



Probability

Probability

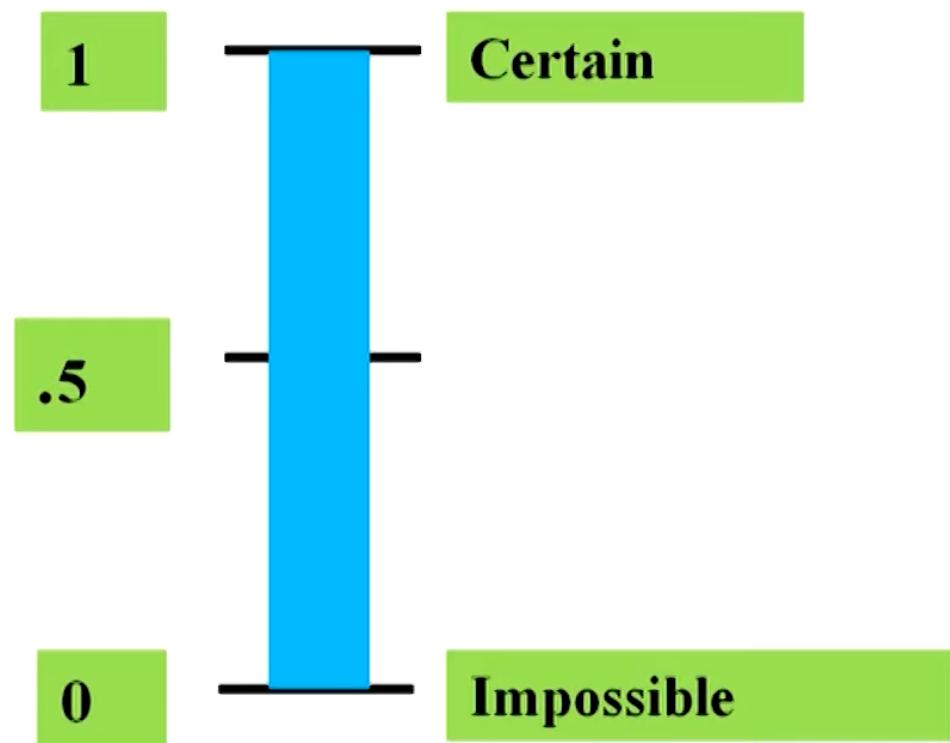
Lecture objectives

- Comprehend the different ways of assigning probability
- Understand and apply marginal, union, joint, and conditional probabilities
- Solve problems using the laws of probability including the laws of addition, multiplication and conditional probability
- Revise probabilities using Bayes' rule

Probability

- Probability is the numerical measure of the likelihood that an event will occur.
- The probability of any event must be between 0 and 1, inclusively
 - $0 \leq P(A) \leq 1$ for any event A.
- The sum of the probabilities of all mutually exclusive and collectively exhaustive events is 1.
 - $P(A) + P(B) + P(C) = 1$
 - A, B, and C are mutually exclusive and collectively exhaustive

Range of Probability



Method of assigning probability

- Classical method of assigning probability (rules and laws)
- Relative frequency of occurrence (cumulated historical data)
- Subjective Probability (personal intuition or reasoning)

Classical probability

- Number of outcomes leading to the event divided by the total number of outcomes possible
- Each outcome is equally likely
- Determined *a priori* -- before performing the experiment
- Applicable to games of chance
- Objective -- everyone correctly using the method assigns an identical probability

Classical probability

$$P(E) = \frac{n_e}{N}$$

Where:

N = total number of outcomes

n_e = number of outcomes in E

Relative frequency probability

- Based on historical data
- Computed after performing the experiment
- Number of times an event occurred divided by the number of trials
- Objective -- everyone correctly using the method assigns an identical probability

Relative frequency probability

$$P(E) = \frac{n_e}{N}$$

Where :

N = total number of trials

n_e = number of outcomes
producing E

Relative frequency probability

Relative Frequency

We calculate **relative frequency** using the following formula:

$$\text{relative frequency} = \frac{\text{no. of times an outcome happened}}{\text{total no. of all outcomes}}$$

Example: A coin is flipped 100 times, the coin lands on heads 48 times.

Calculate the relative frequency for heads.

$$\text{relative frequency} = \frac{48}{100} = 0.48$$

for landing on heads.

Relative frequency probability

Bias

- One of the key uses of relative frequency is in testing for bias. We say that an experiment is biased when the probability of a particular outcome is unfairly bigger or smaller than it should be.

- Example: A biased coin might be cleverly designed so that it's lands on tails more than 50% of the time. If we suspect a coin of being biased, then we test it by flipping it a lot of times and recording the results. Then, if the relative frequency of tails is noticeably more than 50%, we might suspect the coin of being biased

Subjective probability

- Comes from a person's intuition or reasoning
- Subjective -- different individuals may (correctly) assign different numeric probabilities to the same event
- Degree of belief
- Useful for unique (single-trial) experiments
 - New product introduction
 - Initial public offering of common stock
 - Site selection decisions
 - Sporting events

Probability -Terminology

- Experiment
- Event
- Elementary Events
- Sample Space
- Unions and Intersections
- Mutually Exclusive Events
- Independent Events
- Collectively Exhaustive Events
- Complementary Events

Experiment, Trial, Elementary Event and Event

- **Experiment:** a process that produces outcomes
 - More than one possible outcome
 - Only one outcome per trial
- **Trial:** one repetition of the process
- **Elementary Event:** cannot be decomposed or broken down into other events
- **Event:** an outcome of an experiment
 - may be an elementary event, or
 - may be an aggregate of elementary events
 - usually represented by an uppercase letter, e.g., A, E1

An Example Experiment

- Experiment: randomly select, without replacement, two families from the residents of Tiny Town
- Elementary Event: the sample includes families A and C
- Event: each family in the sample has children in the household
- Event: the sample families own a total of four automobiles

Tiny Town Population		
Family	Children in Household	Number of Automobiles
A	Yes	3
B	Yes	2
C	No	1
D	Yes	2

Sample Space

- The set of all elementary events for an experiment
- Methods for describing a sample space
 - roster or listing
 - tree diagram
 - set builder notation
 - Venn diagram

Sample Space-Roster Example

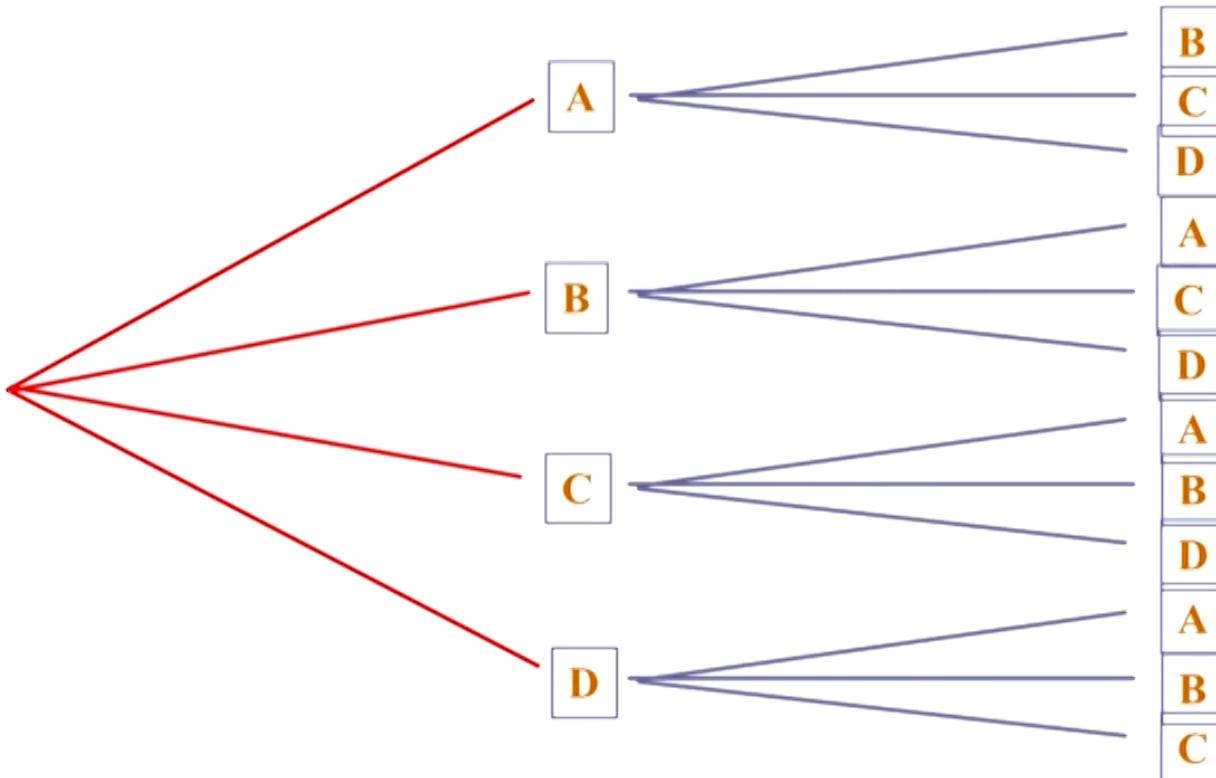
- Experiment: randomly select, without replacement, two families from the residents of Tiny Town
- Each ordered pair in the sample space is an elementary event, for example -- (D,C)

Family	Children in Household	Number of Automobiles
A	Yes	3
B	Yes	2
C	No	1
D	Yes	2

Listing of Sample Space
(A,B), (A,C), (A,D), (B,A), (B,C), (B,D), (C,A), (C,B), (C,D), (D,A), (D,B), (D,C)



Sample Space-Tree Diagram for Random sample of Two Families





Sample Space-Set Notation for Random sample of Two Families

- $S = \{(x,y) \mid x \text{ is the family selected on the first draw, and } y \text{ is the family selected on the second draw}\}$
- Concise description of large sample spaces

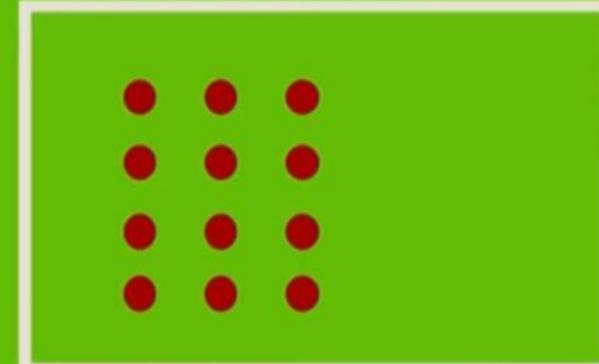
Sample Space

- Useful for discussion of general principles and concepts

Listing of Sample Space

(A,B), (A,C), (A,D),
(B,A), (B,C), (B,D),
(C,A), (C,B), (C,D),
(D,A), (D,B), (D,C)

Venn Diagram



union of sets

- The union of two sets contains an instance of each element of the two sets.

$$X = \{1,4,7,9\}$$

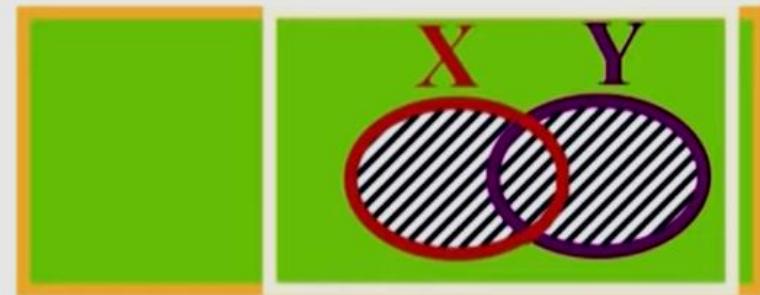
$$Y = \{2,3,4,5,6\}$$

$$X \cup Y = \{1,2,3,4,5,6,7,9\}$$

$$C = \{IBM, DEC, Apple\}$$

$$F = \{Apple, Grape, Lime\}$$

$$C \cup F = \{IBM, DEC, Apple, Grape, Lime\}$$



intersection of sets

- The intersection of two sets contains only those element common to the

$$X = \{1, 4, 7, 9\}$$

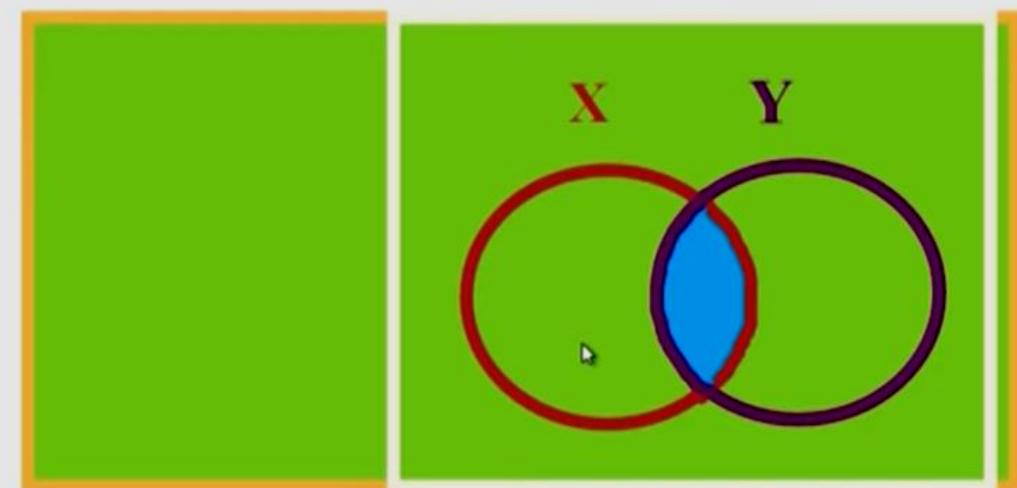
$$Y = \{2, 3, 4, 5, 6\}$$

$$X \cap Y = \{4\}$$

$$C = \{IBM, DEC, Apple\}$$

$$F = \{Apple, Grape, Lime\}$$

$$C \cap F = \{Apple\}$$



Mutually Exclusive Events

- Events with no common outcomes
- Occurrence of one event precludes the occurrence of the other event

$$C = \{IBM, DEC, Apple\}$$

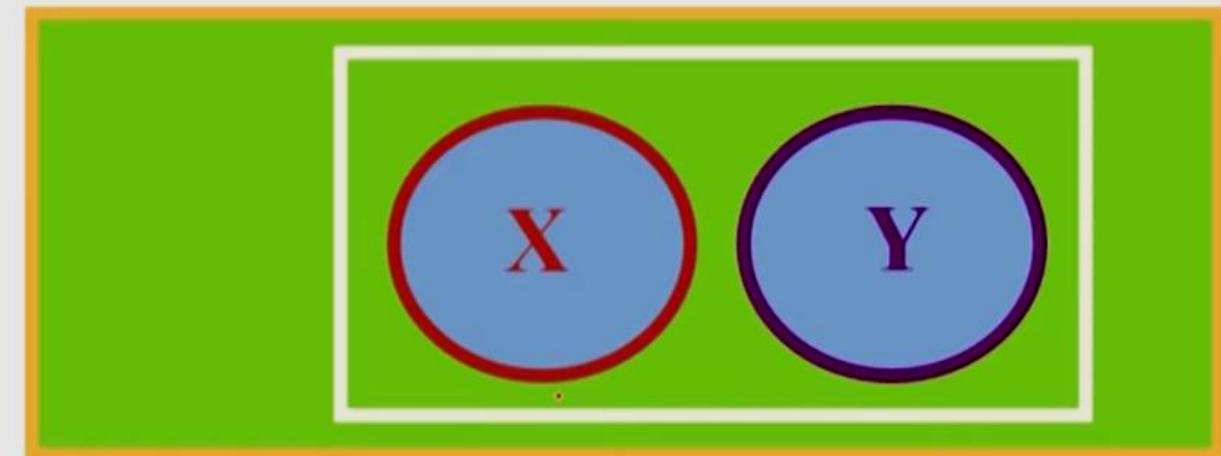
$$F = \{Grape, Lime\}$$

$$C \cap F = \{\}$$

$$X = \{1, 7, 9\}$$

$$Y = \{2, 3, 4, 5, 6\}$$

$$X \cap Y = \{\}$$



$$P(X \cap Y) = 0$$

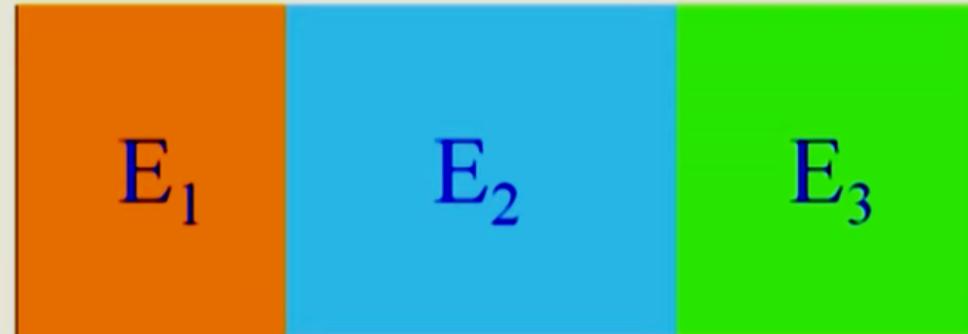
Independent Events

- Occurrence of one event does not affect the occurrence or nonoccurrence of the other event
- The conditional probability of X given Y is equal to the marginal probability of X.
- The conditional probability of Y given X is equal to the marginal probability of Y.

$$P(X|Y) = P(X) \text{ and } P(Y|X) = P(Y)$$

Collectively exhaustive Events

- Contains all elementary events for an experiment

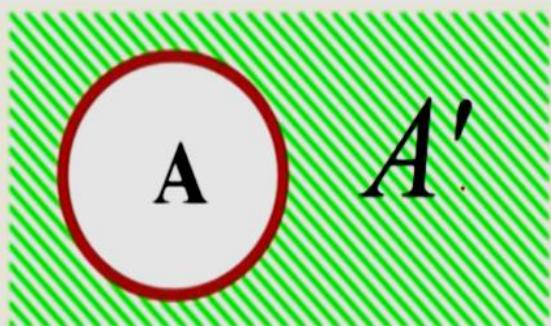


Sample Space with three
collectively exhaustive events

Complementary Events

- All elementary events not in the event 'A' are in its complementary event.

Sample
Space



$$P(\text{Sample Space}) = 1$$

$$P(A') = 1 - P(A)$$

counting the possibilities

- mn Rule
- Sampling from a Population with Replacement
- Combinations: Sampling from a Population without Replacement

mn Rule

- If an operation can be done m ways and a second operation can be done n ways, then there are mn ways for the two operations to occur in order.
- This rule is easily extend to k stages, with a number of ways equal to $n_1 \cdot n_2 \cdot n_3 \dots n_k$
- Example: Toss two coins . The total umber of simple events is $2 \times 2 = 4$

Sampling from a Population with Replacement

- A tray contains 1,000 individual tax returns. If 3 returns are randomly selected **with replacement** from the tray, how many possible samples are there?
- $(N)^n = (1,000)^3 = 1,000,000,000$

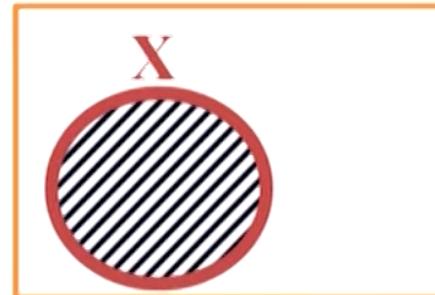
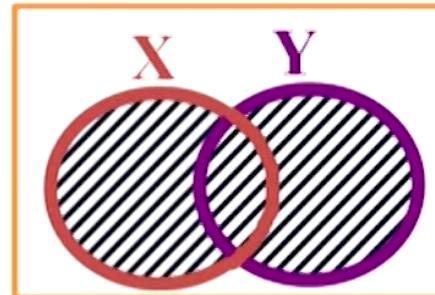
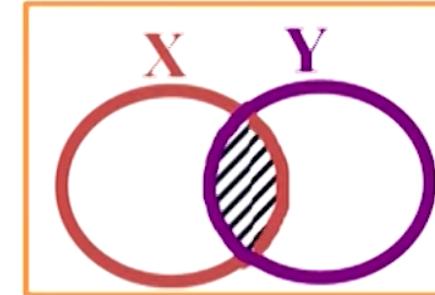
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Combinations

- A tray contains 1,000 individual tax returns. If 3 returns are randomly selected **without replacement** from the tray, how many possible samples are there?

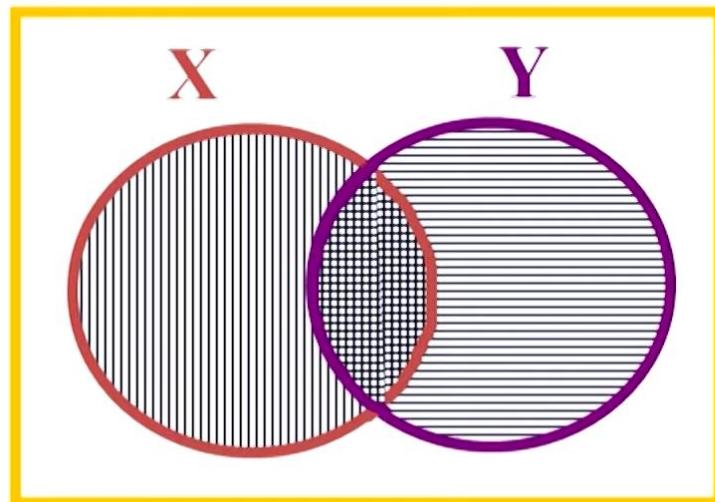
$$\binom{N}{n} = \frac{N!}{n!(N-n)!} = \frac{1000!}{3!(1000-3)!} = 166,167,000$$

Four Types of Probability

Marginal	Union	Joint	Conditional
$P(X)$ The probability of X occurring 	$P(X \cup Y)$ The probability of X or Y occurring 	$P(X \cap Y)$ The probability of X and Y occurring 	$P(X Y)$ The probability of X occurring given that Y has occurred 

General Law of Addition

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$



Design for improving the productivity



Problem

- A company conducted a survey for the American Society of Interior Designers in which workers were asked which changes in office design would increase productivity.
- Respondents were allowed to answer more than one type of design change.

Reducing noise would increase productivity	70 %
More storage space would increase productivity	67 %

Problem

- If one of the survey respondents was randomly selected and asked what office design changes would increase worker productivity,
 - what is the probability that this person would select reducing noise or more storage space?

Solution

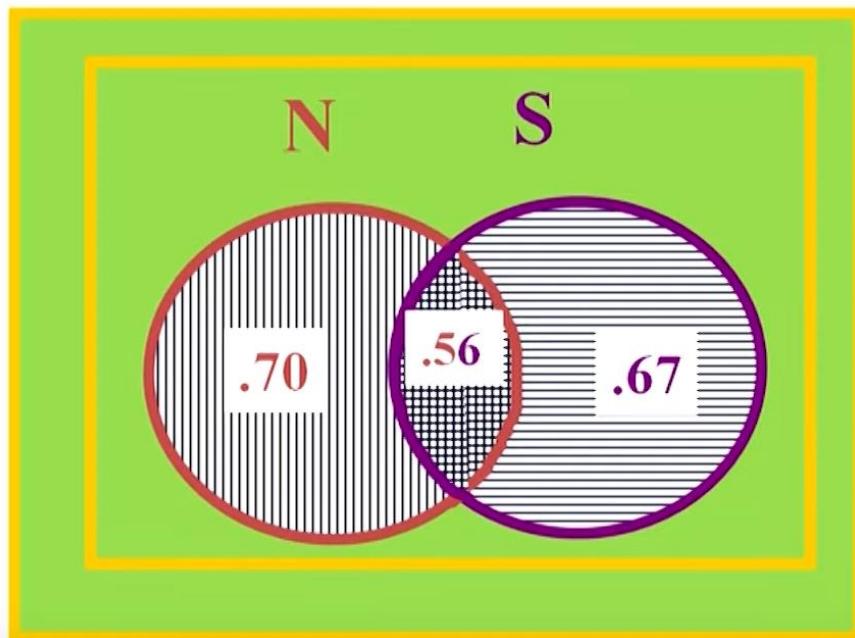
- Let N represent the event “reducing noise.”
- Let S represent the event “more storage/ filing space.”
- The probability of a person responding with N or S can be symbolized statistically as a union probability by using the law of addition.

$$P(N \cup S)$$

.

Solution- General Law of Addition

$$P(N \cup S) = P(N) + P(S) - P(N \cap S)$$



$$P(N) = .70$$

$$P(S) = .67$$

$$P(N \cap S) = .56$$

$$\begin{aligned}P(N \cup S) &= .70 + .67 - .56 \\&= 0.81\end{aligned}$$

Office Design Problem

Probability Matrix

		Increase Storage Space		Total
		Yes	No	
Noise Reduction	Yes	.56	.14	.70
	No	.11	.19	.30
Total	.67	.33	1.00	

Joint Probability Using a Contingency Table

Event	Event		Total
	B ₁	B ₂	
A ₁	P(A ₁ and B ₁)	P(A ₁ and B ₂)	P(A ₁)
A ₂	P(A ₂ and B ₁)	P(A ₂ and B ₂)	P(A ₂)
Total	P(B ₁)	P(B ₂)	1

Joint Probabilities

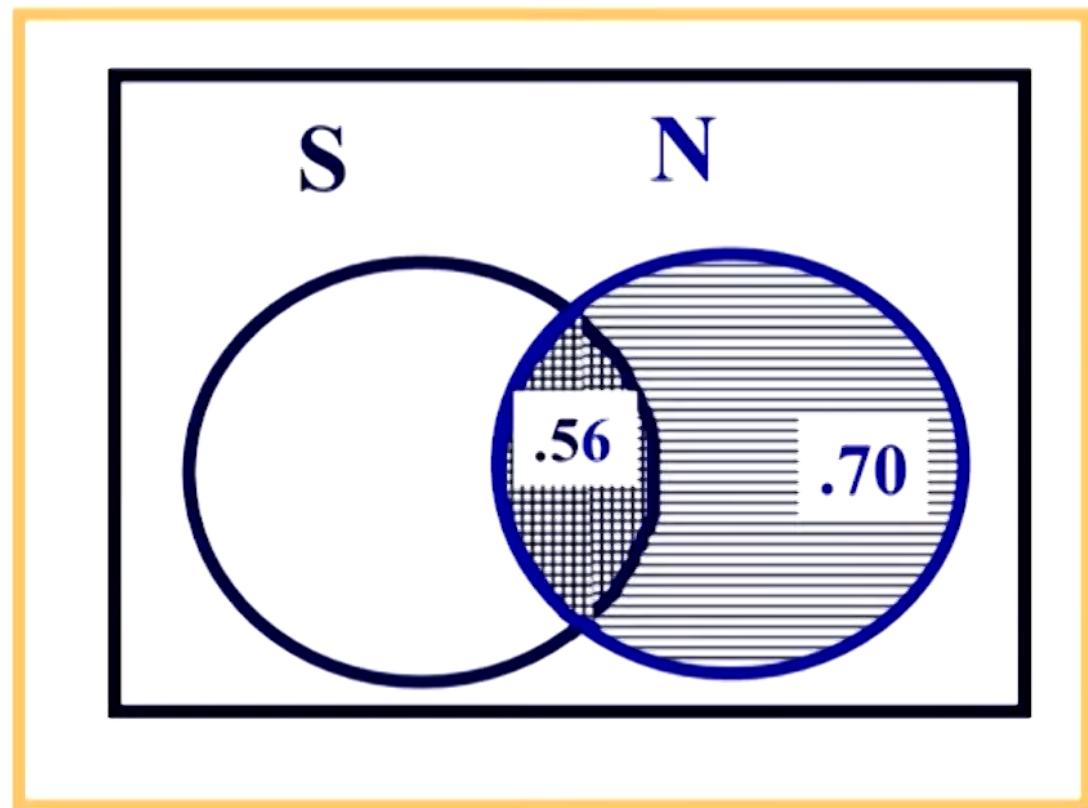
Marginal (Simple) Probabilities

S Office Design Problem - Probability Matrix

		Increase Storage Space		Total
		Yes	No	
Noise Reduction	Yes	.56	.14	.70
	No	.11	.19	.30
	Total	.67	.33	1.00

$$\begin{aligned}P(N \cup S) &= P(N) + P(S) - P(N \cap S) \\&= .70 + .67 - .56 \\&= .81\end{aligned}$$

Law of Conditional Probability



$$\begin{aligned}P(N) &= .70 \\P(N \cap S) &= .56 \\P(S|N) &= \frac{P(N \cap S)}{P(N)} \\&= \frac{.56}{.70} \\&= .80\end{aligned}$$

Solution

Office Design Problem

		Increase Storage Space		Total
		Yes	No	
Noise Reduction	Yes	.56	.14	.70
	No	.11	.19	.30
	Total	.67	.33	1.00

$$P(\bar{N} | S) = \frac{P(\bar{N} \cap S)}{P(S)} = \frac{.11}{.67}$$
$$= .164$$

Problem

- A company data reveal that 155 employees worked one of four types of positions.
- Shown here again is the raw values matrix (also called a contingency table) with the frequency counts for each category and for subtotals and totals containing a breakdown of these employees by type of position and by sex.

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Contingency Table

COMPANY HUMAN RESOURCE DATA

		<i>Sex</i>		
		<i>Male</i>	<i>Female</i>	
<i>Type of Position</i>	<i>Managerial</i>	8	3	11
	<i>Professional</i>	31	13	44
	<i>Technical</i>	52	17	69
	<i>Clerical</i>	9	22	31
	100	55	155	

- If an employee of the company is selected randomly, what is the probability that the employee is female or a professional worker?

$$P(F \cup P) = P(F) + P(P) - P(F \cap P)$$

$$P(F \cup P) = .355 + .284 - .084 = .555.$$

Solution