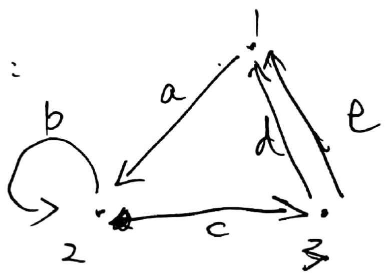
Quivers

Def: A quiver Q consists of two sets Q_0, Q_1, and two maps $h, t: Q_1 \to Q_0$. The elts of Q_0 are called vertices, while those of Q_1 are called arrows.
The head map h and tail map t assign a head ha and a tail ta to every arrow $a \in Q_1$.

Graphically, Q can be represented as a directed graph with one vertex per elt in Q_0 and one edge (ta, ha) per elt $a \in Q_1$. ~~Note that~~

e.g:

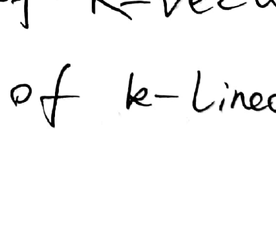


$Q_0 = \{1, 2, 3\}$ $Q_1 = \{a, b, c, d, e\}$
$h: (a, b, c, d, e) \mapsto (2, 2, 3, 1, 1)$
$t: (a, b, c, d, e) \mapsto (1, 2, 2, 3, 3)$

Q is called finite if both sets Q_0, Q_1 are finite.

If the graph of Q is connected, then we say Q is connected.

Def: A representation Q over a field k is a pair $\mathbb{V} = (V_i, V_a)$ consisting of a set of k-vector spaces $\{V_i \mid i \in Q_0\}$ together with a set of k-linear maps $\{V_a : V_{ta} \to V_{ha} \mid a \in Q_1\}$.
Note that the vector spaces and linear maps in $\mathbb{V}$ can be arbitrary, if $a, b, c \in Q_1$ s.t. $tc = ta$, $hc = hb$, $ha = tb$ but $V_c \neq V_b \circ V_a$.



$\mathbb{V}$ is finite-dim if the sum of the dims of its constituent spaces V_i is finite.

Def: $\mathbb{W} = (W_i, w_a)$ is a <u>subrep</u> of $\mathbb{V} = (V_i, v_a)$ if W_i is a subspace of V_i, $\forall i \in Q_0$ and $w_a = v_a|_{W_{ta}}$, $\forall a \in Q_1$.

Def: A morphism ϕ between two k-reps. $\mathbb{V}, \mathbb{W}$ of Q is a set of k-linear maps $\phi_i : V_i \to W_i$ st.

$$\begin{tikzcd} V_{ta} \arrow[r, "v_a"] \arrow[d, "\phi_{ta}"'] \arrow[dr, phantom, "\circlearrowleft"] & V_{ha} \arrow[d, "\phi_{ha}"] \\ W_{ta} \arrow[r, "w_a"] & W_{ha} \end{tikzcd} \qquad \forall a \in Q_1$$

we say $\mathbb{V} \cong \mathbb{W}$ if $\forall \phi_i$ is bijective.

If $\phi : \mathbb{U} \to \mathbb{V}$, $\psi : \mathbb{V} \to \mathbb{W}$

then $\psi \circ \phi : \mathbb{U} \to \mathbb{W}$, defined by $(\psi \circ \phi)_i = \psi_i \circ \phi_i$ $\forall i \in Q_0$. And for each $\mathbb{V}$, $1_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}$ defined by $(1_{\mathbb{V}})_i = 1_{V_i}$ $\forall i \in Q_0$.

$\Rightarrow$ the <u>rep</u> of Q is a category, called $\mathrm{Rep}_k(Q)$

The subcategory of finite-dim <u>rep</u> is called $\mathrm{rep}_k(Q)$

Both categories are abelian.

trivial rep : $(0, 0)$ all spaces and all maps are 0

the direct sum $\mathbb{V} \oplus \mathbb{W}$: $V_i \oplus W_i$ $\forall i \in Q_0$

$$v_a \oplus w_a = \begin{pmatrix} v_a & 0 \\ 0 & w_a \end{pmatrix} \quad \forall a \in Q_1. \text{ defined by pointwise}$$

$\mathbb{V}$ is decomposable $\Leftrightarrow \exists\, \mathbb{U}, \mathbb{W} \neq 0$ st. $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$.

Thm: any ptwise finite-dim rep of $\mathbb{Z}$ over k is a direct sum of interval reps.

$$\mathbb{Z}: \cdots \longrightarrow \underset{-2}{\bullet} \longrightarrow \underset{-1}{\bullet} \longrightarrow \underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots$$

rep $\mathbb{V}$ of $\mathbb{Z}$ is called ptwise finite-dim if each V_i has finite-dim.

Reps of poset $(T, \leq)$: let $T \subseteq \mathbb{R}$ be index set.

Regarding $(T, \leq)$ as a category: with one object per elt $i \in T$ and a single morphism per couple $i \leq j$.

$F: (T, \leq) \longrightarrow$ (vector spaces, linear maps) functor.

~~then~~ i.e. vector spaces: $(V_i)_{i\in T}$ linear maps $(V_i^j : V_i \to V_j)_{i \leq j \in T}$

with $V_i^i = 1_{V_i}$ $\forall i \in T$ and $V_i^k = V_j^k \circ V_i^j$, $i \leq j \leq k \in T$.

Thm: $T \subseteq \mathbb{R}$, any ptwise finite-dim rep of $(T, \leq)$ over k is a direct sum of interval reps.

(不仅仅是 A_n-type)

Def: Given $T \subseteq \mathbb{R}$, a persistence module over T is a rep of the poset $(T, \leq)$.

Def: Given $n \geq 1$, a zigzag module of n is a rep of an A_n-type quiver.

例子: A_n-types: $\underset{1}{\bullet} - \underset{2}{\bullet} \cdots - \underset{n-1}{\bullet} - \underset{n}{\bullet}$

$$Q = \underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xleftarrow{b} \underset{3}{\bullet} \xleftarrow{c} \underset{4}{\bullet} \xrightarrow{d} \underset{5}{\bullet}$$

then we can have two reps of Q:

(1) $k \xrightarrow{\binom{1}{0}} k^2 \xleftarrow{\binom{1}{1}} k \xleftarrow{(0\ 1)} k^2 \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}} k^2$

$\mathbb{V} =$

(2) $k \xrightarrow{0} 0 \xleftarrow{0} k \xleftarrow{I} k \xrightarrow{0} 0$

$\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$, 其中 $\mathbb{U} = 0 \xrightarrow{0} 0 \xleftarrow{0} k \xleftarrow{1} k \xrightarrow{0} 0$

$\mathbb{W} = k \xrightarrow{0} 0 \xleftarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} 0$

Thm: let Q be a finite quiver, and k field. Then, every finite-dim rep $\mathbb{V}$ of Q over k decomposes as

$$\mathbb{V} = \mathbb{V}^1 \oplus \cdots \oplus \mathbb{V}^r$$ (finite gen (graded) module over PID structure thm)

where each $\mathbb{V}^i$ is indecomposable, and the decomposition is unique up to iso and permutation [Krull, Remak, ~~Schmid~~ Schmidt].

Thm: let Q be an A_n-type quiver, and let k be a field. Then every indecomposable finite-dim rep of Q over k is iso to some interval rep $\mathbb{I}_Q[b,d]$

$$\underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{[1,\, b-1]} \xrightarrow{0} \underbrace{k \xrightarrow{1} \cdots \xrightarrow{1} k}_{[b,\, d]} \xrightarrow{0} \underbrace{0 \xrightarrow{0} 0 \cdots \xrightarrow{0} 0}_{[d+1,\, n]}$$

$\exists !$ up to

可以推广至无穷A_n-type, 可直和无限个.

Artin algebra

persistence:

Def: Given a subset $T \subseteq \mathbb{R}$, a filtration $\mathbb{X}$ over T is a family of Top. spaces X_i, s.t. $X_i \subseteq X_j$ whenever $i \leq j \in T$.

$\mathbb{X}$ is a special type of rep. of the poset $(T, \leq)$ in the category of Top spaces.

$\mathbb{X}$ is called (finitely) simplicial if the spaces X_i are (finite) simplicial complexes and X_i is a subcomplex of X_j whenever $i \leq j$, $i \leq j$.

~~when~~ if $\bigcap_{i \in T} X_i = \phi$, $\bigcup_{i \in T} X_i = X$, then we say $\mathbb{X}$ is called a filtration of X.

$$\phi = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n$$

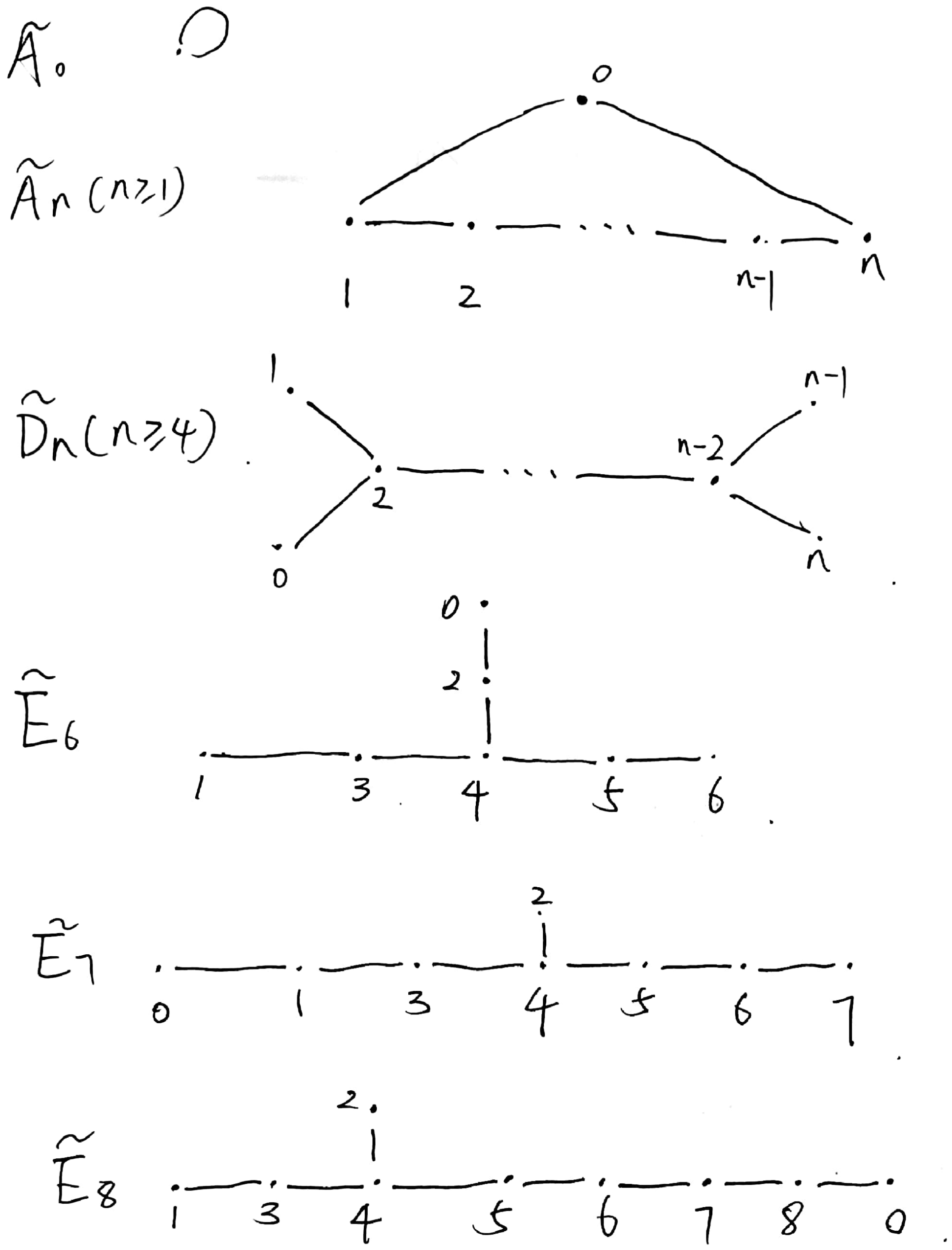
$\downarrow$ homology functor.

$$H_p(X_0) \to H_p(X_1) \subseteq \cdots \subseteq H_p(X_n)$$

induced by the inclusion. $X_i \hookrightarrow X_j$.

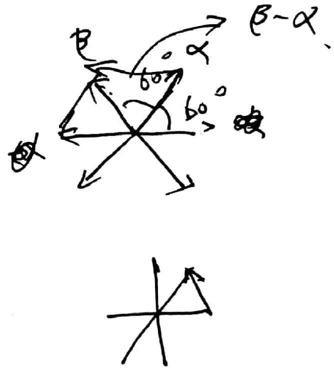
$$f_p^{i,j}: H_p(X_i) \to H_p(X_j),$$

the p-th persistent homology gp. of $\mathbb{X}$.



$\beta-\alpha \overset{?}{\in} \phi$

A_2.



A.3: The Euclidean diagrams.

root system: A root system ϕ consists of finitely many nonzero vectors $\{v_i\}_{i=1}^n$ is a set which satisfy.

(1) These roots $\{v_i\}_{i=1}^n$ can generate $\mathbb{R}^m$.

(2). If $\alpha\in\phi$ then only $\alpha, -\alpha\in\phi$ according to all scale $k\alpha$. i.e. $2\alpha\notin\phi$

(3). $\forall\, \alpha,\beta\in\phi$, we have $\beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha\in\phi$ and

$$2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}.$$

zigzag 指的是：$V_1 \longrightarrow V_2 \longleftarrow V_3 \longrightarrow \cdots\cdots$

箭头来回交错

（等价定义）

Def: Given $n \geq 1$, a zigzag module of length n is a rep of the poset $(\{1, \dots, n\}, \preceq)$ where $\preceq$ is any partial order relation whose Hasse diagram is of type A_n

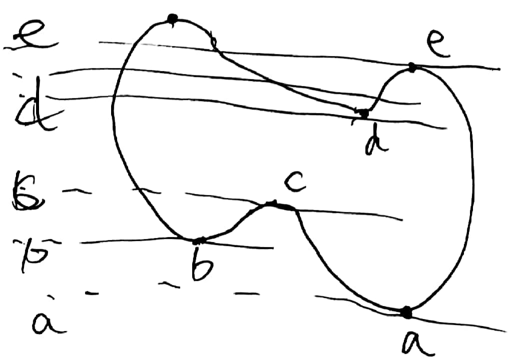
Thm（区间分解定理）A rep $\mathbb{V}$ of the poset $(T, \leq)$ can be decomposed as a direct sum of interval modules in each of the following situations:

(i) T is finite and the $\boxed{\text{Hasse diagram}}$ of $\preceq$ is of A_n-type.

(ii) $\preceq$ 就是 $\leq$ (natural order $\leq$) $\mathbb{V}$ is ptwise fin-dim

$$\mathbb{V} \cong \bigoplus_{j \in J} \mathbb{I}[b_j^{\pm}, d_j^{\pm}]$$

e.g.:



$$H_0(F) \cong \mathbb{I}[a, +\infty) \oplus \mathbb{I}[b, c) \oplus \mathbb{I}[d, e)$$

出现的时间

The Hasse diagram is defined as the graph having $\{1, \dots, n\}$ as vertex set and one edge $i \longrightarrow j$ per couple $i < j$ s.t. $\nexists k$ with $i < k < j$

Classification of quiver reps:

Def: let the vertex set of Q be $Q_0 = \{1, \dots, n\}$. Given a rep $\mathbb{V} \in \mathrm{rep}_k(Q)$, we defined its dim vector.

$$\underline{\dim}\, \mathbb{V} = (\dim V_1, \dots, \dim V_n)^T$$

$$\dim \mathbb{V} = \| \underline{\dim}\, \mathbb{V} \|_1 = \sum_{i=1}^{n} \dim V_i$$

Thm: If Q is finite, $\forall\, \mathbb{V} \in \mathrm{rep}_k(Q)$. $\exists$ indecomposable $\mathbb{V}_1, \dots, \mathbb{V}_r$ s.t. $\mathbb{V} \cong \mathbb{V}_1 \oplus \dots \oplus \mathbb{V}_r$

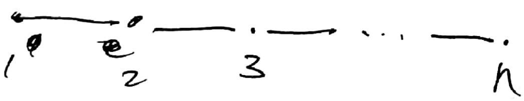
$\exists!$ up to iso and permutation

(the quivers that have a finite number of iso classes of indecomposable finite-dim reps)

Thm: (Gabriel) let Q be a fin connected quiver and let k be a field, Then, Q is of finite type.

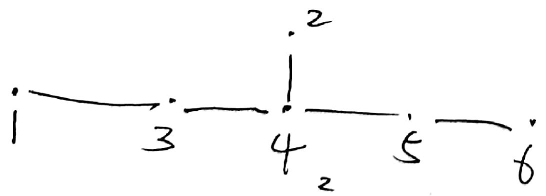
$\Leftrightarrow$ Q is Dynkin:

A_n $(n \geq 1)$

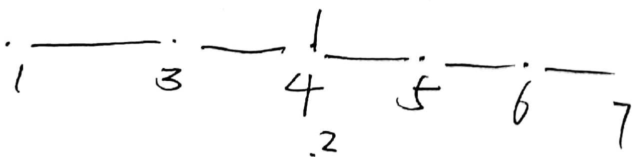


D_n $(n \geq 4)$

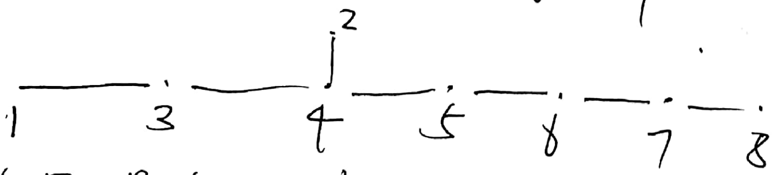
E_6



E_7



E_8



simple
简单 lie gp 分类
也用这个

(还没有画完，还有 B_n, C_n, F_4, G_2.)

Def: A vector in $\mathbb{Z}^n$ is called positive if it belongs to $\mathbb{N}^n - \{0\}$. (The $\underline{\dim}$ vectors of nontrivial reps of Q are such vectors) $\underline{\dim}\,\mathbb{V}$ ($x = \underline{\dim}\,\mathbb{V}_1$, $y = \underline{\dim}\,\mathbb{V}_2$)

$j+1 \xleftarrow{a} i$, $t_a = i$, $h_a = j+1$

Def: The Euler form of Q is the bilinear form $\langle -, - \rangle_Q$: $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ defined by $\langle x, y \rangle_Q = \sum_{i \in Q_0} x_i y_i - \underline{\sum_{a \in Q_1} x_{t_a} y_{h_a}}$

$x_i \; x_{j+1}$

Its symmetrization $(x, y)_Q = \langle x, y \rangle_Q + \langle y, x \rangle_Q$ is called the symmetric Euler form.

Treating elts in $\mathbb{Z}^n$ as col vectors, we can rewrite the symmetric Euler form as follows: $(x, y)_Q = x^T C_Q y$.

where $C_Q = (c_{ij})_{i,j \in Q_0}$, $C_Q^T = C_Q$

$$c_{ij} = \begin{cases} 2 - 2|\{\text{loops at } i\}|, & i = j \\ -|\{\text{arrows between } i \text{ and } j\}|, & i \neq j \end{cases}$$

Def: The Tits form is the quadratic form q_Q associated with the Euler form: $q_Q(x) = \langle x, x \rangle_Q = \frac{1}{2}(x, x)_Q = \frac{1}{2} x^T C_Q x$

(neither symmetric Euler form nor Tits form depend on the orientations of the arrows in Q)

Thm: A finite connected quiver Q is Dynkin iff its Tits form q_Q is positive definite. ($q_Q(x) > 0$ for any nonzero vector $x \in \mathbb{Z}^n$)

Pf: "only if" $\Rightarrow$: consider a quiver Q of type A_n. (For instance)

$$q_Q(X) = \sum_{i\in Q_0} X_i^2 - \sum_{a\in Q_1} X_{ta}X_{ha} = \sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n-1} X_i X_{i+1}$$

$$= \sum_{i=1}^{n-1} \frac{1}{2}\cdot(X_i - X_{i+1})^2 + \frac{1}{2}X_1^2 + \frac{1}{2}X_n^2 \geq 0 .$$

"$=$". iff. $X_1 = \cdots = X_n = 0$.

"$\Leftarrow$". assuming Q is Not Dynkin, we find a nonzero vector $X \in \mathbb{Z}^n$ st. $q_Q(x) \leq 0$. The key obs. is that the underlying undirected graph $\bar{Q}$ must contain one of the diagrams of A.3 as a subgraph.

Now for every such diagram, letting Q' be the corresponding subquiver of Q, it's easy to check by inspection the existence of a nonzero vector $X' \in \mathbb{Z}^{|Q_0'|}$ st. $q_{Q'}(X') = 0$ (take for instance $X = (1,\ldots,1)^T$ when $\bar{Q'} = \hat{A}_n$)

Then one of the following 3 scenarios occurs:

(1) $Q' = Q$. in which case letting $X = X'$ gives $q_Q(X) = q_{Q'}(X') = 0$ and so q_Q is Not positive definite

(2) $Q_0' = Q_0$ and $Q_1' \subsetneq Q_1$, in which case let $X = X'$ gives $q_Q(x) < q_{Q'}(X') = 0$ by Def of the Tits form, so q_Q is indefinite.

(3), $Q_0' \subsetneq Q_0$, in which case let i be a vertex of $Q \setminus Q'$ that is connected to Q' by an edge a (this vertex exists since Q is connected), and let $X = 2X' + b_i$ where b_i is the i-th basis vector in $\mathbb{Z}^n$. This gives $q_Q(X) \leq 4q_{Q'}(X') + X_i^2 - X_{ta}X_{ha}$ $= 0 + 1 - 2 < 0$ $\Rightarrow$ q_Q is indefinite. $\square$

Reflections:

Let Q be a finite connected quiver. We call a vertex $i \in Q_0$ a sink if all arrows incident to i are incoming, that is, if there is no arrow $a \in Q_1$ s.t. $ta = i$.

We call i a source if all arrows incident to i are outgoing, ie. $\nexists\, a \in Q_1$ s.t. $ha = i$.

Ex: $\underset{1}{\cdot} \longrightarrow \underset{2}{\cdot} \longleftarrow \underset{3}{\cdot} \longleftarrow \underset{4}{\cdot} \longrightarrow \underset{5}{\cdot}$ sources: 1, 4. sinks: 2, 5.

The corresponding reflections give the following quivers:

$s_1 Q$: $\underset{1}{\cdot} \longleftarrow \underset{2}{\cdot} \longleftarrow \underset{3}{\cdot} \longleftarrow \underset{4}{\cdot} \longrightarrow \underset{5}{\cdot}$.

$s_2 Q$: $\underset{1}{\cdot} \longleftarrow \underset{2}{\cdot} \longrightarrow \underset{3}{\cdot} \longleftarrow \underset{4}{\cdot} \longrightarrow \underset{5}{\cdot}$.

$s_4 Q$ = $\underset{1}{\cdot} \longrightarrow \underset{2}{\cdot} \longleftarrow \underset{3}{\cdot} \longrightarrow \underset{4}{\cdot} \longleftarrow \underset{5}{\cdot}$.

$s_5 Q$: $\underset{1}{\cdot} \longrightarrow \underset{2}{\cdot} \longleftarrow \underset{3}{\cdot} \longleftarrow \underset{4}{\cdot} \longleftarrow \underset{5}{\cdot}$

Reflection functors: let $i \in Q_0$ be a sink. For every rep $\mathbb{V} = (V_i, v_a) \in \mathrm{rep}_k(Q)$, we define a rep. $R_i^+ \mathbb{V} = (V_i', v_a') \in \mathrm{rep}_k(s_i Q)$ as follows. For all $j \neq i$, set $V_j' = V_j$ and define V_i' to be the kernel of the map

$$\xi_i : \bigoplus_{a \in Q_1^i} V_{ta} \longrightarrow V_i \qquad (x_{ta})_{a \in Q_1^i} \longmapsto \sum_{a \in Q_1^i} v_a(x_{ta})$$

where $Q_1^i = \{a \in Q_1 \mid ha = i\}$.

Define now linear maps between the spaces V_j' as follows. For each arrow $a \in Q_1$, if $a \in Q_1^i$ then set $V_a' = V_a$

if $a \in Q_1^i$ then let b denote the reverse arrow, and set V_b' to be the composition

$$V_{tb}' = V_i' = \ker \xi_i \hookrightarrow \bigoplus_{c \in Q_1^i} V_{tc} \longrightarrow V_{ta} = V_{ta}' = V_{hb}'$$

Given now a morphism $\phi : \mathbb{V} \longrightarrow \mathbb{W}$ between two reps $\mathbb{V}, \mathbb{W} \in rep_k(Q)$

the morphism $\phi' = R_i^+ \phi : R_i^+ \mathbb{V} \rightarrow R_i^+ \mathbb{W}$ is defined by $\phi_j' = \phi_j$, $\forall j \neq i$. and by ϕ_i' being the restriction of the map $\bigoplus_{a \in Q_1^i} \phi_{ta} : \bigoplus_{a \in Q_1^i} V_{ta} \longrightarrow \bigoplus_{a \in Q_1^i} W_{ta}$.

to $V_i' = \ker \xi_i$. It can be readily checked that these mappings define a functor from the category $rep_k(Q)$ to $rep_k(S_i Q)$.

Dually, given a source vertex $i \in Q_0$, to every rep. $\mathbb{V} = (V_i, v_a) \in \mathrm{rep}_k(Q)$, we associate a rep. $R_i^- \mathbb{V} = (V_i', v_a')$ as follows.

For all $j \neq i$, let $V_j' = V_j$, and define V_i' as the cokernel of the map

$$\eta_i : V_i \longrightarrow \bigoplus_{a \in Q_1^i} V_{ha}, \quad x \longmapsto (v_a(x))_{a \in Q_1^i}$$

where $Q_1^i = \{a \in Q_1 \mid ta = i\}$.

Define now linear maps between the spaces V_j' as follows. For each arrow $a \in Q_1$, if $a \notin Q_1^i$ then set $v_a' = v_a$; if $a \in Q_1^i$, then let b denote the reverse arrow, and set v_b' to be the composition

$$V_{tb}' = V_{ha}' = V_{ha} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{hc} \longrightarrow \operatorname{coker} \eta_i = V_i' = V_{hb}'$$

Finally, given a morphism $\phi : \mathbb{V} \to \mathbb{W}$ between two reps $\mathbb{V}, \mathbb{W} \in \mathrm{rep}_k(Q)$, the morphism $\phi' = R_i^- \phi : R_i^- \mathbb{V} \to R_i^- \mathbb{W}$ is defined by $\phi_j' = \phi_j$, $j \neq i$, and by ϕ_i' being the map induced by

$$\bigoplus_{a \in Q_1^i} \phi_{ha} : \bigoplus_{a \in Q_1^i} V_{ha} \longrightarrow \bigoplus_{a \in Q_1^i} W_{ha}$$

on the quotient space $V_i' = \operatorname{coker} \eta_i$, these mappings define a functor from the category $\mathrm{rep}_k(Q)$ to $\mathrm{rep}_k(s_i Q)$.