Evolutionary de Rham-Hodge method

Main idea: Analysis the differential operator on a filtration of manifold to get ~~the~~ topology and geometry information.

eigenvalue Laplacian Betti number Fiedler number

Assumption: M is 3-manifold (with boundary) embedded in $\mathbb{R}^3$.

§ De Rham Complex

Recall: differential form $w^k \in \Omega^k(M)$ ($dx, dy, dy\wedge dz \dots$)

differential operator $d^k: \Omega^k(M) \to \Omega^{k+1}(M)$

function $f \mapsto \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$.

1-form $w = Pdx + Qdy + Rdz \mapsto dP\wedge dx + dQ\wedge dy + dR\wedge dz$.

Thm (Stokes) d^k is diff. op. $S \subset M$ is $(k+1)$-sub mfd. $w^k \in \Omega^k(M)$

then $\int_S d^k w^k = \int_{\partial S} w^k$

Prop: $d^k d^{k+1} = 0$.

Note: By this property, we have de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \Omega^2(M) \xrightarrow{d^2} \Omega^3(M) \xrightarrow{d^3} 0$$

Ie induces de Rham cohomology $H^k_{dR}(M) = \ker d^k / \operatorname{im} d^{k-1}$.

Define the Hodge $*$ operator $*^k: \Omega^k(M) \to \Omega^{n-k}(M)$

Explicitly, for an orthonormal basis $(e_1, \dots, e_n)$,

$$*(e_{i_1}\wedge\cdots\wedge e_{i_k}) = e_{j_1}\wedge\cdots\wedge e_{j_{n-k}}$$

where $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\}$ is even permutation of $\{1, \dots, n\}$.

Now we can define the inner product of diff. forms:

$\alpha, \beta \in \Omega^k(M)$ $\quad \langle \alpha, \beta\rangle \triangleq \int_M \alpha\wedge *\beta = \int_M \beta\wedge *\alpha$.

Define adjoint operator of d (codifferential operator):

$\delta^k: \Omega^k(M) \to \Omega^{k-1}(M)$ $\quad \delta^k \triangleq (-1)^k *^{4-k} d^{3-k} *^k \quad k = 1, 2, 3$

Remark: With δ^k, we have

$$\Omega^0(M) \underset{\delta^1}{\overset{d^0}{\rightleftarrows}} \Omega^1(M) \underset{\delta^2}{\overset{d^1}{\rightleftarrows}} \Omega^2(M) \underset{\delta^3}{\overset{d^2}{\rightleftarrows}} \Omega^3(M)$$

§ Hodge decomposition for manifolds

Hodge theory can be seen as the study of non-integral parts (cohomology) of scalar/vector field through the analysis of diff. ops.

Prop: $\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle + \int_{\partial M} \alpha\wedge *\beta$

Proof: Suppose $\alpha\in\Omega^{k-1}(M)$ $\beta\in\Omega^k(M)$

$d(\alpha\wedge *\beta) = d\alpha\wedge *\beta + (-1)^{k-1}\alpha\wedge d*\beta$

Integrate on both side $\Rightarrow \int_M d(\alpha\wedge *\beta) = \int_M d\alpha\wedge *\beta + (-1)^{k-1}\int_M \alpha\wedge d*\beta$

Stokes' thm $\Rightarrow \int_{\partial M} \alpha\wedge *\beta = \langle d\alpha, \beta\rangle + (-1)^{k-1}\int_M \alpha\wedge d*\beta$.

Note that $\langle\alpha, \delta\beta\rangle = \int_M \alpha\wedge *\delta\beta$

$= \int_M \alpha\wedge (-1)^k **d*\beta$

$**(d*\beta) = d*\beta$ by definition of $*$ (even permutation)

$\Rightarrow \langle\alpha, \delta\beta\rangle = (-1)^k\int_M \alpha\wedge d*\beta = -(-1)^{k-1}\int_M \alpha\wedge d*\beta$ $\square$.

We expect $\int_{\partial M}\alpha\wedge *\beta = 0$. In this case, we will have $\langle d\alpha, \beta\rangle = \langle\alpha, \delta\beta\rangle$ which implies d, δ are adjoint operators.

That is $\ker d = (\operatorname{im}\delta)^\perp$ $\ker\delta = (\operatorname{im} d)^\perp$.

Condition for $\int_{\partial M}\alpha\wedge *\beta = 0$:

① $\partial M = \phi$ (trivial)

② $\alpha|_{\partial M} = 0$ (normal)

③ $*\beta|_{\partial M} = 0$ (tangential)

Explanation :

1-form v^1 tangential if $*v^1(t_1, t_2) = v^2(t_1, t_2) = v(t_1\times t_2) = v\cdot n = 0$

$\rightsquigarrow$ v^1 correspond to vector field tangential to boundary

1-form v^1 normal if $v^1(t_i) = v^1\cdot t_i = 0$ $i=1,2$.

$\rightsquigarrow$ v^1 is normal to the boundary

2-form v^2 tangential if $*v^2(t_i) = v^1(t_i) = 0$ $i=1,2$.

$\rightsquigarrow$ v^2 is normal to the boundary

2-form v^2 normal if $v^2(t_1, t_2) = 0$

$\rightsquigarrow$ v^2 correspond to vector field tangential to boundary

Remark: One can see duality in previous explanation.

① $\partial M = \phi$.

$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle \Rightarrow \Omega^k = \ker \delta^k \oplus d\Omega^{k-1}$ ...... (1)

$\Omega^k = \ker d^k \oplus \delta\Omega^{k+1}$ ...... (2)

Since $d \circ d = 0$, $\operatorname{im} d^{k-1} \subset \ker d^k$. Restrict (1) on $\ker d^k$

$\Rightarrow \ker d^k = (\ker d^k \cap \ker \delta^k) \oplus d\Omega^{k-1} \triangleq \mathcal{H}^k \oplus d\Omega^{k-1}$ ...... (3)

$\mathcal{H}$ called the space of harmonic forms

Put in into (2), $\Omega^k = d\Omega^{k-1} \oplus \delta\Omega^{k+1} \oplus \mathcal{H}$ which is the Hodge decomposition.

Note: From (3) we have $\ker d^k = \mathcal{H}^k \oplus \operatorname{im} d^{k-1}$. Hence there is a bijection between H^k_{dR} and $\mathcal{H}$

Define the Hodge Laplacian $\Delta^k \triangleq d^{k-1}\delta^k + \delta^{k+1}d^k$.

We have $\langle \Delta\alpha, \alpha \rangle = \langle (d\delta + \delta d)\alpha, \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle$

$\Rightarrow \mathcal{H}^k_\Delta \triangleq \ker \Delta^k = \ker d^k \cap \ker \delta^k = \mathcal{H}^k$

Thus we can write $\Omega^k = \operatorname{im} d^{k-1} \oplus \operatorname{im} \delta^{k+1} \oplus \mathcal{H}^k_\Delta$.

② & ③. $\partial M \neq \phi$.

In this case, $\langle \Delta\alpha, \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle + \int_{\partial M} (\delta\alpha \wedge *\alpha - \alpha \wedge *d\alpha)$

We can only get $\mathcal{H} = \ker d \cap \ker \delta \subset \mathcal{H}_\Delta$

But we still want to get "Hodge decomposition"

A natural way: let $*d\alpha|_{\partial M} = 0$ when $*\alpha|_{\partial M} = 0$

$\delta\alpha|_{\partial M} = 0$ when $\alpha|_{\partial M} = 0$

Define $\Omega^k_t \triangleq \{\alpha \in \Omega^k(M) \mid *\alpha|_{\partial M} = 0 \quad *d\alpha|_{\partial M} = 0\}$.

$\Omega^k_n \triangleq \{\alpha \in \Omega^k(M) \mid \alpha|_{\partial M} = 0 \quad \delta\alpha|_{\partial M} = 0\}$.

With these modified tangential / normal forms, we obtain

$\Omega^k = d\Omega^{k-1}_n \oplus \delta\Omega^{k+1}_t \oplus \mathcal{H}^k$.

§ Discrete form and spectral analysis

Goal: discretize the operators d, δ, Δ

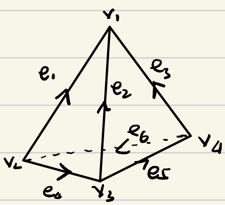
$\mathcal{M} \longrightarrow$ 3D simplicial complex.

k-form $w \longrightarrow$ its integral on oriented k-D elements (k-simplex)
listed as a vector W with length = # k-simplices

i.e. 0-form is assigning a real number per vertex
1 oriented edge
2 oriented triangle
3 oriented tetrahedron

$d^k \longrightarrow D^k$ matrix, transpose of the signed incidence matrix between k-simplices and (k+1)-simplices
sign determined by mutual orientation.

Example:



$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

The computation hold by Stokes' thm.

Remark. ① orientation of k-simplices can be chosen arbitrary, results only change sign.

② $D_k D_{k-1} = 0$ preserve.

Now we want to discrete δ^k.

Let S_k be discretization of L_2-inner product between two discrete k-forms s.t. $(W_1^k)^T S_k W_2^k$ is an approximation of $\langle w_1^k, w_2^k \rangle$

Then set $\delta^k \longrightarrow S_{k-1}^{-1} D_{k-1}^T S_k$.

We can discretize the Hodge Laplacian Δ^k.

$$\Delta^k = d^{k-1}\delta^k + \delta^{k+1} d^k \longrightarrow D_{k-1} S_{k-1}^{-1} D_{k-1}^T S_k + S_k^{-1} D_k^T S_{k+1} D_k$$

$$= S_k^{-1} (S_k D_{k-1} S_{k-1}^{-1} D_{k-1}^T S_k + D_k^T S_{k+1} D_k)$$

$$\triangleq S_k^{-1} L_k.$$

Remark: We are interested in L_k rather than $S_k^{-1} L_k$ since L_k has symmetry.

In the follow we want to analysis the eigenvalues of the operator L_k.

$$L_k W^k = \lambda^k S_k W^k. \quad (S_k^{-1} L_k W^k = \lambda^k W^k \rightsquigarrow \Delta w = \lambda w)$$

It can be written as $\underline{\bar{L}_k} \bar{W}^k = \lambda^k \bar{W}^k$, where $\bar{L}_k = S_k^{-\frac{1}{2}} L_k S_k^{-\frac{1}{2}}$

modified Hodge Laplacian. $\quad \bar{W}^k = S^{\frac{1}{2}} W^k$.

Hence $\bar{L}_k = \bar{D}_k^T \bar{D}_k + \bar{D}_{k-1} \bar{D}_{k-1}^T$ where $\bar{D}_k = S_{k+1}^{\frac{1}{2}} D_k S_k^{\frac{1}{2}}$

semi-positive definite matrix.

The adjoint operator of $\bar{D}_k$ is exactly $\bar{D}_k^T$.

$\Rightarrow$ The entire spectrum of $\bar{L}_k$ can be studied through the singular value decomposition of the discrete diff. op.

$$\bar{D}_k = U_{k+1} \Sigma_k V_k^T.$$

$$\Rightarrow \quad \bar{L}_k = V_k \Sigma_k^2 V_k^T + U_k \Sigma_{k-1}^2 U_k^T.$$

Hence non zero spectra of $\bar{L}_k$ $\longleftrightarrow$ Union of squares of the non zero entries from Σ_k and Σ_{k-1}.

By Hodge decomposition, entire k-form space is spanned by harmonic forms (eigen form with eigenvalue 0) and those column vectors of V_k and U_k.

For domains with boundaries, denote the discrete diff. op. for tangential (normal) k-forms as $D_{k,t}$ ($D_{k,n}$)

$$\Rightarrow \quad L_{k,t} = D_{k,t}^T S_{k+1} D_{k,t} + S_k D_{k-1,t} S_{k-1}^{-1} D_{k-1,t}^T S_k$$

$$L_{k,n} = D_{k,n}^T S_{k+1} D_{k,n} + S_k D_{k-1,n} S_{k-1}^{-1} D_{k-1,n}^T S_k$$

As before, we are interested in singular spectra of $\overline{D}_{0,\cdot}$ $\overline{D}_{1,\cdot}$ $\overline{D}_{2,\cdot}$ $\overline{D}_{3,\cdot}$
$\overline{D}_{3,\cdot}$ is trival. And by duality, there is equivalent between $\overline{D}_{0,t}$ and $\overline{D}_{2,n}^T$, $\overline{D}_{1,t}$ and $\overline{D}_{1,n}^T$ $\overline{D}_{2,t}$ and $\overline{D}_{0,n}^T$. We reduce 8 spectra for Hodge Laplacians to 3 distinct sets of different singular spectra.

Denote the set of singular values of $\overline{D}_{0,t}$ by T
$\overline{D}_{1,t}$ by C
$\overline{D}_{2,t}$ by N

Δ Evolutionary de Rham - Hodge method

Idea: equivalence classes in cohomology group $\xrightarrow{\text{Hodge theory}}$ harmonic diff. form in the null space of Hodge Laplacian.

zero eigenvalues $\longrightarrow$ Harmonic form, topology.

nonzero eigenvalue $\rightsquigarrow$ geometry.

Method: (I) Extend the study of de Rham-Hodge theory to a family of smooth mfd; (II) track the spectral changes in a sequence of mfds.

(I) Discrete data cloud $\longrightarrow$ continuous

$$\rho(r, \eta) = \sum_{j=1}^{N} \Phi(\|r - r_j\|; \eta)$$

Here r is point in space, N is # of particles, r_j is location of j-th particle. η is scalaring parameter. $\Phi(\cdot;\eta)$ is correlation function. Usually, $\Phi(\|r-r_j\|; \eta) = \exp\left(-(\|r-r_j\|/\eta)^k\right)$ $k>0$.

In our case $k=2$. Gaussian function.

Remark. ① $\Phi(\|r-r_j\|;\eta) = \begin{cases} 1 & \|r-r_j\| \to 0 \\ 0 & \|r-r_j\| \to \infty \end{cases}$

② Change η induce a multidimensional persistence homology.

We may assume $\rho(r,\eta)$ Morse function.

$M_c = \{r \mid \rho(r,\eta) > \max \rho(\cdot,\eta) - c\}$ sequence of manifold.

(II) Now we have $\mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \cdots$

and $\Omega^0(\mathcal{M}_i) \to \Omega^1(\mathcal{M}_i) \to \cdots$

Here

$$\begin{array}{ccccccc}
\Omega^0(\mathcal{M}_0) & \xrightarrow{d^0} & \Omega^1(\mathcal{M}_0) & \xrightarrow{d^1} & \Omega^2(\mathcal{M}_0) & \xrightarrow{d^2} & \Omega^3(\mathcal{M}_0) \\
\downarrow C_{0,1} & & \downarrow C_{0,1} & & \downarrow C_{0,1} & & \downarrow C_{0,1} \\
\Omega^0(\mathcal{M}_p) & \xrightarrow{d^0} & \Omega^1(\mathcal{M}_p) & \xrightarrow{d^1} & \Omega^2(\mathcal{M}_p) & \xrightarrow{d^2} & \Omega^3(\mathcal{M}_p) \\
\downarrow C_{1,1} & & \downarrow C_{1,1} & & \downarrow C_{1,1} & & \downarrow C_{1,1} \\
\cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

Question: ① What is $C_{\ell,p}$? i.e. form in $\mathcal{M}_\ell \to$ form in $\mathcal{M}_{\ell+p}$.

② Does the diagram commute?

① $w_{\ell+p} = C_{\ell,p}(w_\ell)$ smooth extension which is minimizing the Dirchlet energy $\langle dw, dw\rangle + \langle \delta w, \delta w\rangle$ in $\mathcal{M}_{\ell+p} \setminus \mathcal{M}_p$.

②. Commute when Ω_t^i or Ω_n^i

For $\Omega_n^\bullet$,

$$\begin{array}{ccccccc}
\Omega_n^0(\mathcal{M}_0) & \xrightarrow{d^0} & \Omega_n^1(\mathcal{M}_0) & \xrightarrow{d^1} & \Omega_n^2(\mathcal{M}_0) & \xrightarrow{d^2} & \Omega_n^3(\mathcal{M}_0) \\
\downarrow C_{0,1} & & \downarrow C_{0,1} & & \downarrow C_{0,1} & & \downarrow C_{0,1} \\
\Omega_n^0(\mathcal{M}_p) & \xrightarrow{d^0} & \Omega_n^1(\mathcal{M}_p) & \xrightarrow{d^1} & \Omega_n^2(\mathcal{M}_p) & \xrightarrow{d^2} & \Omega_n^3(\mathcal{M}_p) \\
\downarrow C_{1,1} & & \downarrow C_{1,1} & & \downarrow C_{1,1} & & \downarrow C_{1,1} \\
\cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

$w_n \in \ker d_\ell \Rightarrow C_{\ell,p}(w_n) \in \ker d_{\ell+p}$.

$\rightsquigarrow$ $\exists$ inj. homo. from $\ker d_\ell$ to $\ker d_{\ell+p}$.

$\longrightarrow$ homo from $\dfrac{\ker d_\ell^k}{\operatorname{im} d_\ell^{k-1}}$ to $\dfrac{\ker d_{\ell+p}^k}{\operatorname{im} d_{\ell+p}^{k-1}}$

$\xrightarrow{\text{de Rham}}$ homo between cohomology.

$\longrightarrow$ homo from $\mathcal{H}^k_{\Delta_n,\ell}$ to $\mathcal{H}^k_{\Delta_n,\ell+p}$

Obtain persistence of normal harmonic forms.

Tangential forms are similarly.

Recall: There are only three independent singular spectra T, C, N

let $\{\lambda^T_{\ell,i}\}$, $\{\lambda^N_{\ell,i}\}$, $\{\lambda^C_{\ell,i}\}$ be eigenvalues of T, C, N for the l-th manifold of the filtration.

Conclusion: multiplicities of the zero eigenvalues in $\lambda_{l,0}^{T}$, $\lambda_{l,0}^{C}$, $\lambda_{l,0}^{N}$ are associated with Betti numbers β_0, β_1, β_2.

$\lambda_{l,1}^{T}$, $\lambda_{l,2}^{N}$, $\lambda_{l,3}^{C}$ are the first non-zero eigenvalues, which are Fiedler values.

\# generators

harmonic $\longleftrightarrow$ cohomology class.

Note: One can refer "graph Laplacian".