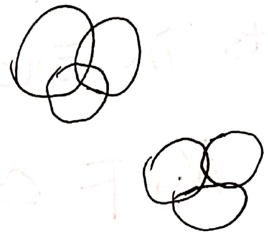
Convex set systems

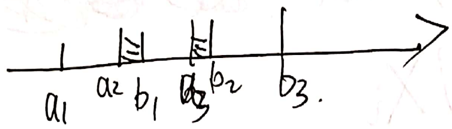
Helly's Thm: let F be a finite collection of closed, convex sets in $\mathbb{R}^d$, then every $d+1$ of the sets have a non-empty common intersection iff they all have a non-empty common intersection.

$\bigcap_{i=1}^{n} X_i \neq \phi \Leftrightarrow \underline{\boxed{\bigcap_{i=1}^{d+1}}}$

$F = \bigcup_{i=1}^{n} X_i$



$d=1$.



每2个区间交非空 $\Leftrightarrow$ 所有的交非空 ✓

Homotopy type: $X \xrightarrow{f,g} Y$ continuous.

we say $f \simeq g$ if $\exists$ continuous $F(x,t) : X \times [0,1] \to Y$

s.t. $F(x,0) = f$, $F(x,1) = g$.

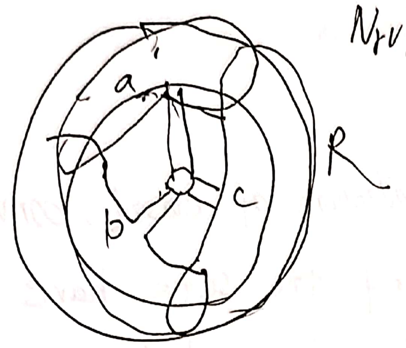
we say $X \simeq Y$ if $\exists$ $f: X \to Y$, $g: Y \to X$

s.t. $g \circ f \simeq id_X$ $f \circ g \simeq id_Y$

Nerves: $Nrv F = \{X \subseteq F \mid \bigcap_{A_i \in X} A_i \neq \phi\}$

all non-empty subcollections of F whose sets have a nonempty common intersection.

e.g.:



$Nrv F = \{a, b, c, R, \{ab\}, \{bc\}, \{ac\}, \{aR\}, \{bR\}, \{cR\}, \{acR\}, \{abR\}, \{bcR\}\}$

$\simeq S^2$

Nerve Thm: let F be a finite collection of closed, convex sets in Euclidean space. Then ~~the~~

$$Nrv F \simeq \bigcup X_i.$$

Čech complexes: $\check{C}ech(r) := \{\sigma \subseteq S \mid \bigcap_{x \in \sigma} B_x(r) \neq \emptyset\}$

where S is a finite set of pts in $\mathbb{R}^d$ and $B_x(r)$ is the closed ball with center x and radius r.

prop: a set of balls has a non-empty intersection iff their centers lie inside a common ball of the same radius.

A pt y belongs to all balls iff $\|x-y\| < r$ $\forall$ center x

Jung's Thm: every $d+1$ pts in S are contained in a common ball of radius r iff all pts in S are.

let $\sigma \subseteq S$ be a subset of the given pts. $\sigma \in \check{C}ech(r)$

Let miniball of σ be the smallest closed ball that contains σ.

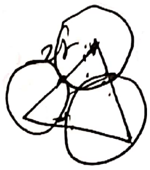
prop: the radius of the miniball is smaller or equal to r.

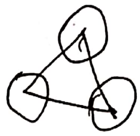
$\Leftrightarrow \quad \sigma \in \check{C}ech(r)$.

Vietoris-Rips complexes: Vietoris-Rips $(r) = \{\sigma \subseteq S \,/\, \operatorname{diam}\sigma \le 2$

here, we consider. σ as a realization ie $\sigma = \{u_0, u_1, u_2\}$

$\operatorname{diam} \sigma = \max\limits_{i,j} \{u_i, u_j\}$.





lem: $\text{Vie}(r) \subseteq \check{C}ech(\sqrt{2}r)$

$\check{C}ech(r) \subseteq \text{Vie}.(r)$.

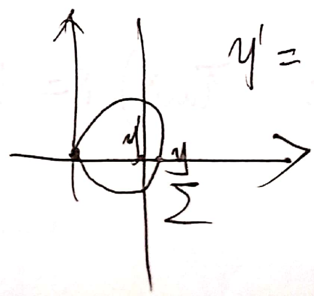
Delaunay complexes

Inversion: $\ell(x) = \dfrac{x}{\|x\|^2}$. $\ell: \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$.

$\ell^2(x) = x$. 该映射将球面 S^d 内外的点. 调换.

而球面上的点. 还会映到球面上.

Thm: let Σ be a d-sphere in $\mathbb{R}^{d+1}$. If $0 \notin \Sigma$, then $\ell(\Sigma)$ is a d-sphere. if $0 \in \Sigma$ then. $\ell(\Sigma)$ is a d-plane.



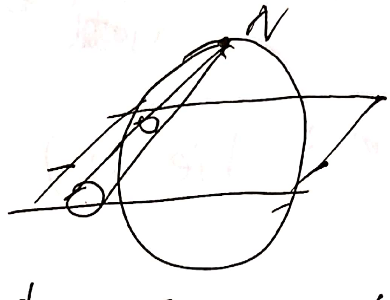
$y' = \ell(y)$.

stereographic proj : for any center $z \in \mathbb{R}^{d+1}$, radius $r>0$,

define ~~$\phi(x)_{z,r}$~~ $\phi_{z,r}(x) = r\,\ell\left(\frac{x-z}{r}\right) + z$

easy to check $\|x-z\|\,\|\phi(x)-z\| = r^2$.

$\zeta(x) : S^d - \{N\} \longrightarrow \mathbb{R}^d$. $\zeta(x) := \phi_{N,\sqrt{2}}(x)$.



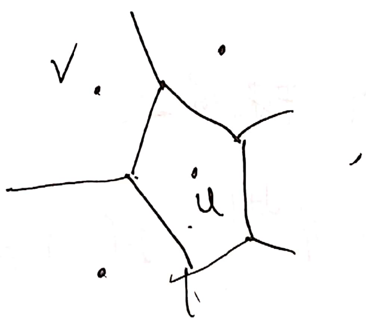
lem : let Σ' be a $(d-1)$-sphere on S^d. If $N \notin \Sigma'$

then $\zeta(\Sigma')$ is a $(d-1)$-sphere. if $N \in \Sigma'$

then $\zeta(\Sigma')$ is a $(d-1)$-plane

Voronoi diagram : $V_u := \{x \in \mathbb{R}^d \mid \|x-u\| \le \|x-v\|, \forall v \in S\}$

$u, v \in S$, 与u距离最近的那些点.



weighted ~~squared~~ Voronoi : $V_u = \{x \in \mathbb{R}^d \mid \pi_u(x) \le \pi_v(x), \forall v$

where $\pi_u(x) := \|x-u\|^2 - w_u$

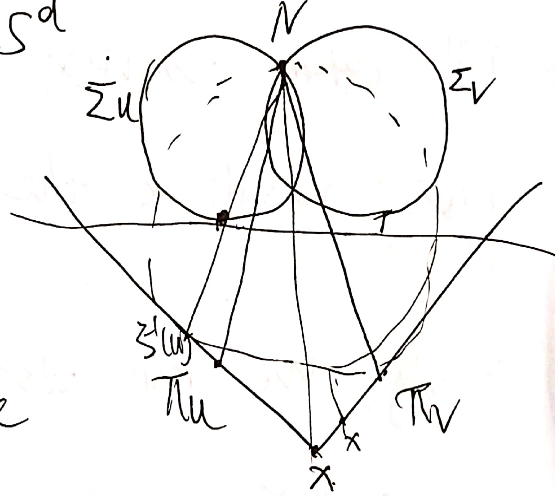
权重

Lifting: we give a different view of the Voronoi diagram by lifting its cells to one higher dim.

let S be a fin. set in $\mathbb{R}^d$, $S \hookrightarrow \mathbb{R}^d$. let $u \in S$.

And Π_u be the d-plane tangent to S^d touching the sphere in the pt $\xi^{-1}(u)$. Using inversion, we map each d-plane Π_u to the d-sphere $\Sigma_u = \varphi_{N,\sqrt{2}}(\Pi_u)$

It passes through the north pole and is tangent to $\mathbb{R}^d$, the preimage of S^d



lem: ① A pt $X \in \mathbb{R}^d$ belongs to the Voronoi cell of $u \in S$ iff the first intersection of the directed line segment from X to N with the d-spheres defined by the pts in S is with Σ_u

一个点 $X \in \mathbb{R}^d$ 属于关于 u 的 Voronoi 胞腔中，iff 射线 XN 首次与 Σ_u 相交（在所有的 d 维球面 Σ 中）

② A pt $X \in \mathbb{R}^d$ belongs to the Voronoi cell of $u \in S$ iff the first inter of the directed line segment from N to X with the d-planes defined by the pts in S is with Π_u.

$\mathrm{Delaunay} := \{ \sigma \subseteq S \mid \bigcap_{u \in \sigma} \bar{V}_u \neq \phi \}$ Delaunay complex

We say the set S is in general position if no d+2 of the pts lie on a comon (d-1)-sphere.

$\Rightarrow$ No d+2 Voronoi cells have a non-empty common intersection $\Rightarrow$ dim Delaunay = d.

Assuming general position, we get a geometric realization by taking convex hulls of abstract simplices. The result is the Delaunay triangulation of S.

Finally, we can also introduce the weighted Delaunay complex, which depends on the Def of $\bar{V}_u$.

Alpha complex: $S \subseteq \mathbb{R}^d$ be a finite pts set $u \in S$, $r \geq 0$.

$R_u(r) := B_u(r) \cap \bar{V}_u$.

$$\mathrm{Alpha}(r) := \{ \sigma \subseteq S \mid \bigcap_{u \in \sigma} R_u(r) \neq \phi \}.$$

Since $R_u(r) \subseteq V_u \Rightarrow \mathrm{Alpha}(r) \subseteq \mathrm{Delaunay}(r)$

For S in general position, we can get a geometric realization by taking convex hulls of abstract simplices.

Furthermore, $R_u(r) \subseteq B_u(r) \Rightarrow \text{Alpha}(r) \subseteq \check{C}ech(r)$.

Since $R_u(r)$ are closed and convex and together they cover the union, By The Nerve Thm implies the union of balls and $\text{Alpha}(r)$

$$\bigcup_{u \in S} B_u(r) \simeq |\text{Alpha}(r)|.$$

Weighted alpha complex: $R_u = B_u \cap \bar{V}_u$.

$r_u = \sqrt{w_u}$ is the radius of B_u.

and $\bar{V}_u$ is the weighted voronoi cells.

Define $\text{Alpha}'(r) = \{ \sigma \subseteq S \mid \bigcap_{u \in \sigma} R_u \neq \emptyset \}$.

Filtration: Given a finite set $S \subseteq \mathbb{R}^d$, we continuly increase the radius and thus we get a 1-parameter family of nested unions.

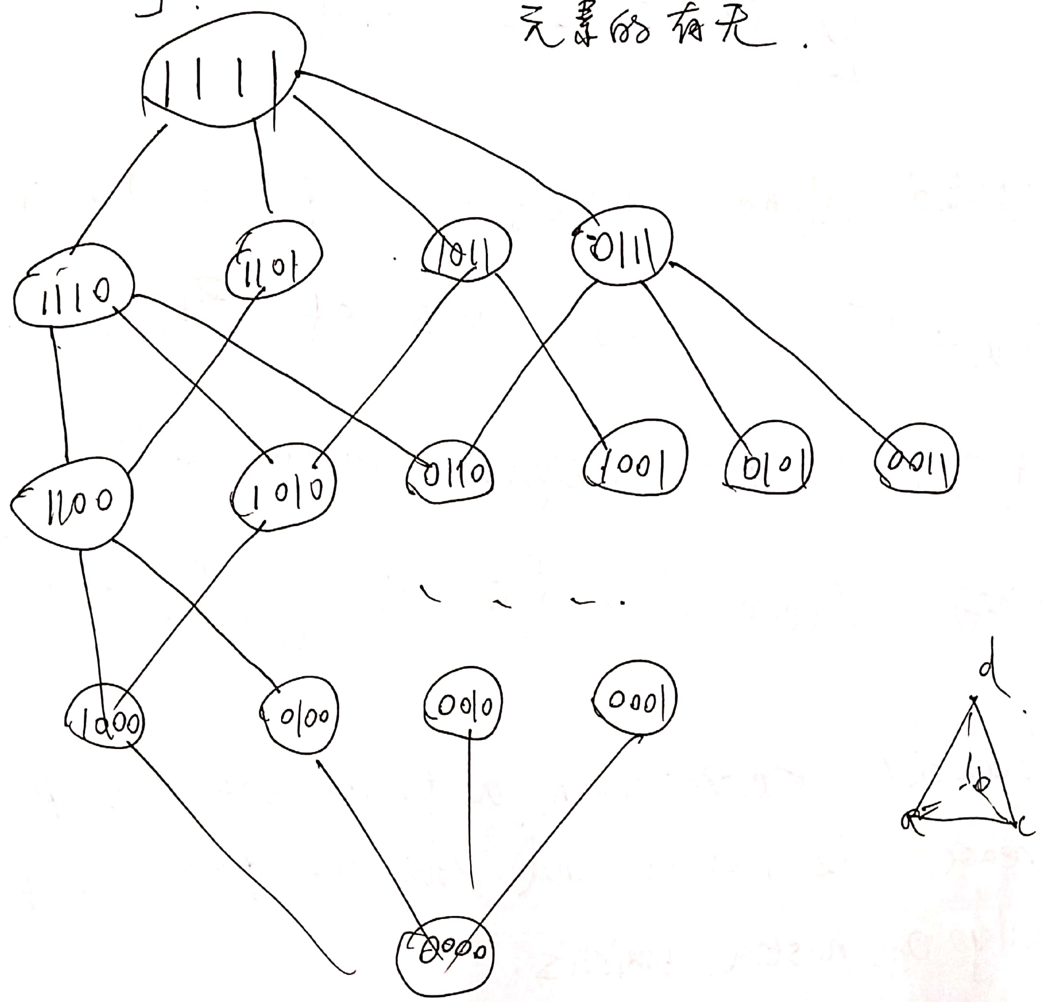
because they're all subcomplexes of a Delaunay complex, which is finite, only finitly many of them are distinct

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = \text{Delaunay}$$

The structure of a simplex: $\boxed{\text{We're interested in the difference between two contiguous complexes in the filtration. } K_{i+1} - K_i}$

Def: For a partially ordered set S, we can draw each elt of S onto a plane. And if $x < y$, and $\nexists\, z$ s.t. $x < z < y$, then we take a ~~line~~ edge to join x and y.

e.g: $S = \{\overset{a}{0}, \overset{b}{1}, \overset{c}{2}, \overset{d}{3}\}$. 用{0,1}编码所有的幂集 {1,0}表示元素的有无.



Hasse diagram.