because P_X(X) = 0 \text{ if } |X| \geq 2)$$

$$\boxed{D_X(x) = r\,p_X(x)}$$

D_X = small if existence prob is small or $P_X(x)$ is small

Poisson PP

X = Poisson RFS with intensity $\lambda(x)$

$$\boxed{D_X(x) = \lambda(x)}$$

$$D_X(x) = D_{X_1}(x) + D_{X_2}(x)$$

union of RFS $\quad \boxed{X = X_1 \cup X_2 \ldots X_N}$

$$E[|X \cap A|] = \sum_{i=1}^{N} E[|X_i \cap A|]$$

(Same) under assumption that prob of ~~2~~ elements in 2 ~~diff~~ sets = 0

Multi-Bern RFS

$$\boxed{D_X(x) = \sum_{i=1}^{N} r_i p_i(x)}$$

(II) Prediction — PHD's are Parametrized by Gaussian mixtures

$$D_{k-1|k-1}(x) = \sum_{h=1}^{H_{k-1|k-1}} w^h_{k-1|k-1} N(x; \mu^h_{k-1|k-1}, P^h_{k-1|k-1})$$

$[D_{k|k-1}$ $H_{k|k-1}$

weights dont sum to 1

$$E[|X_k| / Z_{k-1}] = \int D_{k|k-1}(x).dx$$

$$= \underbrace{\sum_{h=1}^{H_{k|k-1}} w^h_{k|k-1}}$$

$\therefore \int \sum w N()$

$\boxed{\text{Gaussians sum to 1}}$

expected no. of objects from that density : hence dont sum to one

w is independent of x

$$\boxed{\therefore \int D_{k|k-1}(x).dx = \sum_{i=1}^{H_{k|k-1}} w_{k|k-1}}$$

PHD of predicted density may not be gaussian so approximated as GM. (apart from PPP approximations)

Obj Given $\boxed{D_{k-1|k-1}(x)}$ find $D_{k|k-1}(x)$

$$\lambda_{b,k}(x) = \sum_{h=1}^{H_b} w_{b,k}^{h} \, N(x ; \mu_{bk}^{h}, P_{bk}^{h})$$ $\boxed{GM}$

captures where we expect objects to appear

$$\pi_k(x_k|\{x_{k-1}\}) = \begin{cases} P_s N(x_k ; F x_k, Q_{k-1}) & \text{if } x = \{x\} \\ 1 - P_s & \text{if } x = \phi \end{cases}$$

assumed constant. Not actually the case

$x_k | Z_{1:k-1}$ is a PPP with PHD

$$\boxed{D_{k|k-1}(x) = D_{k|k-1}^{s}(x) + \lambda_{b,k}(x)}$$

GM with parameters

$H_{k|k-1}^{s} = H_{k-1|k-1}$

$w_{k|k-1}^{s} = P_s \, w_{k|k-1}^{h}$

μ, P = predicted using standard formula

$x_k | Z_{k-1}$ is a PPP with GM-PHD no approximation

Algorithm

1. set $H_{k|k-1} = H_k^b + H_{k-1|k-1}$
2. for $h = 1$ to H_k^b

   $w_{k|k-1}^{h} = w_{b,k}^{h}$; $\mu_{k|k-1}^{h} = \mu_{bk}^{h}$, $P_{k|k-1}^{h} = P_{bk}^{h}$

end for

3. for $h = 1$ to $H_{k-1|k-1}$ do

   set

   $w_{k|k-1}^{h+H_b} = P^{s} w_{bk}^{h}$ $\quad \boxed{\mu_{k|k-1}^{h+H_b} = F\mu^{h}}$

   $P_{k|k-1}^{h+H_b} = FPF + Q$

end for

See video for clear example

(18) PHD Filter update Part 1

not PPP after update ~~step~~

After Predicted step, we have PPP with PHD

GM $D_{k|k-1} = \sum_{h=1}^{H_{k|k-1}} w^{h}_{k|k-1} N(\mu^{h}_{k|k-1}, P^{h}_{k|k-1})$

soln:

1) find $p(x_k | z_{1:k})$ (true posterior)

2) Find GM-PHD $D_{k|k}(x)$ of $p(x_k|z_k)$ and its params $\{w^{h}_{k|k}, \mu^{h}_{k|k}, P^{h}_{k|k}\}_{h=1}^{H_{k|k}}$

3) Approximate $x_k|z_k$ as PPP with $\boxed{D_{k|k}}$ as PHD

$$\boxed{g_k(z_k|\{x_k\}) = \begin{cases} P_D N(z_k) & \text{if } z = \{z\} \\ 1-P_D & \text{if } z = \emptyset \\ 0 & \text{if } |z_k| > 1 \end{cases}}$$

Bern RFS

$\boxed{\lambda_{c,k}(z)}$ must be easy to evaluate

PPP has intensity $\boxed{\lambda_{k|k}(x) = (1-P_D) D_{k|k-1}(x)}$

MB has M_k components with parameters

$r^{i}_{k|k}$ - $P^{i}_{k|k}(x)$

$\propto (P_D\, N(z, Hx)\, D_{k,k-1})$ $(\propto D_{k|k-1} N(z, Hx))$

doesn't contain Σ over hypotheses

$$\boxed{D_{k|k}(x) = \lambda_{k|k}(x) + \sum_{i=1}^{m_k} \gamma^i_{k|k}\, p^i_{k|k}(x)}$$

$$\boxed{H_{k|k} = H_{k|k-1} \times (m_k+1)}$$

for every predicted component, we obtain (m_k+1) new components.

(it avoids explosion of hypo by not creating a global hypo thing)

---

(19) PPP exact posterior has pare with PHD

$$(1-P_D)\, D_{k|k-1}(x) = \sum_{h=1}^{H_{k|k-1}} (1-P_D)\, \underset{\substack{\downarrow \\ \text{weights} \\ \text{scalar}}}{w^h_{k|k-1}}\, N(x; \mu, P)$$

represents objects that are undetected @ time k.

store these as first $H_{k|k-1}$ components in $\boxed{D_{k|k}}$

GM-PHD update (1)

for $h=1$ to $H_{k|k-1}$ do

$w^h_{k|k} = (1-P_D)\, w^h_{k|k-1}$

$\mu_{k|k}$ = same $\mu_{k|k-1}$

$P_{k|k}$ = same $P_{k|k-1}$

m_k other components

$$\gamma^i_{k|k}\, p^i_{k|k} = \sum_{h=1}^{H_{k|k-1}} w^{w^i}_k\, N(x; \mu_k, P_k)$$

$$\propto N(z; H_k x, R_k)\, N(x; \mu_{k|k-1}, P_{k|k-1})$$

GM PHD update (2)

for $h=1$ to $H_{k|k-1}$ do

$\left.\begin{array}{l} \hat{z}^h_{k|k-1} = H_k \mu_{k|k-1} \\ S^h \\ K^h_k = \text{\_\_\_\_\_\_} \\ P_k \end{array}\right\}$ normal KF equations

end

for $i=1$ to m_K do

for $h=1$ to $H_{K|K-1}$ do

$$\mu_K^{iH_{K|K-1}+h} = \mu^h_{K|K-1} + K^h_K(z^i_K - \hat{z}^h_{K|K-1})$$

$$P_{K|K} = P^h_{K}$$

$\boxed{w_t}$ ← temp. term

end for

for $h=1$ to $H_{K|K-1}$ do

$$w^{iH_{K-1}+h} = \frac{\cancel{w}}{\lambda_c(z) + \boxed{\sum_{h'=1}}}$$

To ensure sum ~~of to~~ of one Bernoi com ≤ 1

---

(20) Mixture reduction (GM-PHD)

→ pruning, merging, capping.

Prediction $H_{K|K-1} = H_{K-1|K-1} + H^b_K$

$H_{K|K} = (m_{K+1}) \times H_{K|K-1}$

~~no~~ normalisation after pruning

1) rm $wt < \gamma$

2) merging $\dot{x}$.

3) N_{max} (Cap).

include $M^h_{K|K}$ Largest weights in set $\boxed{\hat{X}_K}$

Estimator of GM-PHD

$$\hat{n}_{K|K} = \text{round}\left(\sum_{h=1}^{H_{K|K}} w^h_{K|K}\right)$$

Input $\hat{n}$, w^h - y^h - $h = 1 \dots H$
output $\hat{X}$
[out, ind] = sort ([$w^1 \dots w^H$]), 'descend')
initialise $\hat{X} = \phi$
for $i = 1$ to $\hat{n}$ do
set $\hat{X} = \hat{X} \cup \{y^{ind[i]}\}$
end for.

Example GM-PHD simulation @ video end

As $n \uparrow$ with time, PHD becomes more uncertain as in PPP, $\boxed{y = \text{variance} = \lambda}$

As $n \uparrow$, missed false estimates $\uparrow$.

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(21) Metrics evaluate performance which algo is best?
all aspects in 1 number.

$\boxed{\hat{X}_k, X_k \text{ are sets}}$ (NO order in sets).

→ localisation error for detected objects
→ # missed objects
→ # false objects.

GOSPA = generalised optimal sub-optimal assignment metric

informal definition: GOSPA = local. error + $\frac{c}{2}$ [# missed + false]

All metrics must satisfy

1) $d(x,y) \geq 0$ non negative
2) $d(x,y) = d(y,x)$ symmetric
3) $d(x,y) = 0$ if only $y = x$
4) $d(x,y) \leq d(x,z) + d(z,y)$ Δ inequality

LP norm (in euclidean space)

GOSPA (P=1) Euclidean distance (L^2 norm)

1) Find optimal assignment b/w sets [set of pairs of indices]

→ Pairs are unassigned is $d(x, \hat{x}) > c$

→ unassigned elements are false/missed

eg:- $\boxed{c=40}$ $\gamma^* = \{(1,3)\}$

2) Assigned pairs cost $d(x, \hat{x})$

3) Unassigned elements cost ($c/2$)

$\boxed{15 + 3 \times \frac{c}{2} = 75}$

Formal definition $\alpha = 2$.

$$d_p^{(c,2)}(x, \hat{x}) = \left[\min_{\gamma \in \Gamma}\left(\sum_{(i,j) \in \gamma} d(x^i, \hat{x}^j)^p + \frac{c^p}{2}\left(|x| - |\gamma| + |\hat{x}| - |\gamma|\right)\right.\right.$$

where Γ = set of possible assignment sets.

$\alpha = 1 \Rightarrow$ OSPA $\quad p$ = generally 1 or 2

Hungarian/auction algos ✗.

---

(22) GOSPA example

$p = 1$

$$d_1^{(c,2)}(x, \hat{x}) = \min_{\gamma \in \Gamma}\left(\sum_{(i,j) \in \gamma} d(x^i, \hat{x}^j) + \frac{c}{2}\left(|x| - |\gamma| + |\hat{x}| - |\gamma|\right)\right.$$

$\boxed{\gamma^* = \{(2,1)\}}$ optimal assignment, examples shown

(23) GOSPA for RFS (we defined for just two sets)

set of objects & estimates are generally RFS s,
∴ GOSPA is to be defined b/w RFS.

for $\boxed{1 \le p < \infty}$ $\sqrt[p']{E\left[d_p^{(c,2)}(X,\hat{X})^{p'}\right]}$

where $X, \hat{X}$ are RFS, is a metric

$\boxed{p = p' = 1 \text{ or } 2 \text{ are generally used}}$

$\boxed{\text{Mean GOSPA } (p=1) = p' \quad E\left[d_1^{(c,2)}(X,\hat{X})\right]}$

RMS - GOSPA

$p = p' = 2$ $\sqrt{E\left[d_2^{(c,2)}(X,\hat{X})^2\right]}$

(mean - square GOSPA is not a metric)

Decomposing GOSPA for RFS.

γ^* = optimal assignment = deterministic fn $(X, \hat{X})$

γ^* is also a RFS of pairs of elements.

GOSPA = $\sqrt{\underbrace{(\quad\quad)}_{\text{localisation}} + \underbrace{\quad\quad}_{\text{missed P}} \quad \underbrace{\quad\quad}_{\text{false P}}}$