Overview:

1. Hopf invariants and Adams spectral sequences (25 mins)

   Adams spectral sequences.

   Pontriyagin - Thom construction

   The geometry of $\overline{MO}$

3. Arf - Kervaire invariants & Adams spectral sequences (30 mins)

   Browder's formulation

   Lanne's reformulations via secondary operations.

   h_i^2 in Adams spectral sequences.

§. Hopf invariants

Definition 3.1. Let $f\colon \mathbb{S}^n \to \mathbb{S}^0$ be a map in the stable stem. Then the cofiber C_f is a two-cell complex with the cohomology

$$\mathrm{H}^0(C_f) = \mathbb{F}_2\{\alpha\}$$
$$\mathrm{H}^{n+1}(C_f) = \mathbb{F}_2\{\beta\}$$

The **Hopf invariant** $H(f)$ is a value in $\mathbb{F}_2$ such that $\mathrm{Sq}^{n+1}\alpha = H(f)\cdot\beta$. Equivalently, if we let $[C_f] \in \mathrm{H}_{n+1}C_f$ to be the class of the top cell, then we have

$$H(f) = \langle \mathrm{Sq}^{n+1}\alpha, [C_f]\rangle$$

Question: When the Hopf invariant is 1?

Answer: using Adams spectral sequence.

Consider $\mathbb{S} \xrightarrow{e} H \longrightarrow \overline{H}$, and the Adams tower

$$0 \leftarrow \mathbb{F}_2 \leftarrow \mathcal{A}^* \leftarrow I(\mathcal{A}) \leftarrow 0$$
$$0 \to \mathbb{F}_2 \to \mathcal{A}_* \to \overline{\mathcal{A}}_* \to 0$$

$$\begin{tikzcd}
\Sigma^{-3}\overline{H}^{\wedge 3} \arrow[r] \arrow[d] & H\wedge\Sigma^{-3}\overline{H}^{\wedge 3} \arrow[dl, dashed, "\partial"'] \\
\Sigma^{-2}\overline{H}^{\wedge 2} \arrow[r] \arrow[d] & H\wedge\Sigma^{-2}\overline{H}^{\wedge 2} \arrow[dl, dashed, "\partial"'] \\
\Sigma^{-1}\overline{H} \arrow[r] \arrow[d] & H\wedge\Sigma^{-1}\overline{H} \arrow[dl, dashed] \\
\mathbb{S}^0 \arrow[r] & H
\end{tikzcd}$$

$$E_1^{s,t} \cong \pi_{t-s}(H\wedge(\Sigma^{-1}\overline{H})^{\wedge s})$$

Proposition 3.10. There exists $f \in \pi_n(\mathbb{S}^0)$ with Hopf invariant one if and only if there exists $[h] \in \mathrm{E}_1^{1,n+1}$ in the (homological) $\mathbb{F}_2$-Adams spectra sequence such that $h(\mathrm{Sq}^{n+1}) = 1$ and survives to E_∞-page.

"⇒":

$$\begin{tikzcd}
 & \Sigma^{-1}\overline{H} \arrow[r] \arrow[d] & H\wedge\Sigma^{-1}\overline{H} \\
\mathbb{S}^n \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & \mathbb{S}^0 \arrow[r] & H
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathbb{S}^n \arrow[r] \arrow[d, "\tilde{f}"'] & \mathbb{S}^0 \arrow[r] \arrow[d] & C_f \arrow[r] \arrow[d] & S^{n+1} \arrow[d, "\Sigma\tilde{f}"] \\
\Sigma^{-1}\overline{H} \arrow[r] & \underbrace{\mathbb{S}^0 \arrow[r] & H \arrow[r] & \overline{H}}_{\text{pass to cohomology}}
\end{tikzcd}$$

claim: $\Sigma\tilde{f}^*(Sq^{n+1}) = 1$

$$\begin{tikzcd}
\mathbb{F}_2 & H^*C_f \arrow[l] & \mathbb{F}_2[n+1] \arrow[l] \\
\mathbb{F}_2 \arrow[u, equal] & \mathcal{A}^* \arrow[l] \arrow[u, "\text{structure map}"] & I(\mathcal{A}^*) \arrow[l, hook'] \arrow[u, "\Sigma\tilde{f}^*"']
\end{tikzcd}$$

$\leftsquigarrow$

Then by the pairing property, $\mathbb{S}^n \xrightarrow{\tilde{f}} H\wedge\Sigma^{-1}\overline{H} \in \mathcal{A}$ is the claim one.

§. The geometric meaning.

The key tool to understanding the geometry of the Adams spectral sequence is the Pontriyagin-Thom construction:

Thm: $\underset{\underline{\text{framed cobordism class}}}{\underset{[M^n, \nu, t]}{\Omega_*^{fr}}} \cong \pi_*$

$$\begin{array}{l} M^n \hookrightarrow \mathbb{R}^{n+p}, \quad \nu \text{ the normal bundle}, \quad t \text{ the trivialization}. \\ \downarrow \nearrow \\ N(\nu) \\ \Downarrow \text{ collapse map } + - \\ S^{n+p} \longrightarrow Th(\nu) \xrightarrow{t} M_+ \wedge S^p \xrightarrow{c} S^p \in \pi_n \end{array}$$

In this case, let $[M^n, \nu, t]$ be the framed cobordism class $\sim f: S^n \to S$

Question: What $H(f)$ means for $[M^n, \nu, t]$?

Answer: Consider the map $S \to MO \rightsquigarrow \Omega_*^{fr} \to \Omega_*^O$

Since M^n is stably parallesible $\rightsquigarrow$ its Stiefel Whitney numbers vanish $\rightarrow$ it is null-cobordant i.e. we can find N^{n+1} with normal bundle ν_N s.t. $\partial N^{n+1} = M$ and $\nu_N|_M = \nu_M$.

Let
$$\begin{tikzcd} N^{n+1} \arrow[r, "\nu_N"] \arrow[d] & BO_r \\ C_M N \arrow[ur, "\nu_N^t"'] & \end{tikzcd}$$
} assume $p >> n+1$.

$W_{n+1}^t(\nu_N) = (\nu_N^t)^* W_{n+1}$

$\quad\quad \llcorner$ Stiefel-Whitney class

Claim that. $H(f) = \langle W_{n+1}^t(\nu_N), [N, M]$.

First, $MO = \bigvee_i \Sigma^i H$, $i \neq 2^l - 1$ and the Thom class $MO \xrightarrow{u} H$ is one of the summand. Then $H^* MO$ is a free A^*-module and thus the MO-tower is equivalent to the H-Adams tower.

For $n>0$: } Explain it in Adams towers

$$
\begin{tikzcd}
 & \Sigma^{-1}\overline{MO} \arrow[r] \arrow[d] \arrow[dr, "\text{Hurewicz}"] & \Sigma^{-1}\overline{H} \arrow[d] \arrow[drr, "\text{Hurewicz}"] & & \\
\mathbb{S}^{n} \arrow[ur, "{[N,M,t]}"] \arrow[r, "{[M,t]}", "0"'] & \mathbb{S} \arrow[r, equal] & \mathbb{S} & & \\
 & & MO \wedge \Sigma^{-1}\overline{MO} \arrow[rr] & & H \wedge \Sigma^{-1}\overline{H}
\end{tikzcd}
$$

induced "quasi-isomorphism" between the Adams tower

Thom class.

$MO \xrightarrow{u} H$

$$
\begin{tikzcd}
S^{n+r} \arrow[r, "\cong"] & (D^{n+r}, S^{n+r}) \arrow[r, "\bar{f}"] \arrow[d, "P_N"', shift right] \arrow[d, "P_M", shift left] & (MO_r, S^r) \arrow[r, "Sq^{n+1}u"] \arrow[d, equal] & \Sigma^{n+r}\mathrm{H}\mathbb{F}_2 \arrow[d, equal] \\
 & (\mathrm{Th}_N(\nu), \mathrm{Th}_M(\nu_M)) \arrow[r, "\mathrm{Th}(\nu)"] & (MO_r, S^r) \arrow[r] & \Sigma^{n+r}\mathrm{H}\mathbb{F}_2 \\
 & (N, M) \arrow[r, "\nu"] \arrow[u, dashed, "\Theta"] \arrow[d] & (BO_r, *) \arrow[r, "W_{n+1}", "W_\omega"'] \arrow[u, dashed, "\Theta"] \arrow[d] & \Sigma^{n}\mathrm{H}\mathbb{F}_2 \arrow[u, dashed, "\Theta"] \\
S^n \arrow[r, "{[N,M]}"] & \mathrm{H}\mathbb{F}_2 \wedge (N, M) \arrow[r, "\mathrm{id}\wedge\nu"] & \mathrm{H}\mathbb{F}_2 \wedge (BO_r, *) \arrow[r, "\mathrm{id}\wedge W_\omega"] & \mathrm{H}\mathbb{F}_w \wedge \Sigma^{n}\mathrm{H}\mathbb{F}_2 \arrow[u, "m"']
\end{tikzcd}
$$

$[N,M,t]$

$$
\begin{tikzcd}
{[(\mathbb{D}^{n+1}, \mathbb{S}^{n}), (MO, \mathbb{S})]} \arrow[rr, "\text{Huricwz}"] \arrow[d, "\cong"] & & H_{n+1}(\mathbb{D}^{n+1}, \mathbb{S}^{n}) \arrow[d, "{[N,M,t]}"] & \\
\pi_{n+1}(\overline{MO}) \arrow[r] \arrow[d, "u"] & MO_{n+1}(\overline{MO}) \arrow[r] & H_{n+1}(\overline{MO}) \arrow[r, "Sq^{n+1}u", "W_{n+1}u"'] \arrow[d, "u"] & \mathbb{F}_2 \arrow[d, equal] \\
\pi_{n+1}(\overline{H}) \arrow[rr] \arrow[d] & & H_{n+1}(\overline{H}) \arrow[r, "Sq^{n+1}"'] & \mathbb{F}_2 \\
\pi_n(\mathbb{S}) & & &
\end{tikzcd}
$$

classical Hopf map. this is equivalent to

$$F^1\pi_n(\mathbb{S}) \longrightarrow E_\infty^{1,n} \longrightarrow E_1^{1,n} = \mathrm{Hom}^{n+1}(I(A^*), \mathbb{F}_2) \xrightarrow{Sq^{n+1}} 1$$

$$E_1^{1,n} = \pi_n(H \wedge \Sigma^{-1}\overline{H})$$

§. Browder's Arf-Kervaire invariants:

$[M^{2n}, t]$ a framed manifold, $\nu: M^{2n} \longrightarrow \mathbb{R}^{2n+p}$

$x \in H^n(M)$ classified by $\varphi_x: M_+ \longrightarrow \Sigma^n H$

$$\mathbb{S}^{2n+p} \xrightarrow{t \circ P_M} M_+ \wedge \mathbb{S}^p \xrightarrow{\varphi_x \wedge id} \Sigma^{n+p} H \xrightarrow{Sq^{n+1}} \Sigma^{2n+p+1} H$$

forms a Toda braket } functional Steenrod operation.

$$\underbrace{\{\mathbb{S}^{2n+p+1} \longrightarrow \Sigma^{2n+p+1}\}}_{=: q_t(x)} \in \langle t \circ P_M, \varphi_x \wedge id, Sq^{n+1} \rangle$$

$\Sigma^{2n+p+1} \cap \mathbb{F}_2$

claim 1: There is no indeterminence in this Toda bracket i.e. $q_t(x)$ is well-defined.

claim 2: $q_t: H^n(M) \longrightarrow \mathbb{F}_2$ is an $\mathbb{F}_2$-quadratic form on $H^n(M)$

claim 3: $WQ(\mathbb{F}_2) \xrightarrow[\text{Arf}]{\cong} \mathbb{F}_2$ the Witt group of the quadratic space (induced by Arf-invariants).

{ symplectic basis, $\{e_1, f_1, \cdots, e_m, f_m\}$ ~ $\langle e_i, f_j \rangle = \begin{cases} 1, & i=j \\ 0, & \text{otherwise} \end{cases}$

$\langle e_i, e_i \rangle = \langle f_j, f_j \rangle = 0$

$$\text{Arf}(q) := \sum_{i=1}^{m} q(e_i)\, q(f_i)$$

Define: $\text{Kervaire}(M^{2n}, t) := [H^n(M), q_t]$

Let $(N^{2n+1}, M^{2n}, \nu_N)$ be the null-cobordism.

§. Lannes's reformulation:

Lemma: $\exists\, u \in H^n(M)$ s.t. $\mathrm{Arf}(q_t) = q_t(u)$.

Recall: Wu class for vector bundles: $v_i(\xi) = \Phi_\xi^{-1} \chi Sq^i U_\xi$

Let N^{2n+1} s.t. $\partial N^{2n+1} = M$ and $v_N|_M = v_M$.

$$\begin{tikzcd} BO_p\langle v_{n+1}\rangle \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EK(n+1) \arrow[d] \\ BO_p \arrow[r, "v_{n+1}(\gamma_p)"] & K(n+1) \end{tikzcd} \quad \rightsquigarrow \text{ Wu orientation.} \qquad \begin{tikzcd} M \arrow[r, "s"] \arrow[d] & K(n) \arrow[d] \\ N \arrow[r] & BO_p\langle v_{n+1}\rangle \end{tikzcd}$$

Then we let $u = s^* \iota_n$.

Let $f: S^{2n+p} \longrightarrow S^p$ be the map corresponds to

Lannes compute that the Toda bracket

$$Cf \xrightarrow{\psi} MO(p) \xrightarrow{\sum_{i=1}^{n+1} \Theta(v_{n+1}v_{n+1-i})} \bigvee_{i=1}^{n+1} K(\mathbb{F}_2, 2n+2+p-i) \xrightarrow{\sum_{i=1}^{n+1} \mathrm{Sq}^i} K(\mathbb{F}_2, 2n+2+p)$$

$$S^{2n+p+1} \xrightarrow{\Sigma \varphi_u} \Sigma^{p+1} K(n) \xrightarrow{\Sigma^{p+1} \iota_n} K(n+p+1) \xrightarrow{Sq^{n+1}} K(\mathbb{F}_2, 2n+p+2).$$

Remark: $\sum_{i=1}^{n+1} Sq^i \otimes v_{n+1} v_{n+1-i} U \in I(A^*) \otimes H^* MO$ is a cycle the the A^*- resolution of $H^* MO$.

$$\begin{tikzcd} MO(p) \arrow[d, "h"] \\ K(\mathbb{F}_2, p) \times L \\ K(\mathbb{F}_2, p) \arrow[u, "k"'] \end{tikzcd} \qquad a_i = k^*(h^*)^{-1}(\Theta v_{n+1} v_{n+1-i})$$

$$
\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, equal] & MO(p) \arrow[r, "\sum_{i=1}^{n+1}\Theta(v_{n+1}v_{n+1-i})"] & \bigvee_{i=1}^{n+1} K(\mathbb{F}_2, 2n+2+p-i) \arrow[r, "\sum_{i=1}^{n+1}\mathrm{Sq}^i"] & K(\mathbb{F}_2, 2n+2+p) \arrow[d, equal] \\
X \arrow[r, "\alpha"] & K(\mathbb{F}_2, p) \arrow[r, "(a_1,\cdots,a_m)"] & \bigvee_{i=1}^{n+1} K(\mathbb{F}_2, 2n+2+p-i) \arrow[r, "\sum_{i=1}^{n+1}\mathrm{Sq}^i"] & K(\mathbb{F}_2, 2n+2+p)
\end{tikzcd}
$$

and the Toda bracket of the bottom row gives

$$\mathrm{Kervaire}(M,t) = \langle \Phi\alpha, [X] \rangle$$

$$\mathrm{Sq}^{n+1}(\chi\mathrm{Sq}^{n+1}) + \sum_{i=1}^{n} \mathrm{Sq}^i a_i = 0 \rightsquigarrow$$ yields a secondary operation Φ.

Claim: $\langle h_k h_\ell, \mathrm{Sq}^{n+1} \otimes_A \chi\mathrm{Sq}^{n+1} + \sum_{i=1}^{n} \mathrm{Sq}^i \otimes_A a_i \rangle = \begin{cases} 1, & \text{if } 2^k = 2^\ell = n+1 \\ 0, & \text{otherwise} \end{cases}$

[Note $\langle h_k, \chi\mathrm{Sq}^{2^k} \rangle = 1$ since $\underbrace{\chi\mathrm{Sq}^{2^k} \equiv \mathrm{Sq}^{2^k}}_{\text{given by } \sum_{i=1}^{n} \chi\mathrm{Sq}^i \cdot \mathrm{Sq}^{n-i} = 0 \text{ if } n>0}$ mod $I(A)^2$]. then the number is

$$h_k(\mathrm{Sq}^{n+1}) \cdot h_\ell(\chi\mathrm{Sq}^{n+1}) + \sum_{i=1}^{n} h_k(\mathrm{Sq}^i) \otimes h_\ell(a_i)$$

$$\| $$

$$h_k(\mathrm{Sq}^{2^k}) \cdot h_k(\mathrm{Sq}^{2^k}) = 1.$$

$$
\begin{tikzcd}
 & \Sigma^{-2}\overline{MO}^{\wedge 2} \arrow[r] \arrow[d] & MO \wedge \Sigma^{-2}\overline{MO}^{\wedge 2} \\
 & \Sigma^{-1}\overline{MO} \arrow[r] \arrow[d] & MO \wedge \Sigma^{-1}\overline{MO} \\
\mathbb{S}^{2n} \arrow[r, "{[M,t]}"] \arrow[ur, "{[N,M,t]}"] \arrow[uur, bend left] & \mathbb{S} \arrow[r] & MO
\end{tikzcd}
$$

$$
\begin{tikzcd}
\pi_{2n}(\Sigma^{-2}\overline{MO}^{\wedge 2}) \arrow[r] \arrow[ddd] & MO_{2n+2}(\overline{MO}^{\wedge 2}) \arrow[r] & H_{2n+2}(\overline{MO}^{\wedge 2}) \arrow[d] & \\
 & & H_{2n+2}(\overline{H}\wedge\overline{MO}) \arrow[d] & \\
 & & (H_*\overline{H}\otimes H_*\overline{MO}) \arrow[r, "\sum_{i=1}^{n+1} Sq^i \otimes v_{n+1} v_{n+1-i} u"] \arrow[d] \arrow[dr, phantom, "\text{Lannes}"] & \mathbb{F}_2 \arrow[d] \\
\pi_{2n}\Sigma^{-2}\overline{H}^{\wedge 2} \arrow[r] \arrow[d] & H_{2n+2}\overline{H}^{\wedge 2} \arrow[r, phantom, "\cong"] & (H_*\overline{H}\otimes H_*\overline{H})_{2n+2} \arrow[r, "Sq^{n+1}\otimes_A \chi Sq^{n+1} + \sum_{i=1}^{n} Sq^i \otimes a_i"] & \mathbb{F}_2 \\
\pi_{2n}(\mathbb{S}) & & &
\end{tikzcd}
$$