An Introduction to Double Coset Maps

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Given two congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, they can associate with each other by their double coset as a mediator. By this configuration, we can construct a map between their modular forms of weight-k called weight-k double coset map. In particular, the weight-2 double coset map descends to a pullback map between the associated Jacobian varieties.

Analogously, we will construct a homomorphism between the divisor groups of their modular curves in the opposite direction, serving as some interpretation of the aforementioned double coset maps between modular forms. It then descends to a map between the corresponding Picard groups.

1 Double Coset Maps between Modular Forms

Given two congruence subgroups Γ_1, Γ_2 and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, Γ_1 acts on the set $\Gamma_1\alpha\Gamma_2$ by left multiplication, partitioning it into orbits. Now set $\Gamma_1' = \mathrm{SL}_2(\mathbb{Z}) \cap \alpha^{-1}\Gamma_1\alpha$, a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We have the following lemmas.

Lemma 1.1 *Γ_1' is a congruence subgroup.*

p.f. Suppose $\alpha = \dfrac{\beta}{p}$ with $p \in \mathbb{Z}, \beta \in \mathrm{M}_2(\mathbb{Z})$ and $\alpha^{-1} = \dfrac{\gamma}{q}$ with $q \in \mathbb{Z}, \gamma \in \mathrm{M}_2(\mathbb{Z})$. Notice that there is $\Gamma(N) \subset \Gamma_1 \subset \mathrm{SL}_2(\mathbb{Z})$ for some $N \in \mathbb{N}^*$, then $\Gamma(pqN) \subset \Gamma_1 \subset \mathrm{SL}_2(\mathbb{Z})$. Set $\tilde{N} = (pqN)^3$, we have

$$\alpha\Gamma(\tilde{N})\alpha^{-1} \subset \alpha(I + (pqN)^3\mathrm{M}_2(\mathbb{Z}))\alpha^{-1} \subset I + (pqN)\mathrm{M}_2(\mathbb{Z}) \subset \mathrm{M}_2(\mathbb{Z}).$$

Then since $\alpha\Gamma(\tilde{N})\alpha^{-1}$ consists of determinant-1 matrices, we obtain that

$$\alpha\Gamma(\tilde{N})\alpha^{-1} \subset \Gamma(pqN) \subset \Gamma_1,$$

and thus $\Gamma(\tilde{N}) \subset \alpha^{-1}\Gamma_1\alpha$. Hence Γ_1' is a congruence subgroup. □

Lemma 1.2 *Any two congruence subgroups are commensurable.*

p.f. If Γ^1, Γ^2 are two congruence subgroups, then there exists $N \in \mathbb{N}^*$ such that $\Gamma(N) \subset \Gamma^1 \cap \Gamma^2 \subset \Gamma^i \subset \mathrm{SL}_2(\mathbb{Z})$ for $i = 1, 2$. Thus $[\Gamma^i : \Gamma^1 \cap \Gamma^2] \leqslant [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]$ for $i = 1, 2$, which is finite. □

Lemma 1.3 *Left multiplication by α from Γ_2 into $\Gamma_1\alpha\Gamma_2$ induces a bijection from $(\Gamma_1' \cap \Gamma_2)\backslash\Gamma_2$ onto $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$.*

p.f. The induced map $\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ takes γ_2 to $\Gamma_1\alpha\gamma_2$, which equals $\Gamma_1\gamma_1\alpha\gamma_2$ for any $\gamma_1 \in \Gamma_1$, thus it is surjective. On the other hand, for $\gamma_2, \gamma_2' \in \Gamma_2$, $\Gamma_1\alpha\gamma_2 = \Gamma_1\alpha\gamma_2'$ if and only if $\gamma_2' = \alpha^{-1}\gamma_1\alpha\gamma_2$ for some $\gamma_1 \in \Gamma_1$, which means $(\Gamma_1' \cap \Gamma_2)\gamma_2 = (\Gamma_1' \cap \Gamma_2)\gamma_2'$. This shows that left multiplication by α descends to an injective map $(\Gamma_1' \cap \Gamma_2)\backslash\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. From the first part of argument, it is bijective. □

From above, we conclude that the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ is finite. Hence, after choosing a representative for each orbit, we can sum over some operators induced by them which will be defined below. We will see that this sum does not depend on the choice of representatives.

Definition 1.4 *Given $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, $k \in \mathbb{Z}$, the* ***weight-k β operator*** *$[\beta]_k$ on the function space $\mathcal{F}(\mathbb{H}, \mathbb{C})$ is defined as*

$$(f[\beta]_k)(\tau) := (\det \beta)^{k-1}(c\tau + d)^{-k} f(\beta(\tau)) \qquad \tau \in \mathbb{H}.$$

It is easy to verify that this is well-defined.

Property 1.4.1 *For any $\beta_1, \beta_2 \in \mathrm{GL}_2^+(\mathbb{Q})$, $[\beta_1\beta_2]_k = [\beta_1]_k[\beta_2]_k$.* (That is the reason we write the operator on the right of elements it operates.)

p.f. By direct calculation. □

Definition 1.5 *Given two congruence subgroups Γ_1, Γ_2, $k \in \mathbb{Z}$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, the* ***weight-$k$ double coset map (with respect to α)*** *$[\Gamma_1\alpha\Gamma_2]_k$ from $\mathcal{M}_k(\Gamma_1)$ to $\mathcal{M}_k(\Gamma_2)$ is defined as*

$$f[\Gamma_1\alpha\Gamma_2]_k := \sum_j f[\beta_j]_k$$

where $\{\beta_j\}$ is a choice of representatives of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$, i.e. $\Gamma_1\alpha\Gamma_2 = \bigsqcup_j \Gamma_1\beta_j$.

Remark 1.5.1 *It is critical to make sure the above definition is well-defined. There are two issues we need to verify: $[\Gamma_1\alpha\Gamma_2]_k$ takes any element of $\mathcal{M}_k(\Gamma_1)$ to an element of $\mathcal{M}_k(\Gamma_2)$ and the value is independent of the choice of representatives.*

p.f. If $\{\beta_j\}, \{\beta'_j\}$ are two choices of representatives, then by a certain rearrangement, we can assume $\beta'_j = \gamma_{1j}\beta_j$ with $\gamma_{1j} \in \Gamma_1$ for each j. Given $f \in \mathcal{M}_k(\Gamma_1)$, notice that we have $f[\gamma_1]_k = f$ for any $\gamma_1 \in \Gamma_1$. Then for any $\gamma_2 \in \Gamma_2$,

$$\sum_j f[\beta'_j]_k = \sum_j f[\gamma_{1j}]_k[\beta_j] = \sum_j f[\beta_j]_k.$$

Hence the definition is independent of the choice of representatives. It then implies that

$$(f[\Gamma_1\alpha\Gamma_2]_k)[\gamma_2] = \sum_j f[\beta_j\gamma_2] = \sum_j f[\beta_j]_k = f[\Gamma_1\alpha\Gamma_2]_k$$

for any $\gamma_2 \in \Gamma_2$, since $\{\beta_j\gamma_2\}$ is another choice of representatives. The holomorphy of $f[\Gamma_1\alpha\Gamma_2]$ is obvious. Thus $f[\Gamma_1\alpha\Gamma_2]_k \in \mathcal{M}_k(\Gamma_2)$. □

After giving the formal definition of a double coset map $[\Gamma_1\alpha\Gamma_2]_k$, we can derive three specific maps where Γ_1, Γ_2 or α are specialized. In effect, all the general double coset maps of weight k can be decomposed into these specific maps. The idea follows from configuring a connection between Γ_1 and Γ_2, a tinge of which already occurred in the previous lemmas. Set $\Gamma'_1 = \mathrm{SL}_2(\mathbb{Z}) \cap \alpha^{-1}\Gamma_1\alpha$, $\Gamma'_2 = \mathrm{SL}_2(\mathbb{Z}) \cap \alpha\Gamma_2\alpha^{-1}$. We have the following relation between congruence subgroups:

$$\Gamma_1 \geqslant \Gamma'_2 \cap \Gamma_1 \cong \Gamma'_1 \cap \Gamma_2 \leqslant \Gamma_2$$

since $\Gamma'_1 \cap \Gamma_2 = \alpha^{-1}(\Gamma'_2 \cap \Gamma_1)\alpha$, the conjugacy by α. Thus the three specific cases are natural as demonstrated below.

(1) Take $\Gamma_1 \geqslant \Gamma_2$, $\alpha = I$. In this case, $\mathcal{M}_k(\Gamma_1) \subset \mathcal{M}_k(\Gamma_2)$ and $[\Gamma_1\alpha\Gamma_2]_k$ is the inclusion from $\mathcal{M}_k(\Gamma_1)$ to $\mathcal{M}_k(\Gamma_2)$. It is an injection.

(2) Take $\Gamma_2 = \alpha^{-1}\Gamma_1\alpha$. In this case, $[\Gamma_1\alpha\Gamma_2]_k = [\alpha]_k$. It is a bijection.

(3) Take $\Gamma_1 \leqslant \Gamma_2$, $\alpha = I$. In this case, $\mathcal{M}_k(\Gamma_1) \supset \mathcal{M}_k(\Gamma_2)$ and $[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\gamma_{2,j}]$ where $\{\gamma_{2,j}\}$ is a choice of representatives of $\Gamma_1\backslash\Gamma_2$. It is a surjection.

Proposition 1.6 *Any double coset map is a composition of three double coset maps of the above three forms respectively.*

p.f. Given a double coset map $[\Gamma_1\alpha\Gamma_2]_k$, we have the following chain of maps:

$$\mathcal{M}_k(\Gamma_1) \longrightarrow \mathcal{M}_k(\Gamma'_2 \cap \Gamma_1) \longrightarrow \mathcal{M}_k(\Gamma'_1 \cap \Gamma_2) \longrightarrow \mathcal{M}_k(\Gamma_2)$$

that sends

$$f \longmapsto f \longmapsto f[\alpha]_k \longmapsto \sum_j f[\alpha\gamma_{2,j}]_k$$

where $\{\gamma_{2,j}\}$ is a choice of representatives of $(\Gamma_1' \cap \Gamma_2)\backslash\Gamma_2$. They compose $[\Gamma_1\alpha\Gamma_2]_k$ and are of the above three forms, respectively. $\square$

An important fact about double coset maps is that they take cusp forms to cusp forms, as illustrated by the following proposition.

Proposition 1.7 *Given two congruence subgroups Γ_1, Γ_2, $k \in \mathbb{Z}$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, $[\Gamma_1\alpha\Gamma_2]_k$ maps $\mathcal{S}_k(\Gamma_1)$ into $\mathcal{S}_k(\Gamma_2)$.*

p.f. For any $f \in \mathcal{S}_k(\Gamma_1)$ and $\beta \in \mathrm{GL}_k^+(\mathbb{Q})$, $f[\beta]_k$ vanishes at its cusps. Thus $f[\Gamma_1\alpha\Gamma_2]_k$ vanishes at its cusps. Hence $f[\Gamma_1\alpha\Gamma_2]_k \in \mathcal{S}_k(\Gamma_2)$. $\square$

2 Double Coset Maps between Divisor Groups of Modular Curves and Jacobian Varieties

Proposition 1.7 guarantees us to introduce a similar double coset map between Jacobian varieties. Since for any congruence subgroup Γ, there is

$$\mathrm{Jac}(X(\Gamma)) = \Omega^1_{\mathrm{hol}}(X(\Gamma))^*/\mathrm{H}_1(X(\Gamma), \mathbb{Z}) \cong \mathcal{S}_2(\Gamma_i)^*/\mathrm{H}_1(X(\Gamma), \mathbb{Z})$$

we can induce an weight-2 double coset map between modular forms to a map between Jacobian varieties.

Given two congruence subgroups Γ_1, Γ_2 and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, as constructed before, $[\Gamma_1\alpha\Gamma_2]_2$ maps $\mathcal{S}_2(\Gamma_1)$ to $\mathcal{S}_2(\Gamma_2)$. The dual map

$$[\Gamma_1\alpha\Gamma_2]_2^* : \mathcal{S}_2(\Gamma_2)^* \longrightarrow \mathcal{S}_2(\Gamma_1)^*$$

descends to a map between Jacobian varieties, still denoted by $[\Gamma_1\alpha\Gamma_2]^*$, as follows:

$$[\Gamma_1\alpha\Gamma_2]^* : \mathrm{Jac}(X(\Gamma_2)) \longrightarrow \mathrm{Jac}(X(\Gamma_1)).$$

By configuration, we have the following definition.

Definition 2.1 *Given two congruence subgroups Γ_1, Γ_2 and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, the* **double coset map** *$[\Gamma_1\alpha\Gamma_2]$ from $\mathrm{Jac}(X(\Gamma_2))$ to $\mathrm{Jac}(X(\Gamma_1))$ is defined as*

$$[\Gamma_1\alpha\Gamma_2]^*(\overline{\psi}) := \overline{\psi \circ [\Gamma_1\alpha\Gamma_2]_2} \qquad \psi \in \mathcal{S}_2(\Gamma_2)^*$$

where the overlines represent equivalence classes modulo homology group.

We now move to another path. Every congruence subgroup has a modular curve consisting of orbits that serves as the geometric interpretation of its modular form. Given two congruence subgroups Γ_1, Γ_2 and $\alpha \in \mathbb{Q}$, we want to construct such an interpretation of the double coset map between modular forms which, analogous to the Definition 1.4, is a map of summation form. It naturally suffices to conduct it between divisor groups of the modular curves, i.e. their free abelian groups.

The idea is ignited from the configuration of maps between congruence subgroups

$$\Gamma_1 \xleftarrow{\iota_1} \Gamma_2' \cap \Gamma_1 \xrightarrow{\sim} \Gamma_1' \cap \Gamma_2 \xrightarrow{\iota_2} \Gamma_2$$

where ι_1, ι_2 represent the inclusions while $\sim$ is the conjugacy by α. As Proposition 1.6 illustrates, a double coset map between modular forms can be seen as being constructed by such a configuration. Then the construction of our object could be also by an corresponding configuration of maps between modular curves, which is

$$X(\Gamma_1) \xleftarrow{\pi_1} X(\Gamma_2' \cap \Gamma_1) \xleftarrow{\sim} X(\Gamma_1' \cap \Gamma_2) \xrightarrow{\pi_2} X(\Gamma_2)$$

where π_1, π_2 represent the natural projections while $\sim$ is given by $\overline{\tau} \longmapsto \overline{\alpha(\tau)}$. It is easy to verify that $\sim$ is well-defined and bijective. It induces a chain of maps from the mudular curve of Γ_2 to the divisor group of the modular curve of Γ_1 as

$$X(\Gamma_2)) \longrightarrow \mathrm{Div}(X(\Gamma_1' \cap \Gamma_2)) \longrightarrow \mathrm{Div}(X(\Gamma_2' \cap \Gamma_1)) \longrightarrow \mathrm{Div}(X(\Gamma_1)))$$

that sends

$$\overline{\tau} \longmapsto \sum_j \overline{\gamma_j(\tau)} \longmapsto \sum_j \overline{\alpha\gamma_j(\tau)} \longmapsto \sum_j \overline{\alpha\gamma_j(\tau)}$$

where $\{\gamma_j\}$ is a choice of representatives of $(\Gamma_1' \cap \Gamma_2)\backslash\Gamma_2$.

Therefore we have the following well-defined definition.

Definition 2.2 *Given two congruence subgroups Γ_1, Γ_2 and $\alpha \in \mathbb{Q}$, the* ***double coset map (with respect to** α**)*** $[\Gamma_1\alpha\Gamma_2]^\wedge$ *from* $\mathrm{Div}(X(\Gamma_2)))$ *to* $\mathrm{Div}(X(\Gamma_1)))$ *is defined as*

$$[\Gamma_1\alpha\Gamma_2]^\wedge \sum_{\tau\in\mathcal{H}} n_\tau \Gamma_2\tau := \sum_{\tau\in\mathcal{H}} n_\tau \sum_j \Gamma_1\beta_j(\tau) \qquad only\ finitely\ many\ n_\tau\ are\ nonzero$$

where $\{\beta_j\}$ is a choice of representatives of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$, i.e. $\Gamma_1\alpha\Gamma_2 = \bigsqcup_j \Gamma_1\beta_j$.

Proposition 2.3 *Given two congruence subgroups Γ_1, Γ_2 and $\alpha \in \mathbb{Q}$, $[\Gamma_1\alpha\Gamma_2]^\wedge$ maps* $\mathrm{Div}^0(X(\Gamma_2))$ *to* $\mathrm{Div}^0(X(\Gamma_1))$*.*

p.f. This follows from the observation that the degree of a divisor is preserved by this map. $\square$

This proposition allows us to descend the double coset map to Picard groups as demonstrated below.

Definition 2.4 *Given two congruence subgroups Γ_1, Γ_2 and $\alpha \in \mathbb{Q}$, the* ***double coset map (with respect to** α**)*** $[\Gamma_1\alpha\Gamma_2]^\wedge$ *is defined as*

$$[\Gamma_1\alpha\Gamma_2]^\wedge \overline{\sum_{\tau\in\mathcal{H}} n_\tau \Gamma_2\tau} := \overline{\sum_{\tau\in\mathcal{H}} n_\tau \sum_j \Gamma_1\beta_j(\tau)} \qquad only\ finitely\ many\ n_\tau\ are\ nonzero$$

where the overlines represent equivalence classes modulo principal divisor subgroups.

In summary, double coset maps act between Jacobian varieties as composition with its action on modular forms in the pullback direction, while such maps between Picard groups are also in this direction. Actually, these two forms of double coset maps are in some sense correspondent with each other by *Abel's Theorem*, which states that Picard groups and Jacobian varieties are isomorphic, and they have many of their own applications and substantial results on specific congruence subgroups.