Piecewise linear Morse theory (e.g. triangulation of a mfd)
PL

(Discrete Morse theory, Forman's combinatorial approach)
K. Knudsen, Morse theory: smooth and discrete, 2015)

Lower star filtration ($\approx$ sublevel sets $M_a = f^{-1}(-\infty, a]$)

PL functions

K = finite simplicial complex, such as

Assign a distinct real value to each vertex of K.

Extend this assignment linearly over the entire K and obtain generic PL $f : |K| \to \mathbb{R}$

$$f(x) := \sum b_i(x) f(u_i)$$

barycentric coordinate of x

Now, for f generic, order the vertices so that

$$f(u_1) < f(u_2) < \cdots < f(u_n)$$

Define $K_i :=$ full subcpx of K defined by the first i vertices, $0 \le i \le n$ ($K_0 = \emptyset$)

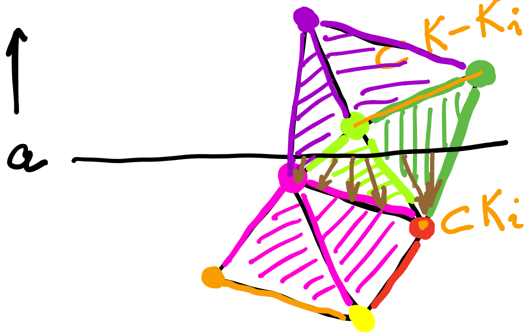
and obtain the "lower star" filtration of f:

$$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_n = K$$

where each K_i is the union of the first i lower stars

$$St\_u_j := \{ \sigma \in St\, u_j \mid x \in \sigma \Rightarrow f(x) \le f(u_j) \}$$

cofaces of u_j



- The lower stars partition K
- $|K_i|$ is a deformation retract of $|K|_a = f^{-1}(-\infty, a]$ for each $f(u_i) \le a < f(u_{i+1})$
- Each vertex u_i viewed as a critical/regular point.

Q: How to characterise "nondegeneracy"?

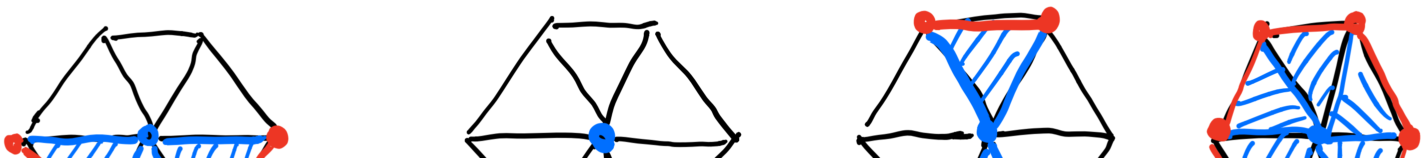
Let us describe the attaching map from K_{i-1} to K_i in the lower star filtration:

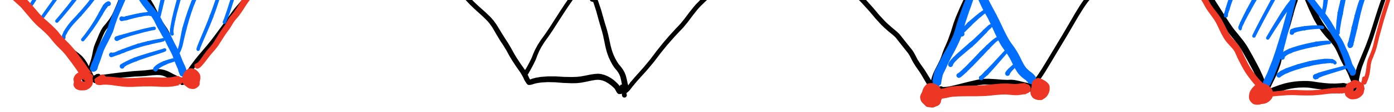
K_i is obtained from K_{i-1} by attaching the closed lower star $\overline{St}\_u_i$ along the lower link
(subcomplex)

$$Lk\_u_i := \{\sigma \in Lk\, u_i = \overline{St}\, u_i - St\, u_i \mid x \in \sigma \Rightarrow f(x) < f(u_i)\}$$

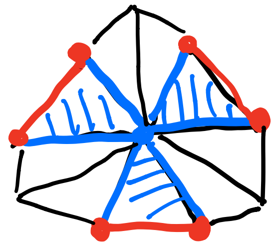
Now assume that K triangulates a closed d-mfd, so that every vertex star lower is an open d-ball and every vertex link lower is a $(d-1)$-sphere.

There are four "good" cases: $d = 2$





regular vertex minimum saddle maximum

(There are other cases:  "monkey saddle"

Monkey saddle: $f(x,y) = x^3 - 3xy^2$

$f_x = 3x^2 - 3y^2$ $\quad$ $f_y = -6xy$

$f_{xx} = 6x$ $\quad$ $f_{xy} = -6y$

$f_{yx} = -6y$ $\quad$ $f_{yy} = -6x$ $\quad \Rightarrow H(0) = 0$)

These good cases can be characterized neatly using reduced Betti numbers of the lower links:

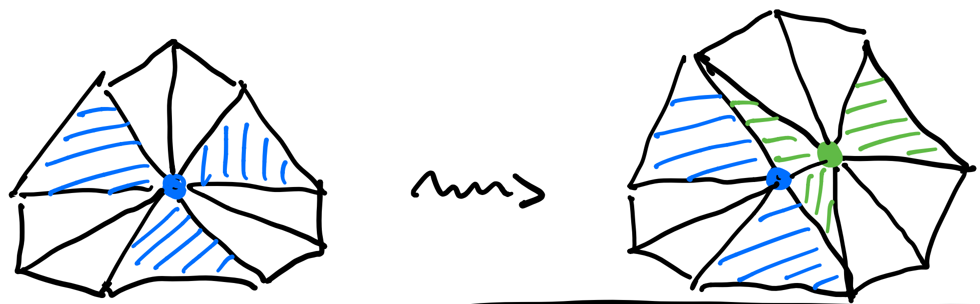
	$\tilde{\beta}_{-1}$	$\tilde{\beta}_0$	$\tilde{\beta}_1$		nonvanishing dim +1 " index
regular	0	0	0	PL regular vertex	
minimum	1	0	0	simple PL critical vertex	0
saddle	0	1	0		1
maximum	0	0	1		2
					⋮

Def A generic PL $f: \overset{\text{manifold}}{|K|} \longrightarrow \mathbb{R}$ is a PL Morse function if each vertex is either PL regular or simple PL critical.

<u>Note</u> PL Morse functions are <u>not</u> dense among the class of all PL functions.

Can sometimes alter the triangulation locally to get a PL Morse function: e.g.,

Monkey saddle cont'd



What do PL Morse functions tell us about the topology of K?

<u>Prop</u> As above, let K be a triangulation of a closed d-manifold and $f: |K| \to \mathbb{R}$ be a PL Morse function. Then the Euler characteristic

$$\chi(K) = \sum_{u \text{ PL critical}} (-1)^{\operatorname{index}(u)}$$

<u>Pf</u> By induction along the lower star filtration.

Recall that $K_i = K_{i-1} \cup_{Lk_- u_i} \overline{St_- u_i}$, $\chi(K_0) = \chi(\emptyset) = 0$.

Observe that

$$\chi(Lk\_u_i) := \sum_{q\geq 1} (-1)^{q-1}\beta_{q-1}(Lk\_u_i)$$

$$= 1 + \sum_{q\geq 0} (-1)^{q-1}\tilde{\beta}_{q-1}(Lk\_u_i)$$

$$= \begin{cases} 1 & \text{if } u_i \text{ is PL regular} \\ 1+(-1)^{\text{index}(u_i)-1} & \underline{\qquad\qquad} \text{ critical} \end{cases}$$

and

$$\chi(Lk\_u_i) = -\left(\chi(St\_u_i) - \underset{u_i}{\underset{\uparrow}{1}}\right)$$

so that adding St_u_i increases χ by

$$\chi(St\_u_i) = 1 - \chi(Lk\_u_i)$$

$$= \begin{cases} 0 \\ (-1)^{\text{index}(u_i)} \end{cases}$$

$\square$

<u>Note</u> This is the special case ($j=d$) of a more general statement.

<u>Thm</u> (PL Morse inequalities) With the above assumptions, write $c_q :=$ # PL critical points of index q. Then we have

(i) weak form: $c_q \geq \beta_q(K)$ for all q;

(ii) strong form: $\sum_{q=0}^{j} (-1)^{j-q} c_q \geq \sum_{q=0}^{j} (-1)^{j-q}\beta_q(K)$ for all $0 \leq j \leq \underset{\dim|K|}{\underset{\|}{d}}$

<u>Idea of pf</u> Induction up along the lower star filtration of K and apply the Mayer-Vietoris sequence to the pair $(K_{i-1}, \mathrm{St}\_u_i \cup \mathrm{Lk}\_u_i)$ across each u_i. $\square$

<u>Reeb graphs</u>

<u>Idea</u>

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{R} \\ & R(f) \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

$R(f)$ — set of "contours" equipped with the quotient topology; connected components of each level set

<u>Applications</u> Accelerate the extraction of level sets (e.g., iso-surfaces for 3-dimensional density data $f: [0,1]^3 \to \mathbb{R}$ in medical imaging).

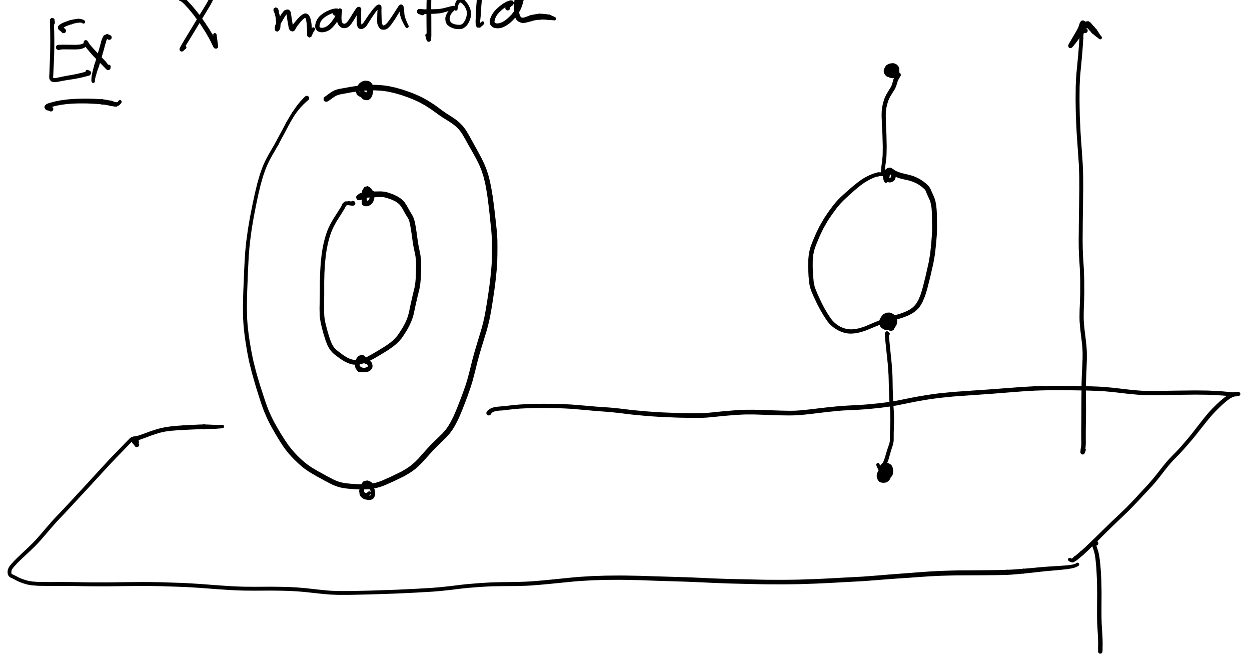
Topology of $R(f)$ as a <u>graph</u>

$$\beta_0(R(f)) = \beta_0(X)$$

$\beta_0(R(f)) = \beta_0(X)$
$\beta_1(R(f)) \leq \beta_1(X)$

X contractible $\Rightarrow$ $R(f)$ is a tree.
(e.g. $[0,1]^3$)

Ex X manifold



Cor (of strong Morse inequality) The Reeb graph of a Morse function on a connected 2-manifold of genus g has g loops if the manifold is orientable and at most $\frac{g}{2}$ loops if it is non-orientable.

$m\log_2 m$ (balanced search tree).
(Fast) algorithm for constructing Reeb graph from a PL Morse function on a 2-manifold.