Vector Bundles on Compact Riemann Surfaces

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Lecture 1. Topological Vector Bundles.

1 Basic definitions Real and complex vector bundle on a topological space, rank, local triviality, trivialization, section, basis, restriction, pull-back, description by transition functions, description by clutching function, operations on vector bundles, direct sum, exterior product, determinant bundle, subbundle, quotient bundle, Euclidean and Hermitian metrics, supplement to a subbundle defined by a Euclidean or Hermitian metric. Homomorphism of vector bundles.

2 Basic notation. If E is a vector bundle on X, and (U_i) is an open cover of X, then a basis for E over U_i is denoted by $e_i = (e_{i,1}, \ldots, e_{i,n})$, to be regarded as an $n \times 1$-matrix in $\Gamma(U_i, E)$. The transition function $g_{i,j} : U_i \cap U_j \to GL(n, \mathbb{C})$ satisfies the matrix equation $e_i g_{i,j} = e_j$. Note that $g_{i,j} g_{j,k} = g_{i,k}$ over $U_i \cap U_j \cap U_k$.

3 Exercise Any vector bundle on a closed interval I is trivial. Any complex vector bundle on S^1 is trivial. More generally, any complex vector bundle on a 1-dimensional finite simplicial complex is trivial. What are the real line bundles on S^1 and more generally what are the real vector bundles on a 1-dimensional finite simplicial complex?

4 Theorem If X is any compact hausdorff space (more generally, if X is a paracompact space), and E is a vector bundle on $X \times I$ where I is a closed interval, then the restrictions E_t are all isomorphic as bundles on X, as t varies over I.

5 Basic Lemma defining degree. Let X be a connected compact hausdorff oriented topological 2-manifold. Let $P \in X$, let U be an open neighbourhood of P with a chosen homeomorphism $\phi : U \to U_r \subset \mathbb{C}$ with $\phi(P) = 0$ where $U_r = \{|z| < r\}$ denotes the open disc of radius $r > 1$ around 0. Suppose that ϕ preserves orientation on X. Let $D_0 \subset U$ be the inverse image of the unit closed disc $\{|z| \leq 1\}$, let $D_\infty \subset X$ be the closed set $X - \phi^{-1}(U_1)$, and let $S = D_0 \cap D_\infty$ which is homeomorphic to S^1 under ϕ. Then D_0 is contractible, while D_∞ is homotopic to the space of a finite simplicial complex of dimension ≤ 1. In particular, any complex vector bundle on D_0 or on D_∞ is trivial. Let L be any complex line bundle on X. Let e_0 be a nowhere vanishing section (basis) of $L|_{D_0}$ and let e_∞ be a nowhere vanishing section (basis) of $L|_{D_\infty}$. Let $g_{0,\infty} : S \to \mathbb{C}^*$ be the function defined by

$$e_0 g_{0,\infty} = e_\infty.$$

Then the winding number of $g_{0,\infty}$ depends only on L and is independent of the choice of P and of $\phi : U \to U_r$.

6 Definition With notation as above, the **degree** of a complex line bundle L on X is the winding number $\deg(L) \in \mathbb{Z}$ of the map $g_{0,\infty} : S \to \mathbb{C}^*$. If E is a complex vector bundle on X, the degree of E is defined to be the degree of the line bundle $\det(E)$.

7 Theorem Sending $L \mapsto \deg(L)$ defines a bijection between the set of all isomorphism classes of complex line bundles on X and the set $\mathbb{Z}$ of integers. The set of all isomorphism classes of complex line bundles on X is a group $TopPic(X)$ under tensor product, and we have $\deg(L \otimes M) = \deg(L) + \deg(M)$, so degree defines a group isomorphism $TopPic(X) \to \mathbb{Z}$.

8 Theorem Any complex vector bundle E on X of rank $n \geq 1$ is topologically isomorphic to $L \oplus \mathbf{1}^{\oplus n-1}$ where L is the line bundle $\det(E)$. Mapping $E \mapsto (\mathrm{rank}(E), \deg(E))$ gives a bijection between the set of all isomorphism classes of non-zero vector bundles on X and the set $\mathbb{Z}^+ \times \mathbb{Z}$ where $\mathbb{Z}^+$ is the set of positive integers.

Lecture 2. Sheaf Cohomology.

9 Standard sheaves on a Riemann surface X:

$\mathbb{Z}_X$ and $\mathbb{C}_X$ the constant sheaves corresponding to $\mathbb{Z}$ and $\mathbb{C}$.

$\mathcal{C}_X$ the sheaf of continuous complex functions on X.

$\mathcal{C}_X^*$ the sheaf of no-where vanishing continuous complex functions on X.

$\mathcal{O}_X$ the sheaf of holomorphic functions on X.

$\mathcal{O}_X^*$ the sheaf of no-where vanishing holomorphic functions on X.

Ω_X^1 the sheaf of holomorphic 1-forms on X.

10 Definition A sheaf of $\mathcal{C}_X$-modules. A sheaf of $\mathcal{O}_X$-modules. Locally free sheaves (of finite ranks).

11 Proposition If E is a continuous (resp. holomorphic) vector bundle on X, then the sheaf of continuous (resp. holomorphic) sections of E is a locally free sheaf of $\mathcal{C}_X$-modules (resp. $\mathcal{O}_X$-modules) of the same rank as rank(E). This defines a functor which is an equivalence of categories from the category of topological (resp. holomorphic) vector bundles on X to the category locally free sheaves of $\mathcal{C}_X$-modules (resp. $\mathcal{O}_X$-modules) of finite ranks.

For any holomorphic line bundle L, we denote by the same letter L the sheaf of its holomorphic sections (which is an invertible sheaf). Conversely, given an invertible sheaf $\mathcal{F}$, we denote the corresponding line bundle by the same letter $\mathcal{F}$. In particular, we denote the trivial line bundle by the same notation $\mathcal{O}_X$ that we use for the sheaf of all holomorphic functions.

12 Theorem: Cohomological description of line bundles. Let L be a topological (resp. holomorphic) line bundle on X. Let U_i be an open cover, with topological (resp. holomorphic) bases e_i for $L|_{U_i}$. Let $U_{i,j} = U_i \cap U_j$. Let $g_{i,j} \in \mathcal{C}^*(U_{i,j})$ (resp. $g_{i,j} \in \mathcal{O}^*(U_{i,j})$) be defined by

$$e_i g_{i,j} = e_j$$

(with $g_{i,j} = 1$ when $U_i \cap U_j$ is empty). This is a Cech 1 cocycle with coefficients $\mathcal{C}_X^*$ (resp. $\mathcal{O}_X^*$) for the open cover (U_i) of X. Let $(L) \in H^1(X, \mathcal{C}_X^*)$ (resp. $(L) \in H^1(X, \mathcal{O}_X^*)$ be the corresponding cohomology class. Then (L) depends only on the isomorphism class $[L] \in TopPic(X)$ (resp. $[L] \in Pic(X)$) of L, and the map $[L] \mapsto (L)$ is an isomorphism $TopPic(X) \to H^1(X, \mathcal{C}_X^*)$ (resp. $Pic(X) \to H^1(X, \mathcal{O}_X^*)$). Note that the inclusion $\mathcal{O}_X^* \to \mathcal{C}_X^*$ induces a homomorphism $H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{C}_X^*)$, under which a holomorphically defined (L) maps to the corresponding continuously defined (L). This makes

13 The exponential sequence Let $\exp : \mathcal{C}_X \to \mathcal{C}_X^*$ (resp. $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$) be the map defined at the level of local sections by $f \mapsto e^{2\pi i f}$. This map is surjective at the level of germs, and we get short exact sequences of sheaves

$$0 \to \mathbb{Z}_X \to \mathcal{C}_X \stackrel{\exp}{\to} \mathcal{C}_X^* \to 0$$

and

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^* \to 0$$

called as exponential sequences.

14 Definition: First Chern class Let $\partial : H^1(X, \mathcal{C}_X^*) \to H^2(X, \mathbb{Z}_X)$ be the connecting morphism for the topological exponential sequence. For any $(L) \in H^1(X, \mathcal{C}_X^*)$, the second integral cohomology class defined by

$$c_1(L) = \partial(L) \in H^2(X, \mathbb{Z}_X)$$

is called the **first Chern class** of the topological line bundle L. If L is a holomorphic line bundle, the image of $(L) \in H^1(X, \mathcal{O}_X^*)$ under the connecting morphism for the holomorphic exponential sequence equals the same element $\partial(L)$ as for its underlying topological line bundle, so the above formula $c_1(L) = \partial(L)$ can also be applied in the holomorphic category.

The first Chern class $c_1(E)$ of a vector bundle is defined to be the first Chern class of its determinant line bundle $\det(E)$.

15 Exercise Show that degree (resp. first Chern class) of a direct sum $E \oplus F$ is the sum of the individual degrees (resp. first Chern classes). Deduce that degree (resp. first Chern class) is additive in short exact sequences.

16 Theorem Let X be a connected compact hausdorff oriented topological 2-manifold, $\eta_X \in H^2(X, \mathbb{Z}_X)$ denotes the positive generator of $H^2(X, \mathbb{Z}_X)$, then for any topological complex vector bundle E on X we have

$$c_1(E) = \deg(E)\, \eta_X.$$

Proof As c_1 and deg are both additive, the above follows if we prove the equality $c_1(\mathcal{O}_X(P)) = \eta_X$ for the line bundle $\mathcal{O}_X(P)$ defined by the divisor P for any $P \in X$ (see 17). The equality $c_1(\mathcal{O}_X(P)) = \eta_X$ is the combination of the basic calculations 80 and 81 given in detail at the end of these notes.

17 Divisors and rank 1 locally free sheaves. Let X be a Riemann surface, and $Y \subset X$ a subset. The group $Div(Y)$ of all divisors on Y is the free abelian group over the set of points of Y. For any open subset U and a non-zero meromorphic function f on U, note that f has only finitely many zeros and poles on U, so that we can define an element $\mathrm{div}(f) \in Div(U)$ to be the formal sum

$$\mathrm{div}(f) = \sum_{P \in U} \nu_P(f) P$$

where $\nu_P(f)$ is the order of f at P. Given any divisor $D \in Div(X)$, we define a sheaf $\mathcal{O}_X(D)$ on X as follows. For any open $U \subset X$, we can write uniquely $D = D|_U + D|_{X-U}$ where $D|_U \in Div(U)$, and $D|_{X-U} \in Div(X - U)$. We take $\mathcal{O}_X(D)(U)$ to be the set of all meromorphic functions f on U such that

$$\mathrm{div}(f) + D|_U \geq 0$$

where we say that a divisor is ≥ 0 if all its coefficient integers are ≥ 0. The set $\mathcal{O}_X(D)(U)$ is in fact a $\mathcal{O}(U)$-module, and $\mathcal{O}_X(D)$ is a sheaf of $\mathcal{O}_X$-modules, which is locally free of rank 1.

18 As $\mathrm{div}(f_1 f_2) = \mathrm{div}(f_1) + \mathrm{div}(f_2)$, we have a natural isomorphism

$$\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \to \mathcal{O}_X(D_1 + D_2) : f_1 \otimes f_2 \mapsto f_1 f_2.$$

Because locally free $\mathcal{O}_X$-modules of rank 1 form a group $(Pic(X))$ under tensor product, they are called as **invertible sheaves**. The above defines a homomorphism $Div(X) \to Pic(X)$, which is surjective by Theorem 20.

19 Proposition. Let $\deg : Div(X) \to \mathbb{Z}$ be the homomorphism defined by sending any point P to 1. Then we have

$$\deg(D) = \deg(\mathcal{O}_X(D))$$

where the right-hand-side denotes the degree of the line bundle corresponding to the invertible sheaf $\mathcal{O}_X(D)$.

20 Theorem Every holomorphic line bundle L on a compact Riemann surface admits a non-zero meromorphic section s. If $D = \mathrm{div}(s)$ is the divisor of s (which is well-defined in terms of local trivializations of L), then L is isomorphic to $\mathcal{O}_X(D)$.

21 Rational equivalence A divisor D on a compact Riemann surface X is said to be a **principal divisor** if there exists a non-zero global meromorphic function f on X such that $D = \mathrm{div}(f)$. Such divisors form a subgroup of $Div(X)$, called the group of **principal divisors**, which is contained in the group $Div^0(X)$ of divisors of degree 0. The holomorphic line bundle $\mathcal{O}_X(D)$ defined by a divisor D is trivial if and only if D is a principal divisor. Two divisors D and D' on a compact Riemann surface X are said to be **rationally equivalent** if $D - D'$ is a principal divisor. The holomorphic line bundles $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ are isomorphic if and only if D and D' are rationally equivalent.

Lecture 3. Serre Duality and Riemann Roch theorems.

The results in this and the next two lectures also hold in the algebraic category, where instead of a compact Riemann surface X, we have a non-singular projective curve X over an arbitrary algebraically closed base field k, and in place of the sheaf of holomorphic functions in Euclidean topology we have the sheaf of regular functions in the Zariski topology. In place of holomorphic locally free sheaves (or vector bundles), we will have locally free sheaves (or vector bundles) in the algebraic category. If the base field is $\mathbb{C}$, then the algebraic and the holomorphic objects 'coincide'.

All vector bundles and their homomorphisms considered below are assumed to be holomorphic.

22 Theorem Let X be a compact Riemann surface. For any holomorphic vector bundle E on X, the complex vector spaces $H^i(X, E)$ are finite dimensional, and $H^i(X, E) = 0$ for all $i \geq 2$.

23 Exercise Let L be a holomorphic line bundle with $\deg(L) < 0$. Then show that $H^0(X, L) = 0$.

24 Serre Duality Let X be a compact Riemann surface. Then there is an isomorphism $res : H^1(X, \omega_X) \to \mathbb{C}$ where $\omega_X = \Omega^1_X$ is the sheaf of holomorphic 1-forms on X. For any holomorphic vector bundle E, the bilinear pairing

$$H^0(X, E) \times H^1(X, \omega_X \otimes E^*) \to H^1(X, \omega_X) \overset{res}{\to} \mathbb{C}$$

is non-degenerate.

25 Poincare duality and residue map By Poincaré duality, we have an isomorphism $H^2(X, \mathbb{C}_X) \to \mathbb{C}$. The isomorphism $res : H^1(X, \Omega^1_X) \to \mathbb{C}$ is the composite

$$H^1(X, \Omega^1_X) \overset{\partial}{\to} H^2(X, \mathbb{C}_X) \overset{D}{\to} \mathbb{C}$$

of the connecting homomorphism $\partial : H^1(X, \Omega^1_X) \to H^2(X, \mathbb{C}_X)$ of the differential sequence $0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to 0$ (which is an isomorphism) with the Poincaré duality isomorphism $D : H^2(X, \mathbb{C}_X) \to \mathbb{C}$. *(This feature is not available when working in the algebraic category!)*

26 In particular, $H^0(X, \omega_X)$ and $H^1(X, \mathcal{O}_X)$ are dual to each other. The non-negative integer $g_X = \dim H^0(X, \omega_X) = \dim H^1(X, \mathcal{O}_X)$ is called the genus of X.

27 If X is a compact Riemann surface then

$$\dim H^1(X, \mathbb{C}_X) = 2 \dim H^0(X, \omega_X).$$

In particular, g_X as defined above is the same as the topological genus of X regarded as an orientable connected compact hausdorff topological manifold of dimension 2.

28 Riemann-Roch Theorem Let X be a compact Riemann surface of genus g_X. For any holomorphic line bundle L on X we have

$$\dim H^0(X, L) - \dim H^1(X, L) = \deg(L) + 1 - g_X.$$

29 The line bundle ω_X has degree $2g_X - 2$ where g_X is the genus of X. If L is any line bundle with $\deg(L) > 2g_X - 2$, then $H^1(X, L) = 0$.

30 If E is a holomorphic vector bundle and s is a non-zero meromorphic section, then there exists a unique holomorphic line subbundle $L \subset E$ such that s is a meromorphic section of E.

31 Exercise Every holomorphic vector bundle on a compact Riemann surface admits a filtration by holomorphic vector subbundles

$$0 \subset E_1 \subset \ldots \subset E_r = E$$

such that each E_i/E_{i-1} is a line bundle. *(Hint: Shift to the algebraic category, take a full flag in the generic fiber, treat is as a section of the flag variety of E, and prolong it by valuative criterion of properness.)*

32 Riemann-Roch for vector bundles. For any holomorphic vector bundle E on X we have

$$\dim H^0(X, E) - \dim H^1(X, E) = \deg(E) + \mathrm{rank}(E)(1 - g_X).$$

33 If E is a holomorphic vector bundle and L is a holomorphic line bundle with $\deg(L) > 0$, then

$$H^0(X, E \otimes L^{-n}) = 0 \text{ and } H^1(X, E \otimes L^n) = 0 \text{ for } n \gg 0.$$

Lecture 4 part 1. Vector bundles on $\mathbf{P}^1$.

34 The genus of the complex projective line $\mathbf{P}^1$ is 0, and any compact Riemann surface of genus 0 is isomorphic to $\mathbf{P}^1$. The tautological line bundle $\mathcal{O}(-1)$ on $\mathbf{P}^1$ has degree -1, and the hyperplane bundle $\mathcal{O}(1)$ has degree 1 (and more generally the bundle $\mathcal{O}(n)$ has degree n for any $n \in \mathbb{Z}$).

35 Two holomorphic line bundles L and L' on $\mathbf{P}^1$ are isomorphic if and only if $\deg(L) = \deg(L')$. Consequently, the group homomorphism $Pic(X) \to \mathbb{Z} : (L) \mapsto \deg(L)$ is an isomorphism when $g_X = 0$.

36 Let X be a compact Riemann surface, and let $P, Q \in X$ be two distinct points. If the line bundles $\mathcal{O}_X(P)$ and $\mathcal{O}_X(Q)$ are isomorphic, then X is isomorphic to $\mathbf{P}^1$.

37 Example. The tautological exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^2 \to \mathcal{O}(1) \to 0$$

on $\mathbf{P}^1$ does not split.

38 Extensions via cohomology. Let E and F be holomorphic vector bundles on a compact Riemann surface X. Then we have a natural isomorphism of vector spaces

$$Ext^1(E, F) \cong H^1(X, F \otimes E^*).$$

39 Grothendieck's theorem. Any holomorphic vector bundle on $\mathbf{P}^1$ is a direct sum of line bundles. If $n \geq 1$ and $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ are n-tuples of integers such that the rank n holomorphic vector bundles $\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_n)$ and $\mathcal{O}(b_1) \oplus \ldots \oplus \mathcal{O}(b_n)$ are isomorphic, then $a_i = b_i$ for all $i = 1, \ldots, n$.

Proof. Let F be the given vector bundle, which we can assume to be of rank $n \geq 2$, and let $E = F(m) = F \otimes \mathcal{O}(m)$ where m is so chosen that $H^0(\mathbf{P}^1, E(-1)) = 0$ while $H^0(\mathbf{P}^1, E) \neq 0$. Therefore any non-zero global section s of E does not vanish at any point of $\mathbf{P}^1$. This implies that the natural homomorphism

$$\phi : H^0(\mathbf{P}^1, E) \otimes \mathcal{O} \to E$$

makes $H^0(\mathbf{P}^1, E) \otimes \mathcal{O}$ a subbundle of E. Let E' be the corresponding quotient bundle, so that we have a short exact sequence

$$0 \to \mathcal{O}^r \to E \to E' \to 0$$

where $r = \dim H^0(\mathbf{P}^1, E)$. By tensoring by $\mathcal{O}(-1)$, this gives an exact sequence

$$0 \to \mathcal{O}^r(-1) \to E(-1) \to E'(-1) \to 0.$$

Note that we have

$$H^0(\mathbf{P}^1, \mathcal{O}(-1)) = H^1(\mathbf{P}^1, \mathcal{O}(-1)) = 0.$$

Hence from the long exact cohomology sequence it follows that

$$H^0(\mathbf{P}^1, E'(-1)) = 0.$$

As $r > 0$, the rank of E' is less than rank of E. By induction on rank, we must have $E' = \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_k)$ where $k = n - r$ is the rank of E'. As $H^0(\mathbf{P}^1, E'(-1)) = 0$, we must have $d_i \leq 0$ for each i. This implies $-d_i \geq 0$, and so

$$H^1(\mathbf{P}^1, \mathcal{O}(-d_i)) = 0$$

for each i. Now note that

$$Ext^1(\mathcal{O}(d), \mathcal{O}) = H^1(\mathbf{P}^1, \mathcal{O}(-d)).$$

Hence it follows that

$$\begin{aligned} Ext^1(E', \mathcal{O}^r) &= Ext^1(E', \mathcal{O})^r \\ &= Ext^1(\oplus_i \mathcal{O}(d_i), \mathcal{O})^r \\ &= \oplus_i H^1(\mathbf{P}^1, \mathcal{O}(-d_i))^r \\ &= 0. \end{aligned}$$

Hence the short exact sequence $0 \to \mathcal{O}^r \to E \to E' \to 0$ splits, proving the existence of the direct sum decomposition. The uniqueness follows from the following stronger statement.

40 Uniqueness of filtration Let $E = \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_n)$ where $a_1 \geq \ldots \geq a_n$, and for any $a \in \mathbb{Z}$, let $E_a \subset E$ be the subbundle $\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_r)$ where $a_r \geq a > a_{r+1}$. Then under any automorphism $\phi : E \to E$, the subbundle E_a is carried into itself. If $c_1 > \ldots > c_m$ are the distinct integers among $a_1 \geq \ldots \geq a_n$, then we get a canonical filtration

$$0 \subset E_{c_1} \subset \ldots \subset E_{c_m} = E$$

of E, which is known as the **Harder-Narasimhan** filtration of E.

Proof The subbundle E_a can be defined as $F \otimes \mathcal{O}(a)$ where $F \subset E(-a)$ the image sheaf of the natural homomorphism $H^0(\mathbf{P}^1, E(-a)) \otimes \mathcal{O} \to E(-a)$.

Lecture 4 part 2. Atiyah-Krull-Remak-Schmidt theorem.

41 Let E be an holomorphic vector bundle on a connected compact complex manifold X, and let $\phi : E \to E$ be a holomorphic endomorphism. For any $P \in X$, let $f_P(t) \in \mathbb{C}[t]$ be the characteristic polynomial of the restriction of f to the fiber E_P. Then $f_P(t)$ is independent of P. We denote it by $f(t)$. Let $f(t)$ factor as $f(t) = g(t)h(t)$ where $g(t)$ and $h(t)$ are co-prime. Then the endomorphisms $g(\phi), h(\phi) : E \to E$ are of constant ranks. Let $E' \subset E$ and $E'' \subset E$ be the kernel subbundles of $g(\phi)$ and $h(\phi)$. Then $E = E' \oplus E''$.

42 A non-zero holomorphic vector bundle E on a connected complex manifold is called **indecomposable** if it is not the direct sum of two non-zero subbundles.

43 **Exercise** Any holomorphic vector bundle on a connected complex manifold is a direct sum of indecomposable subbundles.

44 **Lemma** Let E be an indecomposable holomorphic vector bundle on a connected compact complex manifold X. Then any endomorphism $\phi : E \to E$ is of the form

$$\phi = \lambda 1_E + N$$

where $\lambda \in \mathbb{C}$ and N is a nilpotent endomorphism of E.

45 **Exercise** Let E be an indecomposable holomorphic vector bundle on a connected compact complex manifold X. If ϕ and ψ are endomorphisms of E such that $\phi + \psi$ is an isomorphism, then at least one of ϕ and ψ is an isomorphism.

46 **Theorem** Let $E_1, \ldots, E_m$ and $F_1, \ldots, F_n$ be indecomposable holomorphic vector bundles on a connected compact complex manifold X, such that $E_1 \oplus \ldots \oplus E_m$ is isomorphic to $F_1 \oplus \ldots \oplus F_n$. Then $m = n$, and up to a permutation of indices, $E_1, \ldots, E_m$ are isomorphic to $F_1, \ldots, F_n$.

Proof Let $E_1 \oplus \ldots \oplus E_m = E = F_1 \oplus \ldots \oplus F_n$ where the E_i and the F_j are all indecomposable. Let $A_{j,i} : E_i \to F_j$ be the composite of the inclusion $E_i \to E$ and the projection $E \to F_j$, and let $B_{i,j} : F_j \to E_i$ be the composite of the inclusion $F_j \to E$ and the projection $E \to E_i$. Then we have

$$\sum_j B_{i,j} A_{j,i} = 1_{E_i}.$$

As E_i is indecomposable, and the above sum of endomorphisms is an isomorphism, it follows from an above result that at least one of the terms $B_{i,j}A_{j,i} : E_i \to E_i$ is an isomorphism. Now as F_j is indecomposable, the endomorphism $A_{j,i}B_{i,j} : F_j \to F_j$ is either nilpotent or an isomorphism. If $(A_{j,i}B_{i,j})^p = 0$ for some $P \geq 1$, then $(B_{i,j}A_{j,i})^{p+1} = B_{i,j}(A_{j,i}B_{i,j})^p A_{j,i} = 0$, which shows that $A_{j,i}B_{i,j} : F_j \to F_j$ is also an isomorphism. Hence $A_{j,i} : E_i \to F_j$ is an isomorphism. As $A_{j,i}$ is injective, the intersection of E_i with $\oplus_{k\neq j}F_k$ is zero. Hence, E is the direct sum

$$E = E_i \oplus (\oplus_{k\neq j}F_k)\,.$$

Hence E/E_i is isomorphic to both $\oplus_{r\neq i}E_r$ and $\oplus_{k\neq j}F_k$. As the rank of E/E_i is strictly less than that of E, the result now follows by induction on rank of E.

47 General form The Atiyah-Krull-Remak-Schmidt theorem holds more generally for any coherent sheaf E over any complete variety X over any base field k (not necessarily algebraically closed). The existence of such a decomposition is clear by considering the maximum dimension of the fibers. The uniqueness (up to isomorphism types and multiplicities of summands) is shown as follows. By the usual Krull-Remak-Schmidt theorem (see Lang *Algebra*), if M is a finite length module over a ring A (not necessarily commutative), then M has a decomposition as a finite direct sum of indecomposable submodules, and given any two such decompositions, the isomorphism classes and multiplicities of these indecomposable summands is unique, consequently there exists an automorphism ϕ of M which carries the first decomposition to the second (the proof uses the fact that any endomorphism of an indecomposable module of finite length is either an isomorphism or is nilpotent). Let $A = End(E)$ be the ring of global endomorphisms of the coherent sheaf E on X. Then A is a finite dimensional k-algebra, so A is a finite-length right A-module. If $E = E' \oplus E''$, then the idempotent projection $p' : E \to E$ with image E' and kernel E'' generates a right ideal $p'A \subset A$ which can be seen to be indecomposable. Hence any indecomposable decomposition $E = \oplus E_i$ of E gives a direct sum decomposition of A by indecomposable right ideals $A = \oplus p_i A$. If $E = \oplus F_j$ is another such decomposition, and $A = \oplus q_j A$ is the corresponding decomposition of A, then there exists an automorphism ϕ of the right A-module A which carries the first decomposition to the second, so there is a permutation σ on indices with $\phi(p_iA) = q_{\sigma(i)}A$. But such a ϕ is left-multiplication by an invertible element $\phi \in A$, therefore ϕ is an automorphism of E with $\phi(E_i) \subset F_{\sigma(i)}$ as $E_i = \mathrm{im}(p_i)$ and $F_i = \mathrm{im}(q_j)$. Hence we must have $\phi(E_i) = F_j$. This completes the proof of the theorem.

Lecture 5. Local systems, π_1-representations, and holomorphic connections.

48 Local system A complex local system $\mathcal{E}$ on a topological space X is a sheaf of complex vector spaces such that each $x \in X$ has an open neighbourhood U such that $\mathcal{E}|_U$ is isomorphic to the constant sheaf $\mathbb{C}^n_U$ for some $n \geq 0$ (note that n is locally constant). The number n is called the rank of $\mathcal{E}$ at x. It is constant if X is connected.

49 Example Let $\mathbb{C}_0$ denote complex numbers with **discrete topology**, so that $\mathbb{C}^n_0$ is the vector space $\mathbb{C}^n$ with discrete topology. Consider the projection $X \times \mathbb{C}^n_0 \to X$. This is a covering projection. The sheaf of its continuous sections is the constant local system $\mathbb{C}^n_X$ on X of rank n.

50 Let X be a connected, locally path connected, semi-locally simply connected, hausdorff topological space, let $x_0 \in X$ be a chosen base point, let $\widetilde{X} \to X$ be a universal cover of X, and let $\widetilde{x_0} \in \widetilde{X}$ be a chosen base point above x_0. Let $\pi_1(X, x_0)$ denote the fundamental group of X with base point x_0. Recall that it has a natural right action on $\widetilde{X}$, which is fixed point free with quotient X. If $U \subset X$ is simply connected, then its inverse image in $\widetilde{X}$ is uniquely expressible as a disjoint union of open subsets which map homeomorphically to U under the projection, and the action of $\pi_1(X, x_0)$ permutes these open sets.

51 Exercise Let $\mathcal{E}$ be a local system over the closed interval I. Then $\mathcal{E}$ is a constant sheaf.

52 Let $\mathcal{E}$ be a local system on base X, and let $\gamma : I \to X$ be a path joining $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. Then we get a linear isomorphism

$$\gamma^* : \mathcal{E}_{x_1} \to \mathcal{E}_{x_0}$$

as follows. The pull-back sheaf $\gamma^{-1}\mathcal{E}$ is a constant sheaf on I, so we have an isomorphism of stalks $(\gamma^{-1}\mathcal{E})_1 \to (\gamma^{-1}\mathcal{E})_0$. As $\mathcal{E}_{x_1} = (\gamma^{-1}\mathcal{E})_1$ and $\mathcal{E}_{x_0} = (\gamma^{-1}\mathcal{E})_0$, we get the desired isomorphism $\gamma^* : \mathcal{E}_{x_1} \to \mathcal{E}_{x_0}$.

53 Exercise If γ and δ are two paths in X with $\gamma(0) = \delta(0) = x_0$ and $\gamma(1) = \delta(1) = x_1$ which are path-homotopic, then $\gamma^* = \delta^* : \mathcal{E}_{x_1} \to \mathcal{E}_{x_0}$.

54 Exercise If γ and δ are two paths in X with $\gamma(1) = \delta(0)$, then for the composite path $\gamma * \delta$ we have $(\gamma\delta)^* = \delta^* \circ \gamma^*$.

55 Monodromy representation of a local system Let X be a connected, locally path connected, semi-locally simply connected, hausdorff topological space, and let $\mathcal{E}$ be a local system on X, of rank n. Let $x_0 \in X$ be a chosen base point and let $\mathcal{E}_{x_0}$ be the stalk of $\mathcal{E}$ over x_0 (which is an n-dimensional complex vector space with discrete topology). We attach to it a representation $\rho : \pi_1(X, x_0) \to GL(\mathcal{E}_{x_0})$ under which for any loop $\gamma : I \to X$ at x_0 we put

$$\rho(\gamma) = (\gamma^*)^{-1} : E_{x_0} \to E_{x_0}.$$

For any two local systems $\mathcal{E}$ and $\mathcal{F}$ and a sheaf homomorphism $\phi : \mathcal{E} \to \mathcal{F}$, the linear map

$$\phi_{x_0} : \mathcal{E}_{x_0} \to \mathcal{F}_{x_0}$$

commutes with the monodromy action of $\pi_1(X, x_0)$, so is an intertwining operator. Attaching its monodromy representation $\rho_{\mathcal{E}}$ to each local system $\mathcal{E}$ defines a fully faithful linear functor from the $\mathbb{C}$-linear category of local systems on X to the $\mathbb{C}$-linear category of finite dimensional complex linear representations of $\pi_1(X, x_0)$. This is in fact an equivalence of categories: the essential surjectivity of this functor is shown next.

56 Local system attached to π_1-representation Given any finite dimensional complex linear representation $\rho : \pi_1(X, x_0) \to GL(V)$, we define a local system $\mathcal{E}_\rho$ on X as follows. Let V be given the discrete topology. Consider the right action of $\pi_1(X, x_0)$ on the product $\widetilde{X} \times V$ defined by

$$(y, v) \cdot \gamma = (y\gamma, \rho(\gamma)^{-1} v).$$

This action commutes with the projection on X, and is linear in v. The quotient $\mathbf{E}_\rho = (\widetilde{X} \times V)/\pi_1(X, x_0)$ is again a covering of X, and $\mathcal{E}_\rho$ is the sheaf of sections of $\mathbf{E}_\rho$. Note that $\mathbf{E}_\rho$ is the sheaf space (*espace étalé*) of $\mathcal{E}_\rho$.

57 If $U \subset X$ is an open subset, and if $s : U \to \widetilde{X}$ is a section of $\widetilde{X} \to X$, then for any vector $v \in V$ we get a section (s, v) of $\mathcal{E}_\rho$ over U. If e_i are a basis of V, then the sections (s, e_i) define a basis of E_ρ over U. This shows that $\mathcal{E}_\rho$ is indeed a local system of rank n.

58 Theorem Let X be a connected, locally path connected, semi-locally simply connected, hausdorff topological space. The association $\rho \mapsto \mathcal{E}_\rho$ is functorial, and defines an equivalence of categories from the category of all finite dimensional complex linear representations of $\pi_1(X, x_0)$ to the category of all complex local systems on X. The functor which associates to $\mathcal{E}$ its monodromy representation $\rho_{\mathcal{E}}$ is a quasi-inverse to the above functor.

59 From local systems to locally free sheaves If X is a Riemann surface and $\mathcal{E}$ is a local system of rank n on X, then the tensor product $\mathcal{E} \otimes_{\mathbb{C}_X} \mathcal{O}_X$ is a locally free sheaf of rank n on X.

60 Given any open cover of X by simply connected open sets (U_i), the vector bundle E associated to a local system $\mathcal{E}$ can be described by locally constant transition functions $g_{i,j} : U_i \cap U_j \to GL(n, \mathbb{C})$. Conversely, if a vector bundle E on X is defined by locally constant transition functions with respect to some open cover (U_i), then there exists a local system $\mathcal{E} \subset E$ which is constant over (U_i) such that the induced map $\mathcal{E} \otimes \mathcal{O}_X \to E$ is an isomorphism of locally free sheaves.

61 Let $E = \mathcal{E} \otimes_{\mathbb{C}_X} \mathcal{O}_X$ be a locally free sheaf of rank n on X, attached to a local system $\mathcal{E}$. We now define a $\mathbb{C}$-linear sheaf homomorphism

$$\nabla : E \to E \otimes \Omega^1_X$$

where Ω^1_X is the sheaf of holomorphic 1-forms on X as follows. Any local section of E is a sum of terms $u \otimes f$ where u is a local section of $\mathcal{E}$ and f is a local holomorphic function. We put

$$\nabla(u \otimes f) = u \otimes df.$$

Caution. Note that ∇ is **not** linear over $\mathcal{O}_X$, so it is **not** a homomorphism of vector bundles.

Note that the subsheaf $\mathcal{E} \subset E$ is simply the kernel of ∇. Moreover, note that for any local section e of E and any local holomorphic function f, ∇ satisfies the equality $\nabla(fe) = f\nabla(e) + e \otimes df$.

62 Let E be a locally free sheaf on a Riemann surface X. A **holomorphic connection** on E is a $\mathbb{C}$-linear homomorphism of sheaves

$$\nabla : E \to E \otimes \Omega^1_X$$

which satisfies the identity $\nabla(fe) = f\nabla(e) + e \otimes df$ (called the 'Leibniz rule') for any local section e of E and any local holomorphic function f. All such pairs (E, ∇) form a $\mathbb{C}$-linear category, in which morphisms $(E, \nabla) \to (E', \nabla')$ are $\mathcal{O}$-linear homomorphisms of sheaves $\phi : E \to E'$ which commute with the respective connections.

63 Let (U_i, e_i) be a local trivialization of E. Let ∇ be a connection on E. Then we must have a matrix equality

$$\nabla(e_i) = e_i \Gamma_i$$

where Γ_i is a unique $n \times n$-matrix of holomorphic 1-forms over U_i, called the **connection matrix**. A local section σ of E can be expressed over U_i as $e_i f_i$ where f_i is an $n \times 1$-matrix (column vectors) of holomorphic functions. Then $\nabla(sigma) = e_i(df_i + \Gamma_i f_i)$. Hence σ lies in $\ker(\nabla)$ if and only if the following system of linear first order ODE is satisfied:

$$df_i + \Gamma_i f_i = 0.$$

64 Given any pair (E, ∇) as above, let $\mathcal{E} = \ker(\nabla)$. Then the existence and uniqueness theorem for a system of homogeneous linear first order ordinary differential equations shows that $\mathcal{E}$ is a local system on X, and $E = \mathcal{E} \otimes \mathcal{O}_X$. The association $(E, \nabla) \mapsto \ker(\nabla)$ is a fully faithful $\mathbb{C}$-linear functor from the category of pairs (E, ∇) to the category of local systems on X.

65 Theorem The category of local systems on X and the category of pairs (E, ∇) of vector bundles with holomorphic connections are equivalent, under the pair of functors described above which are quasi-inverse to each other.

66 Lemma Let X be a compact Riemann surface. Let $\mathbb{R}_X \to \mathcal{O}_X$ denote the inclusion of the constant sheaf $\mathbb{R}$ into the structure sheaf. This is a homomorphism of sheaves of real vector spaces. Then the induced $\mathbb{R}$-linear map $H^1(X, \mathbb{R}_X) \to H^1(X, \mathcal{O}_X)$ is an isomorphism.

Proof The real dimensions of $H^1(X, \mathbb{R}_X)$ and $H^1(X, \mathcal{O}_X)$ are equal to $2g_X$. Hence it is enough to show that the above map is injective. Let $a \in H^1(X, \mathbb{R}_X)$ map to zero. We can express a as a Cech cohomology 1-cocycle $(a_{i,j})$ over an open cover (U_i) of X. As $(a_{i,j})$ maps to 0 in $H^1(X, \mathcal{O}_X)$, and as by the general fact[1] that the first Cech cohomology w.r.t. an open cover injects into the first cohomology, (a_i) must be a co-boundary for $\mathcal{O}_X$, that is, $(a_{i,j}) \in B^1((U_i), \mathcal{O}_X)$. This means there exist holomorphic functions $f_i \in \mathcal{O}_X(U_i)$ such that

$$a_{i,j} = f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j}.$$

But as $a_{i,j}$ is real, this implies the imaginary parts of the f_i agree in the overlaps $U_i \cap U_j$ and so define a global function ψ on X. As ψ attains its maximum at some point $P \in X$ as X is compact, the imaginary part of the holomorphic function f_i, where $P \in U_i$, attains its maximum at a point $P \in U_i$. But this means ψ is constant, and so each f_i is also constant. If b_i is the real part of f_i, this means $(a_{i,j})$ is the co-boundary of $(b_i) \in C^0((U_i), \mathbb{R}_X))$. Hence $a = 0$.

[1]Note: We can avoid using this general property of first Cech cohomology simply by passing to a refinement of (U_i).

67 Narasimhan-Seshadri theorem in rank 1. Corresponding to each line bundle L of degree 0 on a compact Riemann surface, there exists a unique representation $\rho : \pi_1(X) \to U(1)$ (where $U(1) \subset GL(1, \mathbb{C})$ is the unitary group $\{z \,|\, |z| = 1\} \subset \mathbb{C}^*$) such that L is isomorphic to E_ρ. The homomorphism $H^1(X, U(1)) \to Pic^0(X)$ is a group isomorphism, where $Pic^0(X)$ is the group of isomorphism classes of degree 0 holomorphic line bundles on X.

Proof Let $U(1)_X$ be the constant sheaf on X with stalk $U(1)$, and consider the homomorphism $\mathbb{R}_X \to U(1)_X : f \mapsto e^{2\pi i f}$ which is epic with kernel $\mathbb{Z}_X$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathbb{Z}_X & \to \mathbb{R}_X \to & U(1)_X & \to 0 \\ & \| & \downarrow & \downarrow & \\ 0 \to & \mathbb{Z}_X & \to \mathcal{O}_X \to & \mathcal{O}_X^* & \to 0 \end{array}$$

where the rows are exact and the vertical maps are the standard inclusions. Note that $H^2(X, \mathbb{Z}_X) \to H^2(X, \mathbb{R}_X)$ is injective, and hence $H^1(X, \mathbb{R}_X) \to H^1(X, U(1)_X)$ is surjective. As $Pic^0(X)$ is the kernel of the connecting morphism $\partial : H^1(X, \mathcal{O}_X) \to H^2(X, \mathbb{Z}_X)$, we get the following commutative diagram with exact rows:

$$\begin{array}{cccc} H^1(X, \mathbb{Z}_X) & \to H^1(X, \mathbb{R}_X) \to & H^1(X, U(1)_X) & \to 0 \\ \| & \downarrow & \downarrow & \\ H^1(X, \mathbb{Z}_X) & \to H^1(X, \mathcal{O}_X) \to & Pic^0(X) & \to 0 \end{array}$$

As the map $H^1(X, \mathbb{R}_X) \to H^1(X, \mathcal{O}_X)$ is an isomorphism by Lemma 66, it follows by the five lemma that the last vertical map $H^1(X, U(1)_X) \to Pic^0(X)$ is an isomorphism. This means each degree 0 line bundle L corresponds to a unique local system (up to isomorphism) whose monodromy representation takes values in $U(1)$.

68 Exercise By the universal coefficient theorem,

$$Hom(\pi_1(X), U(1)) = H^1_{sing}(X; U(1)) = H^1(X, U(1)_X)$$

(we have omitted mentioning the base point for $\pi_1(X)$ as $U(1)$ is abelian). Verify that under this identification, a representation $\rho : \pi_1(X) \to U(1)$ corresponds to an element of $H^1(X, U(1)_X)$ which defines a local system whose monodromy is ρ.

Lecture 6. Weil's theorem.

The Atiyah class

69 1-cocycle for $End(E)$ in matrix form. Let E be a vector bundle of rank n on X, and let $\alpha \in H^1(X, End(E))$. Let U_i be an open cover of X, over which we can represent α by a 1-cocycle $(\alpha_{i,j}) \in Z^1((U_i), End(E))$. Note that when restricted to $U_i \cap U_j \cap U_k$, we have

$$\alpha_{i,j} + \alpha_{j,k} = \alpha_{i,k}.$$

Let E admit local trivializations over each U_i, with transition functions $g_{i,j} : U_i \cap U_j \to GL(n, \mathbb{C})$. This means the chosen bases e_i of $E|_{U_i}$ transform by $e_i g_{i,j} = e_j$ over $U_i \cap U_j$, and we have $g_{i,j} g_{j,k} = g_{i,k}$. In the basis $e_i|_{U_i \cap U_j}$, let $\alpha_{i,j}$ be described by the $n \times n$-matrix $A_{i,j}$. (*Caution: the indices i, j, k are for the open sets – these are not matrix indices.*) Then the cocycle condition written in matrix form reads

$$A_{i,j} + g_{i,j} A_{j,k} g_{i,j}^{-1} = A_{i,k}.$$

More generally, a cohomology class $\alpha \in H^1(X, End(E) \otimes \Omega^1_X)$ will be represented over a fine enough open cover (U_i) by matrices $A_{i,j}$ of 1-forms over $U_i \cap U_j$, which satisfy the above equality.

70 The Atiyah obstruction class. Let E be a vector bundle of rank n on X, locally trivialized over an open cover U_i, with e_i the basis over U_i and transition functions $g_{i,j}$. Let $\alpha_{i,j} \in \Gamma(U_i \cap U_j, End(E) \otimes \Omega^1_X)$ be defined in terms of the basis e_i by

$$A_{i,j} = dg_{i,j}\, g_{i,j}^{-1}$$

which is an $n \times n$-matrix of 1-forms over $U_i \cap U_j$. From $g_{i,j} g_{j,k} = g_{i,k}$, we get $dg_{i,j}\, g_{j,k} + g_{i,j}\, dg_{j,k} = dg_{i,k}$, which gives

$$dg_{i,j}\, g_{i,j}^{-1} + g_{i,j}(dg_{j,k}\, g_{j,k}^{-1}) g_{i,j}^{-1} = dg_{i,k}\, g_{i,k}^{-1}.$$

This means $A_{i,j} = dg_{i,j} g_{i,j}^{-1}$ is a Cech 1-cocycle for $End(E) \otimes \Omega^1_X$ over the open cover (U_i). The corresponding cohomology class

$$\alpha_E = (dg_{i,j}\, g_{i,j}^{-1}) \in H^1(X, End(E) \otimes \Omega^1_X)$$

is called the **Atiyah obstruction class** for E.

71 Exercise Show that the Atiyah obstruction class $\alpha_E \in H^1(X, End(E) \otimes \Omega^1_X)$ is well-defined, independent of the choice of local trivialization (U_i, e_i) for E.

72 Theorem. A holomorphic vector bundle E admits a holomorphic connection if and only if its Atiyah obstruction class $\alpha_E \in H^1(X, End(E) \otimes \Omega^1_X)$ is zero.

73 The differential sequence Let X be a compact Riemann surface. Let $d : \mathcal{O}_X \to \Omega^1_X$ be the exterior derivative. This is surjective as a homomorphism of sheaves of complex vector spaces, and the following sequence is exact:

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to 0$$

The induced connecting homomorphism $\partial : H^1(X, \Omega^1_X) \to H^2(X, \mathbb{C}_X)$ is an isomorphism. Recall (statement 25) that the composite of ∂ with the Poincaré duality isomorphism $H^2(X, \mathbb{C}_X) \to \mathbb{C}$ is the residue isomorphism $res : H^1(X, \Omega^1_X) \to \mathbb{C}$.

Let $\alpha : \mathcal{O}^*_X \to \Omega^1_X$ be the homomorphism of sheaves of abelian groups defined at the level of local sections by $f \mapsto \frac{df}{2\pi i f}$. Then the following diagram is commutative, with exact rows.

$$\begin{array}{ccccccccc}
0 \to & \mathbb{Z}_X & \to \mathcal{O}_X \to & \mathcal{O}^*_X & \to 0 \\
 & \downarrow & \| & \downarrow & \\
0 \to & \mathbb{C}_X & \to \mathcal{O}_X \to & \Omega^1_X & \to 0
\end{array}$$

74 Lemma. Let $tr : H^1(X, End(E) \otimes \Omega^1_X) \to H^1(X, \Omega^1_X)$ denote the homomorphism induced by the trace homomorphism $tr : End(E) \to \mathcal{O}_X$. Let $\partial : H^1(X, \Omega^1_X) \to H^2(X, \mathbb{C}_X)$ be the connecting homomorphism for the short exact sequence $0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0$. Then the image of the Atiyah obstruction class of E under the composite

$$\partial \circ tr : H^1(X, End(E) \otimes \Omega^1_X) \to H^2(X, \mathbb{C}_X)$$

is $2\pi i$-times the first Chern class of E, that is,

$$\partial \circ tr(\alpha_E) = 2\pi i \, c_1(E).$$

Proof. By statement 73, we have a commutative diagram

$$\begin{array}{ccc}
H^1(X, \mathcal{O}^*_X) & \xrightarrow{\partial} & H^2(X, \mathbb{Z}_X) \\
\downarrow & & \downarrow \\
H^1(X, \Omega^1_X) & \xrightarrow{\partial} & H^2(X, \mathbb{C}_X)
\end{array}$$

The image of a 1-cocycle $(h_{i,j})$ for $\mathcal{O}^*_X$ under the first vertical map is $(dh_{i,j}/2\pi i h_{i,j})$, and by Definition 14 the image of $(h_{i,j})$ in $H^2(X, \mathbb{Z}_X)$ is $c_1(L)$ where L is the line bundle defined by $(h_{i,j})$. Hence by the commutativity of the above diagram, the image of $(dh_{i,j}/h_{i,j})$ under the connecting map $\partial : H^1(X, \Omega^1_X) \to H^2(X, \mathbb{C}_X)$ is $2\pi i \, c_1(L)$.

This proves the lemma when $\mathrm{rank}(E) = 1$.

In the general case where $rank(E) \geq 1$, choose any filtration $0 \subset E_1 \subset \ldots \subset E_n = E$ of E by vector subbundles such that $\mathrm{rank}(E_r) = r$ (see Exercise 31). Choose bases $e_i = (e_{i,1}, \ldots, e_{i,n})$ for E over U_i which respect this filtration (that is, $(e_{i,1}, \ldots, e_{i,r})$ is a basis for E_r over U_i). Then the transition functions $g_{i,j}$ are upper triangular, and the matrix of 1-forms $dg_{i,j}\, g_{i,j}^{-1}$ is also upper triangular, with diagonal entries $(dh_{i,j}^{(r)}/h_{i,j}^{(r)})$ where $r = 1, \ldots, n$ and $h_{i,j}^{(r)}$ is the transition function for the quotient line bundle E_r/E_{r-1} with respect to bases $\overline{e_{i,r}} \in \Gamma(U_i, E_r/E_{r-1})$. Note that we have an isomorphism

$$\det(E) \cong E_1 \otimes (E_2/E_1) \otimes \ldots \otimes (E_n/E_{n-1})$$

and hence $c_1(E) = \sum_r c_1(E_r/E_{r-1})$. We therefore have

$$\begin{aligned} \partial \circ tr(\alpha_E) &= \partial \left(\sum dh_{i,j}^{(r)}/h_{i,j}^{(r)} \right) \\ &= 2\pi i \sum c_1(E_r/E_{r-1}) \\ &= 2\pi i\, c_1(E). \end{aligned}$$

Category of holomorphic bundles

75 The category $Vect_X$ which has all holomorphic vector bundles on a compact Riemann surface X as objects, and all holomorphic homomorphisms (not necessarily with constant rank!) as arrows, is not an abelian category. For example, the inclusion $\mathcal{O}_X(-P) \subset \mathcal{O}_X$ is both monic and epic in this category, but it is not an isomorphism.

76 Let A be a discrete valuation ring (we just need the case where A is the ring of convergent power series in a variable z with complex coefficients). Let E and F be finitely generated free A-modules, and let $\phi : M \to N$ be A linear. Then its kernel $\ker(\phi)$ and image $\mathrm{im}(\phi)$ are again finitely generated free A-modules. However, the quotient $F/\mathrm{im}(\phi)$ need not be free. Take $F' \subset F$ to be the inverse image under $F \to F/\mathrm{im}(\phi)$ of the torsion submodule $T \subset F/\mathrm{im}(\phi)$. Then F/F' is free, and $F'/\mathrm{im}(\phi) = T$ is torsion.

77 Let $\phi : E \to F$ be an $\mathcal{O}_X$-linear map of locally free sheaves (holomorphic vector bundles) on a Riemann surface X. Then the subsheaf $E' = \ker(\phi) \subset E$ is a holomorphic subbundle, with quotient bundle $E'' =\in (\phi)$. The bundle F admits a unique holomorphic subbundle F' such that the the induced map $E'' \to F'$ has a torsion cokernel. The map $\phi : E \to F$ is of constant rank if and only if $E'' = F'$. Let ϕ_x denote the restriction of ϕ to the fiber E_x of the bundle E at x (which in terms of the stalk $E_{(x)}$ equals the quotient $E_{(x)}/\mathfrak{m}_x E_{(x)}$ where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal). Then X has a discrete closed subset $Y \subset X$ such that $\mathrm{rank}(\phi_x)$ is a constant r on $X - Y$ while $\mathrm{rank}(\phi_y) < r$ for $y \in Y$. Then on $X - Y$ we have $\ker(\phi_x) = E'_x$ and $\mathrm{im}(\phi_x) = E''_x = F'_x$. For $y \in Y$ we have strict inclusion $E'_y \subset \ker(\phi_y)$, the map $E''_y \to F'_y$ is not injective, and $\mathrm{im}(\phi_y) \neq F'_y$.

78 Exercise Let $N : E \to E$ be a nilpotent endomorphism of a holomorphic bundle E on a compact Riemann surface X. Then E admits a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$ of E by vector subbundles such that $\mathrm{rank}(E_r) = r$ and $N(E_r) \subset E_{r-1}$ for each $r = 1, \ldots, n$. *(Hint: Apply Exercise 31 to* $\ker(N)$ *and use induction on rank for* $E/\ker(N)$*.)*

Atiyah's proof of Weil's theorem

79 Weil's theorem. A vector bundle E on a compact Riemann surface is defined by a representation of the fundamental group of X if and only if each direct summand of E has degree zero.

Proof If exists a representation $\rho : \pi_1(X, x_0) \to GL(n, \mathbb{C})$ such that $E = E_\rho$, then by Theorem 58 and Theorem 65, E admits a holomorphic connection ∇. Then each direct summand F of E admits a holomorphic connection, for if $E = F \oplus G$, then the composite

$$F \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega^1_X \to F \otimes \Omega^1_X$$

is a holomorphic connection on F, where the last map above is the projection. As F admits a connection, its Atiyah obstruction class α_E is zero by Theorem 72. This means $c_1(E) = 0$ by Lemma 74, and so $\deg(E) = 0$ by Theorem 16.

Conversely, let each direct summand of E have degree 0. Let $E = E_1 \oplus \ldots \oplus E_n$ where each E_i is indecomposable. Then Each E_i has degree 0. If E_i is associated to a representation ρ_i, then E is associated to the direct sum of the ρ_i. Hence to complete the proof of the theorem, it is enough to prove that if E is indecomposable of degree 0, then E admits a holomorphic connection, that is, $\alpha_E = 0$ in $H^1(X, End(E) \otimes \Omega^1_X)$.

Note that $End(E)$ is self-dual under the trace pairing $tr : End(E) \otimes End(E) \to \mathcal{O}_X$. Hence by Serre duality and by the expression for residue in terms of Poincaré duality (see statements 24 and 25), we have a non-degenerate pairing

$$D \circ \partial \circ tr : H^0(X, End(E)) \otimes H^1(X, End(E) \otimes \Omega^1_X) \xrightarrow{tr} H^1(X, \Omega^1_X) \xrightarrow{\partial} H^2(X, \mathbb{C}) \xrightarrow{D} \mathbb{C}$$

where D is the Poincaré duality isomorphism. Hence to show that $\alpha_E = 0$, it is enough to show that

$$\partial \circ tr(\phi \alpha_E) = 0$$

for all $\phi \in H^0(X, End(E))$.

By Lemma 44, $\phi = \lambda 1_E + N$ where N is nilpotent. By Lemma 74 we have

$$\partial \circ tr(\lambda 1_E \alpha_E) = \lambda \partial \circ tr(\alpha_E) = \lambda 2\pi i\, c_1(E) = 0.$$

Hence it remains just to show that $tr(N\alpha_E) = 0$. As N is nilpotent, it follows by Exercise 78 that E admits a filtration $0 \subset E_1 \subset \ldots E_n = E$ by vector subbundles such that $\mathrm{rank}(E_r) = r$ and

$$N(E_r) \subset E_{r-1}$$

for each $r = 1, \ldots, n$, where we take $E_0 = 0$. Now as in the proof of Lemma 74, we can choose local bases for E w.r.t. the above filtration, to get transition functions $g_{i,j}$ which are upper triangular, and then the cocycle $\alpha_E = (dg_{i,j}\, g_{i,j}^{-1})$ is upper triangular. But as $N(E_r) \subset E_{r-1}$, in this basis the matrix N_i representing N is strictly upper triangular, with all diagonal entries zero. Hence the matrix product $N_i dg_{i,j}\, g_{i,j}^{-1}$ is again strictly upper triangular, with all diagonal entries zero, so its trace is zero. Hence $tr(N\alpha_E) = 0$. This completes Atiyah's proof of Weil's theorem.

Suggestions for further reading

Atiyah : Collected Works.

Griffiths and Harris : Algebraic Geometry.

Hirzebruch : Topological Methods in Algebraic Geometry.

Husemoller : Fiber Bundles.

Milnor : Characteristic Classes.

M.S. Narasimhan : Vector Bundles on Compact Riemann Surfaces. Lecture notes, Vienna 1976.

M.S. Narasimhan : Collected Papers.

Appendix: Some explicit cohomology calculations.

80 Chern class of $\mathcal{O}_X(P)$. Let X be a compact Riemann surface, let $P_0 \in X$ be any chosen point, and let a homeomorphism $z : U \to U'$ define a local holomorphic coordinate patch around P_0, where $U' \subset \mathbb{C}$ is the open disk of radius > 3, and $z(P_0) = 0$. Consider the open cover $\mathcal{V} = (V_0, V_1, V_2, V_3, V_\infty)$ of X where the sets V_i are defined as follows. For any $P \neq P_0$ in U, we can write $z(P) = r(P)e^{i\theta(P)}$ where $\theta(P)$ is unique modulo $2\pi\mathbb{Z}$. Let $0 < \epsilon < 1/2$ be a chosen real number. We put

$$\begin{aligned}
V_0 &= \{P \in U \,|\, r(P) < 1\} \subset U \\
V_1 &= \{P \in U \,|\, 0 < r(P) < 3 \text{ and } -\epsilon < \theta(P) < 2\pi/3 + \epsilon\} \\
V_2 &= \{P \in U \,|\, 0 < r(P) < 3 \text{ and } 2\pi/3 - \epsilon < \theta(P) < 4\pi/3 + \epsilon\} \\
V_3 &= \{P \in U \,|\, 0 < r(P) < 3 \text{ and } 4\pi/3 - \epsilon < \theta(P) < 2\pi + \epsilon\} \\
V_\infty &= \{P \in U \,|\, 2 < r(P)\} \cup (X - U)
\end{aligned}$$

Note that θ defines a single-valued continuous function on each of V_1, V_2, V_3. We denote these functions by

$$\theta_1 : V_1 \to (-\epsilon, 2\pi/3 + \epsilon),\ \theta_2 : V_2 \to (2\pi/3 - \epsilon, 4\pi/3 + \epsilon),\ \theta_3 : V_3 \to (4\pi/3 - \epsilon, 2\pi + \epsilon).$$

We denote by $r : U \to \mathbb{R}$ the function $P \mapsto |z(P)|$.

Note that we have

$$\begin{aligned}
\theta_2|_{V_1\cap V_2} - \theta_1|_{V_1\cap V_2} &= 0, \\
\theta_3|_{V_2\cap V_3} - \theta_2|_{V_2\cap V_3} &= 0, \\
\theta_3|_{V_1\cap V_3} - \theta_1|_{V_1\cap V_3} &= 2\pi i.
\end{aligned}$$

We define a Cech 2-cocycle $(c_{i,j,k}) \in Z^2(\mathcal{V}, \mathbb{Z}_X)$ by putting

$$\begin{aligned}
c_{0,3,1} &= 1 \\
c_{i,j,k} &= 1 \text{ when } (i,j,k) \text{ is an even permutation of } (0,3,1). \\
c_{i,j,k} &= -1 \text{ when } (i,j,k) \text{ is an odd permutation of } (0,3,1). \\
c_{i,j,k} &= 0 \text{ in all other cases.}
\end{aligned}$$

We define a Cech 1-cocycle $(g_{i,j}) \in Z^1(\mathcal{V}, \mathcal{O}_X^*)$ by putting

$$\begin{aligned}
g_{0,i} &= z \text{ for } i = 1,2,3. \\
g_{i,0} &= 1/z \text{ for } i = 1,2,3. \\
g_{i,j} &= 1 \text{ in all other cases.}
\end{aligned}$$

Note that this cocycle defines the line bundle $\mathcal{O}_X(P_0)$.

We define a Cech 1-cochain $(f_{i,j}) \in C^1(\mathcal{V}, \mathcal{O}_X)$ by putting

$$\begin{aligned}
f_{0,i} &= \log r + \sqrt{-1}\theta_i \text{ for } i = 1,2,3. \\
f_{i,0} &= -\log r - \sqrt{-1}\theta_i \text{ for } i = 1,2,3. \\
f_{i,j} &= 0 \text{ in all other cases.}
\end{aligned}$$

Then note that under the map Cech cochain map

$$exp : C^1(\mathcal{V}, \mathcal{O}_X) \to C^1(\mathcal{V}, \mathcal{O}_X^*)$$

induced by the exponential map $\mathcal{O}_X \to \mathcal{O}_X^* : h \mapsto e^{2\pi i h}$ we have

$$\frac{1}{2\pi i}(f_{i,j}) \mapsto (g_{i,j}).$$

Under the Cech coboundary map

$$\partial : C^1(\mathcal{V}, \mathcal{O}_X) \to C^2(\mathcal{V}, \mathcal{O}_X)$$

the image of $(f/2\pi i)$ is the above 2-cocycle

$$\partial(f/2\pi i) = (c_{i,j,k}) \in Z^2(\mathcal{V}, \mathbb{Z}_X) \subset Z^2(\mathcal{V}, \mathcal{O}_X).$$

(For example, $\partial((f/2\pi i)_{0,3,1} = (f_{3,1} - f_{0,1} + f_{0,3})/2\pi i = 1 = c_{0,3,1}$, and so on.)

Therefore by the definition of the connecting homomorphism

$$\partial : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}_X)$$

for the exponential sequence $0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$, we have

$$\partial(g_{i,j}) = (c_{i,j,k}).$$

Hence by definition of first Chern class, we have

$$c_1(\mathcal{O}_X(P_0)) = \partial(g_{i,j}) = (c_{i,j,k}) \in H^2(X, \mathbb{Z}_X).$$

In the calculation 81 below which traces through the isomorphism between that Cech cohomology $H^2(X, \mathbb{R}_X)$ and the de Rham cohomology $H^2_{dR}(X)$, we show that the image of $(c_{i,j,k})$ under $H^2(X, \mathbb{Z}_X) \hookrightarrow H^2(X, \mathbb{R}_X) \to H^2_{dR}(X)$ is the positive integral generator of $H^2_{dR}(X)$, representing by a differential 2-form η on X for which $\int_X \eta = 1$. This will complete the proof that $c_1(\mathcal{O}_X(P_0))$ defined as $\partial(g_{i,j})$ is the positive generator of $H^2(X, \mathbb{Z}_X)$.

81 From Cech cocycle to differential form. Let X and its open cover $\mathcal{V}$ be as above. Let $c = (c_{i,j,k}) \in Z^2(\mathcal{V}, \mathbb{Z}_X)$ be the 2-cocycle defined above. We now explicitly determine the de Rham cohomology class η that corresponds to c.

Let $\mathcal{A}^r$ be the sheaf of real differential r-forms on X, and let $d : \mathcal{A}^r \to \mathcal{A}^{r+1}$ denote the exterior derivative. Under the inclusion $\mathbb{Z}_X \subset \mathcal{A}^0$, we regard the cocycle $(c_{i,j,k})$ defined above to be an element of $C^2(\mathcal{V}, \mathcal{A}^0)$. With define a 1-cochain $(b_{i,j}) \in C^1(\mathcal{V}, \mathcal{A}^0)$ by putting

$$\begin{aligned} b_{0,i} &= \theta_i/2\pi \text{ for } i = 1,2,3. \\ b_{i,0} &= -\theta_i/2\pi \text{ for } i = 1,2,3. \\ b_{i,j} &= 1 \text{ in all other cases.} \end{aligned}$$

Then under the Cech coboundary map $\partial : C^1(\mathcal{V}, \mathcal{A}^0) \to C^2(\mathcal{V}, \mathcal{A}^0)$ we have

$$\partial(b_{i,j}) = (c_{i,j,k}) \in C^2(\mathcal{V}, \mathbb{Z}_X) \subset C^2(\mathcal{V}, \mathcal{A}^0)$$

(for example, $\partial(b)_{0,3,1} = (b_{3,1} - b_{0,1} + b_{0,3})/2\pi = (0 - \theta_1 + \theta_3)/2\pi = 1 = c_{0,3,1}$, and so on).

The exterior derivative d induces a homomorphism $d : C^1(\mathcal{V}, \mathcal{A}^0) \to C^1(\mathcal{V}, \mathcal{A}^1)$ under which db is the co-chain defined by

$$\begin{aligned} db_{0,i} &= d\theta/2\pi \text{ for } i = 1,2,3. \\ db_{i,0} &= -d\theta/2\pi \text{ for } i = 1,2,3. \\ db_{i,j} &= 1 \text{ in all other cases.} \end{aligned}$$

Note that $d\theta$ is independent of the branch of θ chosen, so the above notation is unambiguous.

Now consider a C^∞ function $\rho : \mathbb{R} \to \mathbb{R}$ so defined that $\rho(r) = 0$ for $r \leq 1/3$ and $\rho(r) = 1$ for $r \geq 2/3$. Then on V_0 we have a well-defined 2-form

$$\eta = d\rho \wedge d\theta/2pi$$

which, being supported on the annulus $1/2 \leq r \leq 2/3$, extends by zero outside V_0 to give a global C^∞ 2-form on X which we again denote by η. Note that

$$\int_X \eta = 1$$

which means η is the positive integral generator of the cohomology $H^2(X, \mathbb{Z}_X)$.

Let $(a_i) \in C^0(\mathcal{V}, \mathcal{A}^1)$ be the 0-cochain defined by

$$\begin{aligned} a_0 &= -\rho(r) d\theta/2\pi \\ a_i &= (1 - \rho(r)) d\theta/2\pi \text{ for } i = 1,2,3 \\ a_\infty &= 0. \end{aligned}$$

Then under the coboundary map $\partial : C^0(\mathcal{V}, \mathcal{A}^1) \to C^1(\mathcal{V}, \mathcal{A}^1)$ we have

$$\partial(a_i) = (db_{i,j})$$

(for example, $\partial(a)_{0,1} = a_1 - a_0 = (1 - \rho(r) + \rho(r)) d\theta/2\pi = d\theta/2\pi = db_{0,1}$, and so on). Under the exterior derivative $d : \mathcal{A}^1 \to \mathcal{A}^2$, we have on $V_0 \cap V_i$ for $i = 1,2,3$ the equality

$$da_0 = da_i = -d\rho(r) \wedge d\theta/2\pi = -\eta.$$

This means that $(da_i) \in C^0(\mathcal{V}, \mathcal{A}^2)$ is the image of $-\eta \in \Gamma(X, \mathcal{A}^2)$.

We have the Cech - de Rham double complex $C^{p,q} = C^p(\mathcal{V}, \mathcal{A}^q)$ with horizontal derivations the Cech coboundary maps ∂^p and vertical derivations the exterior derivatives d^q. The total complex $T^n = \oplus_{p+q=n} C^{p,q}$ has derivative δ which acts on the summand $C^{p,q}$ by

$$\delta^{p,q} = \partial^p + (-1)^p d^q.$$

The element $(c_{i,j,k}) \in C^{3,0}$ therefore has the series of homology equivalences

$$\begin{array}{rcl}
(c) & \sim & (c) - \delta^{1,0}(b) \\
 & = & (c) - \partial(b) + d(b) \quad \text{as } \delta^{1,0} = \partial - d \\
 & = & d(b) \quad \text{as } (c) = \partial(b) \\
 & \sim & d(b) - \delta^{0,1}(a) \\
 & = & d(b) - \partial(a) - d(a) \quad \text{as } \delta^{0,1} = \partial + d \\
 & = & -d(a) \quad \text{as } d(b) = \partial(a) \\
 & = & \eta.
\end{array}$$

Note that $\delta(c) = 0$ and $\delta(\eta) = 0$, and by the above equivalences, these define the same element of $H^2(T, \delta)$. Now, η is the positive integral generator of the de Rham cohomology $H^2_{dR}(X)$. Hence under the isomorphism between Cech and de Rham cohomologies induced by their respective isomorphisms with $H^*(T, \delta)$, $(c_{i,j,k})$ is the positive generator of $H^2(X, \mathbb{Z}_X)$.

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