Barr-Beck Theorem, Morita theory and Brauer groups in ∞-categories

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Monadicity

In classical monad theory, given a monad $T \in Fun(\mathcal{C}, \mathcal{C})$ it will induce a natural adjunction $\mathcal{C} \rightleftarrows \mathrm{LMod}_T(\mathcal{C})$. The $\mathrm{LMod}_T(\mathcal{C})$ here is often denoted by $Alg_T(\mathcal{C})$ in classical references.

Definition

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. We will say G is monadic iff there exists a monad $T \in Fun(\mathcal{C}, \mathcal{C})$ and an equivalence $G_0 : \mathcal{D} \to \mathrm{LMod}_T(\mathcal{C})$ such that G is equivalent to the composition of G_0 with the forgetful functor $\mathrm{LMod}_T(\mathcal{C}) \to C$.

By remark above, any monadic functor is a right adjoint functor.

Question

Given an adjunction $\mathcal{C} \rightleftarrows^F_G \mathcal{D}$, when is G monadic?

We will see that the Barr-Beck Theorem provides a full answer.

Barr-Beck Theorem

Theorem (Classical Barr-Beck Theorem)

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor which admits a left adjoint. Then the following are equivalent:

1. *The functor G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$.*
2. *There exists a monoidal category $\mathcal{E}^{\otimes}$, a left action of $\mathcal{E}^{\otimes}$ on $\mathcal{C}$, an algebra object $A \in \mathrm{Alg}(\mathcal{E})$ and an equivalence $G' : \mathcal{D} \simeq \mathrm{LMod}_A(\mathcal{C})$ such that G is equivalent to the composition of G' with the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$.*
3. *The functor G satisfies the following 2 conditions:*
   (a) The functor $G : \mathcal{D} \to \mathcal{C}$ is conservative; that is, a morphism $f : Y \to Y_0$ in $\mathcal{D}$ is an equivalence if and only if $G(f)$ is an equivalence in $\mathcal{C}$.
   (b) Let $V_1 \rightrightarrows V_0$ be a pair of morphisms of $\mathcal{D}$ which is G-split. Then it admits a colimit in $\mathcal{D}$, and that colimit is preserved by G.

∞-Barr-Beck Theorem

Theorem (∞-Barr-Beck Theorem)

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories which admits a left adjoint. Then the following are equivalent:

1. *The functor G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$.*
2. *There exists a monoidal ∞-category $\mathcal{E}^{\otimes}$, a left action of $\mathcal{E}^{\otimes}$ on $\mathcal{C}$, an algebra object $A \in \mathrm{Alg}(\mathcal{E})$ and an equivalence $G' : \mathcal{D} \simeq \mathrm{LMod}_A(\mathcal{C})$ such that G is equivalent to the composition of G' with the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$.*
3. *The functor G satisfies the following 2 conditions:*
   (a) The functor $G : \mathcal{D} \to \mathcal{C}$ is conservative; that is, a morphism $f : Y \to Y_0$ in D is an equivalence if and only if $G(f)$ is an equivalence in C.
   *(b) Let V_* be a be a **simplicial object** of $\mathcal{D}$ which is G-split. Then it admits a colimit in $\mathcal{D}$, and that colimit is preserved by G.*

Examples of monadicity

Example

1. Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category. Then the forgetful functor ${}_A\mathrm{BMod}_B(\mathcal{C}) \to \mathcal{C}$ is monadic, where the monad T is given by $A \otimes (-) \otimes B$.
2. Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories and $G : \mathcal{D} \to \mathcal{C}$ be a functor which admits a left adjoint. If G is monadic, then so is $Sp(G) : Sp(\mathcal{D}) \to Sp(\mathcal{C})$.
3. By the fact that $\Omega^{\infty} : Sp_{\geq 0} \to \mathcal{S}$ preserves sifted colimits (which includes geometric realization), the forgetful functor $\mathrm{Alg}_{\mathbb{E}_k}(Sp_{\geq 0}) \to Sp_{\geq 0} \to \mathcal{S}$ is monadic.

   In other words, we can identify connective $\mathbb{E}_k$-rings as spaces equipped with some additional structures, i.e. $\mathbb{E}_k$-(semi)ring space structure. Roughly speaking, it consists of an addition and multiplication which satisfy the axioms for a ring (commutative if $k \geq 2$), up to coherent homotopy.

Higher Morita theory

Question

Morita theory began with a natural question: **To what extent does the module category Mod_R determine the ring R itself?** (Also known as recognition principles)

Firstly, we can consider the realization problem: when can a category $\mathcal{C}$ be realized as some module category Mod_R? That was answered by Schwede–Shipley.

Theorem (Schwede–Shipley 2003)

Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is equivalent to RMod_R, for some $\mathbb{E}_1$-ring R, if and only if $\mathcal{C}$ is presentable and there exists a compact object $C \in \mathcal{C}$ which generates $\mathcal{C}$ in the following sense: if $D \in \mathcal{C}$ is an object having the property that $\mathrm{Ext}^n_{\mathcal{C}}(C, D) \simeq 0$ for all $n \in \mathbb{Z}$, then $D \simeq 0$.

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Higher Morita theory

Secondly, we wish characterize functors between module categories. We begin with a classical Morita theorem.

Theorem (classical Morita theorem)

Let R and R' be associative rings, and let $\mathrm{LFun}\left(\mathrm{RMod}_R, \mathrm{RMod}_{R'}\right)$ be the category of functors from RMod_R to $\mathrm{RMod}_{R'}$ which preserve small colimits. Then the relative tensor product functor $\otimes_R : \mathrm{RMod}_R \times_R \mathrm{BMod}_{R'} \to \mathrm{RMod}_{R'}$ induces an equivalence of categories

$$_R\mathrm{BMod}_{R'} \to \mathrm{LFun}\left(\mathrm{RMod}_R, \mathrm{RMod}_{R'}\right).$$

By the theorem above we can see two equivalent module categories does not imply two equivalent rings.

Actually, this leads us to the definition of Morita equivalence between rings.

Definition (Morita equivalence)

Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category compatible with geometric realization. Given $R, R' \in \mathrm{Alg}(\mathcal{C})$, we say that they are Morita equivalent iff there exists ${}_R M_{R'} \in {}_R\,\mathrm{BMod}_{R'}$ and ${}_{R'} N_R \in {}_{R'}\,\mathrm{BMod}_R$ such that ${}_R M \otimes_{R'} N_R \simeq {}_R R_R$ in ${}_R\,\mathrm{BMod}_R$ and that ${}_{R'} N \otimes_R M_{R'} \simeq {}_{R'} R'_{R'}$ in ${}_{R'}\,\mathrm{BMod}_{R'}$.

Definition (Morita category)

Let $\mathcal{K}$ be a small collection of simplicial sets which includes $\mathrm{N}(\mathbf{\Delta})^{op}$ and $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a monoidal ∞-category compatible with $\mathcal{K}$-colimits. For every algebra object $A \in \mathrm{Alg}(\mathcal{C})$, the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ is left-tensored over $\mathcal{C}$, and can therefore be identified with a left $\mathcal{C}$-module object of $\mathcal{C}at_{\infty}(\mathcal{K})$.

We let $\mathcal{M}orita(\mathcal{C})$ denote the full subcategory of $\mathrm{LMod}_{\mathcal{C}}\left(\mathrm{Cat}_{\infty}(\mathcal{K})\right)$ spanned by objects of the form $\mathrm{RMod}_A(\mathcal{C})$, where $A \in \mathrm{Alg}(\mathcal{C})$. We will refer to $\mathcal{M}orita(\mathcal{C})$ as the Morita ∞-category of $\mathcal{C}$.

Formal and general arguments

Definition

1. Let $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K})$ be the large ∞-category (informally) described as follows:
   - objects are pairs $(\mathcal{C}^\otimes, A)$ where $\mathcal{C}^\otimes$ is a monoidal ∞-category compatible with $\mathcal{K}$-colimits and $A \in \mathrm{Alg}(\mathcal{C})$.
   - a morphism from $(\mathcal{C}^\otimes, A)$ to $(\mathcal{D}^\otimes, B)$ is a monoidal functor $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ preserving $\mathcal{K}$-colimits and a morphism $F(A) \to B$ in $\mathrm{Alg}(\mathcal{D})$.
2. Let $\mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K})$ be the large ∞-category (informally) described as follows:
   - objects are pairs $(\mathcal{C}^\otimes, \mathcal{M})$ where $\mathcal{M}$ is left-tensored over $\mathcal{C}$ where tensor product is compatible with $\mathcal{K}$-colimits.
   - a morphism from $(\mathcal{C}^\otimes, \mathcal{M})$ to $(\mathcal{D}^\otimes, \mathcal{N})$ is a monoidal functor $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ preserving $\mathcal{K}$-colimits and a $\mathcal{C}$-linear functor $\mathcal{M} \to \mathcal{N}$ in $\mathrm{Alg}(\mathcal{D})$.

Proposition

The natural forgetful functors $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K}) \to \mathrm{Mon}_{\mathit{Ass}}^{\mathcal{K}}(\mathrm{Cat}_\infty)$ *and* $\mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K}) \to \mathrm{Mon}_{\mathit{Ass}}^{\mathcal{K}}(\mathrm{Cat}_\infty)$ *are coCartesian fibrations.*

We can define a functor $\Theta : \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K}) \to \mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K})$ by $(\mathcal{C}^\otimes, A) \mapsto \mathrm{RMod}_A(\mathcal{C})$, where the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ of right A-module objects of $\mathcal{C}$ is viewed as an ∞-category left-tensored over $\mathcal{C}$.

Proposition

The construction Θ above is a coCartesian-preserving functor,

$$\begin{tikzcd}
\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K}) \arrow[rr, "\Theta"'] \arrow[dr] & & \mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K}) \arrow[dl] \\
& \mathrm{Mon}_{\mathit{Ass}}^{\mathcal{K}}(\mathrm{Cat}_\infty) &
\end{tikzcd}$$

whose restriction on the fiber of any $\mathcal{C}^\otimes \in \mathrm{Mon}_{\mathit{Ass}}^{\mathcal{K}}(\mathrm{Cat}_\infty)$ gives a functor $\mathrm{Alg}(\mathcal{C}) \xrightarrow{\Theta_{\mathcal{C}}} \mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_\infty^{\mathcal{K}})$.

The essential image of $\Theta_{\mathcal{C}}$ is exactly the Morita category of $\mathcal{C}$. Roughly speaking, Morita theory is to study the properties of the functor Θ.

Lemma

Let $\mathcal{K}$ be a small collection of simplicial sets. Let $\mathcal{S}(\mathcal{K}) \subset \mathcal{S}$ be the maximal full subcategory containing Δ^0 and closed under $\mathcal{K}$-colimits, which inherits a Cartesian monoidal structure from $\mathcal{S}$ and we denote as $\mathcal{S}(\mathcal{K})^\times$. Then the pair $(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})$ is an initial object of $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K})$.

Definition

It follows the Lemma that the forgetful functor $\theta : \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K})_{(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})/} \to \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K})$ is a trivial Kan fibration. We let Θ_* denote the composition

$$\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K}) \simeq \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K})_{(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})} \xrightarrow{\Theta} \mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K})_{\mathfrak{M}/}$$

where the first map is given by a section of θ and $\mathfrak{M} = \Theta(\mathcal{S}(\mathcal{K})^\times, \mathbf{1}) = (\mathcal{S}(\mathcal{K})^\times, \mathcal{S}(\mathcal{K}))$.

An object of the ∞-category $\mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K})_{\mathfrak{M}/}$ is given by a morphism $(\mathcal{S}(\mathcal{K})^\times, \mathcal{S}(\mathcal{K})) \to (\mathcal{C}^\otimes, \mathcal{M})$ in $Cat_\infty^{\mathrm{Mod}}(\mathcal{K})$, given by the unique monoidal functor $\mathcal{S}(\mathcal{K})^\times \to \mathcal{C}^\otimes$ which preserves $\mathcal{K}$-indexed colimits together with a functor $f : \mathcal{S}(\mathcal{K}) \to \mathcal{M}$ which preserves $\mathcal{K}$-indexed colimits. Such a functor is determined uniquely up to equivalence by the object $f\left(\Delta^0\right) \in \mathcal{M}$.

Consequently, we can informally regard $\mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K})_{\mathfrak{M}/}$ as an ∞-category whose objects are triples $(\mathcal{C}^\otimes, \mathcal{M}, M)$, where $(\mathcal{C}^\otimes, \mathcal{M}) \in \mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K})$ and $M \in \mathcal{M}$ is an object.

Theorem

Let $\mathcal{K}$ be a small collection of simplicial sets containing $\mathrm{N}(\boldsymbol{\Delta})^{op}$. Then the functor $\Theta_* : \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K}) \to \mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K})_{\mathfrak{M}/}$ is fully faithful. Therefore when it restricts on any fiber $\mathcal{C}^\otimes \in \mathrm{Mon}_{\mathit{Ass}}^{\mathcal{K}}(\mathrm{Cat}_\infty)$, the

$$\mathrm{Alg}(\mathcal{C}) \to \mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_\infty^{\mathcal{K}})_{\mathcal{C}/}$$

informally given by $A \mapsto \mathcal{C} \xrightarrow{-\otimes A} \mathrm{RMod}_A(\mathcal{C})$, is also fully faithful .

In this sense of the full subcategory, algebras are determined by there module categories.

Essential image of Θ

By Barr-Beck theorem it is not hard to describe the essential image of Θ.

Theorem

Let $\mathcal{C}$ be a monoidal ∞-category. Assume that $\mathcal{C}$ admits $\mathrm{N}(\mathbf{\Delta})^{op}$-colimits and that the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves $\mathrm{N}(\mathbf{\Delta})^{op}$-colimits. Let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}$ and let $M \in \mathcal{M}$ be an object. Then there exists an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ and an equivalence $\mathrm{RMod}_A(\mathcal{C}) \simeq \mathcal{M}$ of ∞-categories left-tensored over $\mathcal{C}$ which carries A to M if and only if the following conditions are satisfied:
(1) The ∞-category $\mathcal{M}$ admits $\mathrm{N}(\mathbf{\Delta})^{op}$-colimits. And the action map $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ preserves $\mathrm{N}(\mathbf{\Delta})^{op}$-colimits.
(2) The functor $F : \mathcal{C} \to \mathcal{M}$ given by $F(C) = C \otimes M$ admits a right adjoint G such that G is conservative and preserves $\mathrm{N}(\mathbf{\Delta})^{op}$-colimits.
(3) For every object $N \in \mathcal{M}$ and every object $C \in \mathcal{C}$, the evident map $F(C \otimes G(N)) \simeq C \otimes G(N) \otimes M \simeq C \otimes FG(N) \to C \otimes N$ is adjoint to an equivalence $C \otimes G(N) \xrightarrow{\sim} G(C \otimes N)$.

In this case, we actually have $A \simeq End_{\mathcal{M}}(M)$.

Universal properties of $RMod_A(C)$

Theorem

Let $\mathcal{K}$ be a collection of simplicial sets which includes $\mathrm{N}(\mathbf{\Delta})^{op}$, let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category, and $\mathcal{M}$ an ∞-category left-tensored over $\mathcal{C}$. Assume that $\mathcal{C}$ and $\mathcal{M}$ admit $\mathcal{K}$-indexed colimits, and that the tensor product functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ preserve $\mathcal{K}$-indexed colimits separately in each variable. Let A be an algebra object of $\mathcal{C}$, and let θ denote the composition

$$\begin{aligned}
&\mathrm{LinFun}^{\mathcal{K}}_{\mathcal{C}}\left(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M}\right) \subseteq \mathrm{LinFun}_{\mathcal{C}}\left(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M}\right) \\
&\xrightarrow{\theta'} \mathrm{Fun}\left(\mathrm{LMod}_A\left(\mathrm{RMod}_A(\mathcal{C})\right), \mathrm{LMod}_A(\mathcal{M})\right) \\
&\xrightarrow{\theta''} \mathrm{LMod}_A(\mathcal{M}),
\end{aligned}$$

where θ'' is given by evaluation at the A-bimodule given by ${}_A A_A$. Then θ is an equivalence of ∞-categories.

Universal properties of $RMod_A(C)$

Corollary

Particularly, when $\mathcal{M} = RMod_B(\mathcal{C})$ we have an equivalence of ∞-categories $\mathrm{LinFun}_{\mathcal{C}}\left(\mathrm{RMod}_A(\mathcal{C}), \mathrm{RMod}_B(\mathcal{C})\right) \simeq \mathrm{LMod}_A\left(\mathrm{RMod}_B(\mathcal{C})\right) \simeq {}_A\mathrm{BMod}_B(\mathcal{C})$..

That is, every $\mathcal{C}$-linear functor from $\mathrm{RMod}_A(\mathcal{C})$ to $\mathrm{RMod}_B(\mathcal{C})$ which preserves $\mathcal{K}$-indexed colimits is given by the formula $M \mapsto M \otimes_A K$, for some bimodule object ${}_AK_B \in {}_A\mathrm{BMod}_B(\mathcal{C})$. It follows from this description that $\mathrm{Morita}(\mathcal{C})$ is independent of the choice of $\mathcal{K}$, so long as $\mathcal{K}$ includes $\mathrm{N}(\boldsymbol{\Delta})^{op}$.

Proposition

The equivalence above satisfies the composition law

$$\begin{array}{ccc}
{}_C\mathrm{BMod}_B(\mathcal{C})^{\simeq} \times {}_B\mathrm{BMod}_A(\mathcal{C})^{\simeq} & \longrightarrow & \mathrm{LinFun}_{\mathcal{C}}\left(\mathrm{RMod}_C(\mathcal{C}), \mathrm{RMod}_B(\mathcal{C})\right) \times \mathrm{LinFun}_{\mathcal{C}}\left(\mathrm{RMod}\right. \\
\downarrow & & \downarrow \\
{}_C\mathrm{BMod}_A(\mathcal{C})^{\simeq} & \longrightarrow & \mathrm{LinFun}_{\mathcal{C}}\left(\mathrm{RMod}_C(\mathcal{C}), \mathrm{RMod}_A(\mathcal{C})\right)
\end{array}$$

Now we can give a good description of Morita category.

Definition

Define a new $h\mathcal{S}$-enriched category $\mathrm{Morita}'(\mathcal{C})$ as follows:
- objects are algebra objects $A \in \mathrm{Alg}(\mathcal{C})$.
- Given a pair of objects $A, B \in \mathrm{Alg}(\mathcal{C})$, the mapping space $\mathrm{Map}_{\mathrm{Morita}'(\mathcal{C})}(A, B)$ can be identified with the Kan complex ${}_A\mathrm{BMod}_B(\mathcal{C})^{\simeq}$.
- Given a triple of objects $A, B, C \in \mathrm{Alg}(\mathcal{C})$, the composition law

$$ {}_C\mathrm{BMod}_B(\mathcal{C})^{\simeq} \times {}_B\mathrm{BMod}_A(\mathcal{C})^{\simeq} \to {}_C\mathrm{BMod}_A(\mathcal{C})^{\simeq} $$

is given by $(M, N) \mapsto M \otimes_B N$.

Corollary

The natural enriched functor $\mathrm{Morita}'(\mathcal{C}) \to \underline{h\,\mathrm{Morita}(\mathcal{C})}$ is an equivalence of $h\mathcal{S}$-enriched categories.

Brauer ∞-group

Actually, $\mathit{Morita}(\mathcal{C})$ is the underlying $(\infty,1)$-category of the $(\infty,2)$-category $\mathit{MORITA}(\mathcal{C})$, whose mapping ∞-categories are ${}_A\mathrm{BMod}_B(\mathcal{C})$.

Definition (Brauer space)

We define the (big) Brauer space $\mathbf{Br}(\mathcal{C})$ with respect to $\mathcal{C}^\otimes$ as $\mathit{Morita}(\mathcal{C})^\simeq$, i.e. the underlying $(\infty,0)$-category, also known as the maximal Kan complex.

It is often know that there is a group structure, called the Brauer group. Actually we will see that $\mathbf{Br}(\mathcal{C})$ has a natural group-like $\mathbb{E}_\infty$-structure, therefore $\pi_0\mathbf{Br}(\mathcal{C})$ is indeed a (big) group. But before that, we need to introduce monoidal Morita theory.

Proposition

The $\Theta : \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K}) \to \mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K})$ can be naturally enhanced to a symmetric monoidal functor $\Theta^\otimes : \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K})^\otimes \to \mathrm{Cat}_\infty^{\mathrm{Mod}}(\mathcal{K})^\otimes$, whose restriction on each fiber of $\mathcal{C}^\otimes$ is a symmetric monoidal functor $\Theta_\mathcal{C}^\otimes : \mathrm{Alg}(\mathcal{C})^\otimes \to \mathrm{LMod}_\mathcal{C}\left(\mathrm{Cat}_\infty(\mathcal{K})\right)^\otimes$, informally given by $A \mapsto \mathit{RMod}_A(\mathcal{C})^\otimes$.

Brauer ∞-group

Definition

The monoidal enhancement makes $\mathbf{Br}(\mathcal{C}) \subset \mathit{Morita}(\mathcal{C}) \subset \mathrm{LMod}_{\mathcal{C}}\left(\mathrm{Cat}_{\infty}(\mathcal{K})\right)$ inherit a natural symmetric monoidal structure, denoted as $\mathbf{Br}(\mathcal{C})^{\otimes}$ and $\mathit{Morita}(\mathcal{C})^{\otimes}$.
By the natural equivalence $\mathrm{CAlg}(\mathcal{S}) \simeq \mathrm{Mon}(\mathcal{S})$ between $\mathbb{E}_{\infty}$-spaces and symmetric monoidal groupoids, the $\mathbf{Br}(\mathcal{C})^{\otimes}$ naturally corresponds an $\mathbb{E}_{\infty}$-space.

Now we turn to the case of $\mathbb{E}_{\infty}$-rings.

Definition

Let R be an $\mathbb{E}_{\infty}$-ring. Taking $\mathcal{C}^{\otimes} = \mathrm{Mod}_R^{\otimes}$ we get a symmetric monoidal functor $\Theta^{\otimes}_{\mathrm{Mod}_R} : \mathrm{Alg}_R^{\otimes} = \mathrm{Alg}(\mathrm{Mod}_R)^{\otimes} \to \mathrm{LMod}_{\mathrm{Mod}_R}\left(\mathrm{Pr}^L\right)^{\otimes} = \mathrm{LMod}_{\mathrm{Mod}_R}\left(\mathrm{Pr}^L_{st}\right)^{\otimes} = \mathrm{Cat}_R^{\otimes}$,
where the latter is often called R-linear categories, informally given by $A \mapsto \mathrm{RMod}_A^{\otimes}$.
Because all RMod_A are compactly generated, $\Theta^{\otimes}_{\mathrm{Mod}_R}$ factors through $\mathrm{Alg}_R^{\otimes} \to \mathrm{Cat}_{R,\omega}^{\otimes}$, where $\mathrm{Cat}_{R,\omega}^{\otimes}$ is the category of compactly generated R-linear categories.

Azumaya algebras

Now we introduce Azumaya algebras and (small) Brauer groups.

Definition

Let R be an $\mathbb{E}_\infty$-ring. An R-algebra A is an Azumaya R-algebra if A is a compact generator of Mod_R and if the natural R-algebra map

$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$$

is an equivalence of R-algebras.

Note that if A is an Azumaya R-algebra, then, by definition, $A \otimes_R A^{\mathrm{op}}$ is Morita equivalent to R. The standard example of an Azumaya algebra is the endomorphism algebra $\mathrm{End}_R(P)$ of a compact generator $P \in \mathrm{Mod}_R$.

Theorem (Toën 2012)

If $R = \mathrm{H}k$, where k is an algebraically closed field, then every Azumaya R-algebra is Morita equivalent to R.

Azumaya algebras

Proposition

Let A be an R-algebra. Then, A is compact in Alg_R if and only if Mod_A is compact in the $\mathrm{Cat}_{R,\omega}$.

Corollary

Compactness in Alg_R is a Morita-invariant property.

Theorem (Antieau–Gepner 2012)

Let $\mathcal{C} \in \mathrm{Cat}_{R,\omega}$. Then it is invertible in $\mathrm{Cat}_{R,\omega}$ if and only if $\mathcal{C}$ is equivalent to Mod_A for an Azumaya R-algebra A.

This leads to small Brauer groups.

Proposition

The image of Azumaya algebras under $\Theta^{\otimes}_{\mathrm{Mod}_R} : \mathrm{Alg}^{\otimes}_R \to \mathrm{Cat}^{\otimes}_{R,\omega}$ is exactly those invertible objects $\mathrm{Cat}_{R,\omega}$.

Definition

We define the (small) Brauer space $Br(R)$ as the essential image of Azumaya algebras in $\mathbf{Br}(R)$. By the previous argument, we have the following chain of symmetric submonoidal categories.

$$Br(R)^{\otimes} \subset \mathbf{Br}(R)^{\otimes} \subset Morita(R)^{\otimes} \subset \mathrm{Cat}_R^{\otimes}$$

That makes $Br(R)$ become an $\mathbb{E}_\infty$-space. And we have $Br(R) \simeq \mathrm{Pic}(\mathrm{Cat}_{R,\omega})$ as group-like $\mathbb{E}_\infty$-spaces.

Theorem (Antieau–Gepner 2012)

For R a connective E_∞ ring, any Azumaya R-algebra A is étale locally trivial: there is an étale cover $R \to S$ such that $A \otimes_R S$ is morita equivalent to S.

Theorem (Antieau–Gepner 2012)

For R a connective E_∞ ring, the functor $Br : \mathrm{CAlg}_R^{\geq 0} \to \mathrm{Gpd}_\infty$ restricting on connective $\mathbb{E}_\infty$-R-algebras is a sheaf for the étale topology.

Calculation

Theorem (Antieau–Gepner 2012)

Let X be an object of $\mathrm{Shv}_R^{\mathrm{et}}$. Then, there is a conditionally convergent spectral sequence

$$\mathrm{E}_2^{p,q} = \begin{cases} \mathrm{H}_{\mathrm{et}}^p\left(X, \pi_q Br\right) & p \leq q \\ 0 & p > q \end{cases} \Rightarrow \pi_{q-p} Br(X)$$

with differentials d_r of degree $(r, r-1)$. If X is affine or discrete, then the spectral sequence converges completely.

Proof. Because the Brauer sheaf Br is hypercomplete, the map from Br to the limit of its Postnikov tower $Br \to \lim_n \tau_{\leq n} Br$ is an equivalence. Taking sections preserves limits, so that

$$Br(X) \to \lim_n \left(\tau_{\leq n} Br\right)(X)$$

is also an equivalence. Thus, $Br(X)$ is the limit of a tower, and to any such tower there is an associated spectral sequence.

Calculation

If X is affine or discrete, then the spectral sequence degenerates at some finite page. And if X is discrete the spectral sequence collapses entirely at the E_2-page. So, suppose that $X = \operatorname{Spec} S$. Then, $Br(X)$ can be computed on the small étale site on $\operatorname{Spec} S$. But, as mentioned above, this site is the nerve of a discrete category, the small étale site on $\operatorname{Spec} \pi_0 S$. Therefore, $\mathrm{H}^p_{\mathrm{et}}(\operatorname{Spec} S, \pi_q Br) \cong \mathrm{H}^p_{\mathrm{et}}(\operatorname{Spec} \pi_0 S, \pi_q Br)$.

Corollary (Antieau–Gepner 2012)

If R is a connective $\mathbb{E}_\infty$-ring, then the homotopy groups of $Br(R)$ are described by

$$\pi_k Br(R) \cong \begin{cases} \mathrm{H}^1_{\mathrm{et}}(\operatorname{Spec} \pi_0 R, \mathbb{Z}) \times \mathrm{H}^2_{\mathrm{et}}(\operatorname{Spec} \pi_0 R, \mathbb{G}_m) & k = 0 \\ \mathrm{H}^0_{\mathrm{et}}(\operatorname{Spec} \pi_0 R, \mathbb{Z}) \times \mathrm{H}^1_{\mathrm{et}}(\operatorname{Spec} \pi_0 R, \mathbb{G}_m) & k = 1 \\ \pi_0 R^\times & k = 2 \\ \pi_{k-2} R & k \geq 3. \end{cases}$$

Corollary (Antieau–Gepner 2012)

The Brauer $\mathbb{E}_\infty$-group $Br(\mathbb{S})$ of the sphere spectrum $\mathbb{S}$ is zero, i.e. its underlying space is contractable.