3.1 n-objects tracking - constant & known n objects

$X_k = [x_k^1, x_k^2 \dots x_k^n]$

later, we'll relax known 'n'.

SOT	MOT with clutter
2 sources = 1 object + clutter	n+1 sources = n objects + clutter

DA more difficult.

Need

1) meas. model for n+1 object (incl. clutter)
2) motion model for n objects
→ prior for n objects

GNN, JPA, MHT filters

3.2 Measurements

$Z_k = \Pi(O_k, C_k)$ m_k = no. of measurements,

$C_k = Po(\lambda:)$ $\lambda_c(c) = \bar{\lambda}_c f_c(c)$

$O_k^i = []$ with prob $1 - P^D(x_k^i)$

$O_k^i = o_k^i$ with prob, $g(o_k^i | x_k^i)$ likelihood.

$P(Z_k | X_k) \rightarrow$ clutter, detections, missed detections

$$p(Z|\cancel{X}) = \sum_{\theta \in \Theta} p(Z, \theta | \cancel{X}) = \sum_{\theta \in \Theta} p(Z | \cancel{X}, O, \theta) \, p(\theta, m | X)$$

θ, how to define for n objects

?

(33) Q for n objects

$$Z_k = [z_k^1, z_k^2 \dots z_k^{m_k}]$$

$$\theta_k^i = \begin{cases} j & \text{if object } i \text{ is assigned measurement } j \\ 0 & \text{if undetected} \end{cases}$$

$$\theta_k = [\theta_k^1, \theta_k^2 \dots \theta_k^n]$$

$n=2,\ m=2$

2 objects $X = [x^1, x^2]$

2 measurements $Z = [z^1, z^2]$

$[1, 0]$

$[2, 1]$

Set of valid associat

Θ_k

1) each object is either detected or mis-detection

$0 \leq \theta_k^i \leq n$.

2) any pair of detected objects can't be associated to same measurement

$\forall\, i, i' \in \{1, \dots n\},\ i \neq i',$

if $\theta_k^i \neq 0,\ \theta_k^{i'} \neq 0,$

$\Rightarrow \theta_k^i \neq \theta_k^{i'}$

Point object assumption

Given Z_k, θ_k, we know

$$O_k^i = \begin{cases} z_k^{\theta_k^i} & \text{if } \theta_k^i \neq 0 \\ [\,] & \text{if } \theta_k^i = 0 \end{cases}$$

$$m_k^c = m_k - m_k^o$$

other than those in θ_k, other z_k^i are clutter

$$j \in \{1, \dots m_k\} : \nexists\, i \in \{1 \dots n\}\ \theta_k^i = j$$

Abbreviated $\boxed{\nexists\, \theta_k^i = j}$

example   2 objects $[x_1, x_2]$
2 measure $[z_1, z_2]$

θ	O^1	O^2	C	m^o	m^c
[0, 0]	[ ]	[ ]	$[z_1, z_2]$	0	2
[1, 0]	z_1	[ ]	$[z_2]$	1	1
[2, 0]	z_2	[ ]	$[z_1]$	1	1
[0, 1]	[ ]	z_1	$[z_2]$	1	1
[0, 2]	[ ]	z_2	$[z_1]$	1	1
[1, 2]	z_1	z_2	[ ]	2	0
[2, 1]	z_2	z_1	[ ]	2	0

3.4 Association prior & Associat. condition likelihood

$$p(\theta, m \mid X) = \underbrace{\prod_{i:\theta_k^i = 0} (1 - P_D(x_k^i)) \prod_{i:\theta_k^i \neq 0} P^D(x_k^i)}_{(1)} \underbrace{P_o(m^c)}_{(2)} \underbrace{\frac{1}{\binom{m}{m^o} m^o!}}_{(3)}$$

(1) prob. of detecting a specific set of m^o objects

(2) probab of $m_c = m - m^o$ clutter

(3) prob of specific association, (ie) 1 by no of ways to select m^o detections and assoc. to spec objects

$$= \frac{m^c!}{m!} Po(m^c / \bar{\lambda}_c) \prod_{i:\theta_k^i = 0} (1 - P^D(x_k^i)) \prod_{i:\theta_k^i \neq 0} P^D(x_k^i)$$

~~Give~~ for likelihood,

given m, θ => measurements are independent

$$P(Z \mid x, \theta, m) = \prod_{j: \nexists \theta^i = j} f_c(z_j) \prod_{i:\theta^i \neq 0} g(z^{\theta^i} \mid x^i)$$

~~Poo~~ x

$$P(Z \mid x) = (\times)(x)'$$

$$= \sum_{\theta \in \Theta} \boxed{\frac{e^{-\bar{\lambda}_c}}{m!} \prod_{j: \nexists \theta^i = j} \lambda_c(z_j)} \prod_{i:\theta^i = 0} (1 - P^D(x^i))$$

$$\prod P^D(x^i) g(z^{\theta} \mid x^i)$$

account for all measurement

all n objects

can be simplified to j taken over m

$$\therefore P(Z \mid x) \propto \sum_{\theta = \Theta} \underbrace{\prod_{i:\theta^i = 0} (1 - P^D(x^i))}_{\text{missed detections}} \underbrace{\prod_{i:\theta^i \neq 0} \frac{P^D(x^i) g(z^{\theta^i} \mid x_k)}{\lambda_c(z_{\theta^i})}}_{\text{detection}}$$

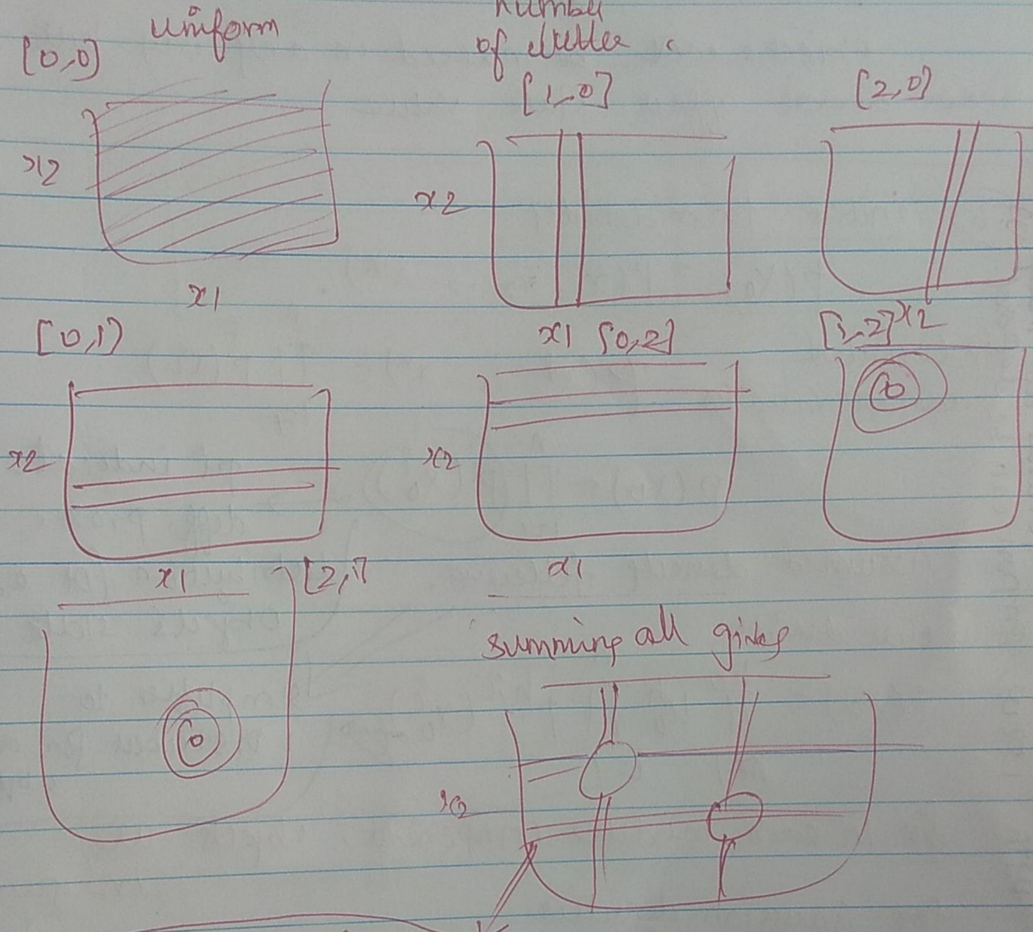
for all possible data hypothesis - we consider each object detection & miss detection

5.3 Linear gaussian model, : $P_D(x) = P_D$

$\lambda_c = \bar{\lambda}_c / V$ (uniform clutter)

$$\Rightarrow p(Z|X) \propto \sum_{\theta \in \Theta} (1-P_D)^{(n-M^{\theta})} \prod_{i:\theta_i \neq 0} \frac{P_D \, \mathcal{N}(z^{\theta_i}; Hx_i, R)}{(\bar{\lambda}^c / V)}$$

number of clutter

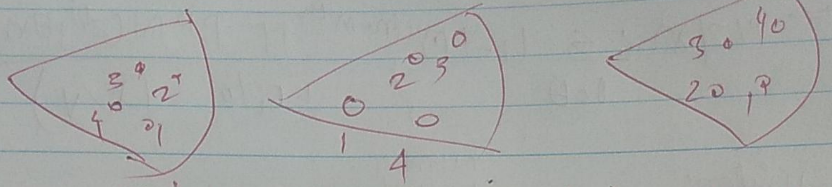


circles are bigger, lines are thinner for P_D >>

$p(Z/x)$ is symmetric w.r.t $\boxed{x_1 = x_2}$ line

holds in higher dimensions too

Reason: 1) unknown DA
2) x_1, x_2 indexing is arbitary



objects cant be indexed in any way but index doesnt affect state value

---

3.6 initial prior density

within 1 column components are indep
with 1 row, compon can be depend

$$p(X_0) = p(x_1; x_2 \dots x^n).$$

independent random variables $p(x^1, x_2, \dots x^n) = \prod_{i=1}^{n} p^i(x^i)$

$$p(X_0) = \prod_{i=1}^{n} p^i(x_0^i) \longrightarrow$$ p^i indicates diff. prob. distribution for each object's state

Assumed density filtering.

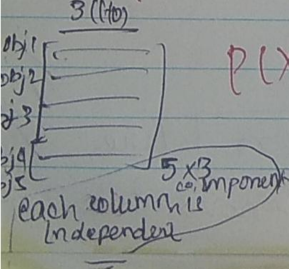
mixture density

$$p(X_0) = \prod_{h=1}^{H_0} w_0^h \prod_{i=1}^{n} p^{h,i}(x_0^i)$$ simplifies to 1 Distribut for all objects

for in each mixture component, objects are independent

for Gaussian densities

$$p(X_0) = \sum_{h=1}^{H_0} w_0^h \prod_{i=1}^{n} \mathcal{N}(x_0^i : \mu_0^{h,i}, P_0^{h,i})$$



each column is independent

H hypothesis n - object

3.7 Posterior density

SOT posterior:

$$p(x_k|Z_{1:k}) \propto p(z_k|x_k)\, p(x_k|Z_{1:k-1})$$

__Structure__

$$\boxed{p(x_k|Z_{1:k}) = \sum_{\theta_k} p(x_k|Z_{1:k},\theta_k)\, \Pr[\theta_k|Z_{1:k-1}]}$$

__n-objects__

$p(X_k|Z_{1:k}) =$ same except ~~x~~X_k is now a matrix not a vector

~~1) uni-modal prior~~

~~2) mixture prior~~ $\sum_h \sum_\theta \tilde{w}^{h,\theta} p^{h,\theta}(x|z)$

3.8 Posterior with unimodal prior

$$p(x) = \prod_{i=1}^{n} p^i(x^i)$$

$$p(z|x) = \sum_{\theta\in\Theta} \underbrace{\prod_{i:\theta^i=0} (1-P^D(x^i))}_{\text{missed detection}} \underbrace{\prod_{i:\theta^i\neq 0} \frac{P^D(x^i)\, g(z^{\theta^i}|x^i)}{\lambda_c(z^{\theta^i})}}_{\text{Detections}}$$

$p(x|z) \propto p(z|x)\, p(x)$

for each ~~hypo~~ DA,

theres a product of (miss. detection prob) (Detection likeli) (prior) missed detection likelihood times prior | Detected times prior

$$= \sum_{\theta \in \Theta} \underbrace{\prod_{i;\theta=0} (1-P_D)\, p^i(x^i)}_{\text{missed objects}} \; \underbrace{\prod_{i,\theta \neq 0} \frac{P_D\, g(z^{\theta^i} \mid x^i)\, p^i(x^i)}{\lambda_c(z^{\theta^i})}}_{\text{detected objects}}$$

unnormalised density

Using SOT logic,

if $p(x) \propto g(x)$ $\quad p(x) = \dfrac{g(x)}{\int g(x)\,dx}$ $\quad \boxed{\theta^i = 0}$

$g(x^i) = (1-P_D(x^i))\,p(x^i) \Rightarrow p^{i,\theta^i}(x^i) = \dfrac{g(x^i)}{\tilde{w}_{\theta^i}}$ $\quad \tilde{w}_{\theta^i} = \int g(x^i)\,dx$

$$= \boxed{\sum_{\theta \in \Theta} \prod_{i=1}^{n} \tilde{w}^{\theta^i}\, p^{i,\theta^i}(x^i)}$$

$$\begin{cases} (1-P_D) & \text{if } \theta = 0 \\ \dfrac{P_D\, g(z \mid \theta)}{\lambda_c} & \text{if } \theta \neq 0 \end{cases}$$

$$\begin{cases} \dfrac{1}{\tilde{w}^{\theta^i}} \cdot (1-P_D)(x^i) \\ \dfrac{1}{\tilde{w}^{\theta^i}} \cdot \dfrac{P_D\, g(z \mid x)}{\lambda_c(z^\theta)} \end{cases}$$

But $p(X/Z)$ is not normalised ~~as~~ n object mixture density

each object is normalised but as 'n object' density not normalised

another normalising is needed.

$$p(Z) = \int p(Z \mid X)\, p(X)\, dX$$

$$= \int \sum_{\theta \in \Theta} p(Z, \theta \mid X)\, dX$$

$$= \sum_{\theta \in \Theta} \int p(z,\theta)\, p(x)\, dx$$

$$= \sum_{\theta \in \Theta} p(z,\theta)$$

Consider gaussian unimodal prior.

$$p(x) = \prod_{i=1}^{n} N(x^i; \mu^i, P^i)$$

$$P(z|x) \propto \sum_{\theta \in \Theta} \prod_{i:\theta^i=0} (1-P^D) \prod_{i:\theta^i \neq 0} \frac{P^D N(z^{\theta^i}; Hx^i, \ldots}{\lambda/V}$$

$\therefore P(x|z)$

$\propto P(z|x)\, P(x)$

$$\propto \sum_{\theta \in \Theta} \prod_{i':\theta^{i'}=0} (1-P^D)\, N(x^{i'}; \mu^{i'}, P^{i'}) \prod_{i:\theta^i \neq 0} \frac{P^D N(z^{\theta^i}, Hx^i, P^i) N}{\lambda V}$$

$$\therefore P(x|z) \propto \sum_{\theta \in \Theta} \tilde{w}^{\theta} \prod_{i} N(x^i; \mu^{i\theta}, P^{i,\theta^i})$$

$$\tilde{w}^{\theta} = (1-P^D)^{n-m\theta} \left(\frac{P^D}{\lambda V}\right)^{m\theta} \prod_{i:\theta^i \neq 0} N(z^{\theta^i}; \hat{z}^i, \ldots$$

Unnormalised weights

for each detected objects there is

$$\mu^{i,\theta^i} = \begin{cases} \mu^i + K^i(z^{\theta^i} - \hat{z}^i) & \text{if } \theta^i \neq 0 \\ \mu^i & \text{if } \theta = 1 \end{cases}$$

$$P^{i,\theta^i} = \begin{cases} P^i - K^i H P^i & \text{if } \theta^i \neq 0 \\ P^i & \text{if } \theta^i = 0 \end{cases}$$

$X = [x^1, x^2]$

$Z = [z^1, z^2] = [-1.6, 1]$

$NA(2,2) = 7.$

measurement model

$P^D(x) = 0.85$

$\lambda_c(c) = 0.3 \quad c \in [-5, 5]$

$g(z|x) = N(z : x ; 0, 2)$

prior

$$P(x) = p^1(x^1)\, p^2(x^2)$$

$p^1(x^1) \rightarrow N(-2.5, 0.36)$, $p^2(x^2) \rightarrow N(2.5; 0.36)$

marginal posterior (individual objects' posterior)

joint posterior (total posterior)

$\theta = [0, 0] \quad \tilde{w}^{\theta} = 5.0409e^{-05}$

$\theta = [1, 0] \quad \tilde{w}^{\theta} = 0.00024629$

$\theta = [1, 2] \quad \tilde{w} = 0.00036$

For gaussian models, associating likelihood closer to priors has higher ~~mean~~ weights

As PD ↑, misdetection weights ↓, number of dominant components ↓

---

9) General exp for pos. density

$$p(x_k | Z_{1:k}) = \sum_{\theta_{1:k}} w^{\theta_{1:k}} p^{\theta_{1:k}}(x_k)$$

sum of all hypothesis

for SOT is a vector

for n-MOT it is a matrix

mixture priors

$$p(x) = \sum_h w^h p^h(x) = \sum_h \Pr[h] \, p(x | h)$$

gives a mixture posterior, with a component for each combin of h, θ

$$p(x|Z) \propto p(Z|X) \, p(X)$$

$$= \left[\sum_{\theta \in \Theta} p[Z; \theta | X] \right] \left[\sum_h w^h p^h(x) \right]$$

$$= \sum_h \sum_{\theta \in \Theta} w^h \, p(Z, \theta | X) \, p^h(x)$$

$$= \sum_{h} \sum_{\theta \in \Theta} w^{h} \tilde{\omega}^{\theta/h} \frac{p(z^{\theta}/x)\, p^{h}(x)}{\tilde{\omega}^{\theta/h}}$$

$$= \sum_{h} \sum_{\theta \in \Theta} \tilde{w}^{h,\theta} p^{h,\theta}(x).$$

Normalised mixture posterior

$$w^{h,\theta} = \frac{\tilde{\omega}^{\theta/h} w^{h}}{\sum_{h} \sum_{\theta \in \Theta} \tilde{\omega}^{\theta/h} \omega^{h}}$$

'n' object posterior @ time k is

$$P_{k|k}(X_k) = P_{X_k|Z_{1:k}}(X_k | Z_{1:k})$$

$$\propto \sum_{\theta_{1:k}} \tilde{w}^{\theta_{1:k}}_{k:k} \; P^{\theta_{1:k}}_{k|k}(X_k)$$

$$= \prod_{i=1}^{n} \prod_{t=1}^{k} \tilde{\omega}^{\theta_t^i | \theta_{1:t-1}}$$

$$\prod_{i=1}^{n} \phi^{i}_{k|k, \theta^{(i)}_{k}} (x_k)$$

$$p(X_k) = \sum_{\theta_1} \sum_{\theta_2} \sum_{\theta_k} p($$

tracking across timesteps in intractable, ∴ approximations.

3.10 Number of associations:

'm' = measurements

n = objects.

1) $m_0 \in \{0, 1, \dots, \min(m, n)\}$ are from objects

2) $\binom{n}{m^0}$ - ways to select m^0 objects from n objects

3) $\binom{m}{m^0}$ - ways to select m^0 detections from m measurem

4) $m^0!$ → ways to assign selected detections & objects

$$N_A(m, n) = \sum_{m^0=0}^{\min(m,n)} \binom{n}{m^0}\binom{m}{m^0} m^0!$$

$$= \sum_{m_0=0}^{\min(m,n)} = \frac{m!\,n!}{m^0!\,(m-m^0)!\,(n-m^0)!}$$

if n=1, it's SOT,

$$N_A(m, 1) = \binom{1}{0}\binom{m}{0}0! + \binom{1}{1}\binom{m}{1}1!$$

$$= (1)(1)(1) + (1)(m)(1)$$

$$= 1+m$$

No. of mixture components @ time k $= N_0 \prod_{t=1}^{k} N_A(m_t, n)$

no. of comp in prior is $N_0 = 1$

$N_0 = 1;\ m_1 = 3,\ m_2 = 5,\ m_3 = 4$

$n = 1;$ (SOT)

$$(1+3)(1+5)(1+4) = 120.$$

$n = 2;$

$$N_A(3,2)\, N_A(5,2)\, N_A(4,2) = 13 \cdot 31 \cdot 21 = 8463$$

3.1 Predicting 'n' object density

Posterior: $p_{k-1|k-1}(X_{k-1})$

Transition density $p(X_k | X_{k-1}) = p(x_k^1, x_k^2 \ldots | x_{k-1}^1, x_{k-1}^2)$

Assumption: each object evolution is independent,

$$p_k(X_k | X_{k-1}) = \prod_{i=1}^{n} \pi_k (x_k^i | x_{k-1}^i)$$

$$\rightarrow \mathcal{N}(x_k;\ f_{k-1}(x_{k-1});\ Q_{k-1}).$$

independent posterior

$$p_{k-1|k-1}(X_{k-1}) = \prod_{i=1}^{n} p^i_{k-1|k-1}(x^i_{k-1})$$

$$\Downarrow$$

$$\prod_{i=1}^{n} p^i_{k|k-1}(x_k^i) \Rightarrow$$ each object predicted independent of others

mixture posterior:

$$p_{k-1|k-1}(x_{k-1}) = \sum_h w^h_{k-1} \, p^h_{k-1}(x_{k-1})$$

$\Downarrow$

$$\sum_h w^h_{k|k-1} \, p^h_{k|k-1}(x_k)$$

each hypo. predicted indep of other hypo.
weights remain same

3.12 independent objects | initial prior | measurements

object motion

are object states independent?
$\boxed{\text{no}}$ at least not in general case

$$p(x) = \sum_h w^h p^h(x) = \sum_h w^h \prod_{i=1}^{n} p^{i,h}(x^i)$$

where each 'h' corresponds to set of 'n' object Data asso.

Conditioned on one h,
objects are indep

$$p(x/h) = \prod_{i=1}^{n} p^i(x^i/h)$$

if objects were independent

$$p(X) = \sum_h w^h \prod_{i=1}^{n} p^{i,h}(x^i) \overset{?}{=} \prod_{i=1}^{n} \left(\sum_{h^i} w^{h^i} p^{i,h^i}(x^i) \right)$$

A ~~multi~~ mixture hypo for each object; multiplied over n objects

General case $\neq$

But exceptions

GNN, JPDA

approx posterior of 'n' objects s.t. they're independent

(3.13) Data association - Introduction.

$$p_{k-1}(X_{k-1}) = \sum_h w^h_{k-1} \, p^h_{k-1}(X_{k-1})$$

Predicted mixture

$$p_{k|k-1}(X_k) = \sum_h w^h_{k|k-1} \, p^h_{k|k-1}(X_k).$$

updated

$$p_{k|k}(X_k) \propto \sum_h \sum_{\theta_h \in \Theta} \tilde{w}^{\theta_k|h} \, w^h_{k|k-1} \, p^{h,\theta_k}_{k|k}(X_k)$$

for each predicted hypo, we get $N_A(m_k, n)$ new hypo

Idea: find subset of $\tilde{\theta}_k \in \theta_k$ such that large weights are found.

challenge: Avoid computing $\theta_k \in \theta_k$ and comparing weights

pose as optimisation problem.
assignment problem

Assignment problem:

assume 3 workers, 3 tasks each task done by 1 worker, ~~task~~ 1 worker can do 1 task

	t_1	t_2	t_3
w_1	5	8	7
w_2	8	12	7
w_3	4	8	5

each worker has a cost for each task

$\therefore$ finding $\boxed{w \to t}$ combo with lowest score

Cost matrix $L^{i,j}$

$$L = \begin{bmatrix} 5 & 8 & 7 \\ 8 & 12 & 7 \\ 4 & 8 & 5 \end{bmatrix}$$

Assignment matrix

$A_{i,j} = 1$ if w^i is assigned t^j

0 otherwise

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{cost} = \sum_i \sum_j A^{ij} L^{ij}$$

$$= tr(A^T L)$$

Given L - Solution A^* with constraned min. problem

min $tr(A^T L)$

subject $A^{i,i} \in (0,1) \; \forall i,j$ → can take values (0,1)

$\sum_j A^{ij} = 1$ → sum across a row = 1

$\sum_i A^{i,j} = 1$ → sum across a colm = 1

objects ⟷ workers

Detection ⟷ tasks

8.14

$$p(x) \propto \sum_h \sum_{\theta \in \Theta} \tilde{w}^{\theta|h} w^h p^{h,\theta}(x)$$

posterior n obj. density

find $\theta^* \subset \Theta$ which has max weight [for each hypo]

$$\theta^* = \arg\max_{\theta \in \Theta} \tilde{w}^{\theta|h} = \arg\max_{\theta \in \Theta} \prod_{i=1}^{n} \tilde{w}^{\theta i|h}$$

Take log

$$\theta^* = \arg\max_{\theta \in \Theta} \log\left(\prod_{i=1}^{n} \tilde{w}^{\theta i|h}\right)$$

$$= \arg\max_{\theta \in \Theta} \sum_{i=1}^{n} \left(\log \tilde{w}^{\theta i|h}\right)$$

$\because$ log is a monotone $\uparrow$ fn.

As a min. problem $\Rightarrow$ $\hat{\theta} = \underset{\theta \in \Theta}{\arg\min} \sum_{i=1}^{n} -\log(w^{\theta^i(h)})$

$$A^* = \underset{A}{\arg\min}\, tr(A^T L).$$

(3.13) 'A' in n object tracking

A must correspond to unique $\theta \in \Theta$

$A = n \times (m+n)$

n objects

m detections n misdetections

$A\{i,j\} = \{0,1\}$

detection $\theta^i = j$ $\quad A^{i,j} = 1$

misdetection $\theta^i = 0$ $\quad A^{i,i+m} = 1.$

all remaining are zero

2 objects 1 measurement

$n=2, \; m=1 \quad NA(1,2) = 3$

$A = 2 \times 3.$

x^1 is associated to z_1

$$\begin{array}{l} \text{obj 1} \rightarrow \\ \text{obj 2} \rightarrow \end{array} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\theta = (0,0)$

$\theta^1 = 1 \Rightarrow A^{1,1} = 1$

$\theta^2 = 0 \Rightarrow A^{2,3} = 1$ $\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\theta = [1, 0]$

$\theta = [0, 1]$ $\quad X^2 \Leftrightarrow Z^2$ $\quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Properties. $\boxed{A^{1,3} = A^{2,2} = 0 \text{ for all } \theta}$

$\sum_j A^{i,j} = 1 \; \forall i \Rightarrow$ row sum $= 1$

$\sum_j A^{i,j} = \{0, 1\}$ for all $j \Rightarrow$ column sum can be 0 or 1

$$A = \left[\begin{array}{cccc|ccc} A^{11} & A^{12} & \cdots & A^{1m} & A^{1,m+1} & 0 & \cdots \\ & & & & & \ddots & \\ & & & & & & A^{n,m+n} \end{array}\right]$$

detections (n×m) submatrix — mis detections (n×n) diagonal

$$A = \left[\begin{array}{c|cc} A^{1,1} & A^{1,2} & 0 \\ A^{1,2} & 0 & A^{2,3} \end{array}\right]$$

needs to be a matrix as all hypo need to be covered.

(3.16) unnormalized association weights

$$\tilde{w}^{\theta^i|h} = \begin{cases} 1 - P^D(x^i)\, p^{i,h}(x^i) & \text{if } \theta^i = 0 \\ \dfrac{P^D(x^i)\, g_k(z \mid x)\, p^{i,h}(x^i)}{\lambda_c(z^{\theta^i})} & \text{if } \theta^i \neq 0 \end{cases}$$

$l^{i,0,h} = \log(\;)$ = log likelihood of misdetecting x^i

$l^{i,j,h} = \log(\;)$ = " of associating x^i to z^j

$$\sum_{i=1}^{n} -\log(\tilde{w}^{\theta^i|h}) = \sum_{i=1}^{n}\sum_{j=1}^{m} A^{i,j}(-l^{i,j,h}) + \sum_{i'=1}^{n} A^{i',m+i'}(-l^{i',0,h})$$

$$L = \left[\begin{array}{cccc|cccc} -l^{1,1} & l^{1,2} & \cdots & l^{1,m} & -l^{1,0} & \infty & \cdots & \infty \\ \vdots & & & & & -l^{2,0} & & \\ & & \ddots & & & & \ddots & \\ & & & -l^{n,m} & & & & -l^{n,0} \end{array}\right]$$

$\downarrow$ detections $\qquad\qquad$ $\downarrow$ misdetections

off-diagonal of right $(n \times n)$ submatrix. $\boxed{-\log(0) = \infty}$

avoids optimisation return multiple A for same $\theta \in \Theta$

$$\tilde{w}^{\theta|h} = \exp\left(-\operatorname{tr}(A^T L^h)\right)$$

3.17 Optimisation algorithms

$L \curvearrowright A = ?$

min $tr(A^T L)$

constraints.

<u>Solvers:</u> Hungarian, Auction, Jonker-Volgenant (one solution) (a stanon (JVC)

Murty's: finds M best assignments.

<u>Gibbs</u> sampling: sub-optimal but comp. efficient.

↓ markov chain monte carlo.

3.18 <u>Gating for N objects</u>

→ reducing no. of possible DA.

· for Gaussian, ellipsoidal gating.

$$d_{ij} = (z^j - \hat{z}^{i,h})^T (S^{-1}) (z^j - \hat{z}^{i,h})$$

if $d_{ij} \leq G$

$d_{ij} \uparrow \longrightarrow$ log likelihood ↓

if z^{ik} falls outside X_k^{ℓ}, we set $\boxed{\ell^{i,j,k} = -\infty}$

nPn

(3!) 8! / 0!

8! × 3!

3 × 4 × 5 × 6 × 7 × 8 × 6

36 56 280

216 / 4

Group by Gating

3 groups.

Using gating to group detections & objects into groups

40

30 : 40

5 : 1

35 : 10

30 10

1

→ Computationally cheaper

$n = 6$, $m = 15$

$NA(15, 6) \geq 6 \cdot 10^6$

Gr1: $n = 1$, $m = 2$ $NA(2,1) = 3$

Gr2: $n = 2$, $m = 4$ $NA(4,2) = 21$

Gr3: $n = 3$, $m = 6$ $NA(6,3) = 229$,

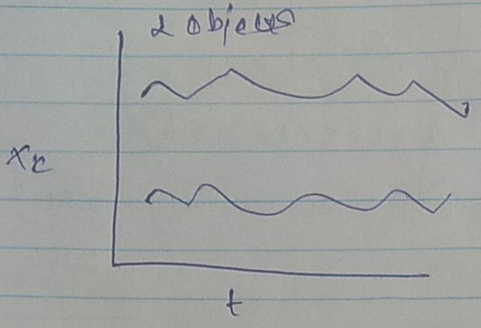
after gating & grouping

$3 \times 21 \times 229 = 14 \cdot 10^3 \ll 6 \cdot 10^6$

Considering gating in groups

$\boxed{3 \times 11 \times 41 = 1353} \ll 14$

3.19 N-object Tracking



Pruning
reduction.

Merging
single.
mixture
density
approximated
by 1
density
kullback-leib
divergence is
minimised

1) Greedy approach (GNN)
best association & prune all others
complete optimal assign ment
1 hypo - posterior
GNN

2) Merge all hypo into 1
(JPDA) - marginal assoc. probabilities

3) Maintain multiple hypo with highest
weights prune rest.

M best assignments
Multi-hypothesis tracker

3.20 GNN basic idea.

find θ^* and prune all other $\theta_k \in \Theta_k$.

$$p^{GNN}_{k|k}(x_k) = p^{\theta^*_{1:k}}_{k|k}(x_k)$$

$\theta_1^*, \quad \theta_2^* | \theta_1^* \quad \ldots \quad \theta_k^* | \theta_{k-1}^*$

Parametrised by object densities

if Gaussian, $p_{k|k}^{\theta^* i_k}(x_k^i) \quad (i = 1 \ldots n)$

$\boxed{\text{mean, } P}$

for $k = 1, 2 \ldots k$

1) Prediction

for $i = 1 \ldots n$, prediction

$$p_{k|k-1}^{i}(x_k^i) = \pi(x_k^i | x_{k-1}^i)$$

2) Update

a) Create cost matrix L_k

b) θ_k^* find

c) for $i = 1 \ldots n$, Do update

$$p_{k|k}^{i}(x_k^i) = \begin{cases} P_D(x_k) g_k(z_k^{\theta_i} | x_k) p_k(x_k^i) & \text{if } \theta \neq 0 \\ (1 - P_D) p_k(x_k) & \text{if } \theta = 0 \end{cases}$$

monitored by expected value, var for gaussian densities

3.21 Examples of GNN

Perform worse when SNR is lower

~~$P_D \ll 1$~~, $P_D \ll 1.0$ then GNN rejects many measurements as clutter & hence poor results

Pros: comput. cheap.
simple to implement
simple scenarios with high SNR (i.e.) high P_D and low $\bar{\lambda}$, small R.

Cons: Not guaranteed $w^{\hat{\theta}_{1:k}} > w^{\theta_{1:k}} \ \forall \theta_{1:k} \in \Theta_{1:k}$
1 n-object hypo may not be sufficient
poor tracking ~~objects~~ performance in moderate to low SNR, in high SNR when objects are close to each other. (many will have ↑ w, but we'll be taking one only)

3.22 JPDA
- merge marginal post. densities
→ compute marginal association probal for all objects.

$$\boxed{p^{JPDA}_{k|k}(x_k) = p^{\beta_{1:k}}_{k|k}(x_k)}$$

$\beta_1, \ \beta_2|\beta_1 \ \ldots \ \beta_k|\beta_{k-1}$

1 hypotheses per timestep

marg. ~~prob~~ association prob
(prob that object i is assigned to measurement j)

$$\beta_k^{i,j} = \Pr(\theta^i = j \mid Z_{1:k-1}) = \sum_{\theta_k \in \Theta_k, \theta_k^i = j} w^{\theta_k} \propto \sum_{\theta_k \in \Theta_k, \theta_k^i = j} \tilde{w}^{\theta_k}$$

sum of ~~the~~ weights of hypo for which $\theta_k^i = j$

$$\beta_k^{i,0} = \Pr(\theta_k^i = 0 \mid Z_{1:k-1}) = 1 - \sum_j \beta_k^{i,j}$$

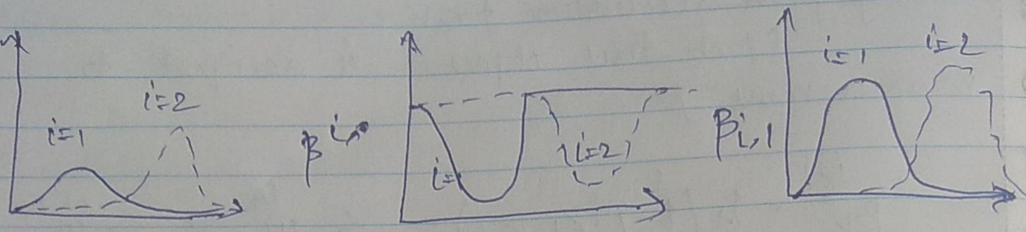
$$= \sum_{\theta_k \in \Theta_k, \theta_k^i = 0} w^{\theta_k} \propto \sum_{\theta_k \in \Theta_k, \theta_k^i = 0} \tilde{w}^{\theta_k}$$

Computed over all valid DA θ_k (intractable, approx. needed)

$\boxed{\beta^{i,j} = \text{joint distr}}$

---

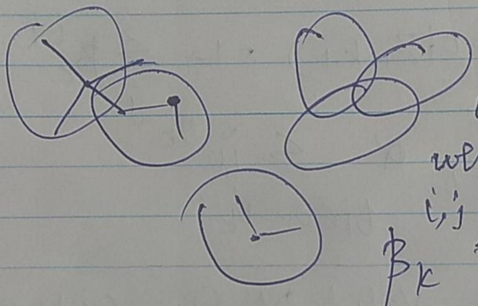
assume $n=1$ to $n=4$ cases and simple linear Gaussian scalar states, Rand. walk and direct measurements



for 2 objects with far means, small effect on $\beta^{i,j}$

ut when objects are closer, profiles are more complex.

2D detection $m=15$ $n=6$



for each predicted state, there are only few measurements which are relevant

$$\beta_k^{i,j} = \Pr[\theta_k^i = j \mid Z_{1:k-1}) = \sum_{\theta_k \in \Theta_k,\ \theta_k^i = j} w^{\theta_k}$$

If gate is large enough, lead to negligible errors

5.25 JPDA, predict - update.

$$\boxed{p^{i}_{k|k}(x^{i}_{k}) = \beta_k^{i,0}\, p_k(x^{i}) + \sum_{j=1}^{m_k} \beta_k^{i,j}\, p^{i,j}(x^{i})}$$

marginal posterior for object i.

Merge marginal post. densities into 1 density using some Merge fn.

usually moment. matching

$$p^{i,\beta}_{k|k}(x^{i}_{k}) = \mathrm{Merge}(\cdot)$$

posterior for 'n' objects is

$$p^{\beta}_{k|k}(X_k) = \sum_{i=1}^{n} p^{i,\beta}_{k|k}(x^{i}_{k})$$

for $k = 1, 2, \ldots K$.

1) prediction

$$p^{i,\beta}_{k|k-1}(x^{i}_{k}) = p^{i,\beta_{1:k-1}}_{k-1}(x_{k-1}),\quad (\text{for } i = 1, \ldots, n)$$

2) Compute marg. asso. prob $\beta_k^{i,j}$

for $i = 1, \ldots, n$ JPDA update

$$p^{i,\beta_{1:k}}_{k|k}(x^{i}_{k}) = \mathrm{Merge}\Big(\beta^{i,0}\, p_{k|k}(x_k) + \sum_{j=1}^{m} \beta^{i,j}\, p^{i,\beta_{1:k,j}}_{k|k-1}$$

Predicted params $\{\mu^{i}_{k|k-1}, P^{i}_{k|k-1}\}^{n}_{i=1}$.

Update mean

$$\varepsilon^{i,j}_{k} = z^{j}_{k} - H\mu^{i}_{k|k-1}$$

$$\varepsilon_{k} = \sum_{j=1}^{m_k} \beta^{i,j}_{k}\, \varepsilon^{i,j}_{k}$$

$$\mu_{k|k} = \mu^{i}_{k|k-1} + K\varepsilon^{i}_{k}$$

Example consider 1 prediction

$\beta^{6,0} = 0.09$

$\beta^{6,15} = 0.14$

$\beta^{6,1} = 0.77$.

Cheap JPDA

Suboptimal JPDA

Fast JPDA

approx. computation of β

$$\beta^{i,j} = \sum_{\theta_k \in \Theta_k,\ \theta^{i}_{k} = j} \omega^{\theta_k}$$

<u>alternate soln:</u> $\beta^{i,j}_{k} \Rightarrow$ M best associations alone

3.24: JPDA examples

SNR is lower JPDA $\gg$ GNN

But when SNR is $\uparrow$, JPDA $\sim$ GNN

Variance is high

Pros:

GNN $\gg$ JPDA	JPDA $\gg$ GNN
Comput. cost,	When SNR $\downarrow$,
simple to implement.	tracking performance

cons: When true objects are closer, difficult to track

3.25 HO prediction, update

MHT $\rightarrow$ HO.

Idea: find M best associations θ_k^m and prune others.

Reduce such that at most Nmax hypo are included in posterior.

$$p_{k|k}^{MHT}(x_k) = \sum_{h_k=1}^{H_k} w_k^{h_k} p_k^{h_k}(x_k)$$

$$H_k \leq N_{max}$$

2 variants of MHT, HO | TO

$\downarrow$ Hypo oriented | $\downarrow$ Track oriented

HO-MHT

parametrised by log weights, densities

$$\ell^{h_k} = \log(w_{k|k}) \quad \{p_{k|k}^{i,h_k}(x_k)\}_{i=1}^{n} \quad \text{for } h = 1,2,..H_k$$

for $k = 1,2,..K$

1) Prediction

for each hypo and object prediction

2) Update:

compute multiple associations, construct post. hypothesis

Prediction:

Input: $\left\{ \ell^{h_{k-1}}, \left\{ p_{k-1|k-1}^{i,h_{k-1}}(x_{k-1}) \right\}_{i=1}^{n} \right\}_{h=1}^{H_{k-1}}$

Output

doesn't change during prediction

$p_{k|k-1}$

Update $\{ l^{h_{k-1}}, \{ p_{k|k-1}^{i, h_{k-1}} \}_{i=1}^{n} \}_{h=1}^{H_{k-1}}$

Initialise $h_k = 0$

for $h_{k-1} = 1, \ldots, H_{k-1}$

Create cost matrix $L^{h_{k-1}}$

Compute $M_{h_{k-1}}$ associations θ_m.

for $m = 1, \ldots, M_{h_{k-1}}$

Increase $h_k \leftarrow h_k + 1$

Compute posterior parameters.

$l^{h_k}, \{ p_{k|k}^{i, h_k} \}_{i=1}^{n}$

Initialise $\tilde{l}_{h_k} = l^{h_{k-1}}$

$$p_{k|k}^{i}(x_k^i) \propto \begin{cases} P^D(x_k)\, g_k(z_k^{\theta^i}|\,)\cdot p_{k|k-1}(x_k) & \theta^i \neq 0 \\ (1-P^D(x_k))\, p_{k|k-1}(x_k) & \theta^i = 0 \end{cases}$$

$\tilde{l}^{h_k} = \tilde{l}^{h_k} + l^{i, \theta_m^i, h_{k-1}}$

log likelihood of θ_m^i

Set $H_k = h_k$

normalise $l_k^h \leftarrow \tilde{l}^{h_k}$.

$M_k \rightarrow$ constant.

$$M_k = \max(1, H_{max} \exp(l^{h_{k-1}}))$$

Working with log weights avoids numerical problems

$$l^{h_k} = \tilde{l}^{h_k} - \tilde{l}^{h_k^{max}} - \log\left(1 + \sum_{h_k \neq h_k^{max}} e^{(\tilde{l}^{h_k} - \tilde{l}^{h_k^{max}})}\right)$$

MHT Reduction:

$$\boxed{l_{k|k}^{h_k} \leq \Gamma}$$

$\Gamma \leq \log(0.01)$ or smaller.

Capping $\boxed{H_k > N_{max}}$

renormalise   merging also possible but out of scope

3.26

$$\boxed{h^* = \max_{h_k} l_{k|k}^{h_k}} \rightarrow$$ common object estimator in MHT

$\boxed{MHT = 1 \Rightarrow GNN}$

Cauchy Schwarz divergence

& KL divergence

Pros: low SNR also ✓.

Cons: comp. cost, complicated,
~~if Nmax~~ not guaranteed that most prob is included in mixture
But if Nmax is >>, this is possible

3.27 Representation of Hypo in MHT.

Global → for all n-objects $\boxed{\theta_{1:k}}$

$$H_k = \prod_{i=1}^{k} N_A(m_t, n)$$

Local → for i object $\boxed{\theta^i_{1:k}}$

$$\boxed{H_k = \prod_{i=1}^{k} \cancel{A}(m_t+1)}$$

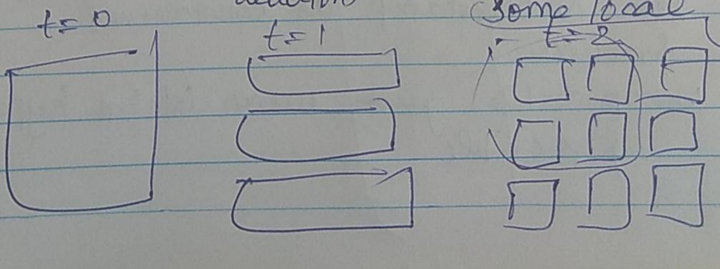
Global
$H_0 = 1$ (1 hypo per object), $H_1 = 3$ (3 DA @ $t=1$), $H_2 = 9$

$H^i_0 = 1$, $H^i_1 = 2$, $\boxed{H^i_2 = 4}$

Local

(mis detection, detection)

Same local hypo occurs in multiple global hyp → local



Inference

each global hypo is indepen of other
local hypo repetition
inefficient mem. & comp. power

∴ → efficient
a) 1 copy of each local hypo
b) look up table of global hyp
which local hypo are included in global hypo
⇓
TO-MHT

3.30

Extensions:
→ Using Gating + Grouping into subsets
n-object MHT into many MHT for < n obje
⇒ parallel execution

→ Complicated when prev. "ind. objects" interact
also as groups separate difficult

N-scan MHT
→ Keep last N-timestamp as constrained assignment problem
→ no lookup table, ~~just~~ many Global hypo possible

Unknown objects (GNN, JPDA, MHT) ?

→ yes

→ Track initiation, track deletion

score based

M/N based

measurement driven initiation emp.

proven to be good.

$\boxed{\text{RFS}}$ → MOT

3.28 Hypothesis tree

|

Local hypo - tracks

H_k^i

look up tables

for global Hypo

H_k

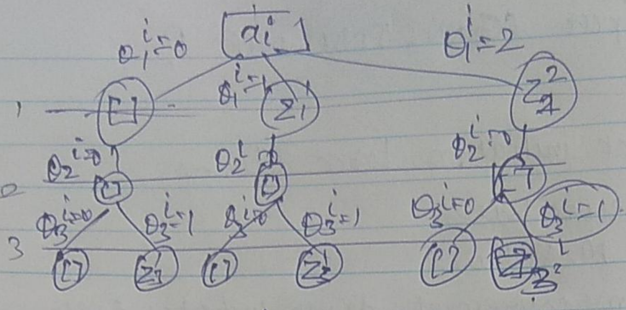
$\boxed{H_k(h_k, i)}$

TO-MHT: a Hypo tree

Leaf = local hypo;

for each leaf, $P_{k|k}(x_k | Z, \theta)$

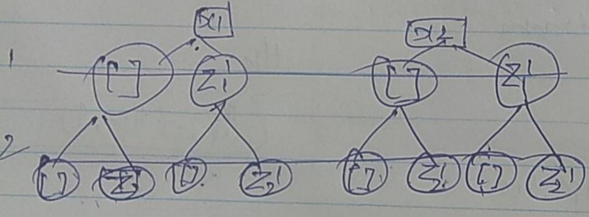
At time k, H_k^i leafs in tree



$$\begin{array}{ll} m_1 = 2 & H_1^i = 3 \\ m_2 = 0 & H_2^i = 3 \\ m_3 = 1 & H_3^i = 6 \end{array}$$

$n = 2 \quad m_1 = 1 \quad m_2 = 1$

Hypothesis trees



$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

3 hypotheses

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \\ 1 & 3 \\ 1 & 4 \\ 3 & 1 \\ 4 & 1 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Occurs 4 times for object 1. So saves mem. & comp by saving just once

3.29 TO-MHT predict & update

Params:

$$\mathcal{H}_k, \{l_k\}_{h_k=1}^{H_k} \quad \left\{ \{ p_{k|k}^{i,h_k}(x_k^i) \}_{h_k=1}^{H_k^i} \right\}_{i=1}^{n}$$

For $k = 1, 2, \ldots K$:

1) prediction, for each local hypo predict
2) Local update, for each object, compute
   update local hypo
3) Global $\mathcal{H}_k$, l_k of post. global hypo
4) pruning, capping

1) Predictions

$$H_{k-1}, \quad (l_{h_{k-1}}), \quad \{ p(x_{k-1}) \}_{i=1}^{n}$$

$\downarrow$ same, $\downarrow$ same, $\downarrow$ for each object Hypo predict

2) Update:

for $i = 1 \ldots n$ $\quad \{ p_{k|k-1}(x_k) \}_{h_k=1}^{H_{k-1}^i}$

for $h_{k-1} = 1, \ldots H_{k-1}$

for $j = 0, \ldots m_k$

Index $h_k = (h_{k-1}^i - 1)(m_k + 1) + 1 + j$

Density $\propto \begin{cases} j \neq 0 \\ j = 0 \end{cases}$

Association log-likelihood

$H_k^i = H_{k-1}^i (m_k + 1)$

Global hypo update

<u>Input:</u>

Set $h_k = 0$

for $h_{k-1} = 1 \dots H_{k-1}$

Create cost matrix $L^{h_{k-1}}$ using $l_{i,j}^{i,a}, h_{k-1}$

Compute $M^{h_{k-1}}$ assignments θm,

for $m = 1 \dots M_{k-1}$

increase $h_k \leftarrow h_k + 1$

initialise $l^{h_k} = l^{h_{k-1}}$

Update lookup table

for $i = 1 \dots n$,

$H_k(h_k, i) = H_{k-1}(h_{k-1}, i-1)(M_k + 1)$

increase log weight $+ l + \theta_m^i$

Normalise log weights

<u>Common to HO-MHT:</u>

→ Deciding no. of DA

→ normalising log wts

pruning & capping - when global hypo are pruned, more local hypo are also pruned, (not included in any global hypo)

$$H_2 = \begin{bmatrix} \\ \\ \end{bmatrix}_{9\times 2} \qquad \overset{\gamma_0}{w_1^{\theta}} = \begin{bmatrix} \\ \\ \end{bmatrix}_{9\times 1}$$

Pros:

TO-MHT - same performance as HO-MHT.

TO-MHT << MO-MHT.

↓ mem.

comp. cost

Cons

TO-MHT >> GNN, JPDA

↑ comp. cost

memory

complexity

TO-MHT - defacto standard