Curves in the Plane

Def A closed curve is a cts map $\gamma: S^1 \to \mathbb{R}^2$.
γ is simple if γ is an embedding.

Thm (Jordan curve theorem) Removing the image of any simple closed curve leaves two connected components. the bounded inside and the unbounded outside.

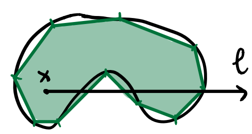
Rmk: Still true for S^2, proved using Mayer-Vietoris seq in homology. Not true for general manifolds.

Parity Algorithm

Problem: Given a simple closed curve γ and a point $x=(x_1, x_2) \in \mathbb{R}^2$, we want to decide whether x lies inside or outside the curve.

Idea: x lies inside the curve if and only if the number of crossing of the curve and a half line emanating from x is odd. (That's where the name comes from.)

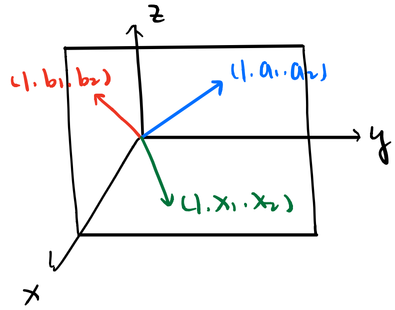
Step 0: First assume a finite approximation of the curve by picking finitely many points on it, then joining the adjacent points by line segment. (For example, we may take $\{\gamma(t) \mid t=0.01, 0.02, \dots, 1\}$



Set int cross = 0.

Step k: Pick the k^{th} line segment ab. $a=(a_1, a_2)$. $b=(b_1, b_2)$ and test whether the horizontal half line ℓ crosses ab.
If it does, cross++.
How to test: we may assume $a_2 < b_2$. First check if $a_2 \le x_2 < b_2$. if not return false.

If true, the the right half line crosses ab if and only if (x, a, b) is right-handed, if and only if $\Delta(x, a, b) := \begin{bmatrix} 1 & x_1 & x_2 \\ 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \end{bmatrix}$ has positive determinant.



boolean Doescross (x, a, b) { if (! $a_2 \le x < b_2$) return false ;

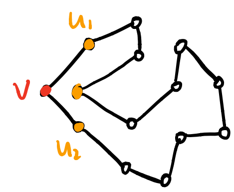
Else return (det $\Delta(x, a, b) > 0$).

What if $x_2 = a_2 = b_2$ for some x, a, b? In this case, we substitute (x_1, x_2) by some $(x_1+\varepsilon_1, x_2+\varepsilon_2)$ such that the two points have the same decision.

Polygon Triangulation

Thm For any polygon in $\mathbb{R}^2$, there is a triangulation of it s.t. all vertices of those triangles are vertices of the polygon.

Pf: We prove by induction on # of vertices n.



When $n \le 3$, nothing to prove.

Suppose $n > 3$. Consider a left-most vertex and its two neighbours u_1, u_2. If the line segment u_1u_2 is in the polygon, then connect them. Otherwise, there must be other vertices inside $\Delta(v u_1 u_2)$. Pick the left-most one and connect it with v. In each case, we seperated the polygon into two polygons, each with at least three vertices. Using induction hypothesis to get the result. □

Winding number

Def Let $\gamma : [0,1] \to \mathbb{R}^3$ be a closed curve, $x \notin \mathrm{Im}(\gamma)$ be a point. Then $\gamma(s) - x$ is never zero. Write it in polar form:

$\gamma(s) - x = R(s)\, \Theta(s)$. Total # of counterclockwise turns of the curve

The winding number $W(\gamma, x)$ of γ and x is defined by
$W(\gamma, x) = \frac{\theta(1)-\theta(0)}{2\pi}$, which is an integer.

Observation: The curve γ divides the plane into different regions, distinguished by the winding number.
Crossing the curve changes the winding number by -1 from left to right and by $+1$ from right to left. (Relative to the curve)

This further implies at least two regions in the decomposition having their boundary arcs consistently oriented by γ:
The region with biggest | smallest winding #.

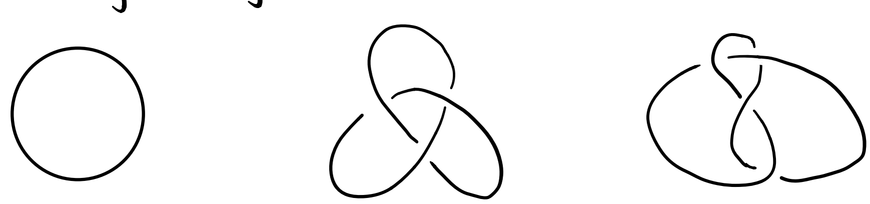
Knots and Links

An embedding $S^1 \hookrightarrow \mathbb{R}^2$ decomposes the space by Jordan curve theorem. But in $\mathbb{R}^3$, this is not the case.

Def A knot is an embedding $K: S^1 \to \mathbb{R}^3$.
Another knot is equivalent to K if it can be continuously deformed to K without crossing itself.

Def A generic projection of a knot is a projection of the image onto some plane with no violation of injectivity, except for finitely many double points of self-intersection.

E.g.



→ This direction | not generic.

The generic projections of the unknot, the trefoil knot and the figure-eight knot.

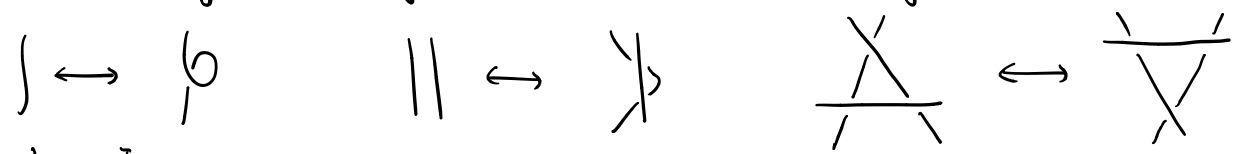
How to distinguish different knots by their generic projections?

"continuous deformation of projections in plane" does not suffice! E.g., a trivial knot can be projected as O, 8, ... not homeomorphic in the plane.

Intuitively we want to drag the upper strand aside, leading to the following definition:

Reidemeister moves

The following three types of transitions of generic projections:



type I type II type III

Prop Two knots are equivalent if and only if their exist generic projections of them which can be continuously deformed to each other after finitely many Reidemeister moves. In particular, different generic projections of one knot is connected by finitely many Reidemeister moves.

Pf: Omitted.

Def A strand of a knot is a piece of the knot from one underpass to another.

Def A tricoloring of a generic projection colors each strand with one of three colors such that

(i) At least two colors are used.

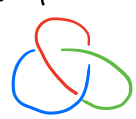
(ii) At each crossing, either all three colors or only one color comes together.

A generic projection is tricolorable if one can find a tricoloring on it.

Prop The Reidemester moves preserve tricolorability.

Pf: List all circumstances. For example, 

Prop The trefoil knot is not equivalent to the unknot.

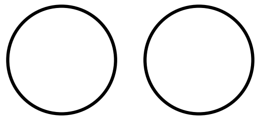
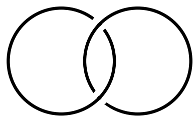
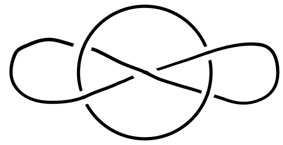
Pf: The unknot is clearly not tricolorable while 

However, although both the figure-eight knot and the trivial knot are not tricolorable (why?), they are not equivalent.

To distinguish them, we may find other invariants. e.g. Jone's polynomial, whose value on the trivial knot is different from all other knots we have known. However, there is possibly one such knot.

Def A link is a collection of two or more disjoint knots.

(Equivalence, generic projection, Reidemester moves defined the same way.)

E.g.   

unlink Hopf link Whitehead link

One can check that only the unlink is tricolorable.

Def Suppose a link has two knots k and λ. Orient both knots arbitrarily, and look at each crossing locally. If the underpass goes from the left or the over pass goes from the right, then count $+1$. otherwise count -1. Let x be a crossing. $\operatorname{sign}(x)$ be the sign at x. define linking number to be

$\operatorname{Lk}(k,\lambda) = \frac{1}{2}\sum_x \operatorname{sign}(x)$.

Note: The linking number is always an integer. (why?)

$\operatorname{Lk}(k,\lambda)$ changes sign if one of the orientations of k, λ is reversed.

Prop The linking number is invariant under Reidemeister moves.

Cor The unlink, the Hopf link and the Whitehead link are pairwisely unequivalent.

Pf: unlink $\neq$ Hopf & Whitehead by tricolorability,

Hopf $\neq$ Whitehead since the former has nonzero linking #

(± 1, depending on the orientation) while the latter has

linking # 0. $\square$

Writhing number

Def Let $k: S^1 \to \mathbb{R}^3$ be a knot, $u \in S^2$. Project k onto the plane perpendicular to u. If the projection is generic, then at each crossing x we count by $\operatorname{sign}(x) = \pm 1$ as in linking number, taking self-crossing into consideration and without divided by 2.

The directional writhing number at direction u is

$DWr(k,u) = \sum_x sign(x)$. summed over all crossing.

The writhing number of k is $Wr(k) := \frac{1}{4\pi} \int_{u\in S^2} DWr(k,u)\, du$

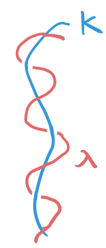
Rmk ① The writhing number describes how contorted the curve is.

② This number is affected by the Reidmeister move type I.

③ Unlike linking number, the writhing number is not necessarily an integer.

④ The fact is that the non-generic directions is of measure 0, so the integral is well-defined.

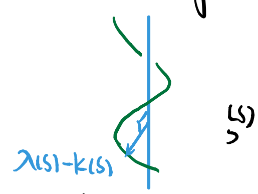
Application in DNA



In the double-helix structure of DNA, two ribbons twist and turn around an imaginary center axis, denoted by k. Denote one of the helix λ.

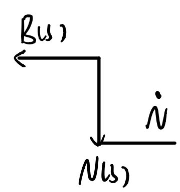
k may also twist itself.

We'd like to define a <u>twisting number</u> to evaluate the average motion of λ relatively to k, regardless of how k twisted.



To formalize, suppose k, λ are parametrized such that $\lambda(s)-k(s)$ has unit length and is perpendicular to the center axis

let $N(s) = \lambda(s) - k(s)$, $T(s) = \frac{\dot{k}(s)}{|\dot{k}(s)|}$ and $B(s) = T(s) \times N(s)$.



Then the twisting number is defined to be

$Tw(k,\lambda) = \frac{1}{2\pi} \int_{s\in S^1} \langle \dot{N}(s), B(s) \rangle ds$.

Note: $\dot{N}(s)$ is actually on the same direction as $B(s)$. for $N(s)$ is a plane vector with constant length.

So the more rapid $N(s)$ change, the more twisted λ is w.r.t. k.

The twisting number of K, λ may be regarded as the number of local crossings of K, λ, since we ignored the effect of global twisting of K.

In contrast, the global crossing is $Wr(K)$, describing how does K twist. Furthermore, $Lk(K, \lambda)$ counts the total number of crossings.

We state an intuitively correct theorem without proof:

Thm Let K be a smooth closed curve in $\mathbb{R}^3$ and λ one of the two ribbons centered along K. Then

$Lk(K, \lambda) = Tw(K, \lambda) + Wr(K)$.

Relation to winding number:

The directional writhing number depends on the direction $u \in S^2$. Actually, for most place on S^2, $DWr(K, u)$ does not change when u go through it. There exist a closed curve in S^2 such that $DWr(K, u)$ changes if and only if u crosses the curve. It is called the curve of critical directions.

For such a curve T in S^2 and $u \in S^2$, we define its winding number to be the net number of counter-clockwise turns formed by T around the directed line defined by u. Denoted $W(T, u)$.

When u cross T, $DWr(K, u)$ and $W(T, u)$ change simutaneously. Actually, by the same amount: $DWr(K, u_0) - DWr(K, u) = W(T, u_0) - W(T, u)$.

Integrating both sides on S^2, getting

$$Wr(K) = DWr(K, u_0) - W(T, u_0) + \frac{1}{4\pi}\int_{u \in S^2} W(T, u)\, du .$$