MTAT.07.003 Cryptology II

Zero-knowledge Proofs

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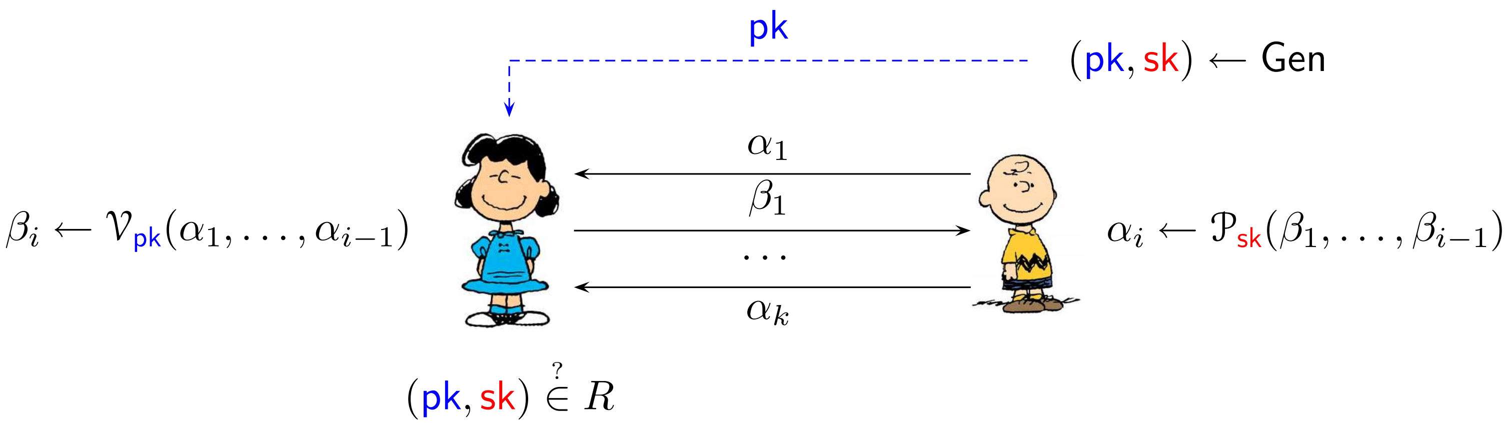
Formal Syntax

Zero-knowledge proofs



In many settings, some system-wide or otherwise important parameters pk are generated by potentially malicious participants.

- ▷ Zero-knowledge proofs guarantee that the parameters pk are correctly generated without leaking any extra information.
- ▷ Often, public parameters pk are generated together with auxiliary secret information sk that is essential for the zero-knowledge proof.
- ▷ The secret auxiliary information sk is known as a *witness* of pk.

A few interesting statements

An integer n is a RSA modulus:

- ▷ A witness is a pair of primes (p, q) such that $n = p \cdot q$.
- ▷ The relation is defined as follows $(n, p, q) \in R \Leftrightarrow n = p \cdot q \wedge p, q \in \mathbb{P}$

A prover has a secret key sk that corresponds to a public key pk:

- ▷ A witness is a secret key sk such that $(\mathsf{pk}, \mathsf{sk}) \in \mathsf{Gen}$.
- ▷ More formally $(\mathsf{pk}, \mathsf{sk}) \in R \Leftrightarrow \forall m \in \mathcal{M} : \mathsf{Dec}_{\mathsf{sk}}(\mathsf{Enc}_{\mathsf{pk}}(m)) = m$.

A ciphertext c is an encryption of m wrt the public key pk:

- ▷ A witness is a randomness $r \in \mathcal{R}$ such that $\mathsf{Enc}_{\mathsf{pk}}(m; r) = c$.
- ▷ The relation is defined as follows $(\mathsf{pk}, c, m, r) \in R \Leftrightarrow \mathsf{Enc}_{\mathsf{pk}}(m; r) = c$.

Two flavours of zero knowledge

An ideal implementation of a zero-knowledge proof

pk

$\xrightarrow{\mathsf{pk}}$ $\xleftarrow{y}$

If $\exists \mathsf{sk} : (\mathsf{pk}, \mathsf{sk}) \in R$

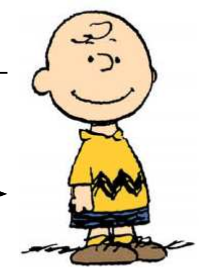
| $y \leftarrow 1$

Else

| $y \leftarrow 0$

$\xleftarrow{\mathsf{pk}}$ $\xrightarrow{\perp}$

pk



An ideal implementation of a zero-knowledge proof of knowledge

pk

$\xrightarrow{\mathsf{pk}}$ $\xleftarrow{y}$

If $(\mathsf{pk}, \mathsf{sk}) \in R$

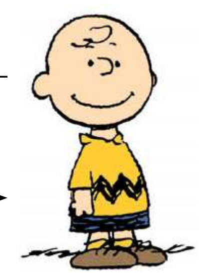
| $y \leftarrow 1$

Else

| $y \leftarrow 0$

$\xleftarrow{\mathsf{pk}, \mathsf{sk}}$ $\xrightarrow{\perp}$

pk, sk



Formal security requirements

Completeness. A zero-knowledge proof is *perfectly complete* if all runs between honest prover and honest verifier are accepting. A zero knowledge protocol is ε_1-*incomplete* if for all $(\mathsf{pk}, \mathsf{sk}) \in R$ the interaction between honest prover and honest verifier fails with probability at most ε_1.

Soundness. A zero-knowledge proof is ε_2-*unsound* if the probability that an honest verifier accepts an incorrect input pk with probability at most ε_2. An input pk is incorrect if $(\mathsf{pk}, \mathsf{sk}) \notin R$ for all possible witnesses sk.

Zero-knowledge property. A zero-knowledge proof is $(t_{\mathrm{re}}, t_{\mathrm{id}}, \varepsilon_3)$-*private* if for any t_{re}-time verifying strategy $\mathcal{V}_*$ there exists a t_{id}-time algorithm $\mathcal{V}_\circ$ that does not interact with the prover and the corresponding output distributions are statistically ε_3-close.

A Simple Example

Quadratic residuosity

$v = s^2$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $s \in \mathbb{Z}_n^*$

$\beta \xleftarrow[u]{} \{0,1\}$

$$\xleftarrow{\alpha = r^2}$$

$$\xrightarrow{\beta}$$

$$\xleftarrow{\gamma = r s^{\beta}}$$

$r \xleftarrow[u]{} \mathbb{Z}_n^*$

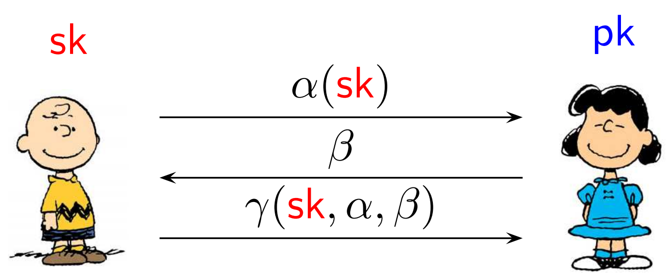
Halt if $\gamma \notin \mathbb{Z}_n^*$

$\gamma^2 = r^2 s^{2\beta} \stackrel{?}{=} \alpha v^{\beta}$

The modified Fiat-Shamir protocol is also secure against malicious verifiers.

- ▷ If we guess the challenge bit β then we can create α such that the transcript corresponds to the real world execution.
- ▷ Random guessing leads to the correct answer with probability $\frac{1}{2}$.
- ▷ By rewinding we can decrease the failure probability. The failure probability decreases exponentially w.r.t. maximal number of rewindings.

Simulation principle

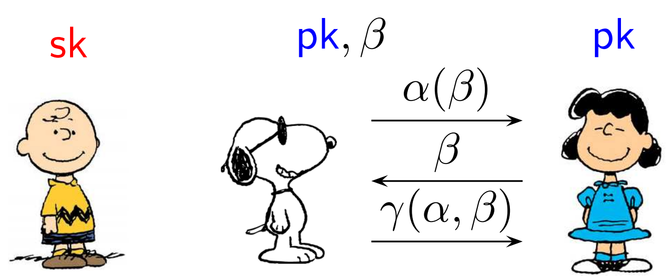


Even if Lucy is *honest*

▷ she might learn something about the secret sk.

since

▷ messages α and γ depend on the secret sk.



As Lucy is *malicious* the value of β is not known by her before the protocol and Snoopy must guess β to simulate the other messages.

Lucy should not be able to distinguish between these two experiments.

Simulation as rejection sampling

- ▷ As the Fiat-Shamir protocol is a sigma protocol, we can construct protocol transcripts $(\alpha_\circ, \beta_\circ, \gamma_\circ) \leftarrow \mathsf{Sim}_{\mathsf{Fiat\text{-}Shamir}}$ for honest verifier.
- ▷ Note that $\alpha_\circ$ has the same distribution than α in the real protocol run.
- ▷ Now consider a modified prover $\mathcal{P}_*$ that
  - ◇ generates $(\alpha_\circ, \beta_\circ, \gamma_\circ) \leftarrow \mathsf{Sim}$ and sends $\alpha_\circ$ to the verifier,
  - ◇ given a challenge β computes the correct reply γ,
  - ◇ outputs Sim-Success if $\beta_\circ = \beta$.

Important observations. Let $\mathcal{D}_\circ$ denote the distribution of the outputs of a verifier $\mathcal{V}_*$ which satisfy the condition $\mathcal{P}_*$ outputs Sim-Success. Then the distribution $\mathcal{D}_\circ$ coincides with the distribution of all outputs of $\mathcal{V}_*$.

- ▷ For each reply β, the condition $\beta = \beta_\circ$ holds with probability $\frac{1}{2}$.
- ▷ The distribution $\mathcal{D}_\circ$ is easily simulatable.

The complete simulator construction

$$
\begin{array}{l}
\mathcal{V}_\circ \\
\left[\begin{array}{l}
\text{For } i \in \{1, \ldots, k\} \text{ do} \\
\left[\begin{array}{l}
(\alpha_\circ, \beta_\circ, \gamma_\circ) \leftarrow \mathsf{Sim}_{\mathsf{Fiat\text{-}Shamir}} \\
\beta \leftarrow \mathcal{V}_*(\alpha_\circ) \\
\mathsf{if}\ \beta = \beta_\circ\ \mathsf{then}\ \textbf{return}\ \mathcal{V}_*(\gamma_\circ)
\end{array}\right. \\
\textbf{return}\ \mathsf{failure}
\end{array}\right.
\end{array}
$$

By the construction the output distribution of $\mathcal{V}_\circ$ is

$$(1 - 2^{-k})\mathcal{D}_\circ + 2^{-k}\mathsf{failure} \equiv (1 - 2^{-k})\mathcal{D} + 2^{-k}\mathsf{failure}$$

and thus the statistical distance between outputs of $\mathcal{V}_*$ and $\mathcal{V}_\circ$ is 2^{-k}.

The corresponding security guarantees

Theorem. The modified Fiat-Shamir protocol is a zero-knowledge proof with the following properties:

- ▷ the protocol is perfectly complete;
- ▷ the protocol is $\frac{1}{2}$-unsound;
- ▷ for any k and t_{re} the protocol is $(t_{\mathrm{re}}, k \cdot t_{\mathrm{re}}, 2^{-k})$-private.

Further remarks

- ▷ Sequential composition of ℓ protocol instances decreases soundness error to $2^{-\ell}$. The compound protocol becomes $(t_{\mathrm{re}}, k \cdot \ell \cdot t_{\mathrm{re}}, \ell \cdot 2^{-k})$-private.
- ▷ The same proof is valid for all sigma protocols, where the challenge β is only one bit long. For longer challenges β, the success probability decreases with an exponential rate and simulation becomes inefficient.

Zero-Knowledge Proofs
and
Knowledge Extraction

Challenge-response paradigm

$v \in \mathsf{QNR}(n)$

$\beta \xleftarrow[u]{} \{0,1\}$

$r \xleftarrow[u]{} \mathbb{Z}_n^*$

$\beta \stackrel{?}{=} \overline{\beta}$

$\xrightarrow{\alpha = r^2 v^\beta}$

$\xleftarrow{\overline{\beta}}$

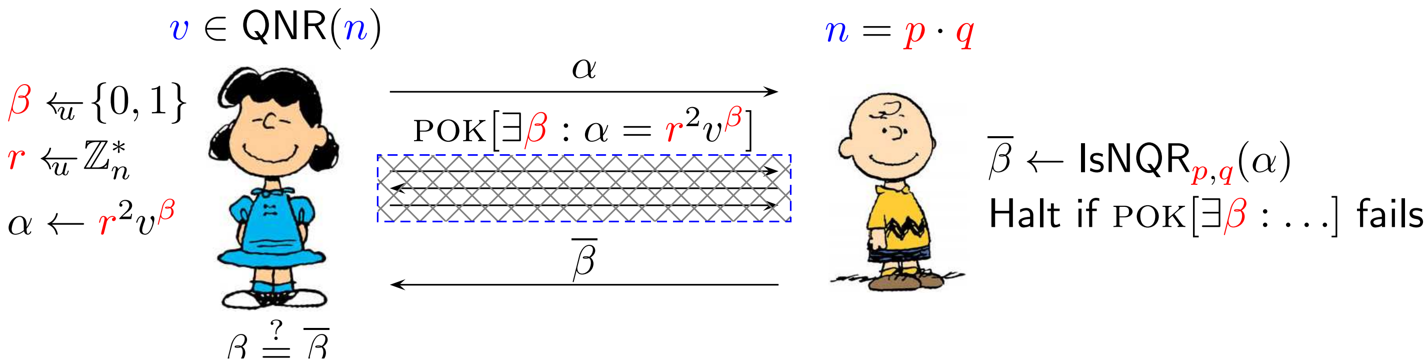
$n = p \cdot q$

$\overline{\beta} \leftarrow \mathsf{IsNQR}_{p,q}(\alpha)$

For semi-honest provers it is trivial to simulate the interaction, since the verifier knows the expected answer $\beta = \overline{\beta}$. To provide security against malicious verifiers $\mathcal{V}_*$, we must assure that we can extract β from $\mathcal{V}_*$:

- ▷ Verifier must prove that she knows (r, β) such that $c = r^2 v^\beta$
- ▷ The corresponding proof of knowledge does not have be zero knowledge proof as long as it does not decrease soundness.

Classical construction



We can use proofs of knowledge to assure that the verifier knows the end result β. The proof must perfectly hide information about witness β.

- ▷ If $v \in \mathsf{QR}$ then α is independent from β and malicious prover can infer information about β only through the proof of knowledge.
- ▷ Hence, we are interested in *witness indistinguishability* of the proof of knowledge, i.e., the proof transcripts should coincide for both β values.

Witness indistinquishability provides soundness

We have to construct a sigma protocol for the following statement

$$\mathrm{POK}\left[\exists\beta\ \exists r : \alpha = r^2 v^\beta\right] \equiv \mathrm{POK}\left[\exists r : r^2 = \alpha\right] \vee \mathrm{POK}\left[\exists r : r^2 = \alpha v^{-1}\right]$$

Both sub-proofs separately can be implemented through the modified Fiat-Shamir protocol. To achieve witness indistinguishability, we just use disjunctive proof construction.

- ▷ For fixed challenge β, the sub-challenge pairs are uniformly chosen from a set $\mathcal{B} = \{(\beta_1, \beta_2) : \beta_1 + \beta_2 = \beta\}$.
- ▷ Hence, the interactions where $\mathcal{V}$ proves $\mathrm{POK}\left[\exists r : r^2 = \alpha\right]$ and simulates $\mathrm{POK}\left[\exists r : r^2 = \alpha v^{-1}\right]$ are indistinguishable form the interactions where $\mathcal{V}$ proves $\mathrm{POK}\left[\exists r : r^2 = \alpha v^{-1}\right]$ and simulates $\mathrm{POK}\left[\exists r : r^2 = \alpha\right]$.
- ▷ If $v = s^2$ then also $\alpha_0 = r^2$ and $\alpha_1 = r^2 v$ are indistinguishable.

Consequently, a malicious adversary succeeds with probability $\frac{1}{2}$ if $v = s^2$.

Simulator construction

$\mathcal{S}^{\mathcal{V}_*}$

Choose randomness ω for $\mathcal{V}_*$ and store α.

Use knowledge extractor to extract β.

Run $\mathcal{V}_*$ once again.

if $\mathrm{POK}_\beta\left[\exists r : \alpha = r^2 v^\beta\right]$ fails then

- Send $\perp$ to $\mathcal{V}$ and output whatever $\mathcal{V}_*$ outputs.

else

- Send β to $\mathcal{V}$ and output whatever $\mathcal{V}_*$ outputs.

The simulation fails only if knowledge extraction fails and $\mathrm{POK}_\beta\left[\cdot\right]$ succeeds. With proper parameter choice, we can achieve failure ε in time $\Theta\left(\frac{t_{\mathrm{re}}}{\varepsilon-\kappa}\right)$.

Optimal choice of parameters

Let ε be the desired failure bound and let κ be the knowledge error of the sigma protocol. Now if we set the maximal number of repetitions

$$\ell = \frac{4\left\lceil \log_2(1/\varepsilon)\right\rceil}{\varepsilon - \kappa}$$

in the knowledge extraction algorithm so that the knowledge extraction procedure fails on the set of good coins

$$\Omega_{\mathrm{good}} = \{\omega \in \Omega : \Pr\left[\textsc{pok}_\beta\left[\cdot\right] = 1|\omega\right] \geq \varepsilon\}$$

with probability less than ε. Consequently, we can estimate

$$\begin{aligned}\Pr\left[\mathsf{Fail}\right] \leq \Pr\left[\omega \notin \Omega_{\mathrm{good}}\right] \cdot \Pr\left[\textsc{pok}_\beta\left[\cdot\right] = 1|\omega\right] \cdot \Pr\left[\mathsf{ExtrFailure}|\omega\right] \\ + \Pr\left[\omega \in \Omega_{\mathrm{good}}\right] \cdot \Pr\left[\textsc{pok}_\beta\left[\cdot\right] = 1|\omega\right] \cdot \Pr\left[\mathsf{ExtrFailure}|\omega\right] \leq \varepsilon \enspace .\end{aligned}$$

Soundness through temporal order

$v \in \mathsf{QNR}(n)$

$\beta \xleftarrow[u]{} \{0,1\}$

$r \xleftarrow[u]{} \mathbb{Z}_n^*$

$\alpha \leftarrow r^2 v^\beta$

$\xrightarrow{\alpha}$

$\xleftarrow{c,\ \mathsf{pk}}$

$\xrightarrow{r,\ \beta}$

$\xleftarrow{d}$

$n = p \cdot q$

$\mathsf{pk} \leftarrow \mathsf{Gen}$

$\overline{\beta} \leftarrow \mathsf{IsNQR}_{p,q}(\alpha)$

$(c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(\overline{\beta})$

Halt if $\alpha \neq r^2 v^\beta$

$\beta \stackrel{?}{=} \mathsf{Open}_{\mathsf{pk}}(c, d)$

Let $(\mathsf{Gen}, \mathsf{Com}, \mathsf{Open})$ is a perfectly binding commitment scheme such that the validity of public parameters can be verified (ElGamal encryption).

▷ Then the perfect binding property assures that the malicious prover $\mathcal{P}_*$ cannot change his reply. Soundness guarantees are preserved.

▷ A commitment scheme must be $(t_{\mathrm{re}} + t, \kappa)$-hiding for t_{re}-time verifier.

▷ By rewinding we can find out the correct answer in time $\Theta(\frac{1}{\varepsilon - \kappa})$, where ε is the success probability of malicious verifier $\mathcal{V}_*$.

Simulator construction

$\mathcal{S}^{\mathcal{V}_*}$

Choose randomness ω for $\mathcal{V}_*$ and store α.

Use knowledge extractor to extract β.

Run $\mathcal{V}_*$ once again with $(c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(\beta)$.

if $\alpha \neq r^2 v^\beta$ then

  Send $\perp$ to $\mathcal{V}$ and output whatever $\mathcal{V}_*$ outputs.

else

  Send d to $\mathcal{V}$ and output whatever $\mathcal{V}_*$ outputs.

Knowledge-extraction is straightforward. We just provide $(c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(0)$ and verify whether $\alpha = r^2 v^\beta$. The choice of parameters is analogous.

Further analysis

The output of the simulator is only computationally indistinguishable from the real protocol run, as the commitment is only computationally hiding. Let $\mathcal{A}$ be a t-time adversary that tries to distinguish outputs of $\mathcal{V}_*$ and $\mathcal{S}^{\mathcal{V}_*}$

- If $\alpha = r^2 v^\beta$ and knowledge extraction succeeds, the simulation is perfect.
- If $\alpha \neq r^2 v^\beta$ then from $(t_{\mathrm{re}} + t, \kappa)$-hiding, we get

$$\left|\Pr\left[\mathcal{A} = 1 | \mathcal{V}_*^{\mathcal{P}} \wedge \alpha \neq r^2 v^\beta\right] - \Pr\left[\mathcal{A} = 1 | \mathcal{S}^{\mathcal{V}_*} \wedge \alpha \neq r^2 v^\beta\right]\right| \leq \kappa \enspace .$$

- Similarly, $(t_{\mathrm{re}} + t, \kappa)$-hiding assures that

$$\left|\Pr\left[\alpha = r^2 v^\beta | \mathcal{V}_*^{\mathcal{P}}\right] - \Pr\left[\alpha \neq r^2 v^\beta | \mathcal{V}_* \wedge (c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(0)\right]\right| \leq \kappa \enspace .$$

Hence, the knowledge extractor makes on average $\frac{1}{\varepsilon - \kappa}$ probes.

Strengthening of Σ-protocols

Strengthening with commitments

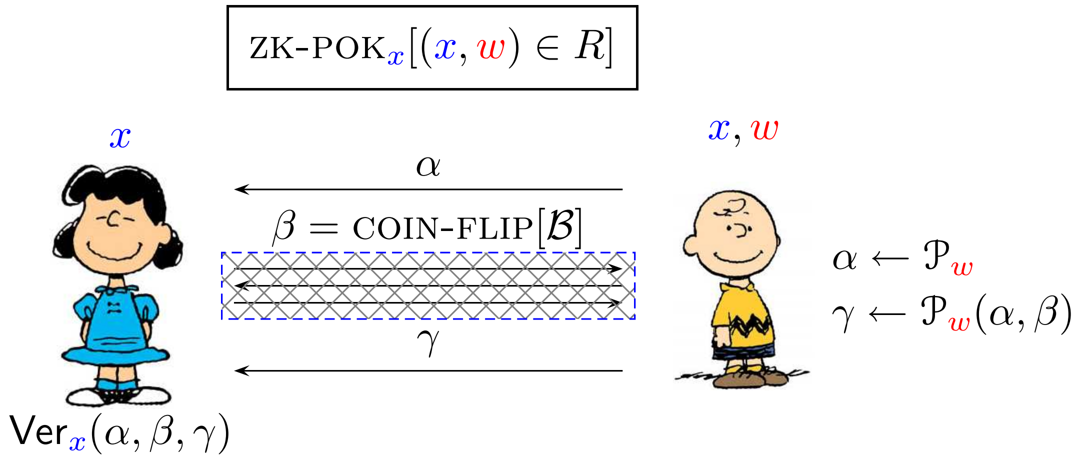
$$\boxed{\mathrm{ZK}_x[(x,w)\in R]}$$

	x		x, w	
		$\xleftarrow{\mathsf{pk}}$		$\mathsf{pk} \leftarrow \mathsf{Gen}$
$\beta \xleftarrow[u]{} \mathcal{B}$		$\xrightarrow{c}$		$\alpha \leftarrow \mathcal{P}_w$
$(c,d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(\beta)$		$\xleftarrow{\alpha}$		$\beta \leftarrow \mathsf{Open}_{\mathsf{pk}}(c,d)$
		$\xrightarrow{d}$		$\gamma \leftarrow \mathcal{P}_w(\alpha,\beta)$
		$\xleftarrow{\gamma}$		
	$\mathsf{Ver}_x(\alpha,\beta,\gamma)$			

If the commitment is statistically hiding then the soundness guarantees are preserved. Again, rewinding allows us to extract the value of β.

- ▷ If commitment scheme is $((\ell+1)\cdot t_{\mathrm{re}}, \varepsilon_2)$-binding then commitment can be double opened with probability at most ε_2.
- ▷ Hence, we can choose $\ell = \Theta(\frac{1}{\varepsilon_1})$ so that simulation failure is $\varepsilon_1 + \varepsilon_2$.
- ▷ The protocol does not have knowledge extraction property any more.

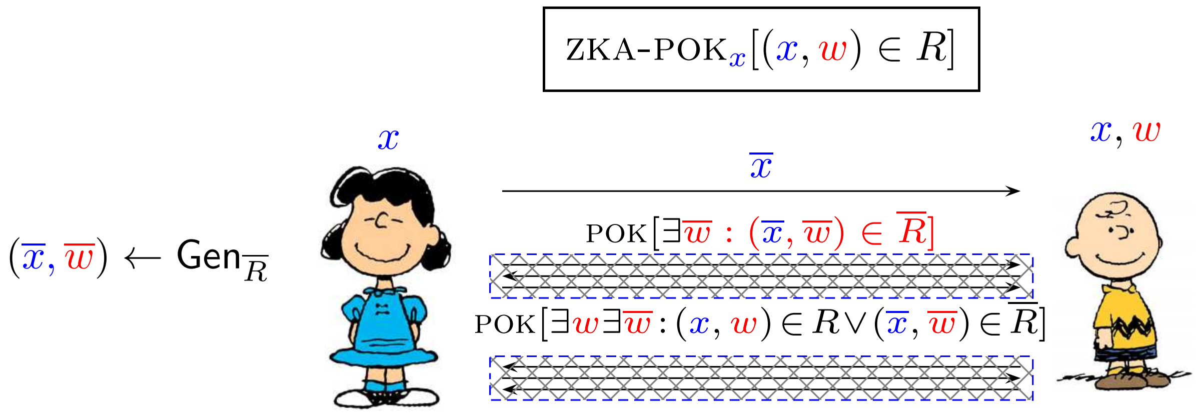
Strengthening with coin-flipping



We can substitute trusted sampling $\beta \xleftarrow[u]{} \mathcal{B}$ with a coin-flipping protocol.

- ▷ To achieve soundness, we need a coin-flipping protocol that is secure against unbounded provers.
- ▷ Statistical indistinguishability is achievable provided that the coin-flipping protocol is secure even if all internal variables become public afterwards.
- ▷ Rewinding takes now place inside the coin-flipping block.

Strengthening with disjunctive proofs



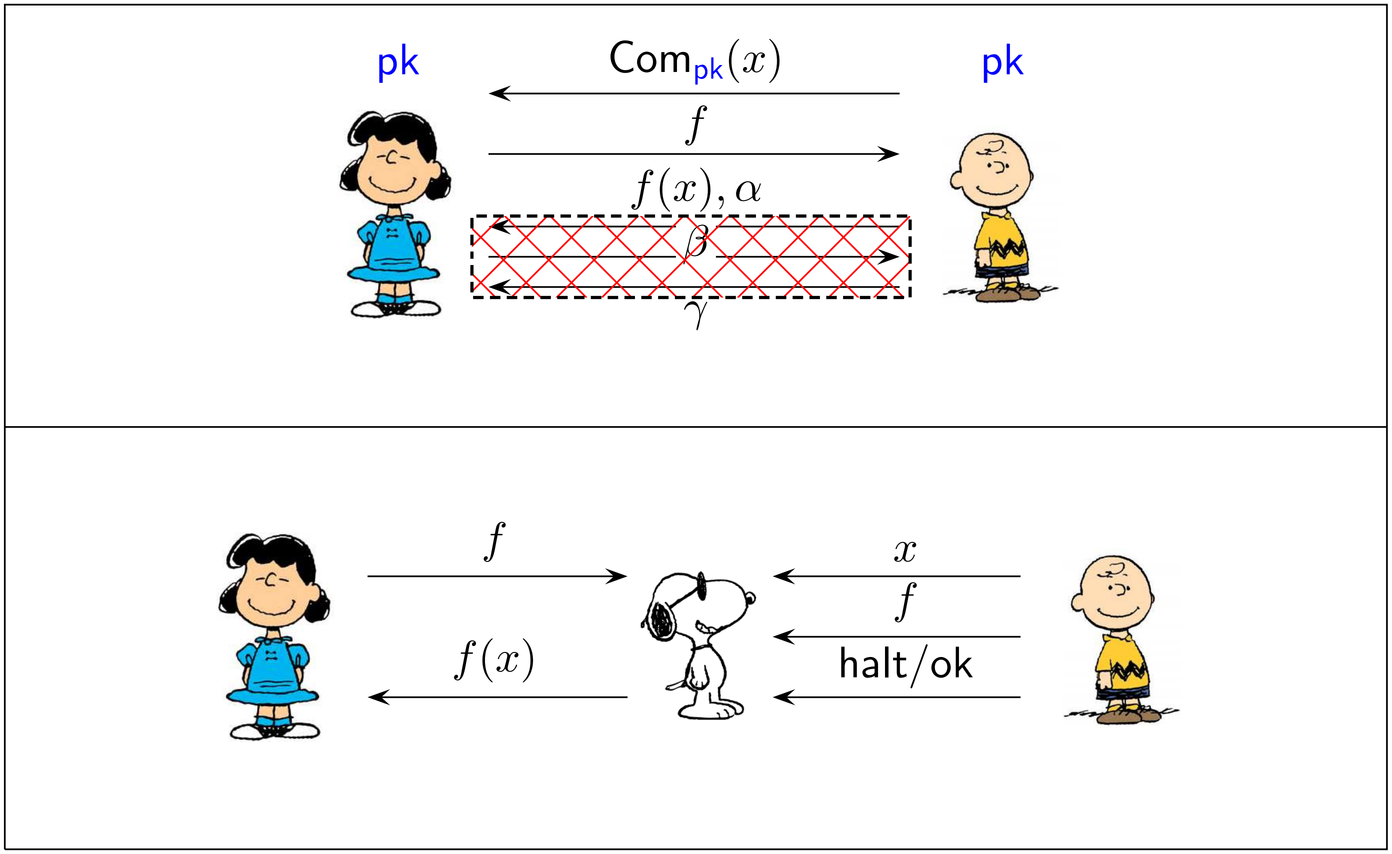
If the relation $\overline{R}$ generated by $\mathsf{Gen}_{\overline{R}}$ is hard, i.e., given $\overline{x}$ it is difficult to find matching $\overline{w}$, then the proof is computationally sound.

The hardness of $\overline{R}$ also guarantees that the second proof is witness hiding. Thus, we can extract first $\overline{w}$ and use it to by-pass the second proof.

Certified Computations

Malicious case

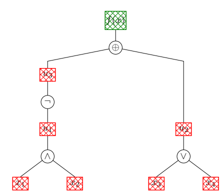
The concept



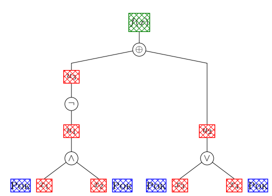
Lucy should learn $f(x)$ and nothing more even if Charlie is malicious.

A quick recap of the semihonest case

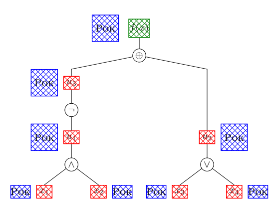
Construct the circuit and commit all values



Construct a sigma protocols for validity of inputs



Construct a sigma protocol for validity of intermediate values



Security against malicious verifiers

We can use several methods to strengthen the protocol.

- ▷ We can restrict challenge space $\mathcal{B}$ to $\{0, 1\}$ and then use sequential composition to achieve reasonable soundness level.
- ▷ We can use commitments to strengthen the sigma protocol.
- ▷ We can use coin-flipping protocol to generate the challenge β.
- ▷ We can use disjunctive proofs to strengthen the sigma protocol.

The resulting construction which is based on a coin-flipping protocol is often referred as $\textsc{Gmw}$-compiler, since it forces semihonest behaviour.