

On the "Secondary power operations and the Brown-Peterson spectrum at the prime 2"

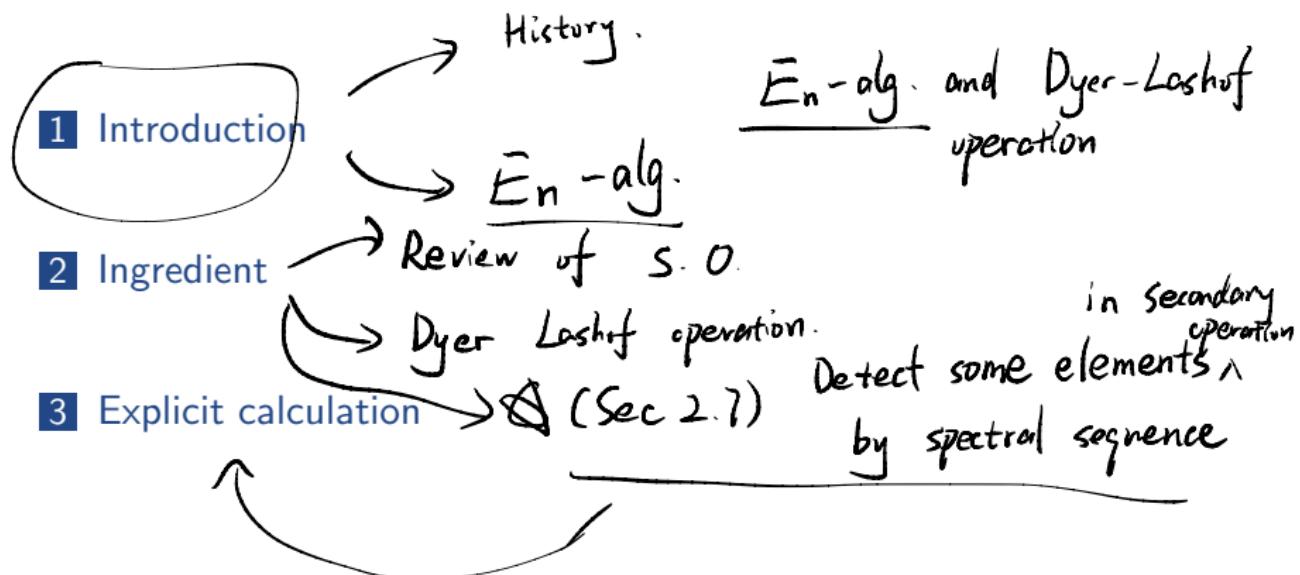
Zhonglin Wu

Southern Univ. of Science and Technology (SUSTech)

June 1st, 2024



Outline



Outline

1 Introduction

2 Ingredient

3 Explicit calculation



Cup- i products

Cup- i products encode the communications of the coherently commutative multiplication structures.

Definition

A cochain complex C_* has cup- i products if it is equipped with operations $(x, y) \rightarrow x \cup_i y$ for $i \geq 0$ such that

$$x \cup y \underset{(?)}{\stackrel{\cong}{\circlearrowleft}} y \cup x$$

\uparrow
cup-1 prod.

$$x \cup_i y \underset{(?)}{\stackrel{\cong}{\circlearrowleft}} y \cup_i x$$

\uparrow
cup- i prod.

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- if $x \in C^p$, $y \in C^q$, then $x \cup_i y \in C^{p+q-i}$

$$x \otimes y \rightarrow x \cup_i y$$

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- $(x + x') \smile_i y = x \smile_i y + x' \smile_i y$ and the same result is true for y .

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- $\delta(x \smile_0 y) = (\delta x) \smile_0 y + x \smile_0 (\delta y)$



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for $i \geq 0$,

$\delta(x \smile_i y) = (\delta x) \smile_i y + x \smile_i (\delta y)$

cup-0 prod

= cup prod

$x \smile_{i-1} y - y \smile_{i-1} x$.

$$\delta(x \smile_0 y) = (\delta x) \smile_0 y + x \smile_0 (\delta y)$$
$$\delta(x \smile_i y) = (\delta x) \smile_i y + x \smile_i (\delta y)$$
$$x \smile_{i-1} y - y \smile_{i-1} x$$

Some facts about cup- i products

① $D_0 : C_* \rightarrow C_* \times C_*$ (Any homology group).
 (cup product).

$$(v \cup v, v)(G) = (v \otimes v)(D_0 G)$$

$$D_{i-1} \Rightarrow D_i \quad \underline{D_i \alpha + \partial D_i} = D_{i-1} + e D_{i-1}$$

(e : switch two elements)

$$(u \cup v, v)(G) \cong (u \otimes v)(D_i G).$$

$T^2 :$

$$\alpha \cup \beta = (\pm)(\alpha + \beta)$$

$\alpha, \beta \in H_*(T^2)$.

$H_*(T^2)$

$$\alpha \xrightarrow{\Delta} \alpha \otimes \alpha.$$



Some facts about cup- i products

$$\textcircled{1} \quad Sg^i(x) = x \cup_{m-i} x \quad m \rightarrow m+i$$

$\deg x = m$ $m + m - (m-i)$:

\textcircled{2} E_1 : asso. prod. $\vdash E_n$?

\bar{E}_∞ : commu. prod

\textcircled{3} $\bar{E}_{n+1}\text{-alg.} \cong E_1\text{-algs in (Cart. of } \bar{E}_n\text{-alg)}$

Some history about the possible E_∞ -structures on

BP

- ## ■ Baas-Suillvan theory

 BP supports an Σ_∞ str.

↳ Give some geo. realization of some spectra.

(with E-str)

Rob 89

La2 O1



Some history about the possible E_∞ -structures on BP

- Baas-Suillvan theory
 - Get E_n -structures by killing the obstructions in E_{n-1} -structures.

[Ric 06] BP $\sim \underline{E_n}$

M-Barterra -

M.A. Mandell

TAQ thy

Mary. \tilde{E}_4

[BM(3)] ① BP $\hookrightarrow MU_{(4)}$, summand
↓ ② BP supports an \tilde{E}_4 -str.

Some history about the possible E_∞ -structures on BP

- Baas-Sullivan theory
 - Get E_n -structures by killing the obstructions in E_{n-1} -structures.
 - If not? E_∞ -structures support power operations

BMMS 86

Hao Ring spectra and their applications

- If not? E_∞ -structures support power operations

100

→ Hu - Kriz - Mey

$$BP \leftrightarrow MU_{(q)}$$

Joel - Niel 2010 MU_(p) → BP



Some history about the possible E_∞ -structures on BP

relation in Dyer-Lashof operation

$$\langle \mathfrak{f}_1^2, Q, R \rangle = \mathfrak{f}_5 \pmod{s^h}$$

- Baas-Sullivan theory
- Get E_n -structures by killing the obstructions in E_{n-1} -structures.
- If not? E_∞ -structures support power operations
- Find a (secondary) operation such that $H_* BP$ is not close under this operation.

($p=2$)

$$\begin{aligned} H_* BP &= F_2 [\mathfrak{f}_1^{(3)}, \mathfrak{f}_2^{(2)}, \dots] & Q(\mathfrak{f}_a^{2a}, \mathfrak{f}_{a_2}^{2b}, \dots) \\ &\downarrow & \downarrow \\ H_* H\bar{E} &= [F_2[\mathfrak{f}_1, \mathfrak{f}_2, \dots]] & \text{SUSTech} \end{aligned}$$

Ingredient

- Operations on $(\text{mod}-p)$ homology: (Dyer-Lashof operations)
 $+ (- - - \cdot \cdot \cdot)$

Ingredient

- Operations on $(\text{mod}-p)$ homology: Dyer-Lashof operations
- Calculating some secondary operations on it by some spectral sequences.

Introduction
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Ingredient
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Explicit calculation
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Outline

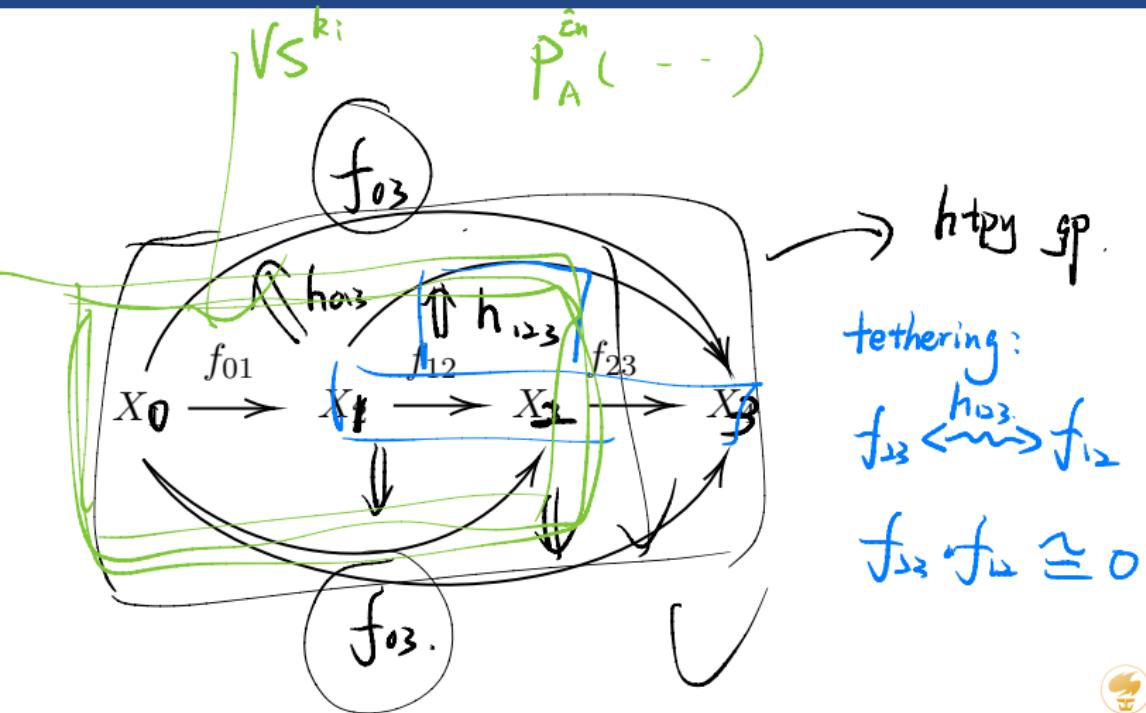
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Secondary opeartions

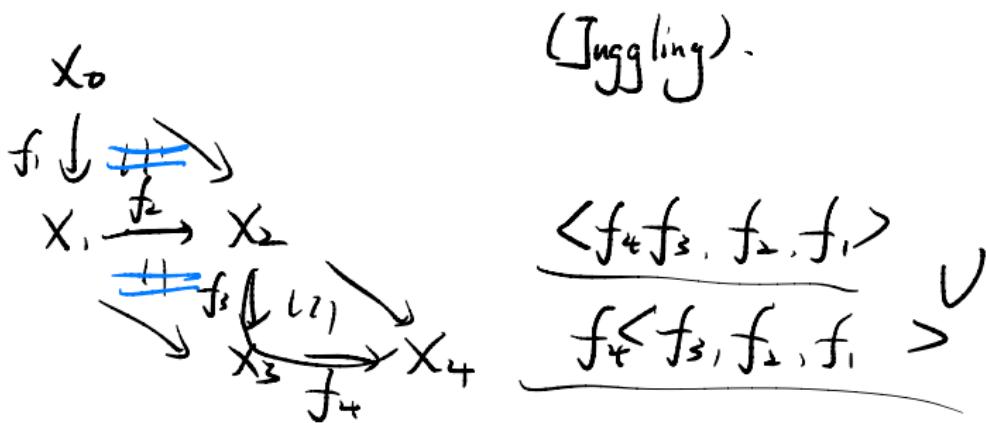


Secondary opeartions

f_{03} : base pt.

The associated secondary composite is the element of
 $\pi_1(Map_D(X_0, X_3), f_{03})$ represented by the path composite
 $(h_{023})^{-1} \cdot (f_{23}h_{012})^{-1} \cdot (h_{123}f_{01}) \cdot h_{013}$

Secondary opeartions



Secondary opeartions

Definition

A tethering of this composite is a homotopy class of nullhomotopy of gf : a homotopy class of path $h : gf \Rightarrow *$ in $\text{Map}_C(X_0, X_2)$. We will write $h : g \rightsquigarrow f$ to indicate such a tethering.

Secondary opeartions

Definition

For the above diagram, if we have a tethering $h : f_{12} \circ f_{23} \rightsquigarrow 0$ and $f_{12} \circ f_{01}$ is nullhomotopic, we write

$$\langle f_{23} \rightsquigarrow f_{12}, f_{01} \rangle \subset \pi_1(Mapc(X_0, X_3), *)$$

for the set of all elements $\langle f_{23} \rightsquigarrow f_{12} \rightsquigarrow f_{01} \rangle$, where the later tethering k ranges over possible tetherings. This also decides an operation for f_{01} such that $f_{12} \circ f_{01}$ is nullhomotopic, which is called the secondary operation determined by the tethering.

The set of maps f_{01} such that $f_{12} \circ f_{01}$ is nullhomotopic is referred to as the domain of definition of this secondary operation, and the possibly multivalued nature of this function is referred to as the indeterminacy of the secondary operation.



Zero and Indeterminacy

Theorem

Zero and Indeterminacy

Theorem

- *Changing the tethering and homotopy class of maps alters the value of a secondary composite by multiplication by loops.*

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- *A secondary operation $\langle f_{23} \leftrightarrow f_{12}, - \rangle$ determines a well-defined map Φ on $\ker f_{12} \subset \pi_0 \text{Map}_c(X_0, X_1)$ whose values are right cosets:*

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- $\ker f_{12} \rightarrow (f_{23}\pi_1 \text{Map}_c(X_0, X_2)) \backslash \pi_1 \text{Map}_c(X_0, X_3)$

Zero and Indeterminacy

Theorem

- *Changing the tethering and homotopy class of maps alters the value of a secondary composite by multiplication by loops.*
- *A secondary operation $\langle f_{23} \leftrightarrow f_{12}, - \rangle$ determines a well-defined map Φ on $\ker f_{12} \subset \pi_0 \text{Map}_c(X_0, X_1)$ whose values are right cosets:*
- $\ker f_{12} \rightarrow (f_{23} \pi_1 \text{Map}_c(X_0, X_2)) \backslash \pi_1 \text{Map}_c(X_0, X_3)$
- *If two tetherings h, h' give rise to operations Φ, Φ' , then there exists an element $u \in \pi_1 \text{Map}_c(X_1, X_3)$ such that $\Phi x = \Phi' x \cdot (ux)$ for all $x \in \ker f_{12} \subset \pi_0 \text{Map}_c(X_0, X_1)$.*

Example: A secondary operation on $\mathbb{C}P^\infty$

$$\begin{array}{c} \phi: H^*(\mathbb{C}P^\infty): x \in H^*(\mathbb{C}P^\infty) & \phi(x^2) \rightarrow H^6 \\ \downarrow & \\ x^3 & \phi(H^2(\mathbb{C}P^\infty)) \text{ zero: } (x, 0) \\ \left. \begin{array}{l} S_9^4 S_9^1 + S_9^2 S_9^3 + S_9^1 S_9^4 \\ S_9^4(x) = 0 \end{array} \right\} \text{indeterminacy: } 0 & \rightarrow [x^3] x^0 \\ \text{Cartan formula} & S_9^4(x^2) = x^4 \end{array}$$

$S_9^4 S_9^1 + S_9^2 S_9^3 + S_9^1 S_9^4$

$S_9^4(x) = 0$ ✓

$S_9^2(x^2)$ → Cartan formula $= 0$

$[x^3] x^0$

$S_9^4(x^2) = x^4$

Homotopy operations

↑ cohomology operations: $x_1 \rightarrow x_2, \dots, x_i \in K_p$

$$\oplus \pi_{k_j}(-) \rightarrow \pi_{\alpha}(-)$$

$$\sqrt{\left[S^k \right], -} \quad \left[S^\alpha \right], -$$

$$S^\alpha \rightarrow \vee S^{k_j}$$



Homotopy operations

Homotopy operations can be represented by maps between spheres and their dot unions.

Definition

Given a commutative ring spectrum A , we let $\mathbb{P}_A^{E_n}$ be the left adjoint to the forgetful functor from $E_n A$ -algebras to spectra; if $n = \infty$, we simply write \mathbb{P}_A , and if $A = \mathbb{S}$, then we will omit A from the notation. What's more,

$$\mathbb{P}_A^{E_n}(X) \cong \bigvee A \wedge (E_n(k)_+ \wedge_{\Sigma_k} X^{\wedge k})$$

$\mathbb{P}_A^{E_n} \text{ Spec.} \rightleftarrows E_n A\text{-alg. F}$

where the spaces $E_n(k)$ are the terms in our chosen E_n -operad

Homotopy operations

Definition

A homotopy operation on E_n A -algebras is a natural transformation of functors

$$\prod \pi_{k_i}(-) \rightarrow \pi_j(-)$$

represented by a homotopy class of map of E_n A -algebras

$$\mathbb{P}_A^{E_n}(S^j) \rightarrow \mathbb{P}_A^{E_n}(\vee S^{k_i})$$



Homotopy operations

$B : E_n A\text{-alg.}$

htpy op. on $E_n A\text{-alg. under } B$

$B \amalg P_A^{\tilde{z}^n}(S^j)$

$E_\infty : E_\infty B\text{-Alg.}$

cupprod.
(in Alg).

$B \amalg P_A^{\tilde{z}^n}(S^j) \xrightarrow{R} B \amalg P_A^{\tilde{z}^n}(V; S^{k;j})$

$\downarrow \quad \quad \quad \downarrow Q$
 $B \longrightarrow B \amalg P_A^{\tilde{z}^n}(V; s^{l;s})$

Dyer-Lashof operations

 $A : H\mathbb{F}_2$

Theorem

For any commutative H -algebra A , there are homotopy operations

$$Q^s : \pi_k \rightarrow \pi_{k+s}$$

for E_n A -algebras when $s < k + n - 1$, called the Dyer-Lashof operations. These satisfy the following relations:

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- *the additivity relation: $Q^s(x + y) = Q^s(x) + Q^s(y)$;*

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- *the instability relations:* $Q^s x = x^2$ when $|x| = s$, $Q^s x = 0$ when $|x| > s$;



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- *the Cartan formula:* $Q^s(xy) = \sum_{p+q=s} Q^p(x)Q^q(y)$;

Dyer-Lashof operations

Theorem

Section 8, 10 → Rezk, Lecture notes of power op.

For any commutative H -algebra A , there are homotopy operations

$$\text{Kuhn} \xrightarrow{Q^s : \pi_k \rightarrow \pi_{k+s}} \begin{array}{l} \textcircled{1} \text{ constr. of DL. operation} \\ \textcircled{2} \text{ DL-operation} \leftarrow \text{Ste. Alg.} \end{array}$$

for E_n A -algebras when $s < k + n - 1$, called the Dyer-Lashof operations. These satisfy the following relations:

- the additivity relation: $Q^s(x + y) = Q^s(x) + Q^s(y)$; homology op. on infinite loop space
- the instability relations: $Q^s x = x^2$ when $|x| = s$, $Q^s x = 0$ (Peter May)
(forgetful map)
- the Cartan formula: $Q^s(xy) = \sum_{p+q=s} Q^p(x)Q^q(y)$; f
- the Adem relations: If $r > 2s$, then $Q^r Q^s(x) = \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i$. $m \leq n$, $\tilde{E}^{\text{alg}} \leftarrow \tilde{E}^{\text{alg}}$
preserve Dyer-Lashof operation.

Dyer-Lashof operations

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Dyer-Lashof operations $\gamma_A : H\mathbb{F}_2$

Theorem

For any commutative H -algebra A , all homotopy operations for $(E_\infty A)$ -algebras C are composites of the following types:

Dyer-Lashof operations

Theorem

For any commutative H -algebra A , all homotopy operations for $E_\infty A$ -algebras C are composites of the following types:

- the constant operation associated to an element $\alpha \in \pi_n A$ which takes no arguments and whose value on C is the image of α under the map $\pi_* A \rightarrow \pi_* C$.

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- *the Dyer-Lashof operations $Q^s : \pi_n(C) \rightarrow \pi_{n+s}(C)$*
- *the binary addition operations $\underline{\pi_n(C) \times \pi_n(C)} \rightarrow \pi_n(C)$*

Dyer-Lashof operations

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- the Dyer-Lashof operations $Q^s : \pi_n(C) \rightarrow \pi_{n+s}(C)$
- the binary addition operations $\pi_n(C) \times \pi_n(C) \rightarrow \pi_n(C)$
- the binary multiplication operations $\pi_n(C) \times \pi_m(C) \rightarrow \pi_{n+m}(C)$.

if: H_∞ ring - - -
↳ [IX.2]

Dyer-Lashof operations

Theorem

The suspension operator, on homotopy operations for E_n
 A -algebras under B takes zero-preserving homotopy operations
 $\prod \pi_{l_s} \rightarrow \pi_k$ to homotopy operations $\prod \pi_{(l_s+1)} \rightarrow \pi_{k+1}$. Suspension
preserves addition, composition, and multiplication by scalars from
 B . Suspension also takes Q^s to Q^s and takes the binary
multiplication operation $\pi_p \times \pi_q \rightarrow \pi_{p+q}$ to the trivial operations.

$$\sum S^j \rightarrow \sum VS^k;$$



Geometric realization of some secondary operations

Some secondary operations can be detected by spectral sequences.

Theorem

Suppose X , Y , and Z are spectra, $X \xrightarrow{f} Y \xrightarrow{g} Z$ is nullhomotopic, and that $\alpha \in \ker(f) \subset \pi_n(X)$ is represented by a map $S^n \rightarrow X$. Given any extension $\tilde{X} \rightarrow Cf \xrightarrow{h} Z$ from the mapping cone representing a tethering, the secondary operation $\langle g \rightsquigarrow f, \alpha \rangle$ is (up to sign) the set $h(\partial^{-1}\alpha)$, where $\delta : \pi_{n+1} Cf \rightarrow \pi_n X$ is the connecting homomorphism in the long exact sequence of homotopy groups.

$$S^n \rightarrow X \rightarrow Y \rightarrow Z$$



Geometric realization of some secondary operations

Corollary X_i : i -skeleton.

Suppose that X_* is a simplicial spectrum with geometric realization X and that F is the homotopy fiber in the sequence $F \xrightarrow{j} X_1 \xrightarrow{d_0} X_0$. Then the composite $F \xrightarrow{d_1 j} X_0 \xrightarrow{i} |X_*|$ has a canonical tethering. If $\alpha \in \pi_n(F) \subset \pi_n X_1$ is in the kernel of d_1 , then in the geometric realization spectral sequence

$$H_p(\pi_q X_*) \Rightarrow \pi_{p+q} |X_*|$$

the secondary operation $\langle i \rightsquigarrow d_1 j, \alpha \rangle$ is represented (up to sign) by the element $[\alpha] \in H_1(\pi_n X)$ in the spectral sequence.

Geometric realization of some secondary operations

$|X_+|$ 1-skeleton.

$$X_+ \vee X_+ \rightarrow X_+ \wedge [0, 1]$$

$$d_{0 \vee 1} \downarrow \quad \downarrow$$

$$X_0 \rightarrow sk^{(1)}(X_+)$$

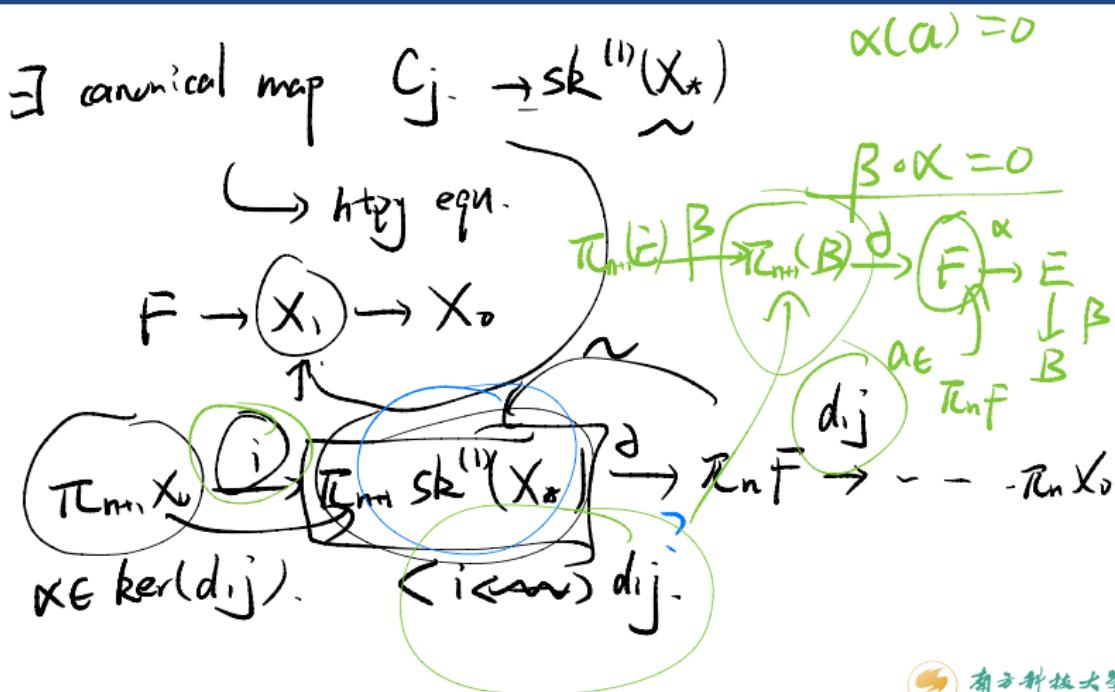
a homotopy between $(id_0) \sim id_1$ is given by the above diagram.

$$d_{0 \vee 1} \Rightarrow 0$$

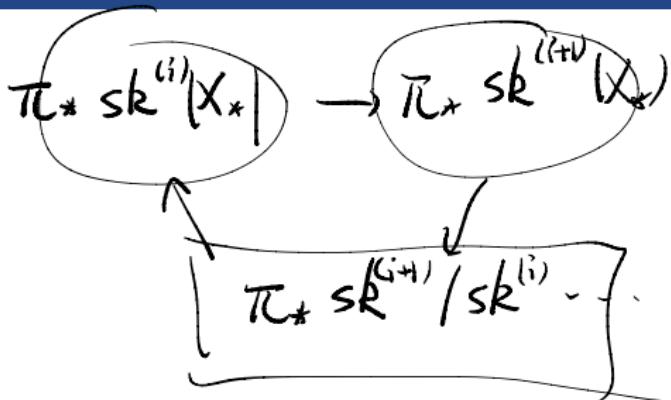
$$id_1 \Rightarrow id_0$$



Geometric realization of some secondary operations



Geometric realization of some secondary operations



Geometric realization of some secondary operations

Theorem

Suppose $f : R \rightarrow S$ is a map of commutative ring spectra, and let $i = 1 \wedge f : S \wedge R \rightarrow S \wedge S$. Then, in the (pointed) category of augmented commutative S -algebras, there is a canonical tethering $p \rightsquigarrow i$ for the composite

$$\overbrace{S \xrightarrow{\quad} S \wedge R \xrightarrow{i} S \wedge S}^{\text{composite}} \xrightarrow{p} S \wedge_R S$$

Let $x \in \pi_n(S \wedge R)$ map to zero in $\pi_n(S \wedge S)$, so that $\sigma x = \langle p \rightsquigarrow i, x \rangle \in \pi_{n+1}(S \wedge_R S)$ is defined. Then σx is detected by the image of x under $\pi_n(S \wedge R) \rightarrow \pi_n(S \wedge R \wedge S)$ in the two-sided bar construction spectral sequence

$$H_p(\pi_q(S \wedge R^{\wedge *} \wedge S)) \Rightarrow \pi_{p+q}(S \wedge_R S)$$



Geometric realization of some secondary operations

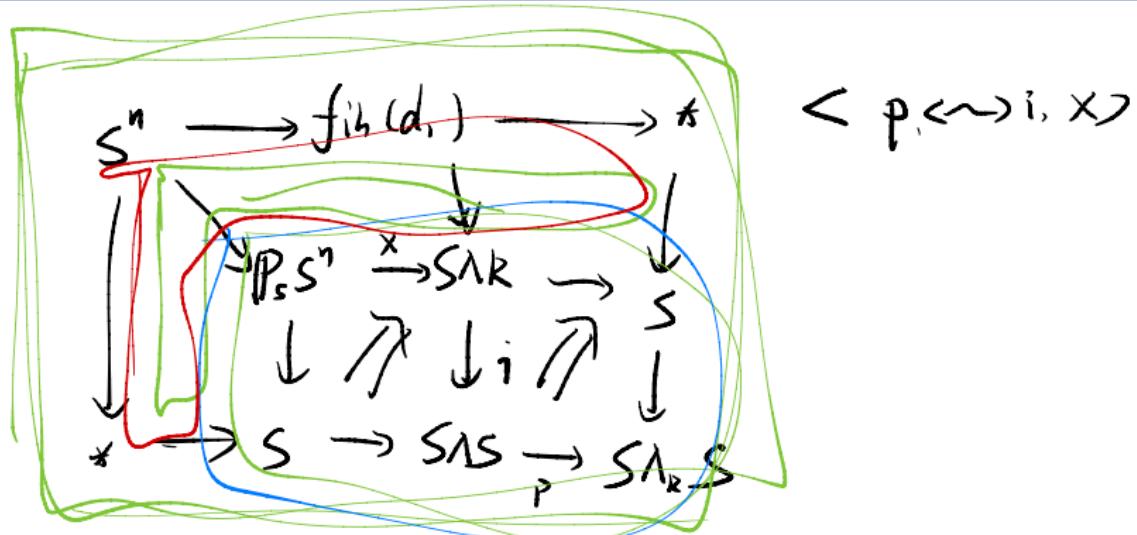
$$\boxed{\text{SARAS}} \rightarrow \text{SAS} \rightarrow \text{S}\Lambda\text{s}$$

$$\text{d}_j: \text{SAR} \rightarrow \text{SAR} \times \text{S} \rightarrow \text{SIS}$$

$$j=0: \text{SAR} \xrightarrow{\eta_L} \text{SAS} \quad \text{null map}$$

$$j=1 \quad \text{SAR} \rightarrow \text{SAS}$$

Geometric realization of some secondary operations



Geometric realization of some secondary operations

If $\pi_*(S \wedge R)$ is flat over S , we can identify the E_2 -term in the two-sided bar construction spectral sequence:

$$E_{**}^2 = \text{Tor}_{**}^{\pi_*(S \wedge R)}(\pi_*(S \wedge S), \pi_*(S)) \Rightarrow \pi_*(S \wedge_R S)$$

The element x gives rise to the corresponding element in $\text{Tor}_{1,n}$. In particular, we have the following result when the target is the mod-2 Eilenberg-Mac Lane spectrum.

$$(S = H\mathbb{F}_2)$$



Geometric realization of some secondary operations

Theorem

Suppose $R \rightarrow H$ is a map of E_∞ -algebras and $x \in H_n R$ maps to zero in the dual Steenrod algebra $H_* H$. Then there is an element $\sigma x = \langle p \leadsto i, x \rangle \in \pi_{n+1}(S \wedge_R S)$ in the R -dual Steenrod algebra $\pi_*(H \wedge_R H)$ that is detected by the image of x in homological filtration 1 of the spectral sequence

$$\mathrm{Tor}_{**}^{H_* R}(H_*, H_* H) \Rightarrow \pi_*(H \wedge_R H)$$

$$R \cong MV$$



Geometric realization of some secondary operations

6 SL C

We now specialize this result to the case where MU is the complex bordism spectrum.

Sec 3.4: Some DL operation act on the MU.

Theorem

Let n be an integer that is not of the form $2^k - 1$ for any k , so that the corresponding generator $b_n \in H_{2n}MU \cong \mathbb{F}_2[b_1, b_2, \dots]$ in mod-2 homology is the Hurewicz image of the generator $x_n \in \pi_{2n}MU \cong \mathbb{Z}[x_1, x_2, \dots]$. Then the diagram of E_∞ H -algebras

$$\mathbb{P}_H S^{2n} \xrightarrow{b_n} H \wedge MU \xrightarrow{p} H \wedge H \xrightarrow{i} H \wedge_{MU} H$$

已被更进一步处理
 $Q^S(6x_n) = ? \pmod{\dots}$

determines a bracket, and $\sigma x_n \equiv \langle p \rightsquigarrow i, b_n \rangle \pmod{\text{decomposables}}$.

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3 Explicit calculation

Actions of Dyer-Lashof operations on MU

[Pr: TS] Dyer-Lashof operations for the classifying space of certain matrix groups.

Theorem

The Dyer-Lashof operations in $H_*MU = H_*BU$ are determined by the following identity:

$$\sum Q^j b_k = \left(\sum_{n=k}^{\infty} \sum_{u=0}^k \binom{n-k+u-1}{u} b_{n+u} b_{k-u} \right) \left(\sum_{n=0}^{\infty} b_n \right)^{-1}$$

Here $b_0 = 1$ by convention. In particular, we have

$$\sum Q^j b_1 = \left(\sum_{n=1}^{\infty} (b_n b_1 + (n-1)b_{n+1}) \right) \left(\sum_{n=0}^{\infty} b_n \right)^{-1}$$

on some explicit homology gp.
DL. op.

Remark: Find out how to calc. explicit actions of Steenrod algebra

Actions of Dyer-Lashof operations on MU

Actions of Dyer-Lashof operations on H

Theorem

The 2-primary Dyer-Lashof operations in the dual Steenrod algebra satisfy the following identities:

$$1 + \xi_1 + Q^1\xi_1 + Q^2\xi_1 + Q^3\xi_1 + \cdots = (1 + \xi_1 + \xi_2 \cdots)^{-1}$$

$$Q^s \bar{\xi}_i = Q^{s+2^i-2} \xi_1 \quad \text{if } s \equiv 0, -1 \pmod{2^i} \quad 0 \quad \text{otherwise}$$

$$Q^{2^i} \bar{\xi}_i = \bar{\xi}_{i+1}$$

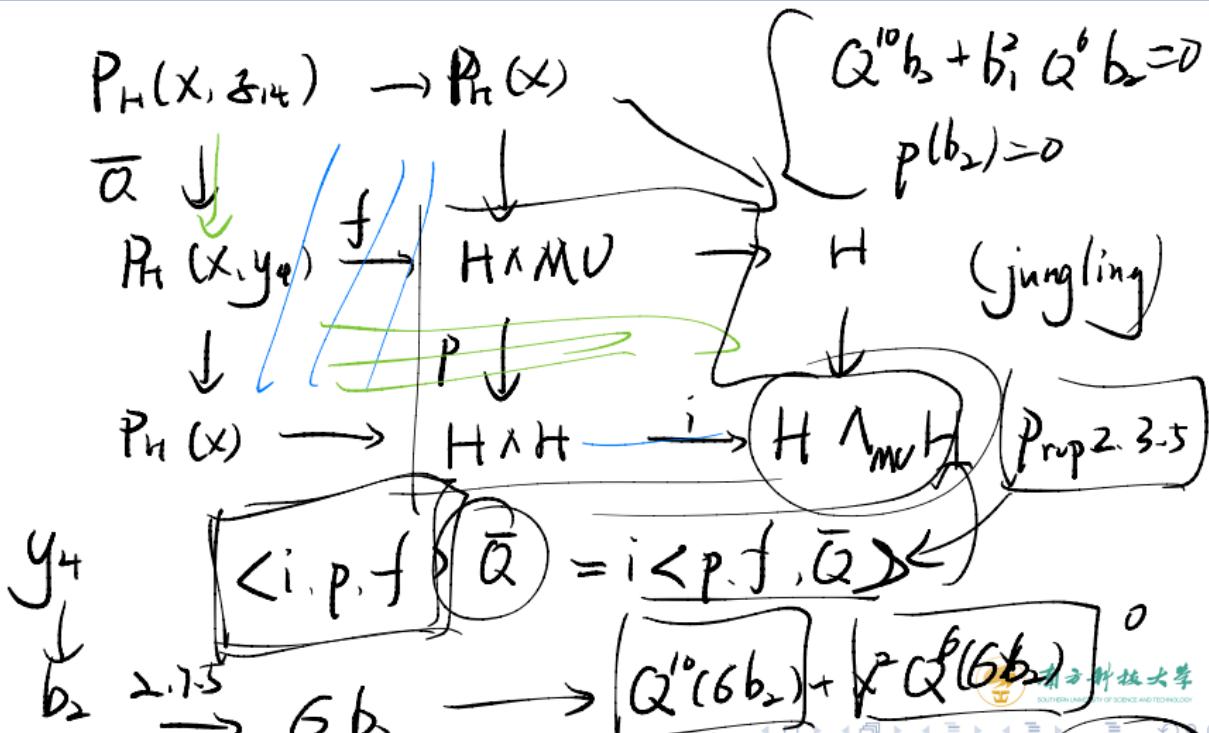
Functional operations for $MU \rightarrow H\mathbb{Z}/2$

Theorem

Consider the maps

$$\mathbb{P}_H(x, z_{14}) \xrightarrow{\bar{Q}} \mathbb{P}_H(x, y_4) \xrightarrow{f} H \wedge MU \xrightarrow{p} H \wedge H$$

in the category \mathcal{C} , where \bar{Q} sends z_{14} to $Q^{10}y_4 + x^2Q^6y_4$ and f sends (x, y_4) to (b_1, b_2) . Then a functional homotopy operation $\langle p, f, \bar{Q} \rangle$ is defined in $\mathbb{P}_H(x)$ -algebras and satisfies $\langle p, f, \bar{Q} \rangle \equiv \xi_4$ mod decomposables.

Functional operations for $MU \rightarrow \mathbb{H}\mathbb{Z}/2$ 

A secondary operation in the dual Steenrod algebra

The diagram illustrates the relationship between the dual Steenrod algebra and the Brown-Peterson spectrum. It features three main components:

- Left Component:** A circle containing $6x_1 \equiv i(S_4)$. Below it is a commutative diagram: $H_* H \xrightarrow{i} H_* MU$.
- Middle Component:** A circle containing $i(p.f)(\bar{Q}) \equiv \langle i.p.f \rangle \bar{Q} \equiv i(S_4)$. Inside this circle, there is a smaller circle labeled $\langle \bar{Q} \rangle$ and another labeled $\langle S_4 \rangle$.
- Right Component:** A circle labeled "(mod decomp.)".

A curved arrow labeled "indeterminacy?" points from the left component towards the middle component.

At the bottom, the equation $\langle p.f. \bar{Q} \rangle \equiv S_4 \pmod{\text{decomp.}}$ is written.

A secondary operation in the dual Steenrod algebra

 (Q, R) $\langle \times, (Q \rightarrow R) \rangle$ mod decomp.

$\mu R = \bar{\alpha} v + \beta \alpha.$

$\langle S^2, Q, R \rangle = \langle p.b, Q, R \rangle$

$\cdot c \langle p.b, Q, R \rangle$

$= \langle p.f.\bar{\mu}, R \rangle$

$\rightarrow \langle p.f.\mu R \rangle$

$\langle p.f.\bar{\alpha} \rangle v$
 $\langle p.f.\beta \rangle k$



A secondary operation in the dual Steenrod algebra

$$\begin{aligned} & \langle \zeta_1^2 Q, R \rangle = \langle p.f. \bar{Q} \rangle v + \underbrace{\langle p.f. \bar{R} \rangle \alpha}_{\equiv 0 \text{ mod decomp}} \\ & \underbrace{\langle p.f. \bar{Q} \rangle (v \zeta_{30})}_{\text{mod decomp}} = Q^{16} (\dots) = \zeta_5. \end{aligned}$$

A secondary operation in the dual Steenrod algebra

In the dual Steenrod algebra, any element in the bracket $\langle \xi_1^2, Q, R \rangle$ is congruent to ξ_5 mod decomposables.

A secondary operation in the dual Steenrod algebra

Theorem

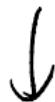
The Brown–Peterson spectrum BP is connective, with $\pi_0 BP \cong \mathbb{Z}_{(2)}$. The map $BP \rightarrow H\mathbb{F}_2$ induces an inclusion $H_* BP \hookrightarrow H_* H\mathbb{F}_2$ whose image is the subalgebra

$$\mathbb{F}_2[\xi_1^2, \xi_2^2, \dots] \subset \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

of the dual Steenrod algebra. The image in positive degrees consists entirely of decomposables.

The End

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Classifying manifold \cong "principle bundle"