Radon Transform and Filtered Back-Projection

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Advanced Topics in Sampling (049029)
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Talk Outline

- Motivation
- Brief physical background
- Radon transform and the slice theorem
- Filtered back-projection
- Extensions
- Haar-based reconstructions
- Discretization with spline convolutions
- Discussion

Motivation (1)

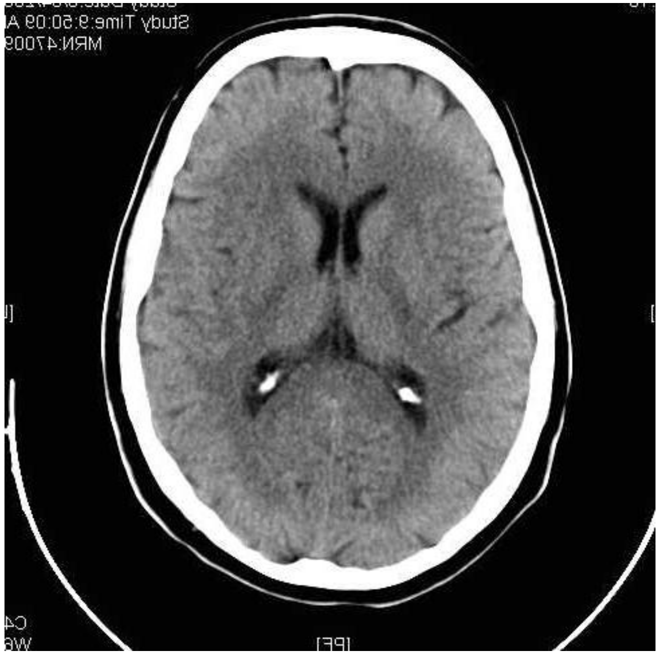
- Primary use: in medical imaging

http://orthoinfo.aaos.org

(American Academy of Orthopaedic Surgeons)

Motivation (2)

- Computerized Tomography (CT)



http://www.southernhealth.org.au

(Casey Hospital CT Department)

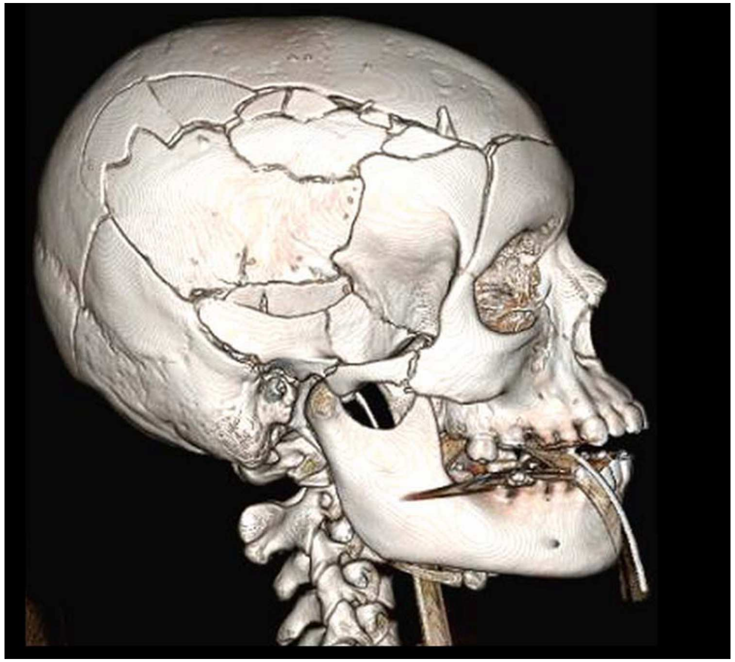
http://path.upmc.edu

(University of Pittsburgh, School of Medicine, Department of Pathology)

Motivation (3)

- 3D imaging, selective display

http://www.medscape.com
(Medscape Today)



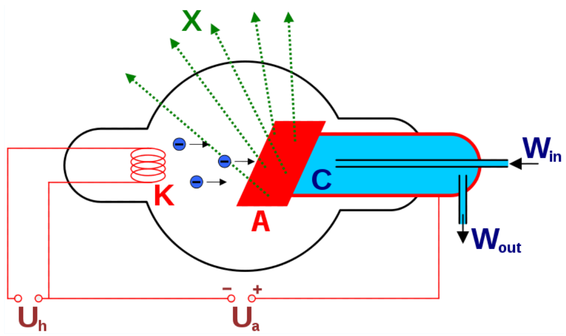
http://www.rsna.org
(Radiological Society of North America)

References

- [Bro66] J. G. Brown, X-rays and their applications, New York: Plenum Press, 1966
- [Kak88] A. C. Kak and M. Slaney, Principles of computerized tomographic imaging, New York: IEEE Press, 1988
- [Gué94] J. P. Guédon and Y. Bizais, Bandlimited and Haar filtered back-projection reconstructions, IEEE Trans. Med. Imaging, vol. 13, no. 3, pp. 340-440, 1994
- [Hor02] S. Horbelt, M. Leibling and M. Unser, Discretization of the Radon transform and of its inverse by spline convolutions, IEEE Trans. Med. Imaging, vol. 21, no. 4, pp. 363-376, 2002

Brief Physical Background (1)

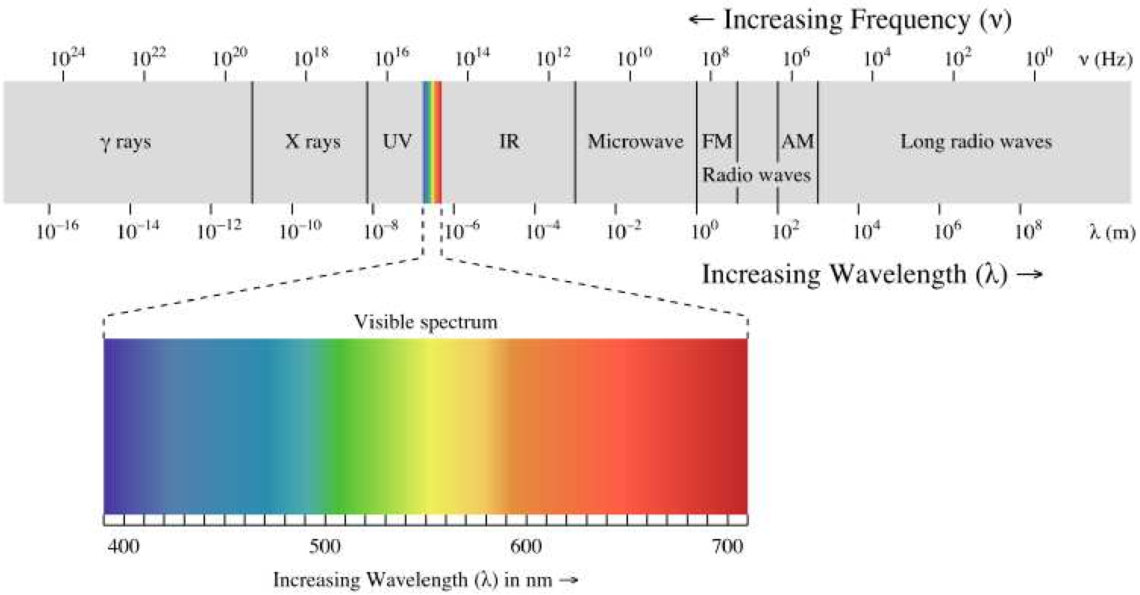
- X-rays (Röntgen)
  - Cathode ray source
  - Accelerating voltage
  - Solid target → electromagnetic radiation



Wikipedia

Brief Physical Background (2)

- X-rays (Röntgen)
  - Very energetic (wavelengths: 0.01-10 [nm])



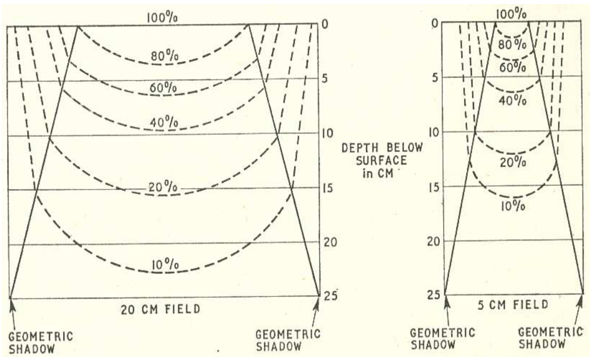
Wikipedia

Brief Physical Background (3)

- Interaction with matter
  - Narrow beam: linear absorption coefficient

$$dI = -\mu I dx \quad \Rightarrow \quad I(x) = I_0 e^{-\mu x}$$

  - Wide beam: no analytical expression



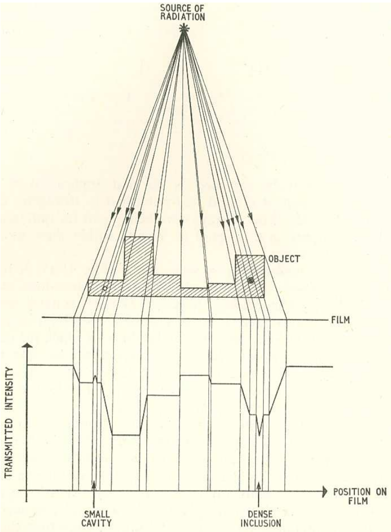
[Bro66]

Brief Physical Background (4)

- Interaction with matter
  - Serial connection of different substances

$$I(x) = I_0 e^{-\mu_1 x_1} e^{-\mu_2 x_2} \cdots e^{-\mu_N x_N} \xrightarrow{x_i \to 0} I_0 e^{-\int_0^x \mu(x')dx'}$$

  - Indication of density - $\mu \propto \rho$

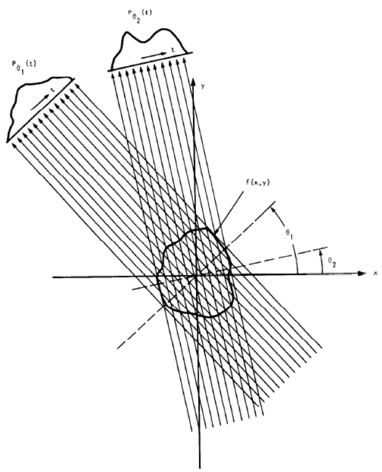


[Bro66]

Radon Transform and The Slice Theorem (1)

- Single beam: $\ln\left(I_0/I_L\right)=\int_0^L \mu(s)\,ds$
- Radon transform (projection) $\left(f(x,y)=\mu(x,y)\right)$

$$P_\theta(t)\triangleq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta\big(x\cos(\theta)+y\sin(\theta)-t\big)\,dxdy$$



[Kak88]

Radon Transform and The Slice Theorem (2)

- Analytical expression for simple ellipse

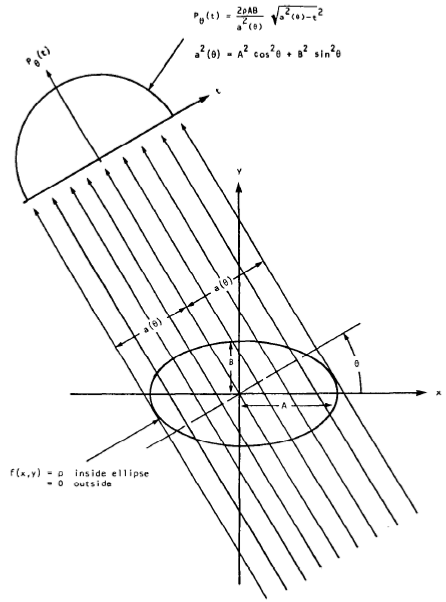
$$P_\theta(t) = \begin{cases} \dfrac{2\rho AB}{a^2(\theta)}\sqrt{(a^2(\theta) - t^2)}, & |t| \le a(\theta) \\ 0, & |t| > a(\theta) \end{cases}$$

$$a^2(\theta) = A^2\cos^2(\theta) + B^2\sin^2(\theta)$$

- Translation + rotation

$$P'_\theta(t) = P_{\theta-\alpha}\left[t - s\cos(\gamma - \theta)\right]$$

$$s = \sqrt{x_1^2 + y_1^2}, \quad \gamma = \arctan(y_1/x_1)$$



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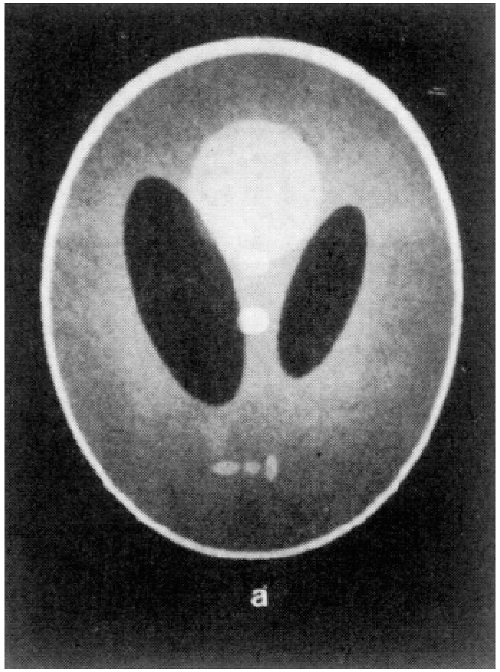
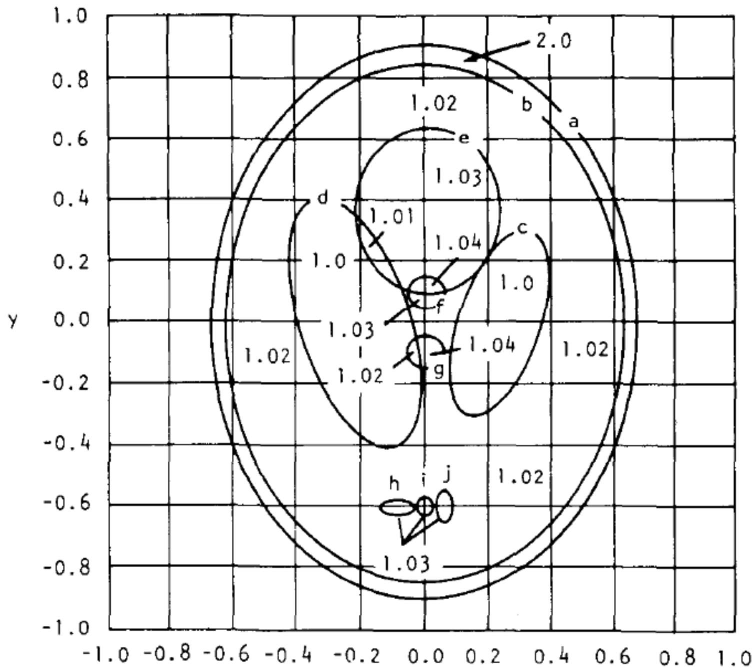
Radon Transform and The Slice Theorem (3)

- Shepp-Logan phantom





[Kak88]

Radon Transform and The Slice Theorem (4)

- Fourier transform of a section

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

- Radon transform in frequency domain

$$\begin{bmatrix} t \\ s \end{bmatrix}=\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

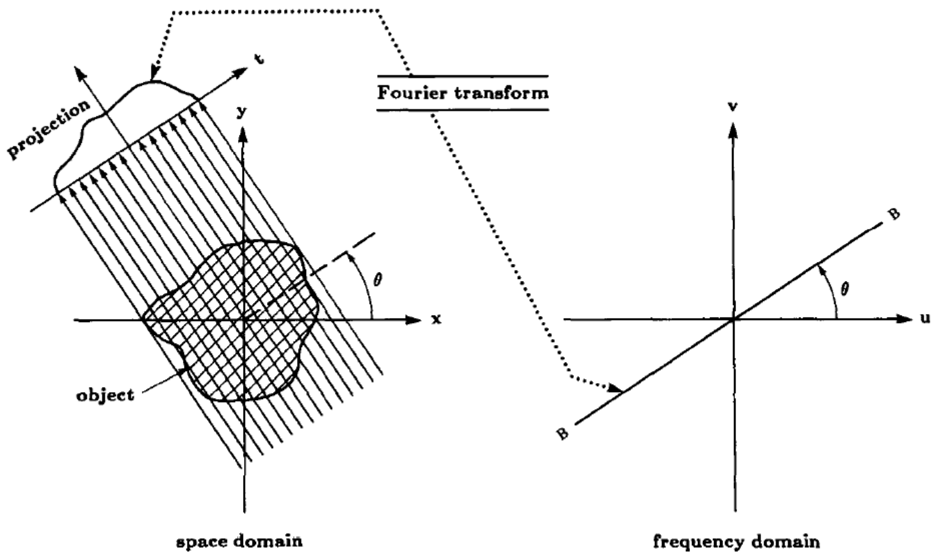
$$P_\theta(t)=\int_{-\infty}^{\infty} f(t,s)ds \quad \Rightarrow \quad S_\theta(w)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,s)e^{-j2\pi wt}dsdt$$

$$S_\theta(w)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi w(x\cos(\theta)+y\sin(\theta))}dxdy \equiv F\left(w\cos(\theta), w\sin(\theta)\right)$$

Radon Transform and The Slice Theorem (5)

- The Slice Theorem:

  The Fourier transform of a parallel projection of an image f(x,y) taken at angle θ gives a slice of the two-dimensional transform, F(u,v), subtending at angle θ with the *u*-axis.



[Kak88]

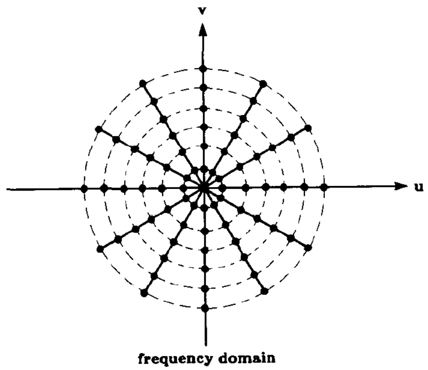
Radon Transform and The Slice Theorem (6)

- Naïve reconstruction of bounded image

$$-A/2 \le x, y \le A/2$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ux+vy)}\, du\, dv = \frac{1}{A^2}\sum_m \sum_n F\left(\frac{m}{A},\frac{n}{A}\right) e^{j2\pi((m/A)x+(n/A)y)}$$

- Approximation: $-N/2 \le m,n \le N/2$
- The catch:



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Filtered Back-Projection (1)

- Reconstruction approaches:
  - Frequency domain interpolation ✖
    - Reconstruction procedure awaits all projections
    - High sensitivity to interpolation error
  - Filtered back-projection ✓
    - Almost real-time reconstruction
    - Low sensitivity to space domain interpolation error

Filtered Back-Projection (2)

- The concept
  - Each slice is obtained by convolving the image with a filter of line-shaped support
  - Desire: filter of sector-shaped support
  - Compromise: ramp filter of line-shaped support

$$S_\theta(w) \rightarrow \left(2\pi|w|/K\right) S_\theta(w)$$

[Kak88]

Filtered Back-Projection (3)

- The mathematics

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{j2\pi(ux+vy)}dudv \overset{\substack{u=w\cos(\theta)\\ v=w\sin(\theta)}}{=} \int_{0}^{2\pi}\int_{0}^{\infty}F(w,\theta)e^{j2\pi w(x\cos(\theta)+y\sin(\theta))}wdwd\theta$$

$$=\int_{0}^{\pi}\int_{0}^{\infty}F(w,\theta)e^{j2\pi w(x\cos(\theta)+y\sin(\theta))}wdwd\theta+\int_{0}^{\pi}\int_{0}^{\infty}\overbrace{F(w,\theta+\pi)}^{F(-w,\theta)}e^{j2\pi \overbrace{w(x\cos(\theta+\pi)+y\sin(\theta+\pi))}^{-w(x\cos(\theta)+y\sin(\theta))}}wdwd\theta$$

$$=\int_{0}^{\pi}\int_{0}^{\infty}F(w,\theta)e^{j2\pi w(x\cos(\theta)+y\sin(\theta))}wdwd\theta+\int_{0}^{\pi}\int_{-\infty}^{0}F(w,\theta)e^{j2\pi w(x\cos(\theta)+y\sin(\theta))}(-w)dwd\theta$$

$$=\int_{0}^{\pi}\left[\int_{-\infty}^{\infty}F(w,\theta)e^{j2\pi w(x\cos(\theta)+y\sin(\theta))}|w|dw\right]d\theta=\left.\int_{0}^{\pi}\overbrace{\left[\int_{-\infty}^{\infty}S_{\theta}(w)|w|e^{j2\pi wt}dw\right]}^{Q_{\theta}(t)}d\theta\right|_{t=x\cos(\theta)+y\sin(\theta)}$$

Filtered Back-Projection (4)

- Discrete formulation
  - Sampling in space: $P_\theta(kT),\ k = -(N/2),\cdots,(N/2)-1$
  - Fourier transform: $S_\theta(w) \overset{w=m/NT}{\approx} T \cdot \sum_{k=-N/2}^{N/2-1} P_\theta(kT) e^{-j2\pi(mk/N)}$
  - Filtering:

$$Q_\theta(kT) \approx \left(\frac{1}{NT}\right) \sum_{m=-N/2}^{N/2} S_\theta\left(\frac{m}{NT}\right) \left|\frac{m}{NT}\right| e^{j2\pi(mk/N)}, \quad k = -\frac{N}{2},\cdots,\frac{N}{2}$$

$$\Rightarrow \quad Q_\theta(kT) \approx \frac{1}{NT} P_\theta(kT) \otimes \phi(kT), \quad \phi(kT) = IDFT\left\{\left|\frac{m}{NT}\right| H\left(\frac{m}{NT}\right)\right\}$$

  - Back-Projecting: $f(x,y) \approx \frac{\pi}{K} \sum_{i=1}^{K} \mathrm{Q}_{\theta_i}\left(x\cos(\theta_i) + y\sin(\theta_i)\right)$

Filtered Back-Projection (5)

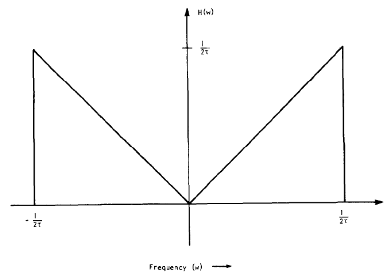
- Problem: Finite projection bandwidth + finite projection order = contradiction!
- Sources of artifacts:
  - Cyclic convolution instead of linear conv.
  - Frequency information deleted in the entire m=0 “cell” instead of w=0
- Optional solution: zero-padding

Filtered Back-Projection (6)

- Alternative computer implementation
  - Finite bandwidth W:

$$Q_\theta(t) = \int_{-\infty}^{\infty} S_\theta(w) H(w) e^{j2\pi wt} dw$$

$$H(w) = \begin{cases} |w| & |w| < W \\ 0 & otherwise \end{cases}$$



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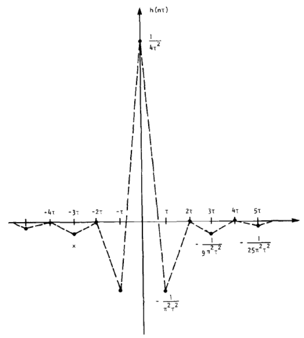
  - Sampled filter:

$$h(nT) = \begin{cases} (2T)^{-2} & n = 0 \\ 0 & n\ even \\ -(\pi nT)^{-2} & n\ odd \end{cases}$$



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Filtered Back-Projection (7)

- Alternative computer implementation
  - Whittaker-Shannon interpolation formula

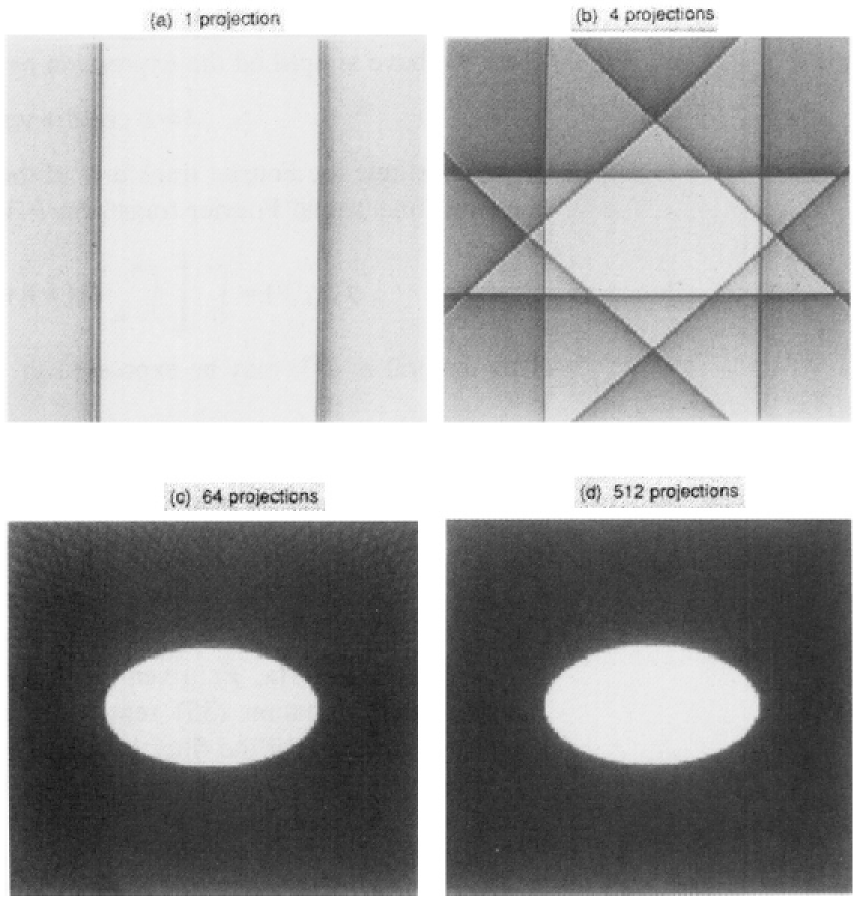
$$\overbrace{P_\theta(t)}^{or\ h(t)} = \sum_{k=-\infty}^{\infty} \overbrace{P_\theta(kT)}^{or\ h(kT)} \frac{\sin\left[2\pi W(t-kT)\right]}{2\pi W(t-kT)}$$

  - The convolution simplifies:

$$Q_\theta(nT) = T\sum_{k=-\infty}^{\infty} P_\theta(kT)h(nT-kT)$$

  - Filter of length 2N+1 suffices

Filtered Back-Projection (8)



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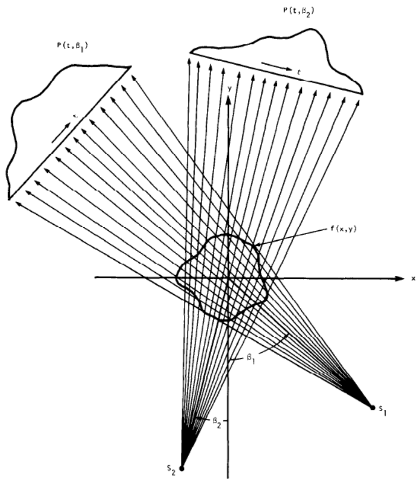
Extensions (1)

- Fan beam
  - Static point source at each angle
  - Simultaneous detection



[Kak88]

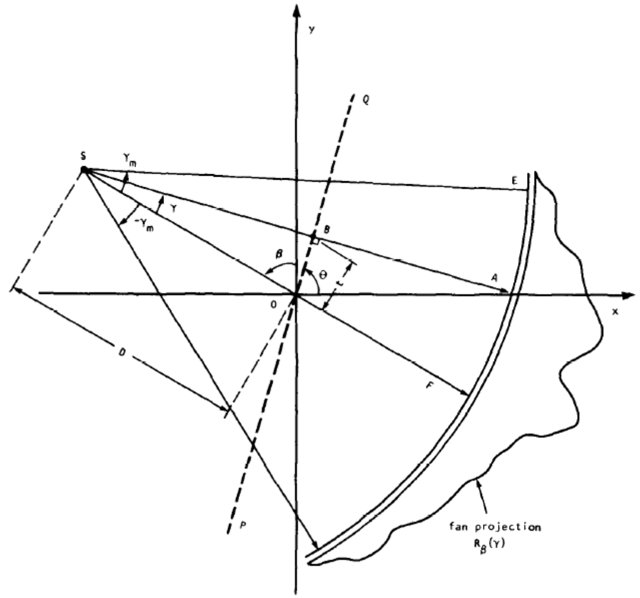
http://smugpuppies.com

Extensions (2)

- Equiangular rays
  - The trick: handle each ray as part of parallel beam

$$R_\beta(\gamma) = P_{\beta+\gamma}\left(D\sin(\gamma)\right)$$

$$(x,y) \to \left(r\cos(\phi), r\sin(\phi)\right)$$



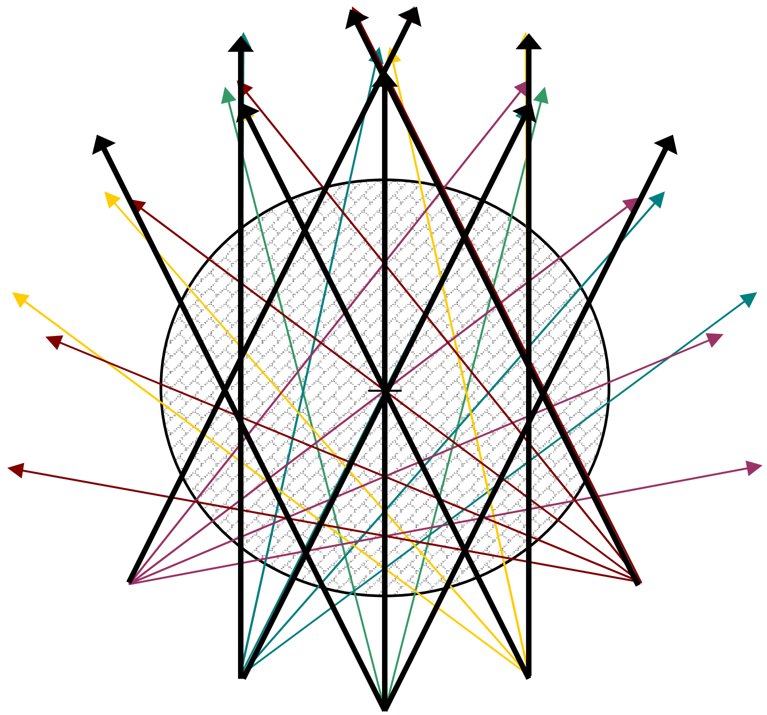
[Kak88]

$$f(r,\phi) = \frac{1}{2}\int_0^{2\pi}\int_{-\gamma_m}^{\gamma_m} R_\beta(\gamma)\, h\left(r\cos(\beta+\gamma-\phi) - D\sin(\gamma)\right) D\cos(\gamma)\, d\gamma\, d\beta$$

  - Equispaced collinear detectors: similar trick

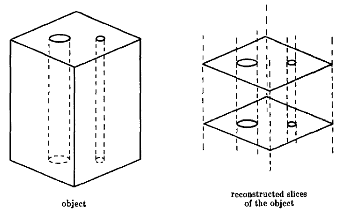
Extensions (3)

- Demonstration



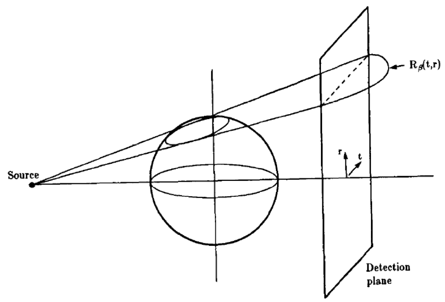
Extensions (4)

- 3D reconstruction
  - Simple straight-forward method



[Kak88]

  - More efficient (time-saving) methods



[Kak88]

Break

- Suggestions:

Haar-Based Reconstructions (1)

- Inverse Radon transform – ill-posed

$$f(x,y) = \overbrace{\mathbf{R}^* \underbrace{\mathbf{K}}_{\substack{ramp\\filter}} \underbrace{\mathbf{R}f(x,y)}_{transform}}^{back-projection}$$

$$k(t,\theta) = \lim_{\varepsilon \to 0} k_\varepsilon(t,\theta), \quad k_\varepsilon(t,\theta) = \begin{cases} \varepsilon^2 \ \ if \ |t| < \varepsilon \\ \dfrac{-1}{t^2} \ \ if \ |t| \geq \varepsilon \end{cases}$$

- Discretization after anti-aliasing filtering

$$\hat{f}(k,l) = b(x,y) ** f(x,y)_{|x=k,y=l}$$

$$\Rightarrow \quad \hat{f}(k,l) = \mathbf{R}^*\left(k_b(t,\theta) * p_f(t,\theta)\right)_{|x=k,y=l}, \quad k_b(t,\theta) = \mathbf{F}^{-1}\left\{\pi |v| P_b(v,\theta)\right\}$$

Haar-Based Reconstructions (2)

- Pixel Intensity Distribution Model (PIDM)

$$s_{0,\Delta}(z) = \begin{cases} 1, & if\ |z| < \frac{\Delta}{2} \\ \frac{1}{2}, & if\ |z| = \frac{\Delta}{2} \\ 0 & if\ |z| > \frac{\Delta}{2} \end{cases}$$

$$s_{0,\Delta}(x-k, y-l) = s_{0,\Delta}(x-k) \cdot s_{0,\Delta}(y-l), \qquad \Delta = pixel\ size$$

- Sampled function → continuum form

$$f_0(x,y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_0(k,l)\, s_{0,\Delta}(x-k, y-l)$$

Haar-Based Reconstructions (3)

- In the space of piecewise constant functions, the optimal filter: $b(x,y) = s_{0,\Delta}(x,y)$

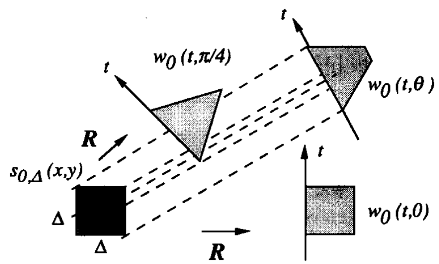
$$f_0(k,l) = \mathbf{R}^*\left(Kw_0(t,\theta) * p_f(t,\theta)\right)_{x=k,y=l}, \quad w_0(t,\theta) = \mathbf{R}\left(s_{0,\Delta}(x,y)\right)$$

- Sampling kernel projections (incorrect!)

$$w_0(t,\theta) = s_{0,\Delta\cos\theta}(t) * s_{0,\Delta\sin\theta}(t)$$

$$W_0(\nu,\theta) = \Delta^2 \mathrm{sinc}(\nu\Delta\cos\theta)\,\mathrm{sinc}(\nu\Delta\sin\theta)$$

$$k_0(t,\theta \neq 0) = \frac{1}{\pi\sin(2\theta)}\left(\ln\left|\frac{4t^2 - \Delta^2(1+\sin(2\theta))}{4t^2 - \Delta^2(1-\sin(2\theta))}\right|\right)$$



[Gué94]

- Oversampling is needed: $\rho = \Delta/\tau > 1$

Haar-Based Reconstructions (4)

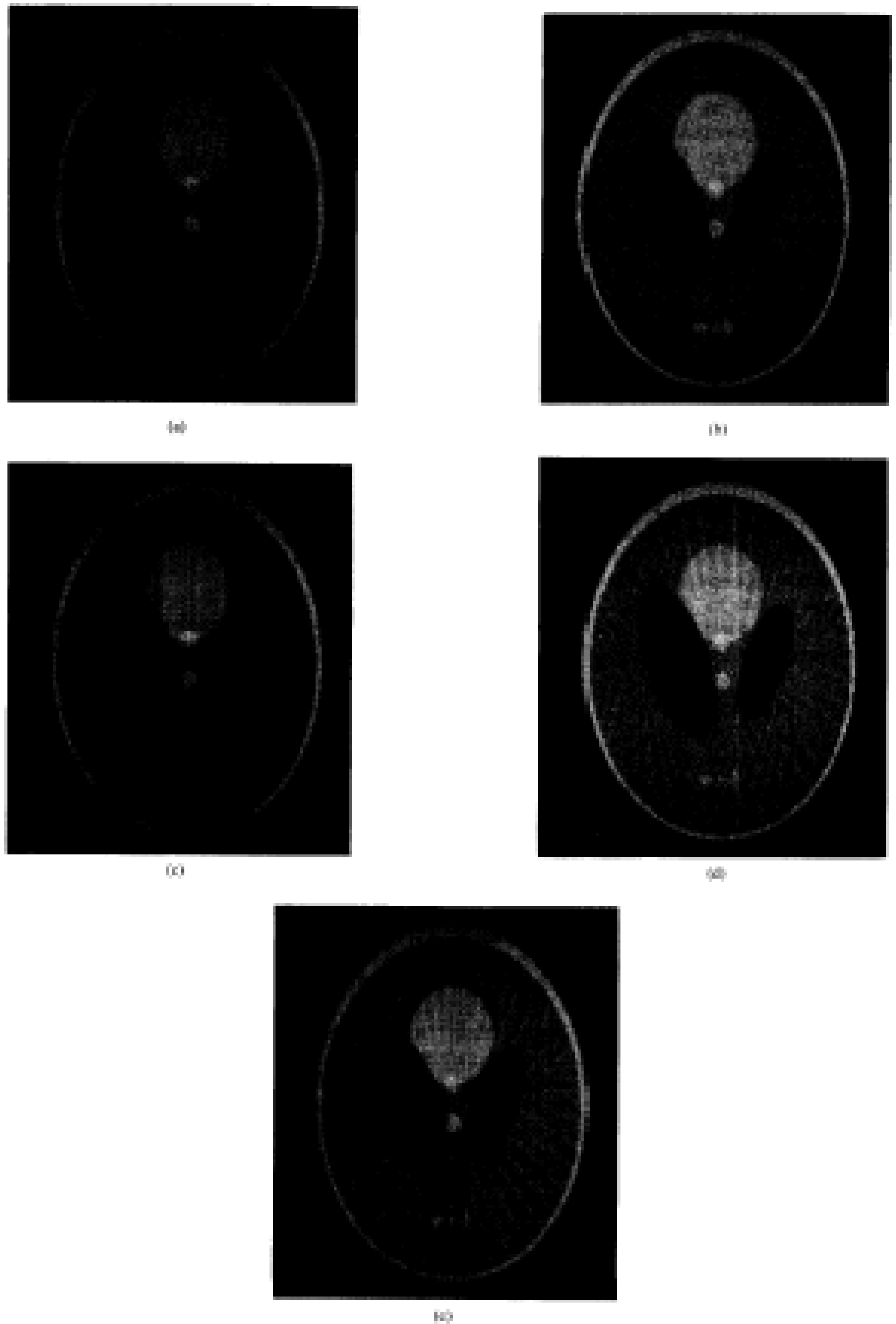
- Digital filter
  - Sampling times: $t = n\tau = \frac{n\Delta}{\rho}$
  - Discretization through averaging

$$k_d(n\tau,\theta) = \left(k_0(t,\theta) * rect(t/\tau)\right)_{t=n\tau} = \int_{(n-1/2)\tau}^{(n+1/2)\tau} k_0(u,\theta)\,du$$

$$k_d(n\tau,\theta) = \frac{1}{\pi\sin(2\theta)}\left[(t-\alpha)\ln|t-\alpha| + (t+\alpha)\ln|t+\alpha| - (t-\beta)\ln|t-\beta| + (t+\beta)\ln|t+\beta|\right]_{(n-1/2)\tau}^{(n-1/2)\tau}$$

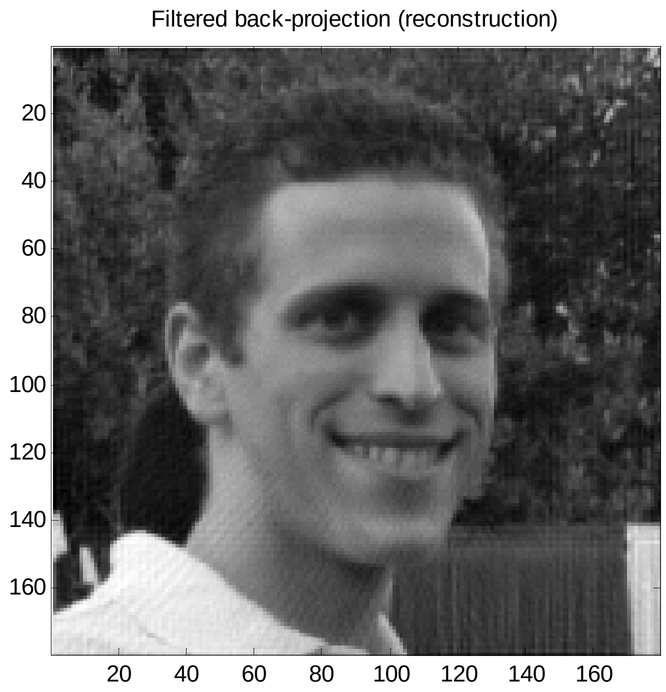
$$\alpha = \frac{\rho\tau}{2}(\cos\theta + \sin\theta), \quad \beta = \frac{\rho\tau}{2}(\cos\theta - \sin\theta)$$

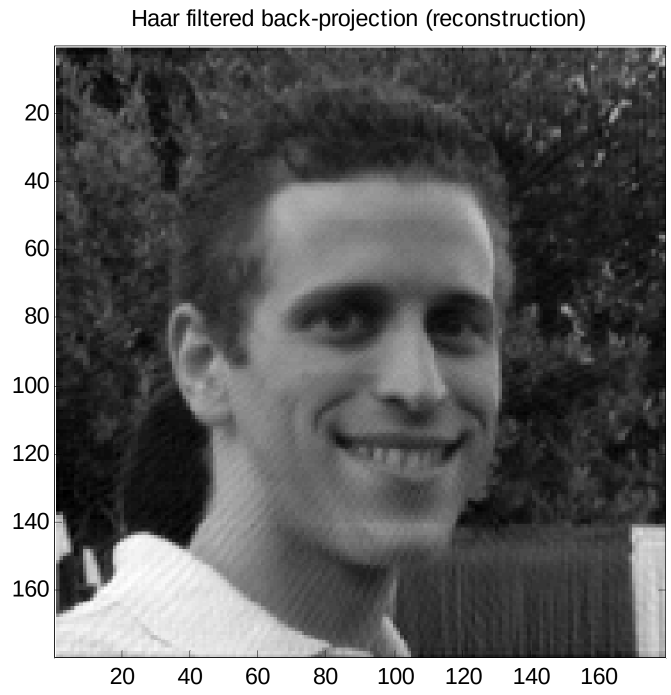
  - Singular case: $k_d(n\tau,0) = \frac{\Delta}{2\pi}\left(\ln\left|n^2 - \left(\frac{1+\rho}{2}\right)^2\right| - \ln\left|n^2 - \left(\frac{1-\rho}{2}\right)^2\right|\right)$
  - Is oversampling enough?


(a)
(b)
(c)
(d)
(e)

Haar-Based Reconstructions (6)

- Demonstration for the ignorant…





Discretization with Spline Convolutions (1)

- B-spline functions

$$\beta_h^0(x) = \begin{cases} \frac{1}{h}, & -\frac{h}{2} \le x < \frac{h}{2} \\ 0, & otherwise \end{cases}$$

$$\beta_h^n(x) = \beta_h^0(x) * \beta_h^{n-1}(x) = \underbrace{\beta_h^0(x) * \ldots * \beta_h^0(x)}_{n+1\ times} \qquad \beta_h^n = \Delta_h^{n+1} * \frac{x_+^n}{n!}$$

- One-sided power function

$$x_+^n = \begin{cases} x^n, & x \ge 0 \\ 0, & otherwise \end{cases} \qquad \frac{x_+^n}{n!} = \overbrace{x_+^0 * \ldots * x_+^0}^{n+1\ times}$$

- Finite difference operator

$$\Delta_h^{n+1} = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{\delta\left(x + h \cdot \left(\frac{n+1}{2} - k\right)\right)}{h^{n+1}} \qquad \Delta_h^{n+1} = \overbrace{\Delta_h^1 * \ldots * \Delta_h^1}^{n+1\ times}$$

Discretization with Spline Convolutions (2)

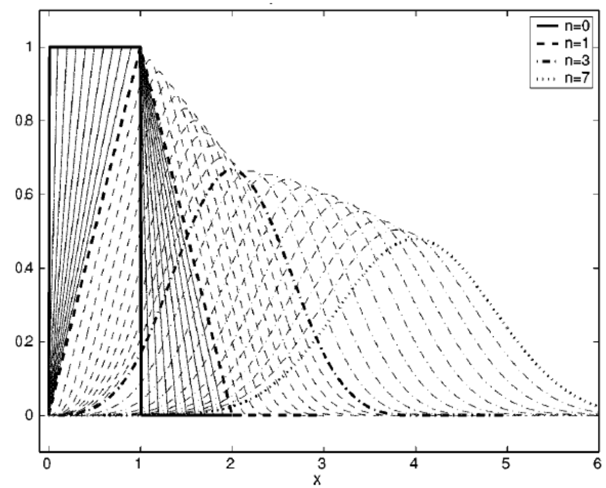
- B-spline convolution kernels
  - Spline *m*-kernel

$$\beta_{h_1,\ldots,h_m}^{n_1,\ldots,n_m}(x) = \beta_{h_1}^{n_1} * \ldots * \beta_{h_m}^{n_m}(x) = \Delta_{h_1,\ldots,h_m}^{n_1+1,\ldots,n_m+1} * \frac{x_+^{N_m}}{N_m!}, \quad N_m = m - 1 + \sum_{i=1}^{m} n_i$$

  - Spline bikernel

$$\beta_{h_1,h_2}^{n_1,n_2}(x) = \Delta_{h_1,h_2}^{n_1+1,n_2+1} * \frac{x_+^{n_1+n_2+1}}{(n_1+n_2+1)!} = \ldots$$

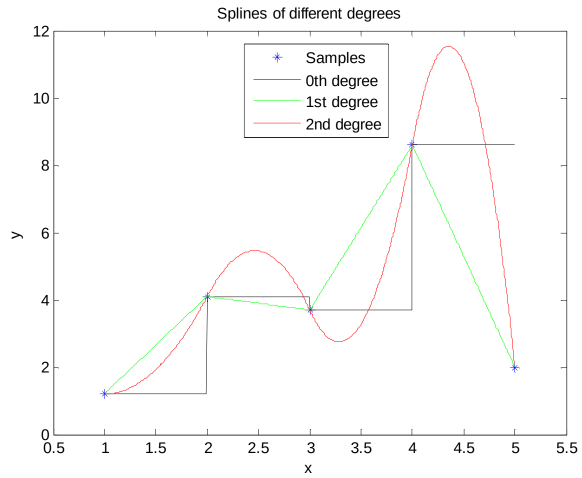
Fig. 1. Spline bikernels by convolution of two causal B-splines $\beta_1^n * \beta_h^n(x)$ with $n = \{0, 1, 3\}$ and $h \in [0.1]$. The kernels generate a smooth transition between B-splines of degree n and degree $2n+1$. For $h = 0$ and $h = 1$, the kernels are B-splines, as $\lim_{h\to 0} \beta_1^n * \beta_h^n(x) = \beta_1^n(x)$ and $\beta_1^n * \beta_1^n(x) = \beta_1^{2n+1}(x)$.



[Hor02]

Discretization with Spline Convolutions (3)

- Polynomial splines
  - Alternative to Shannon's sampling theorem
  - Spline: Smooth Piecewise-Polynomial Line



Discretization with Spline Convolutions (4)

- Representation in B-spline basis

$$f_h(x) = \sum_{k \in \mathbb{Z}} c_k \beta_h^n (x - hk)$$

  - Interpolation approach
  - Least squares approximation
    - Optimal coefficients: $c_k = h \cdot \langle f(x), \tilde{\varphi}_h (x - hk) \rangle$

      $\langle \varphi(x), \tilde{\varphi}(x-k) \rangle = \delta_k$
    - Basis functions and their dual functions can be interchanged

Discretization with Spline Convolutions (5)

- Spline-based Radon transform

$$f_h(\vec{x}) = \sum_{k,l\in\mathbb{Z}} c_{k,l}\beta_h^{n_1}(\vec{x} - h\vec{k})$$

  - Linearity: $R_\theta \beta_h^n(t) = \beta_{h|\cos\theta|}^n * \beta_{h|\sin\theta|}^n(t) = \beta_{h|\cos\theta|,h|\sin\theta|}^{n,n}(t)$
  - Shift-invariance: $R_\theta\{\varphi(\vec{x} - \vec{x}_0)\} = (R_\theta\varphi)(t - t_0), \quad t_0 = \vec{x}_0^T \cdot \vec{\theta}$

$$g_\theta(t) = \sum_{k,l\in\mathbb{Z}} c_{k,l}\beta_{h|\cos\theta|,h|\sin\theta|}^{n_1,n_1}(t - h\vec{k}^T\vec{\theta})$$

  - Sinogram discretization with dual splines

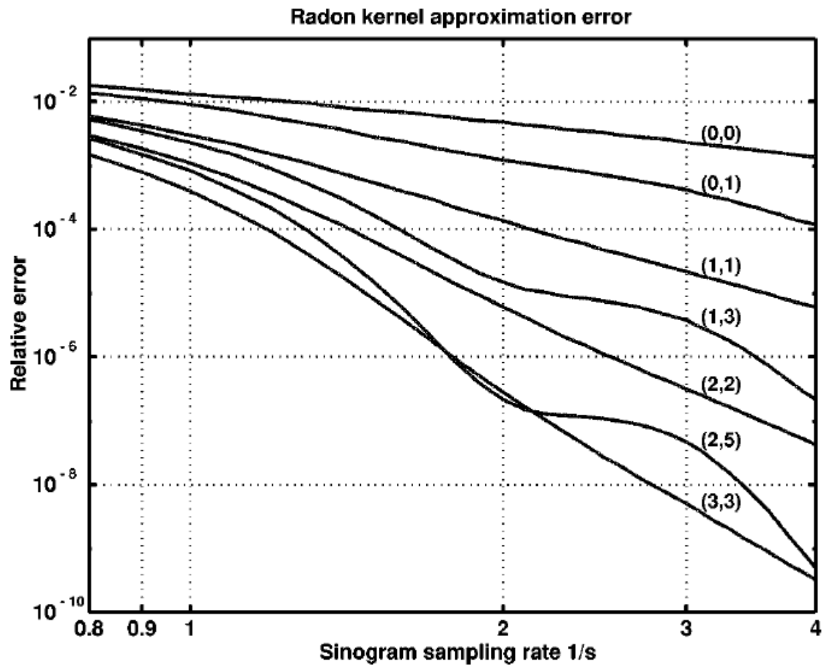
$$g_{\theta,s}(t) = P_s g_\theta(t) = \sum_{i\in\mathbb{Z}} \overbrace{s\left\langle g_\theta(t), \beta_s^{n_2}(t - is)\right\rangle}^{\tilde{e}_{i,\theta}} \cdot \tilde{\beta}_s^{n_2}(t - is)$$

$$\tilde{e}_{i,\theta} = \sum_{k,l\in\mathbb{Z}} d_{i,\theta,k,l} \cdot c_{k,l} \cdot s, \qquad d_{i,\theta,k,l} = \beta_{h|\cos\theta|,h|\sin\theta|,s}^{n_1,n_1,n_2}(h\vec{k}^T\vec{\theta} - is)$$

Discretization with Spline Convolutions (6)

- Error analysis

$$e^2(s) = \| f - f_s \|_{L_2}^2$$



[Hor02]

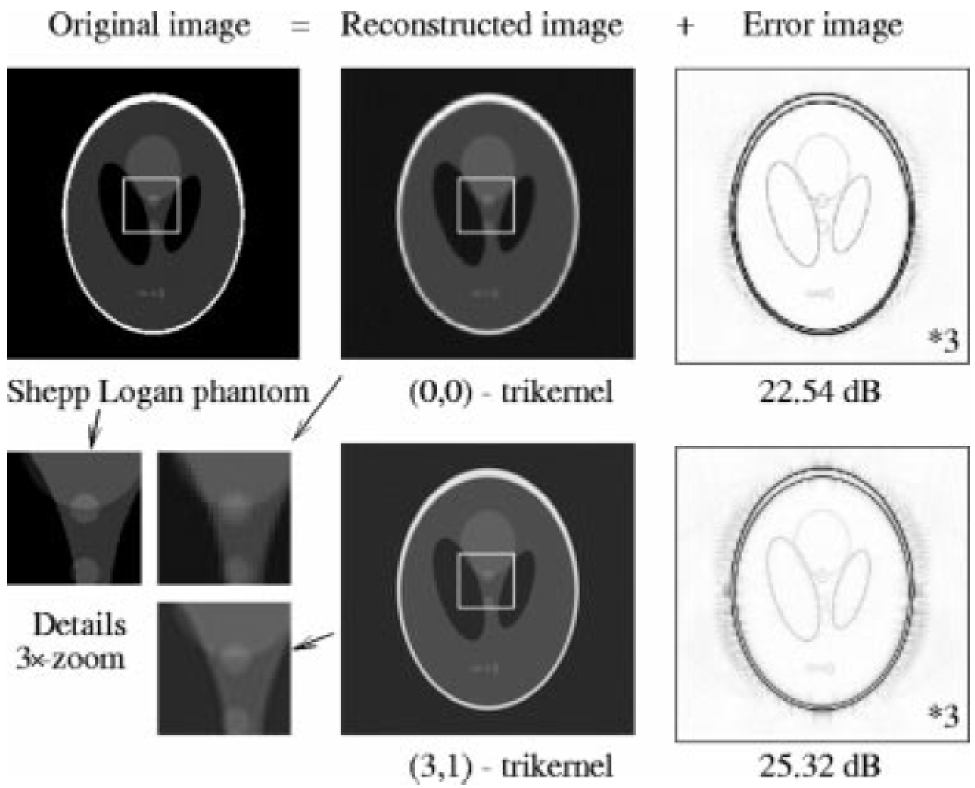
Discretization with Spline Convolutions (7)

- Spline-based back-projection
  - Filtering: $\hat{p}_\theta(\omega) = \hat{g}_\theta(\omega) \cdot \hat{q}(\omega)$
  - Discretization: $p_{\theta,s}(t) = \sum_{i \in \mathbb{Z}} c_{\theta,i} \cdot \beta_s^{n_2}(t - is)$
  - Back-projecting to dual splines space

$$\tilde{f}_h(\vec{x}) = \sum_{k,l \in \mathbb{Z}} \tilde{c}_{k,l} \cdot \tilde{\beta}_h^{n_1}\left(\vec{x} - h\vec{k}\right), \qquad \tilde{c}_{k,l} = \sum_{i \in \mathbb{Z},\theta} c_{\theta,i} \cdot h^2 \cdot d_{i,\theta,k,l}$$

Discretization with Spline Convolutions (8)

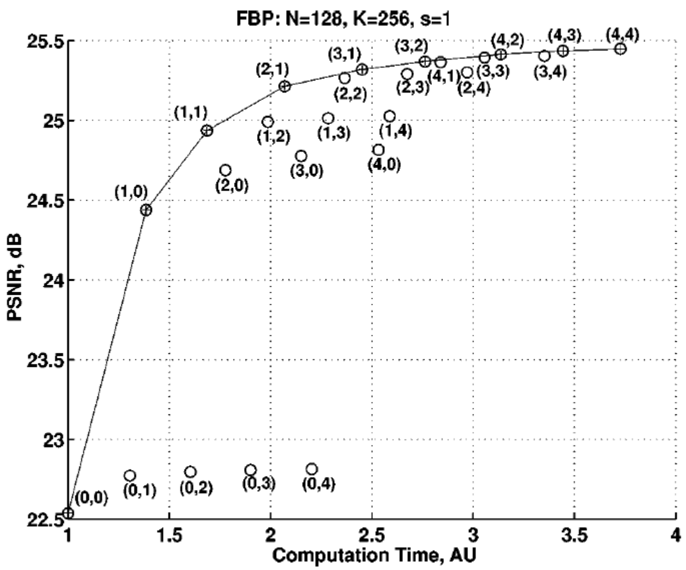
- Results



[Hor02]

Discretization with Spline Convolutions (9)

- Complexity considerations
  - Run time: $t \propto N_x \cdot N_y \cdot K \cdot \frac{1}{s} \cdot \left(\overline{\mathrm{supp}} \left(\beta^{n_1, n_1, n_2}_{h|\cos\theta|, h|\sin\theta|, s} \right) + c \right)$



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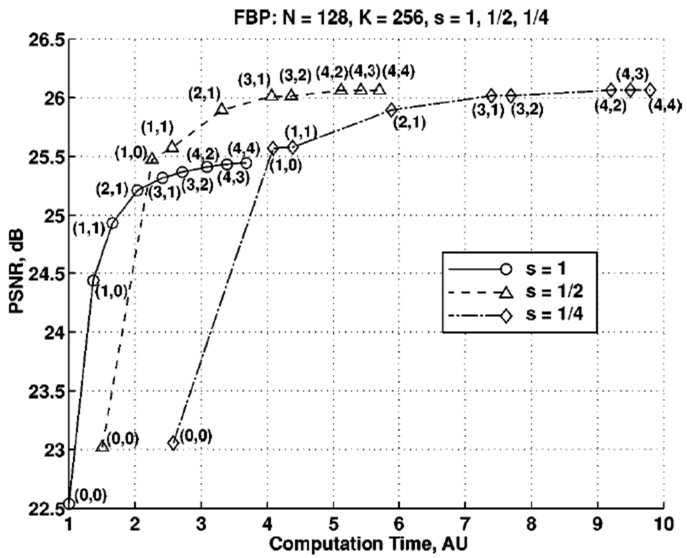

[Hor02]

Discussion

- What we had:
  - Physical method for acquiring projections
  - Derivation of filtering requirements
  - Efficient basic reconstruction algorithm
  - Suggested methods for improvement
    - Pixel shape taken into account
    - Smooth-basis representation replaces sampling

The End