Signal Processing and Linear Systems

Chapter 2
Linear Time-Invariant Systems
(Continuous-Time)

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- Signal Classification
  - Continuous & Discrete
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  - Even & Odd
  - Periodic & Non-periodic
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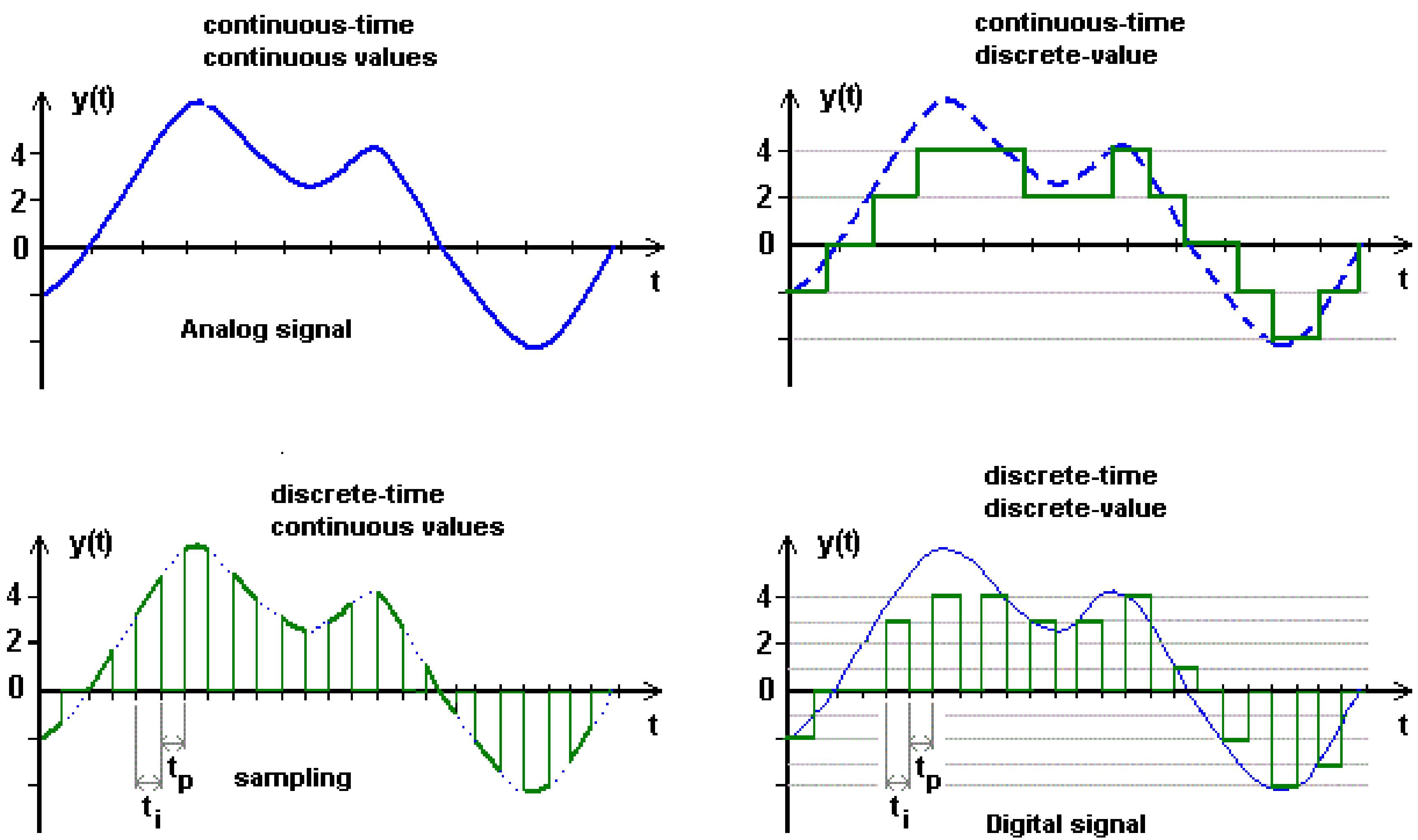
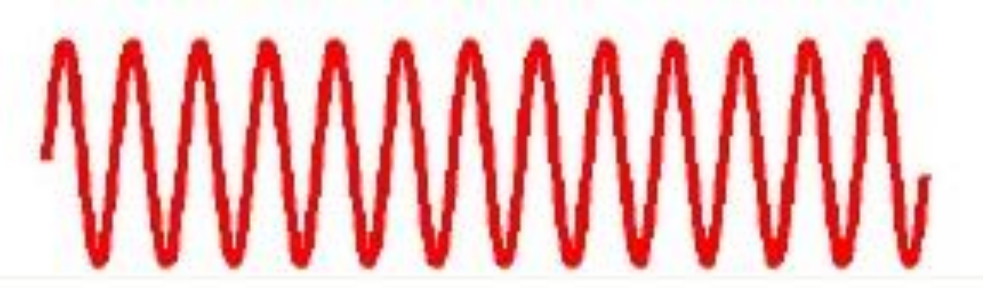
- Convolution Integral

Continuous & Discrete

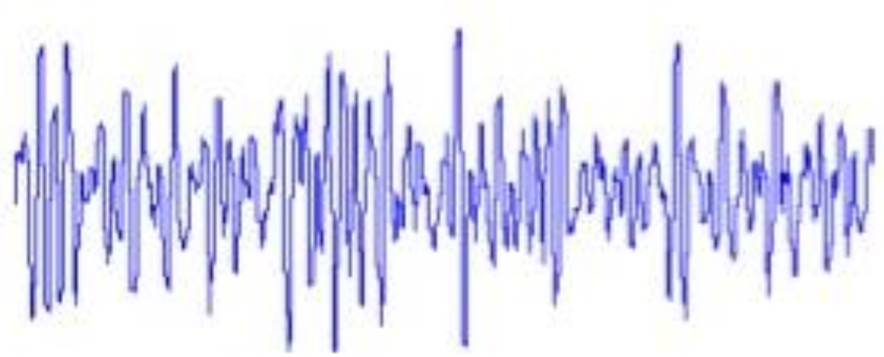


Deterministic & Random

- *Deterministic* signal
  - Signal value is completely determined at a given time *t*



- *Random* signal
  - Signal value is random at a given time *t*



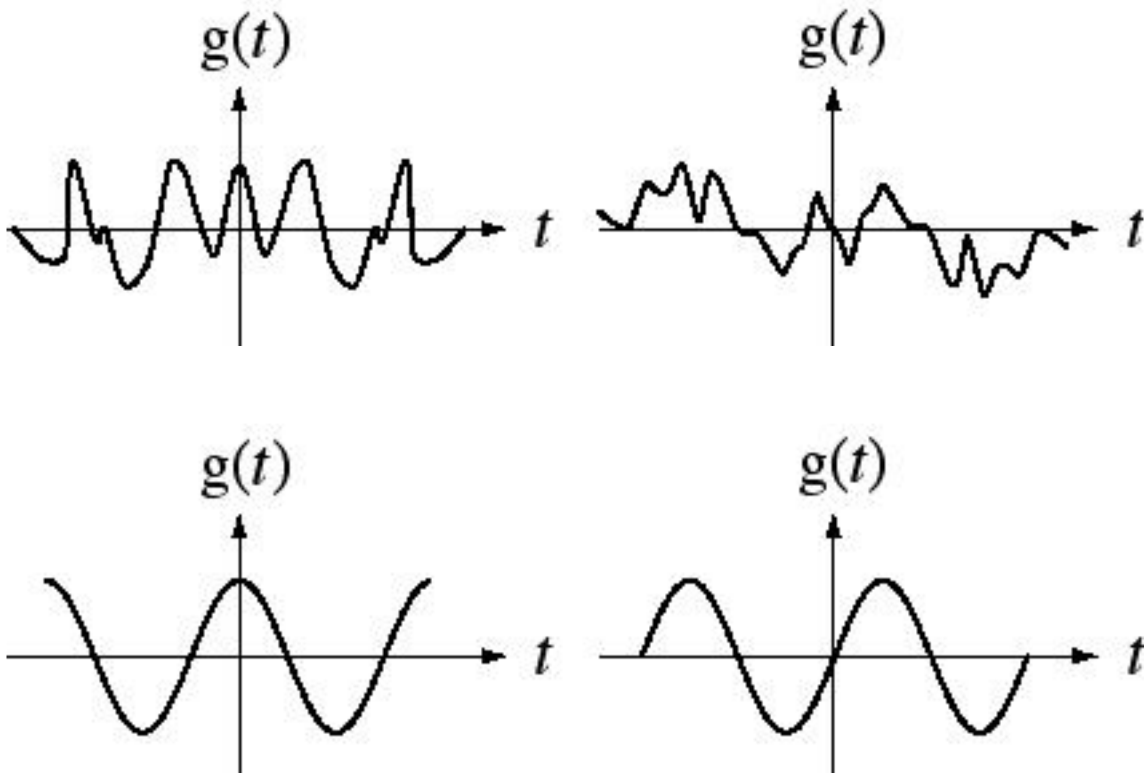
Even & Odd

Even Signals

$$g(t) = g(-t)$$

Odd Signals

$$g(t) = -g(-t)$$



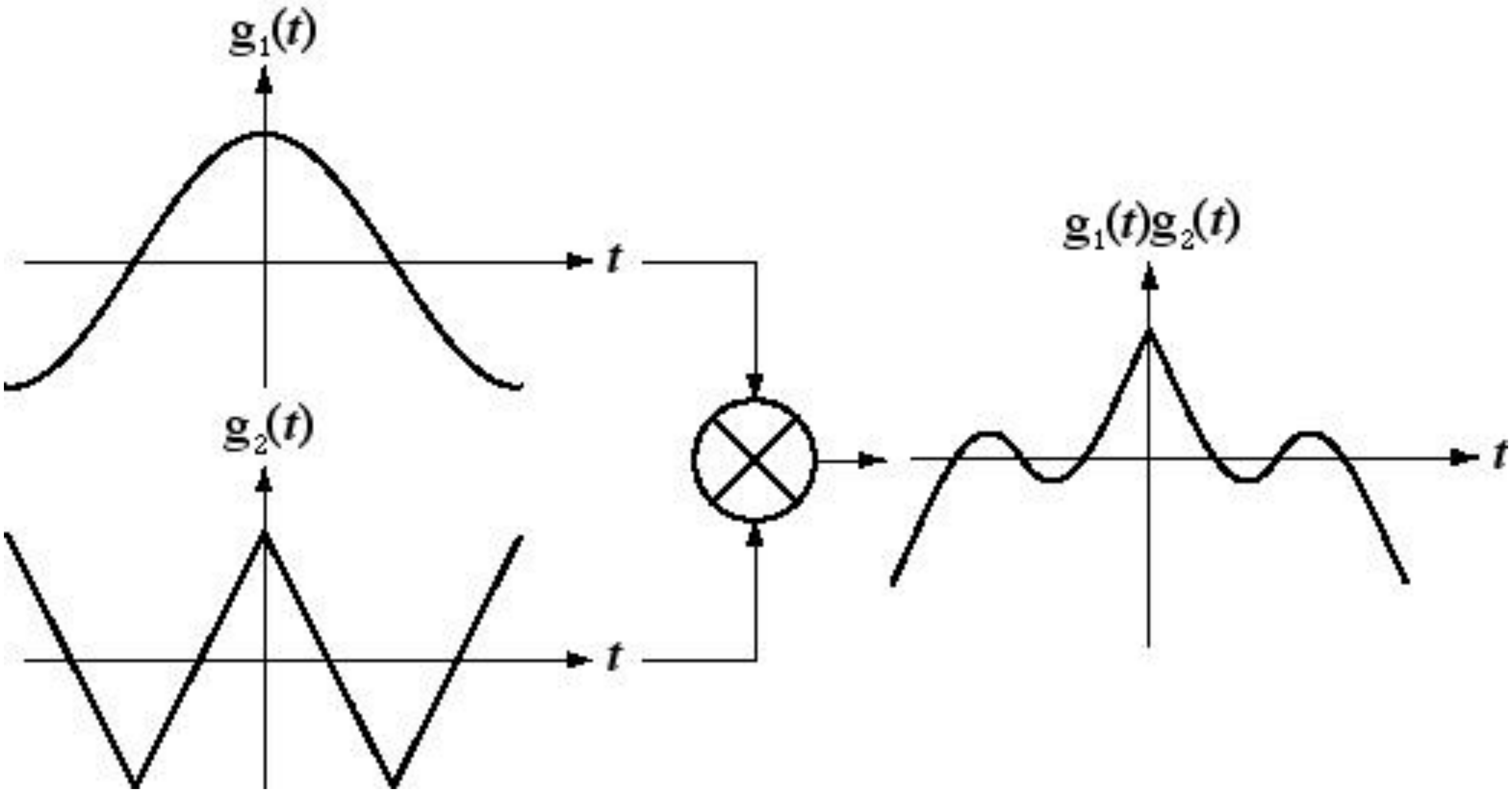
Making of Even or Odd CT Signal

An even CT signal $$g_e(t) = \frac{g(t) + g(-t)}{2}$$

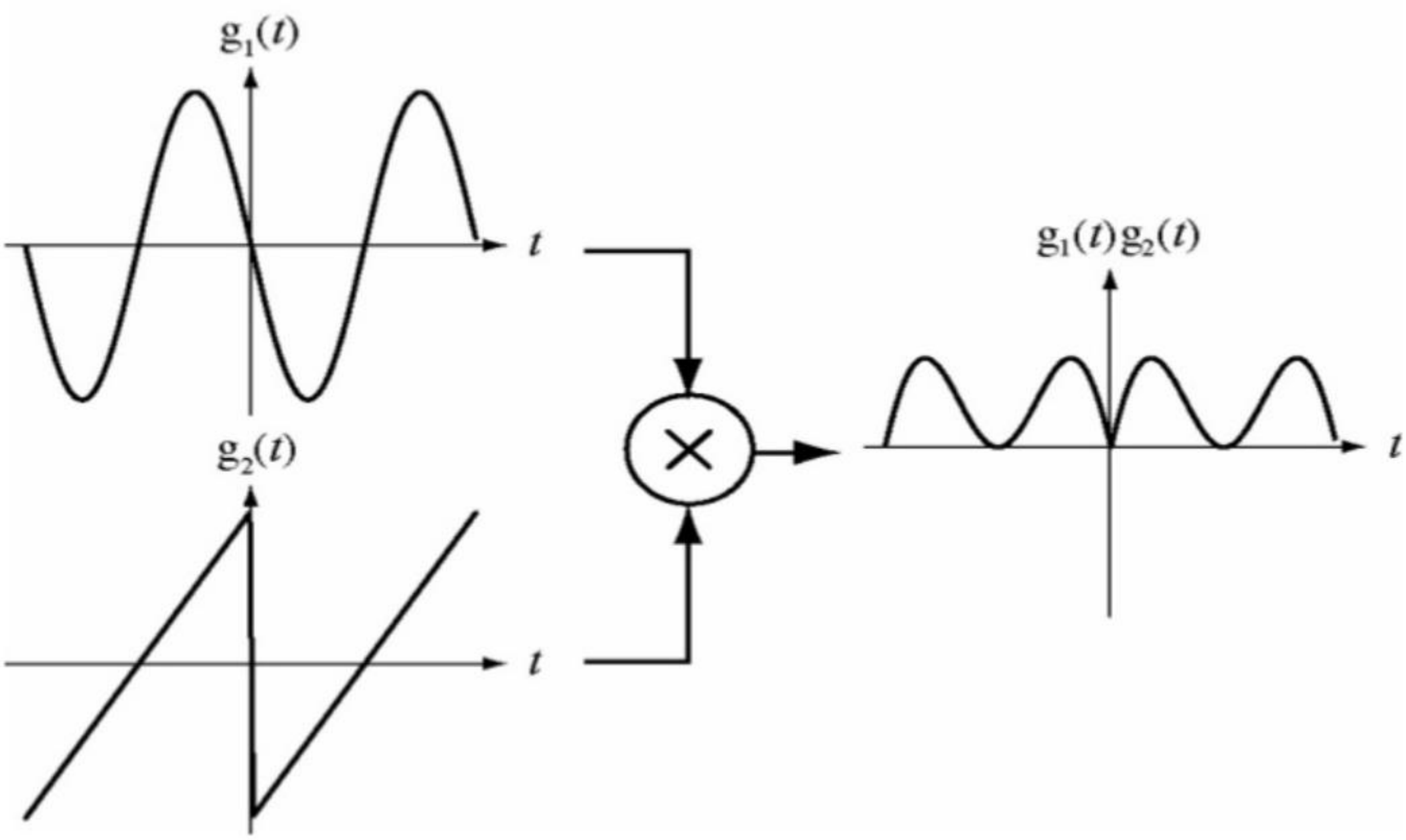
An odd CT signal $$g_o(t) = \frac{g(t) - g(-t)}{2}$$

- The derivative of an odd CT signal is even
- The derivative of an even CT signal is odd
- The integral of an odd CT signal is even
- The integral of an even CT signal is an odd CT signal with a constant

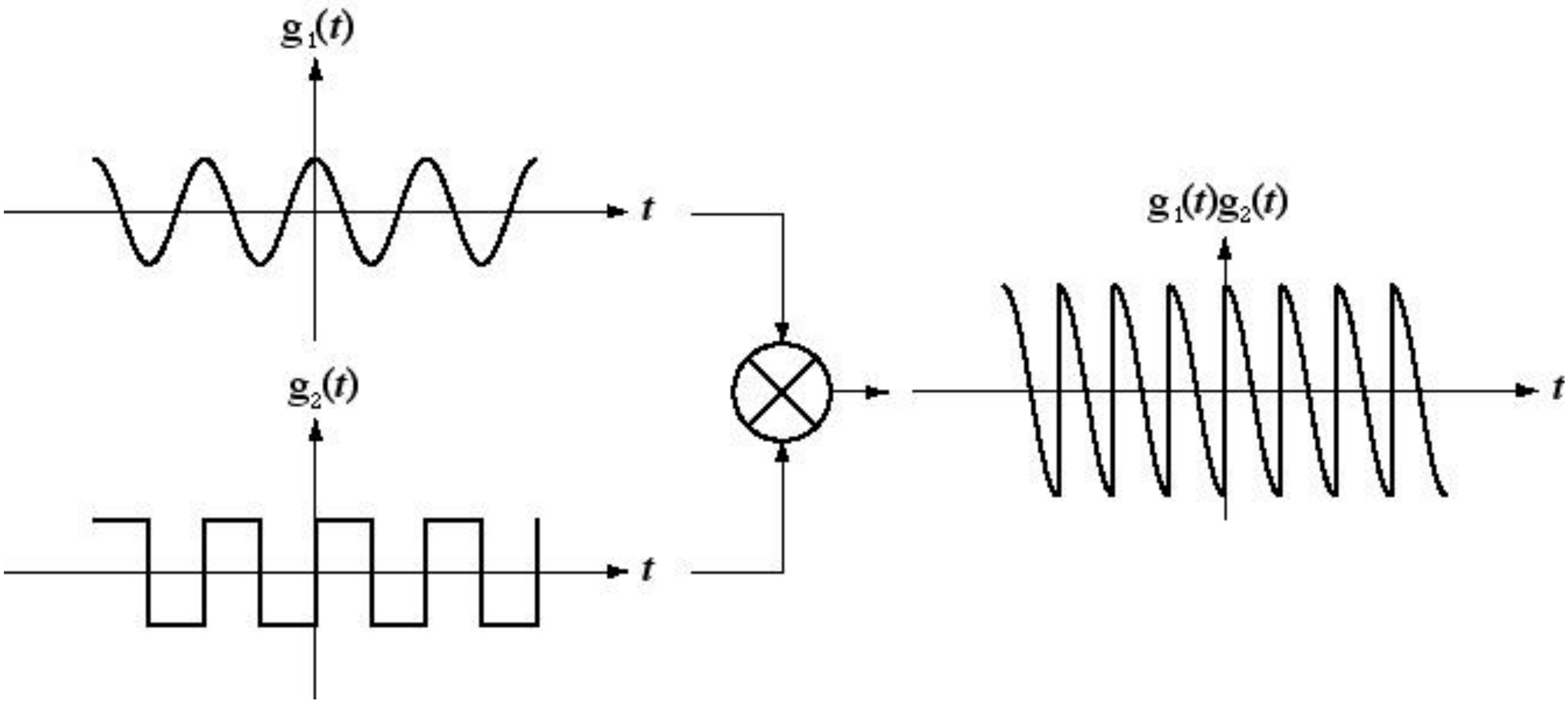
Product of Two Even CT Signals



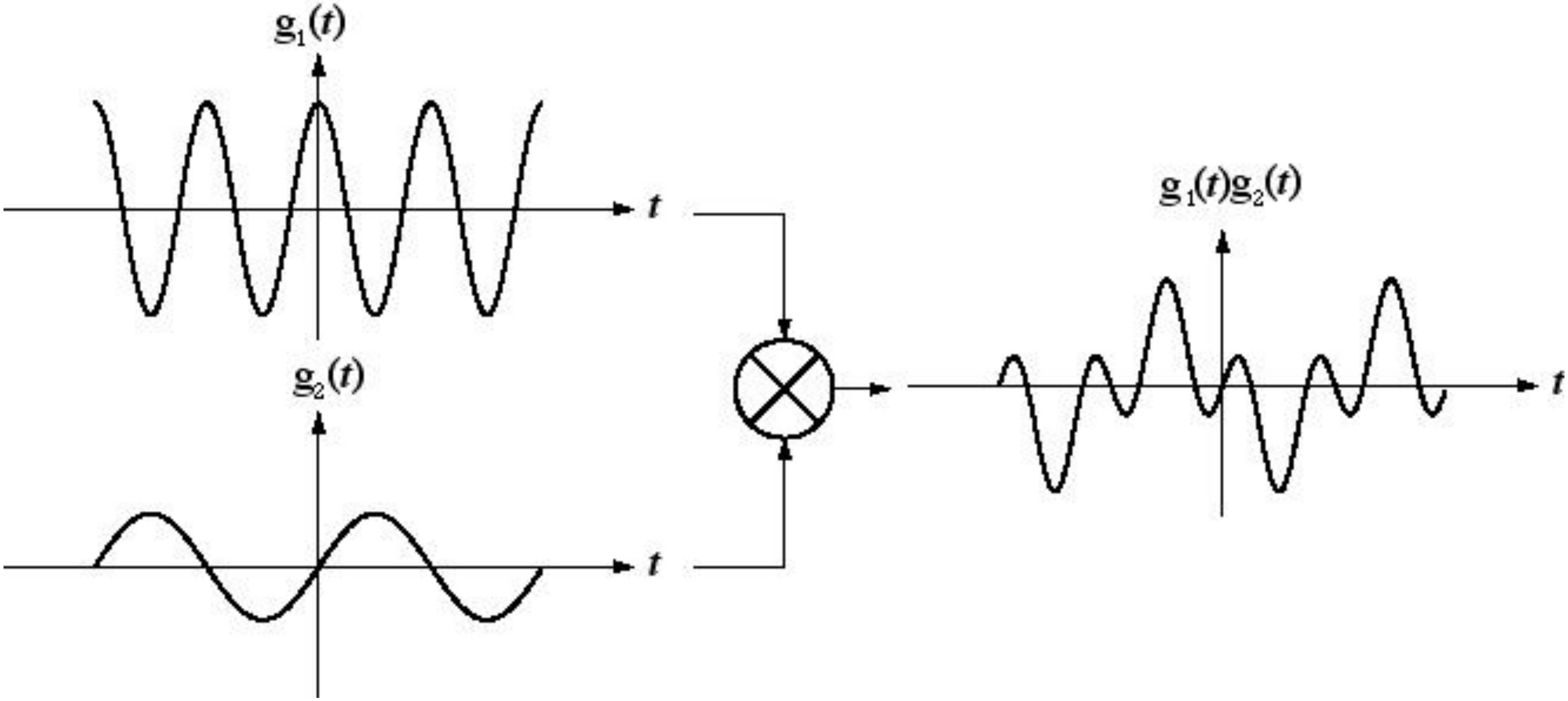
Product of Two Odd CT Signals



Product of Even & Odd CT Signals



Product of Even & Odd CT Signals

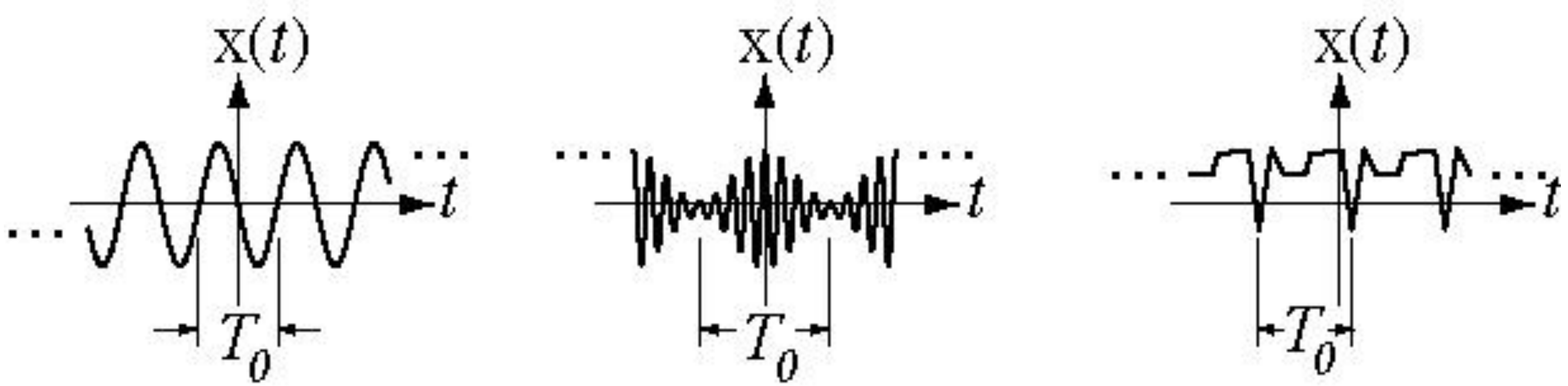


Periodic CT Signals

If a CT signal is periodic $g(t) = g(t + nT)$, where n is any integer and T is a *period* of the signal.

The minimum positive value of T for which $g(t) = g(t + T)$ is called the ***fundamental period*** of the signal, T_0.

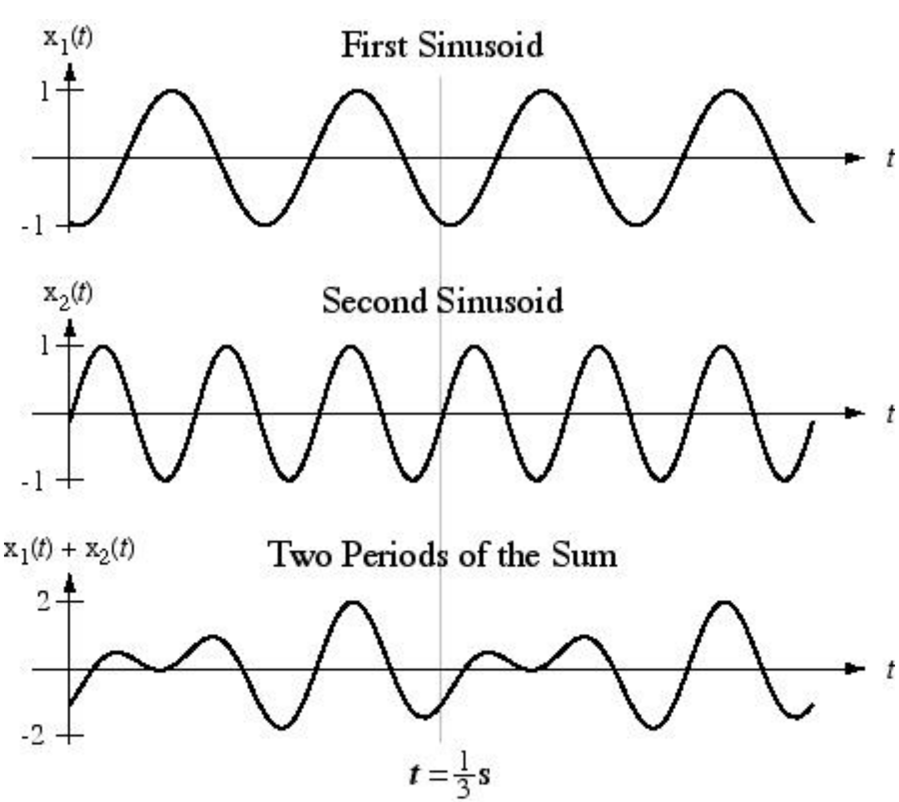
The reciprocal of the fundamental period $f_0 = 1 / T_0$ is the ***fundamental frequency***.

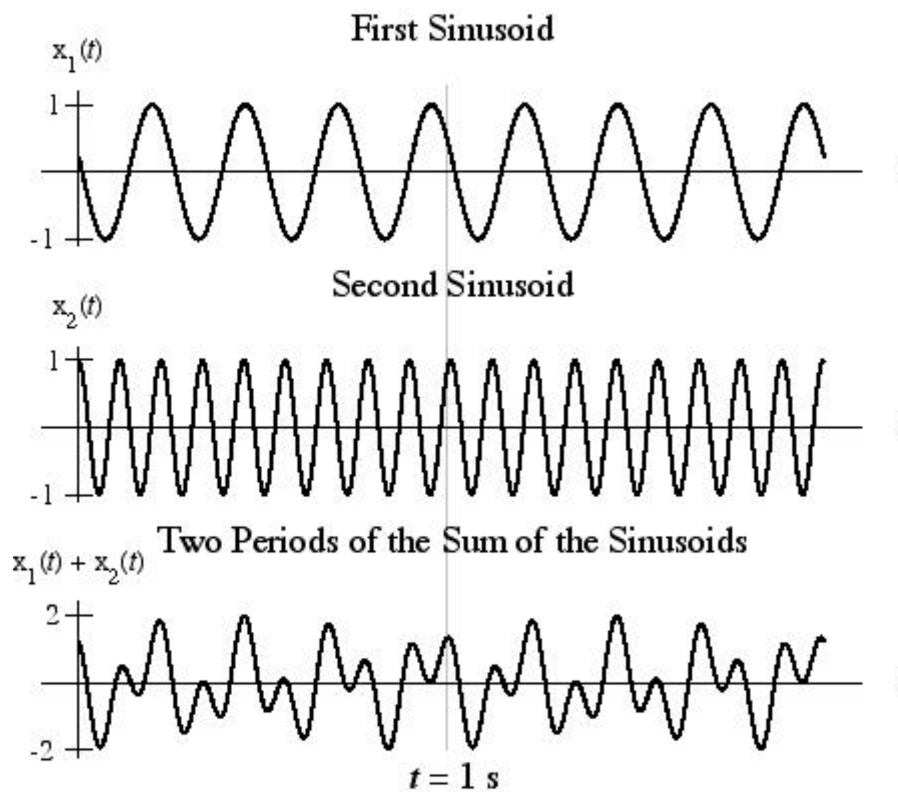


A signal that is not periodic is called *aperiodic* or *nonperiodic*.

Sum of CT Periodic Signals

The period of the sum of CT periodic signals is the *least common multiple* of the periods of the individual signal.
If the *least common multiple* is infinite, the summed signal is aperiodic.

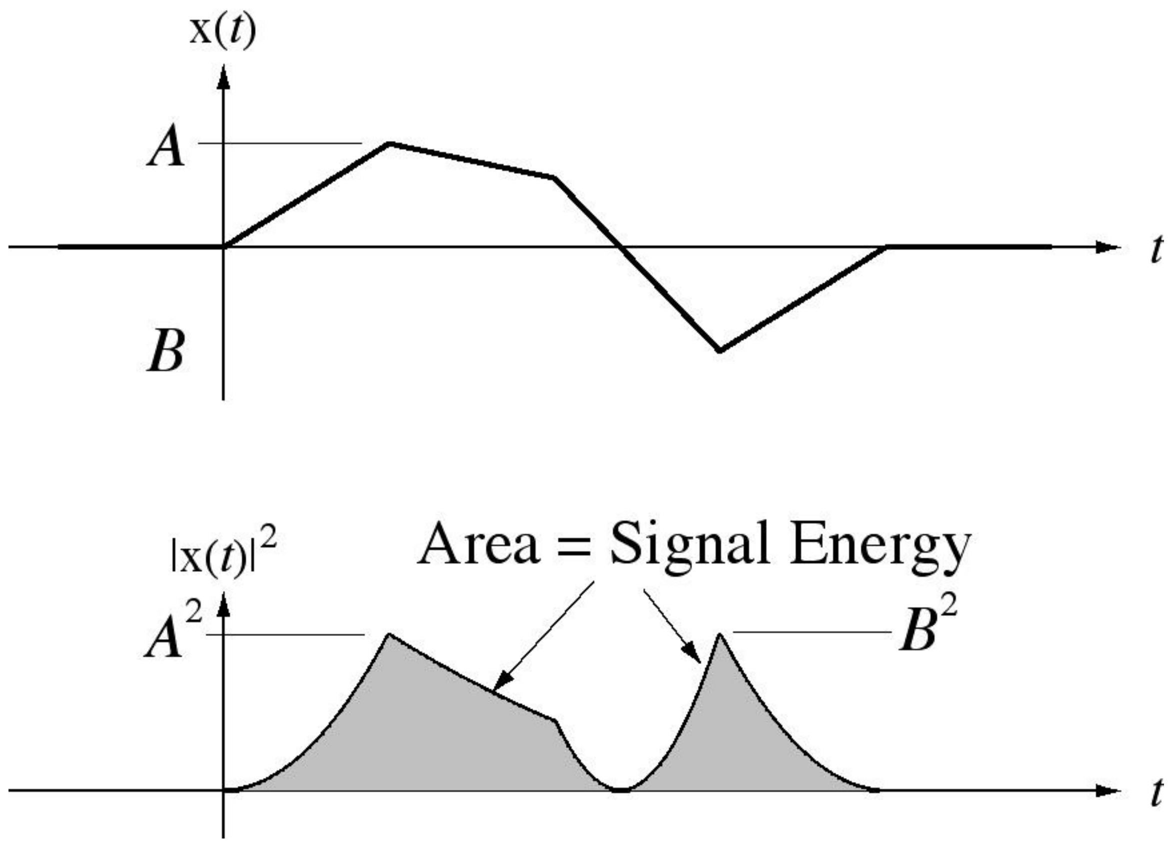




Signal Energy

The *signal energy* of a CT signal, $x(t)$, is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$



Average Signal Power

Some signals have infinite signal energy. In that case, it is more convenient to use *average signal power* instead.

average signal power of a CT signal, $x(t)$, is

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left|\mathrm{x}(t)\right|^2 dt$$

For a periodic CT signal, $x(t)$, the *average signal power* is

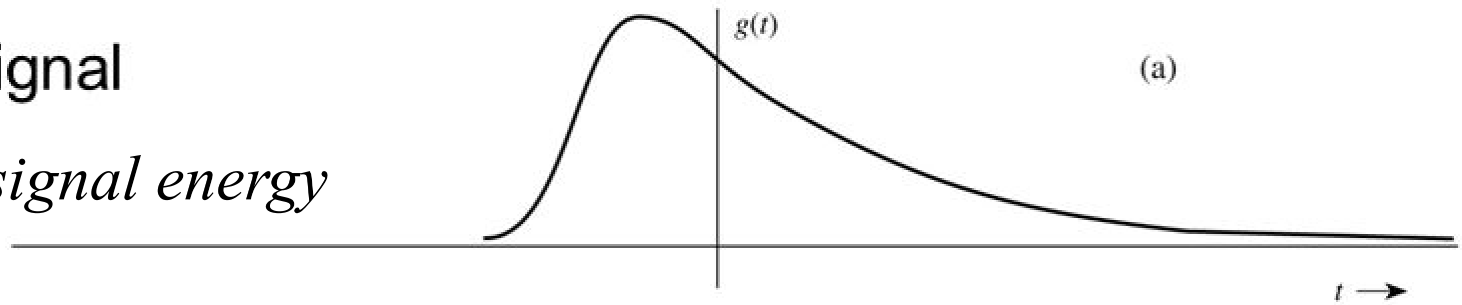
$$P_x = \frac{1}{T} \int_T \left|\mathrm{x}(t)\right|^2 dt$$

where T is any period of the signal.

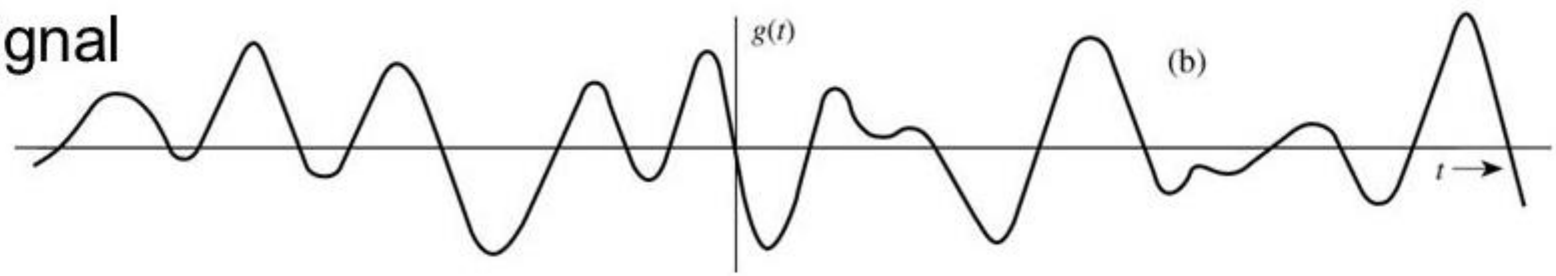
Energy Signal and Power Signal

Energy Signal

finite signal energy



Power Signal

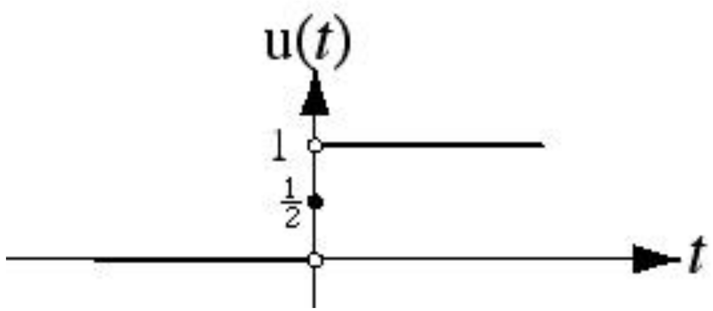


infinite signal energy and *finite average signal power*

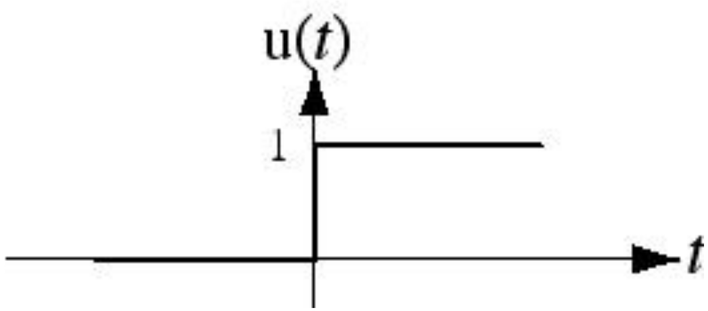
CT Unit Step Signal

$$u(t) = \begin{cases} 1 \ , & t > 0 \\ \frac{1}{2} \ , & t = 0 \\ 0 \ , & t < 0 \end{cases}$$

Precise Graph



Commonly-Used Graph



The product signal of g(t)u(t) can be thought of the signal g(t) was "turned on" at time, t = 0.

CT Unit Impulse Signal

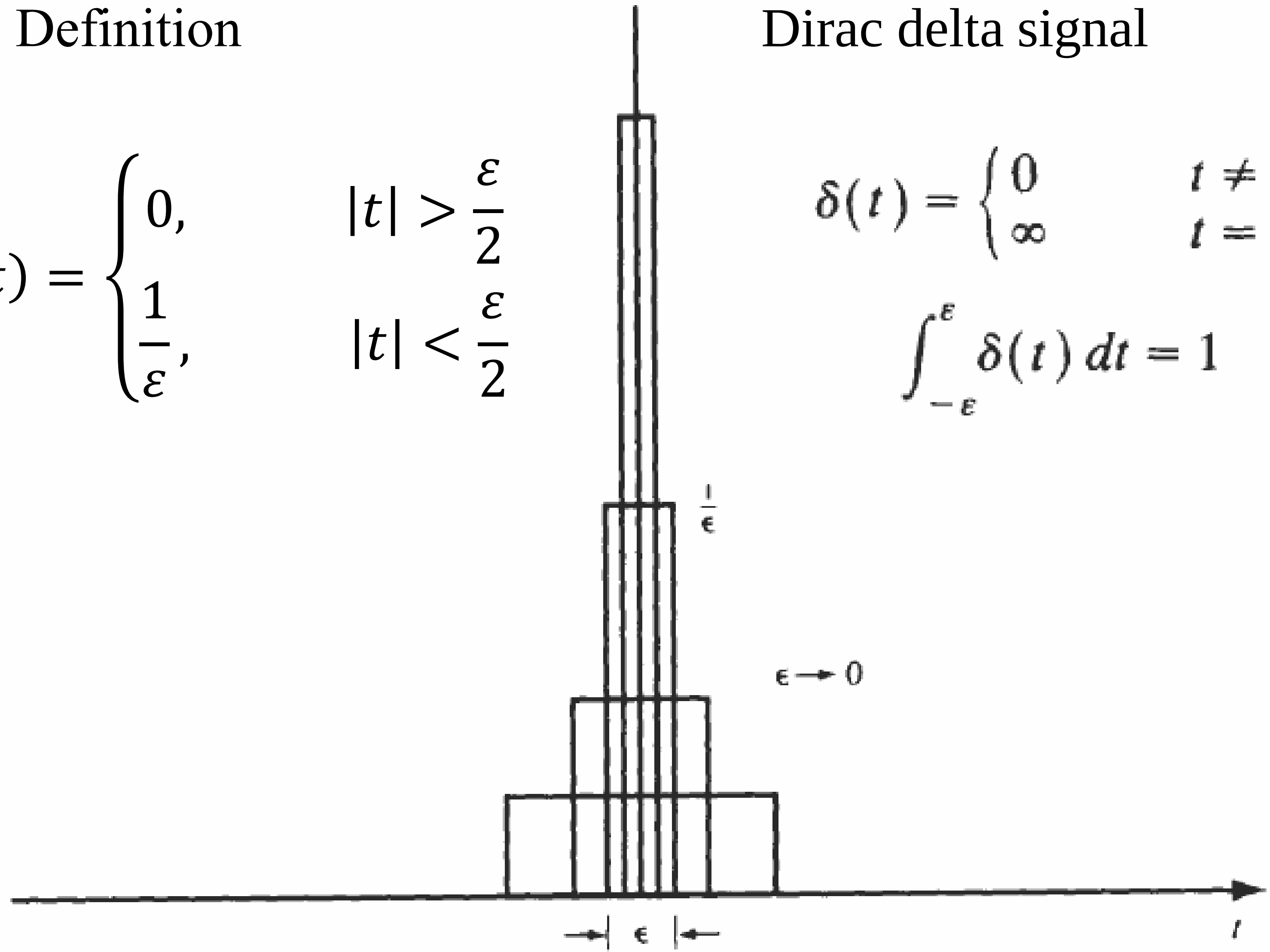
Definition

$$\delta_\varepsilon(t) = \begin{cases} 0, & |t| > \dfrac{\varepsilon}{2} \\ \dfrac{1}{\varepsilon}, & |t| < \dfrac{\varepsilon}{2} \end{cases}$$

Dirac delta signal

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1$$

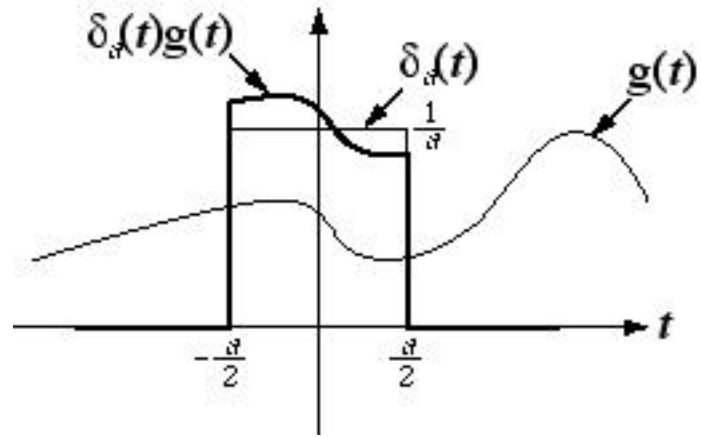


CT Unit Impulse Signal

Let g(t) be finite and continuous at $t = 0$.

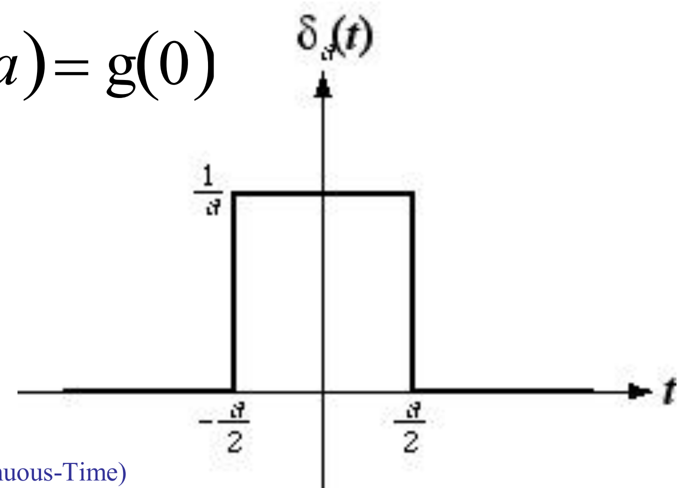
The area under the product of two signals is

$$A = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} g(t)\,dt$$



As the width of $\delta_a(t)$ approaches zero,

$$\lim_{a\to 0} A = g(0)\lim_{a\to 0}\frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} dt = g(0)\lim_{a\to 0}\frac{1}{a}(a) = g(0)$$

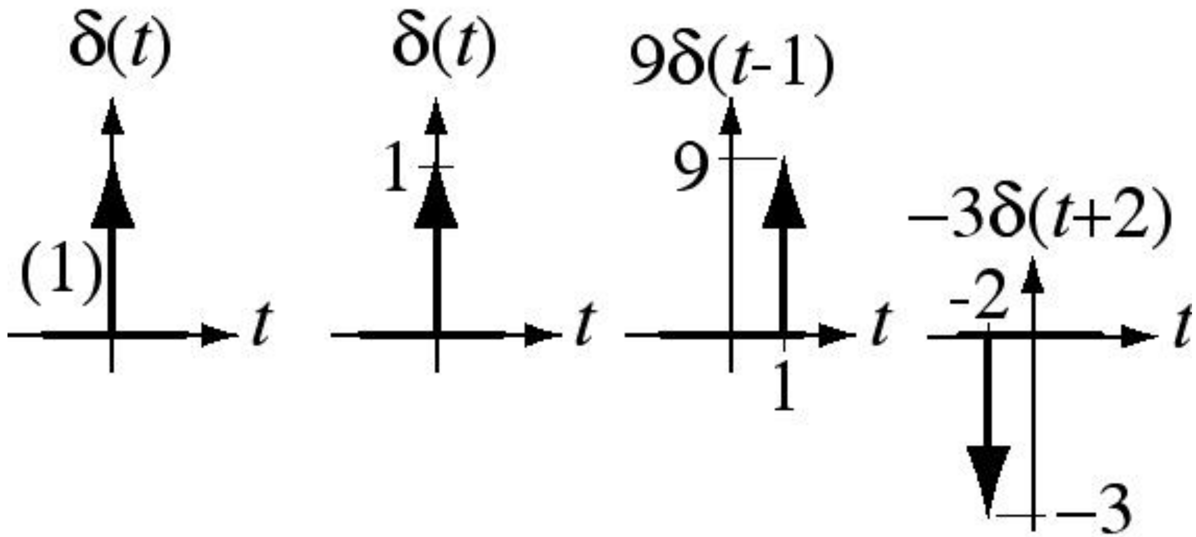


CT unit impulse is implicitly defined by

$$g(0) = \int_{-\infty}^{\infty} \delta(t)\,g(t)\,dt$$

Graphical Representation of CT Unit Impulse Signal

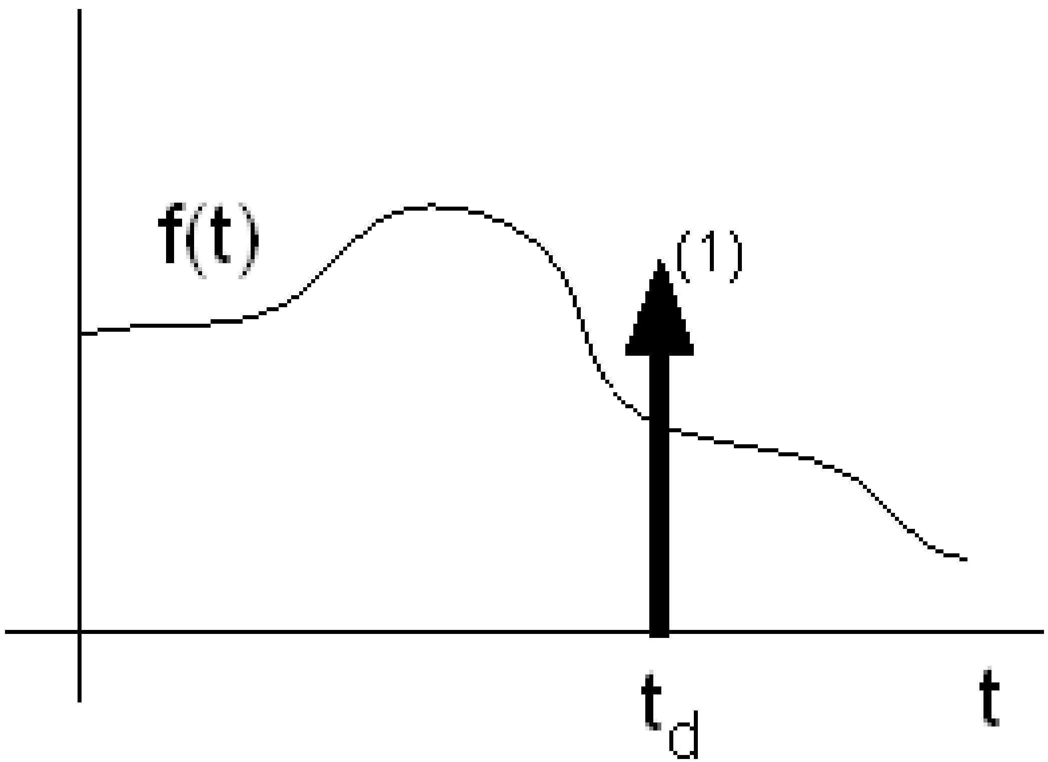
CT unit impulse signal is not a signal in the ordinary sense because its value at the time of occurrence is not defined.
It is represented a vertical arrow. Its strength is either written beside it or is represented by its length.



Sampling Property of a CT Unit Impulse Signal

CT unit impulse signal is used to sample a CT signal at a time t_d.

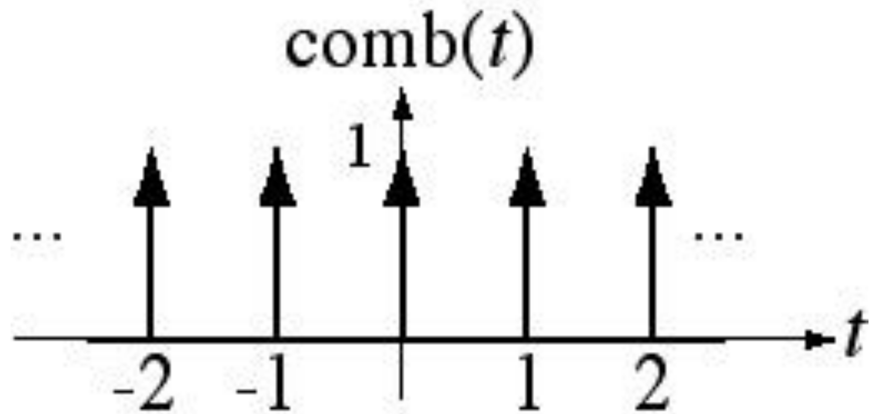
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_d)dt = f(t_d)$$



CT Unit Comb

CT unit comb is defined by

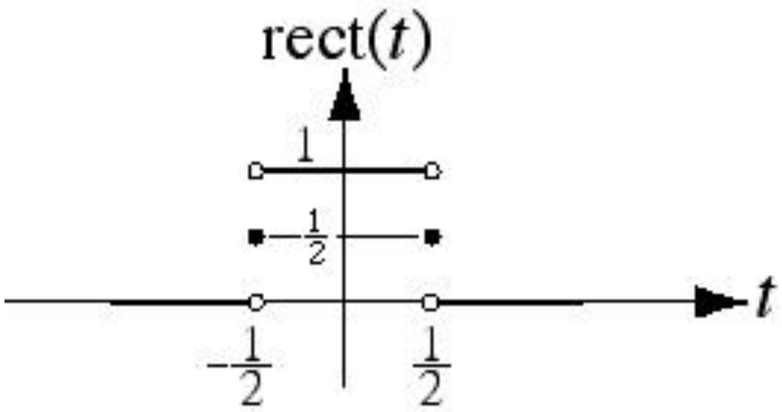
$$\text{comb}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n) \ , \quad n \text{ an integer}$$

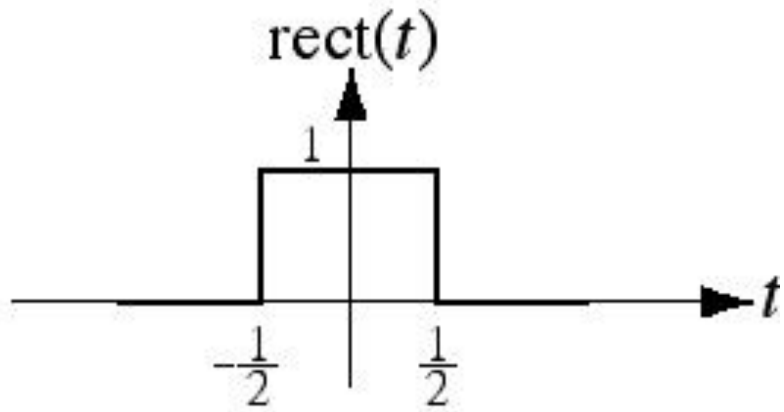


The comb is a sum of uniformly-spaced impulses.

CT Unit Rectangle Signal

$$\operatorname{rect}(t) = \begin{cases} 1\ , & |t| < \dfrac{1}{2} \\ \dfrac{1}{2}\ , & |t| = \dfrac{1}{2} \\ 0\ , & |t| > \dfrac{1}{2} \end{cases}$$

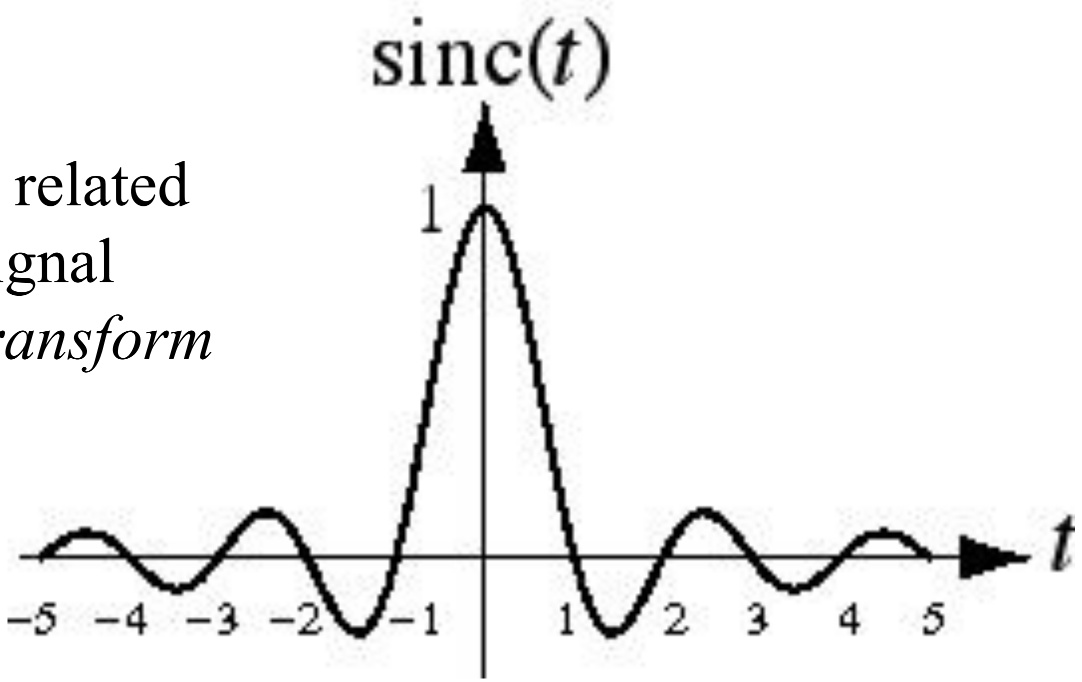




The product signal, g(t)rect(t), can be thought of as the signal, g(t), "turned on" at time t = -1/2 and "turned off" at time t = 1/2.

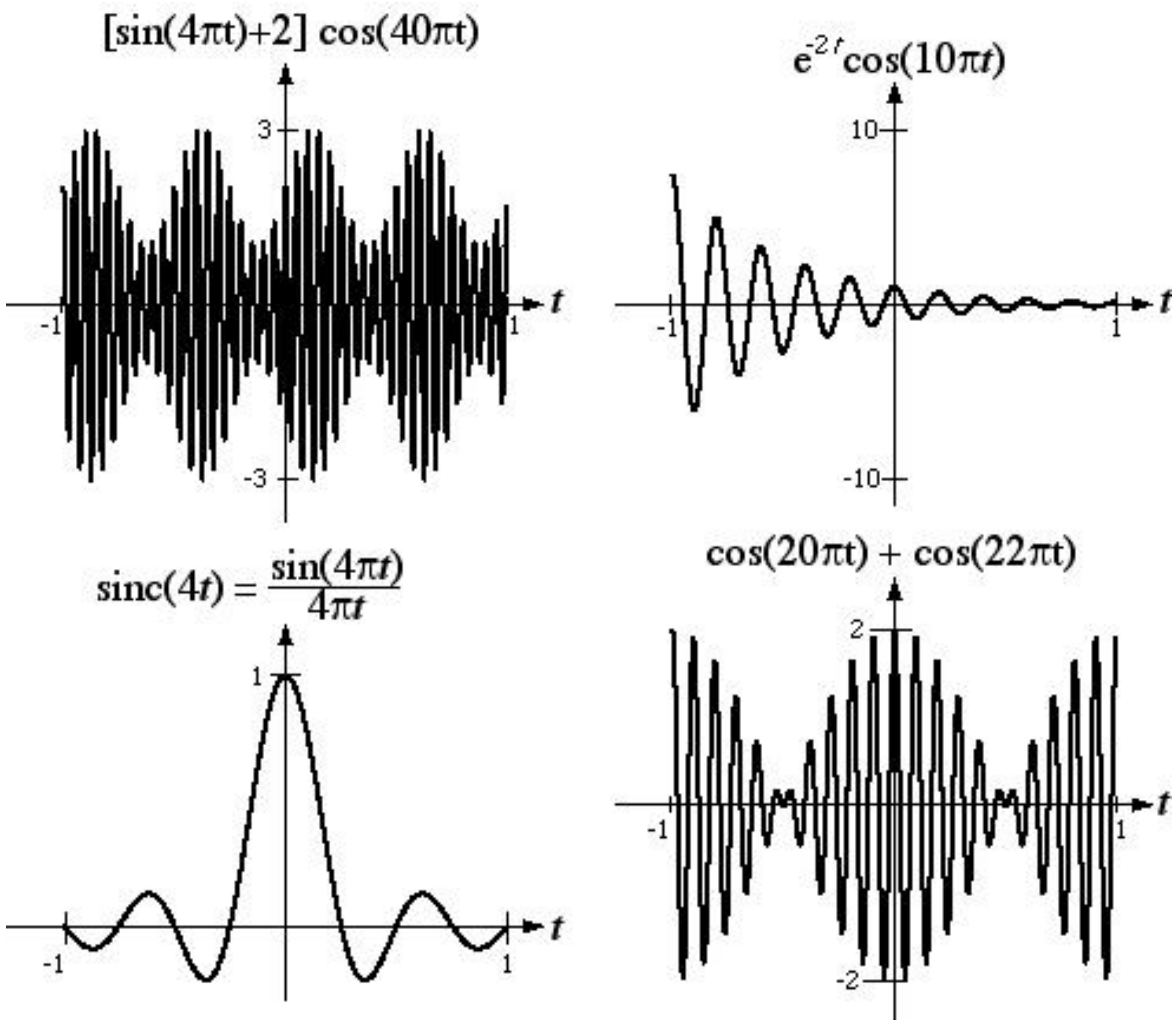
CT Unit *sinc* Signal

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



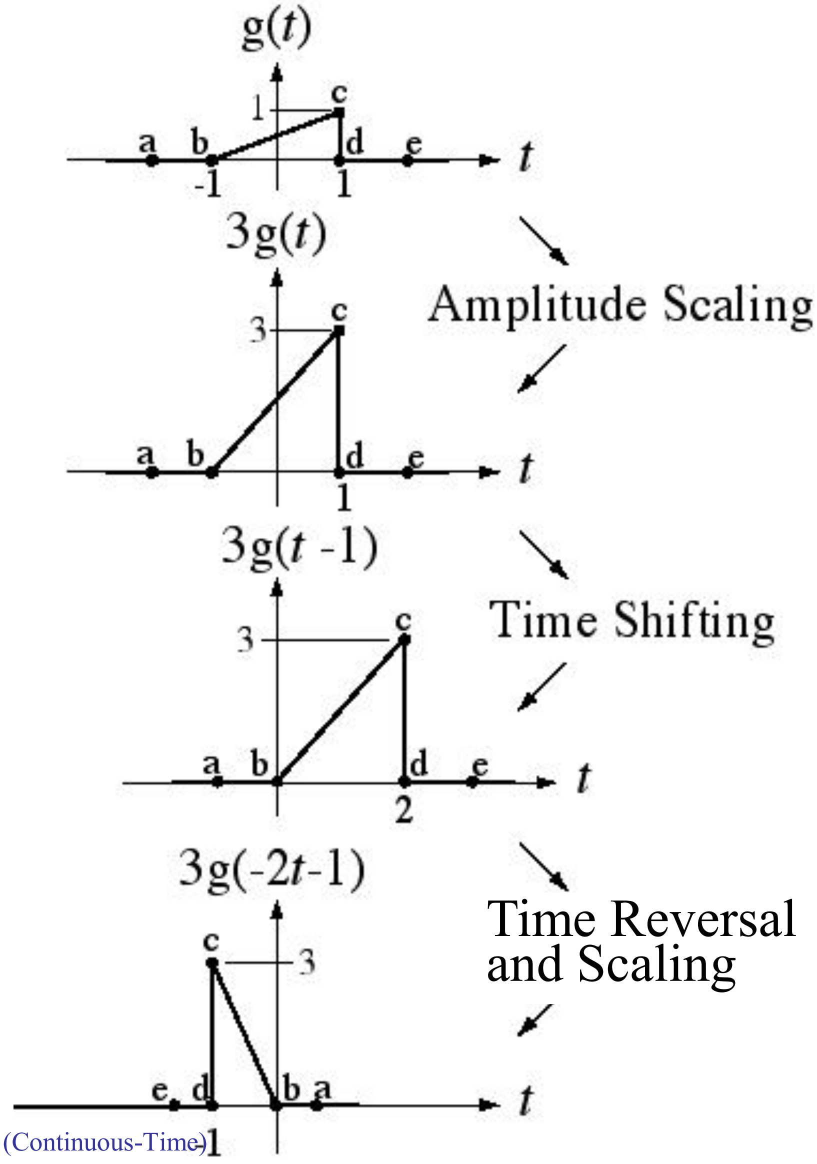
The unit *sinc* signal is related to the unit rectangle signal through the *Fourier transform*

Combinations of CT Signals



Operations of CT Signals

- Time reversal $y(t) = g(-t)$
- Time shifting $y(t) = g(t - t_d)$
- Time scaling $y(t) = g(at)$
- Amplitude scaling $y(t) = Bg(t)$
- Addition $y(t) = g_1(t) + g_2(t)$
- Multiplication $y(t) = g_1(t).g_2(t)$

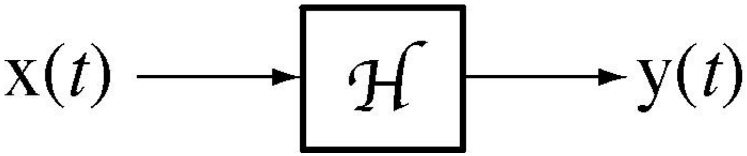


Remarks on Signals

- A signal is a function of some variables
- Variable
  - Continuous time t or discrete integer n
- Function (Signal value)
  - Deterministic or Random
  - Continuous or Discrete
  - Periodic or Non-periodic
  - Real or Complex
  - Even or Odd
- Analog vs Digital

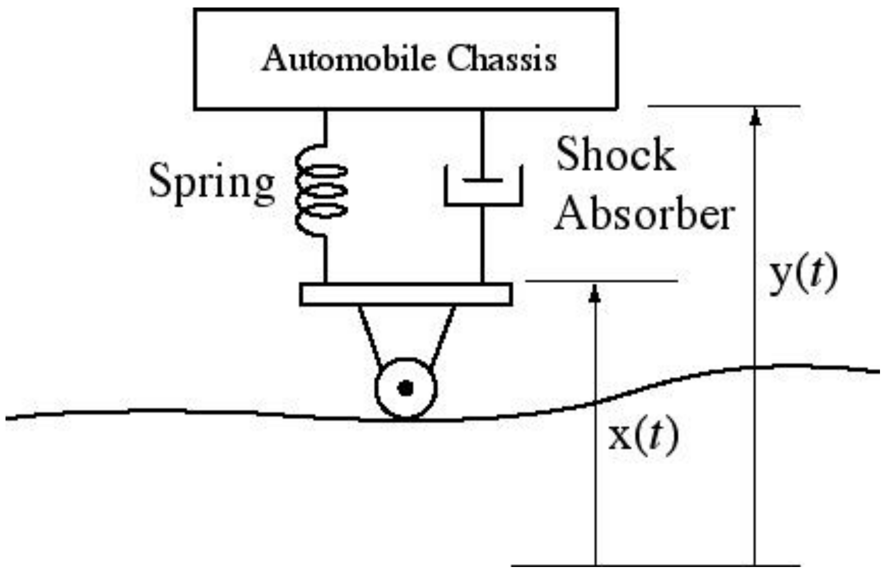
System

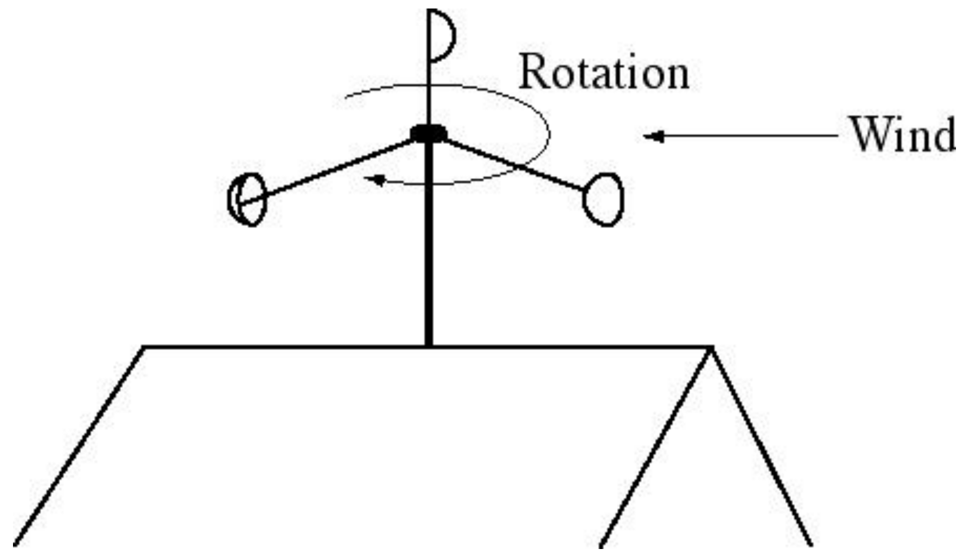
- A system has inputs and outputs
- A system accepts excitation signals at its inputs and produces response signals at its outputs
- A system is often represented by *block diagrams*

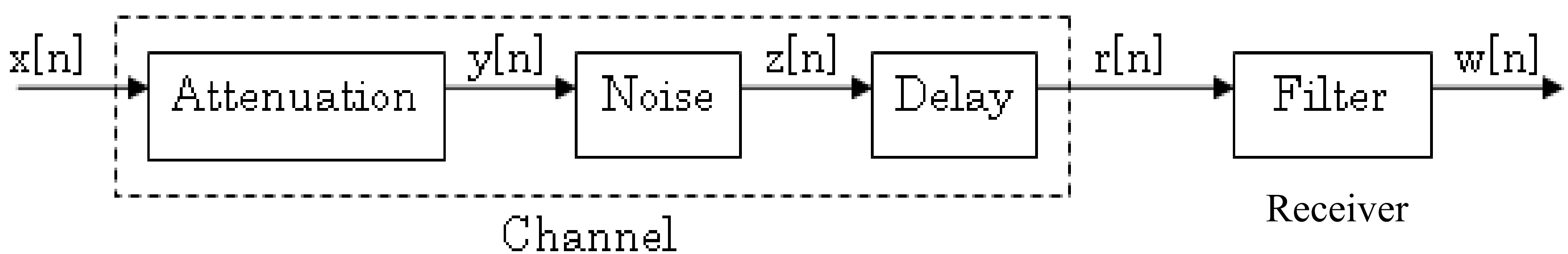


A single-input, single-output system block diagram

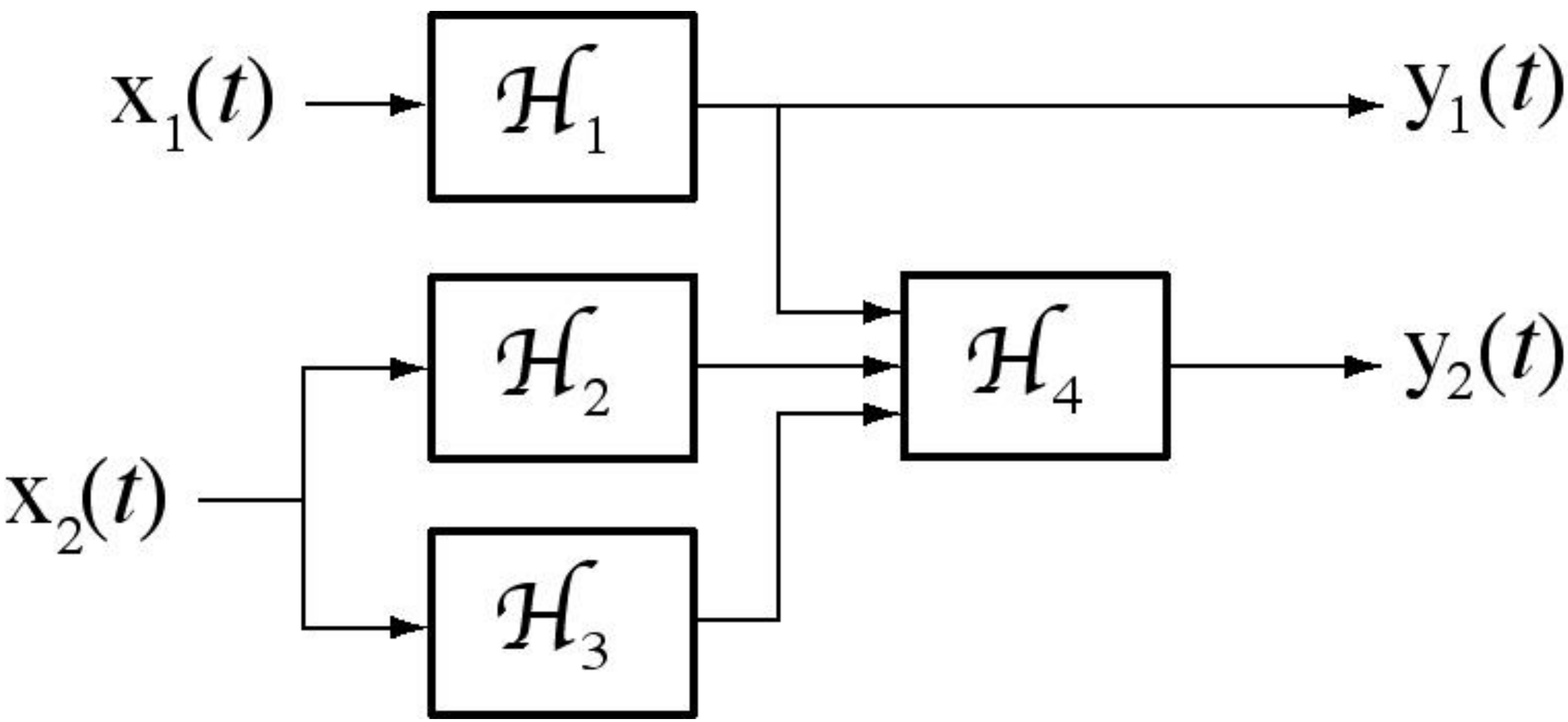
System Examples





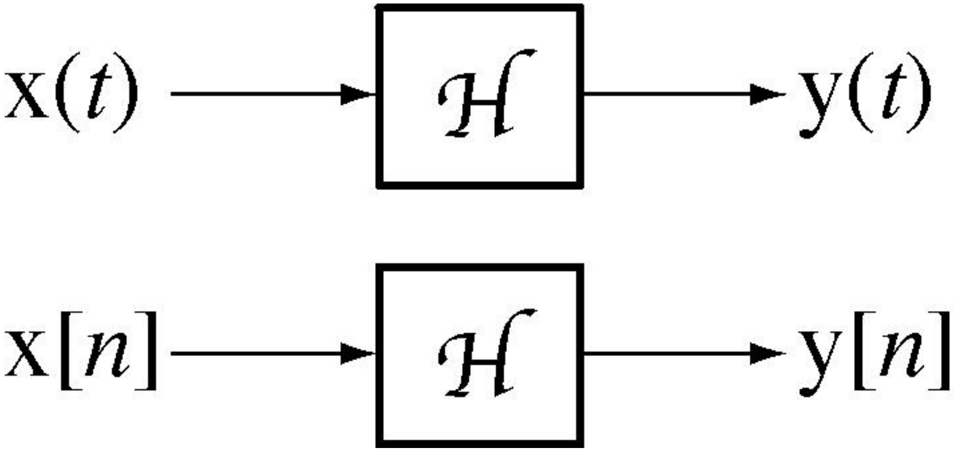


A Multiple-Input, Multiple-Output System Block Diagram



CT and DT Systems

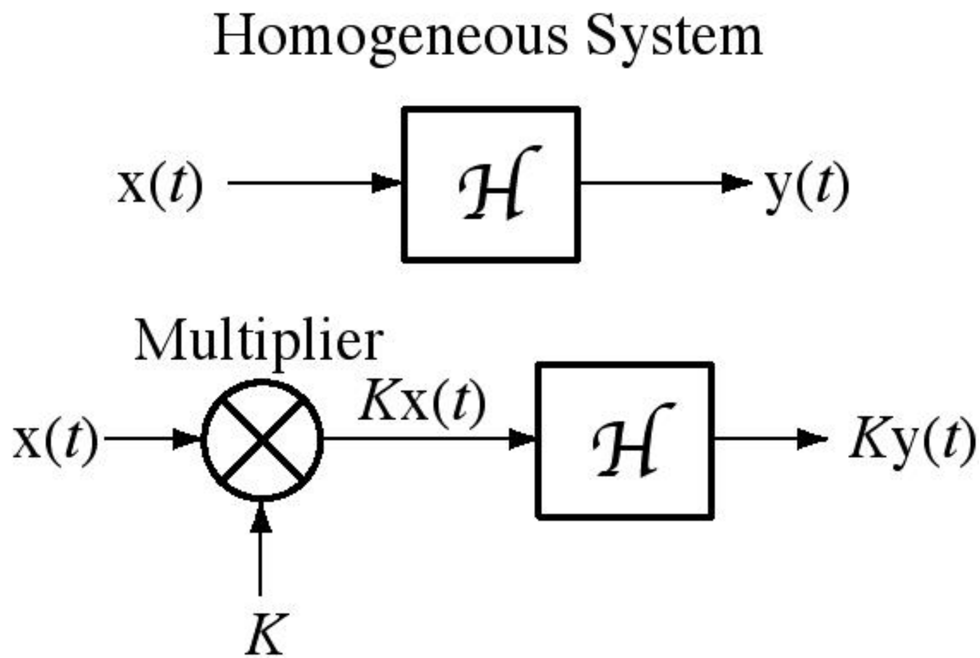
CT systems respond to and produce CT signals



DT systems respond to and produce DT signals

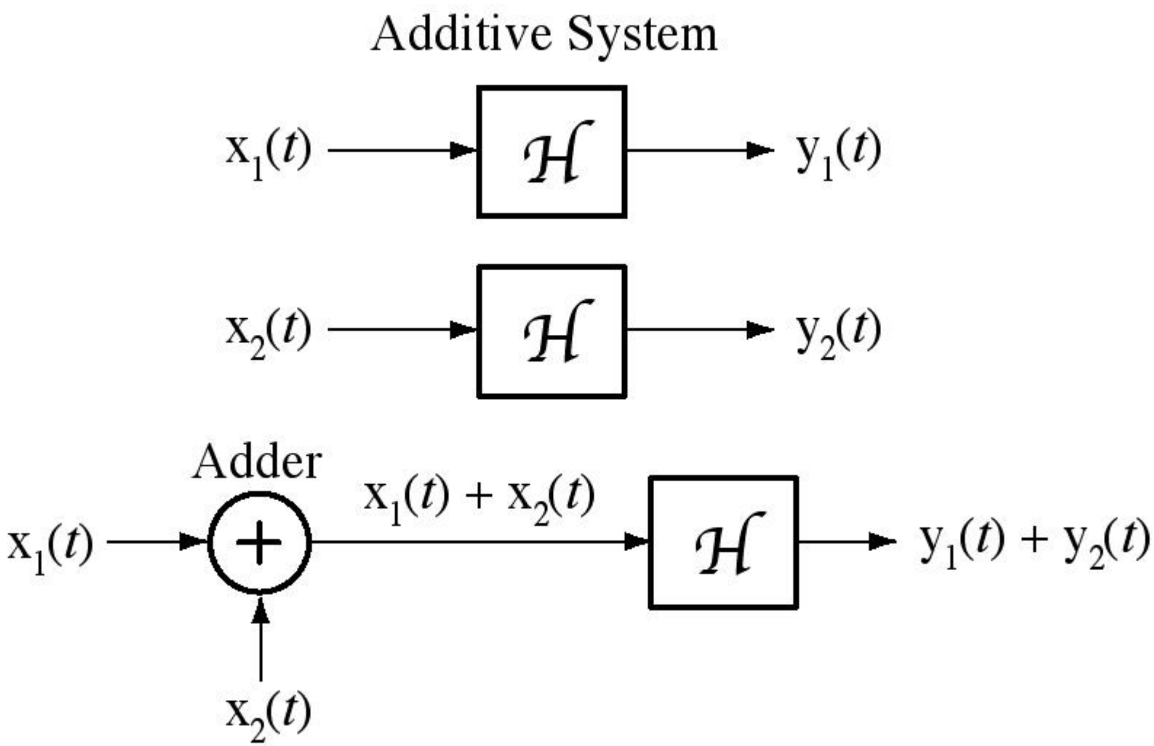
Homogeneity

- In a *homogeneous* system, multiplying the input by any constant (including *complex* constants), the response is multiplied by the same constant.



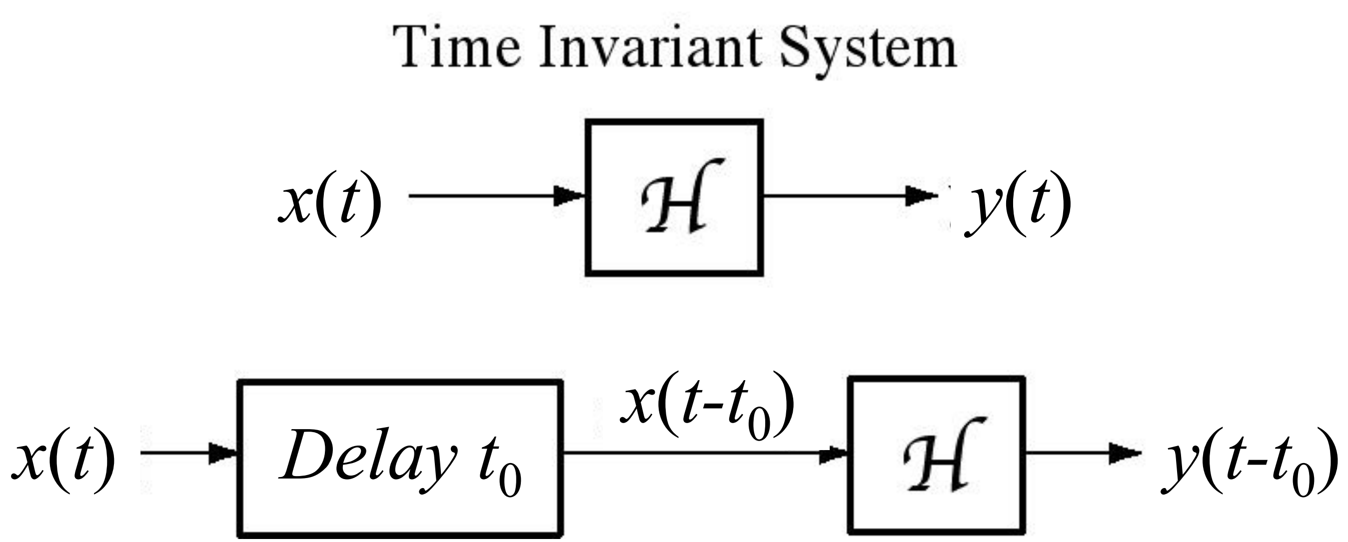
Additivity

- If one input causes a response and another input causes another response and if, for any arbitrary inputs, the sum of the two inputs causes a response which is the sum of the two responses, the system is said to be *additive*



Time Invariance

- If an excitation causes a response and delaying the excitation simply delays the response by the same amount of time, regardless of the amount of delay, then the system is *time invariant*



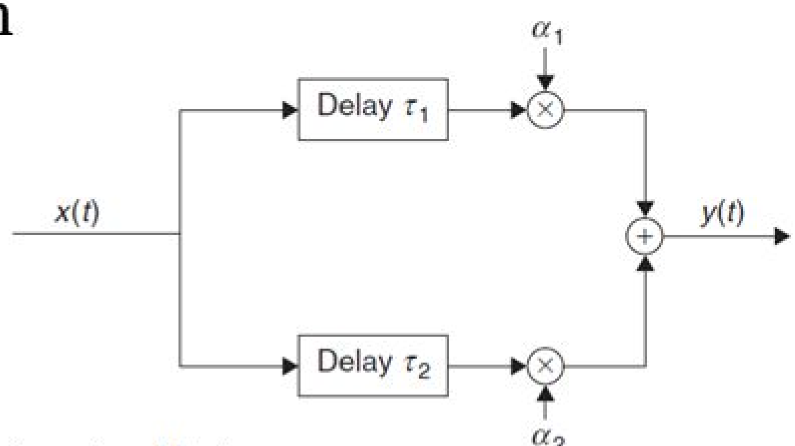
Linearity and LTI Systems

- If a system is both homogeneous and additive, it is *linear*.
- If a system is both linear and time-invariant, it is called a *Linear* and *Time-Invariant* (*LTI*) system.
- Some non-linear systems can be approximated by linear systems in the case of small excitations for analytical purposes.

Stability

- Any system for which the response is bounded for any arbitrary bounded excitation, is called a *bounded-input-bounded-output* (BIBO) stable system.

- Example: Multi-echo path system



$$|y(t)| \leq |\alpha_1||x(t-\tau_1)| + |\alpha_2||x(t-\tau_2)| < [|\alpha_1| + |\alpha_2|]M$$

Causality

- Any system for which the response occurs only during or after the time in which the excitation is applied is called a *causal* system.
- Strictly speaking, all real physical systems are causal.

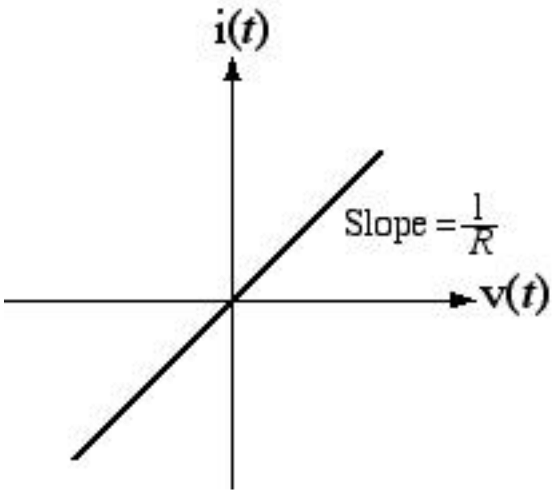
Memory

- If a system's response at any arbitrary time depends only on the excitation at that same time and not on the excitation or response at any other time is called a *static* system and is said to have no *memory*.
- A system whose response at some arbitrary time does depend on the excitation or response at another time is called a *dynamic* system and is said to have *memory*.

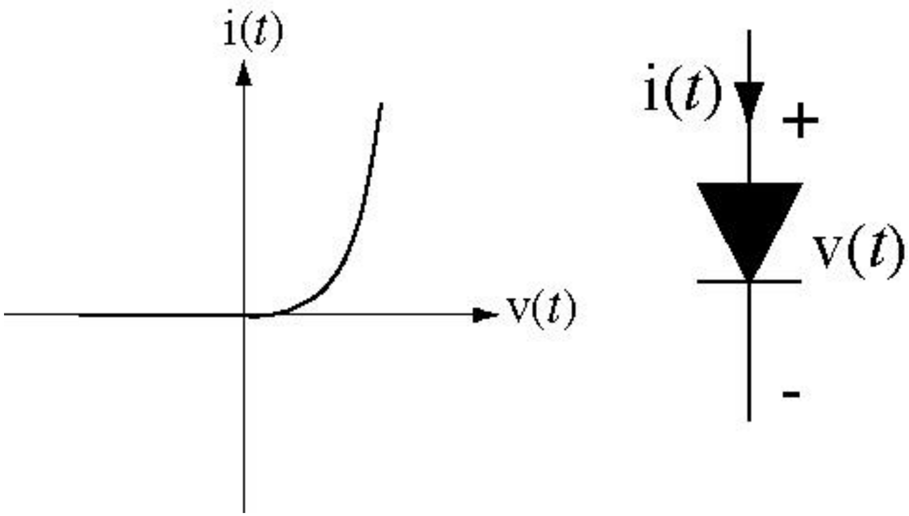
Static Non-Linearity

- Many real systems are non-linear because the relationship between excitation and response is non-linear.

V-I Diagram for a Linear Resistor



V-I Diagram for a Diode



System Representation

1. A system is represented as an "*abstract*" operator **T**{x}
   – Examples: Amplifier, Noisy channel, etc.

2. A system is represented in a *general manner*
   – Impulse response
   – Convolution
   – Time-domain

Our Purpose

- Given an ***arbitrary input***, how to compute the output?
- We will describe/model the input-output relationship for LTI Systems
  - General model
  - Not just amplifier, noisy channel

Convolution Integral - 1

- In a LTI system
  - $\delta(t) \rightarrow h(t)$
  - $h(t) = \mathbf{T}[\delta(t)]$
  - Unit impulse input → the impulse response

- It is possible to use $h(t)$ to solve for any input-output relationship

- One way to do it is by using the Convolution Integral

Convolution Integral - 2

- Remember

- Any input can be expressed using the unit impulse signal

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

Convolution Integral - 3

$x(t)$ → LTI System → $y(t)$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

$$y(t) = T\{x(t)\}$$

$$h(t) = T\{\delta(t)\}$$

$$h(t-\tau) = T\{\delta(t-\tau)\}$$

$$y(t) = T\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right]$$

$$Linearity: y(t) = \int_{-\infty}^{\infty} x(\tau)T\left[\delta(t-\tau)\right]d\tau$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$y(t)=x(t)*h(t)$$

Do not confuse convolution with multiplication!

Convolution Integral - 4

A system can be characterized using its impulse response

$$x(t) \longrightarrow \boxed{\text{LTI System: } h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

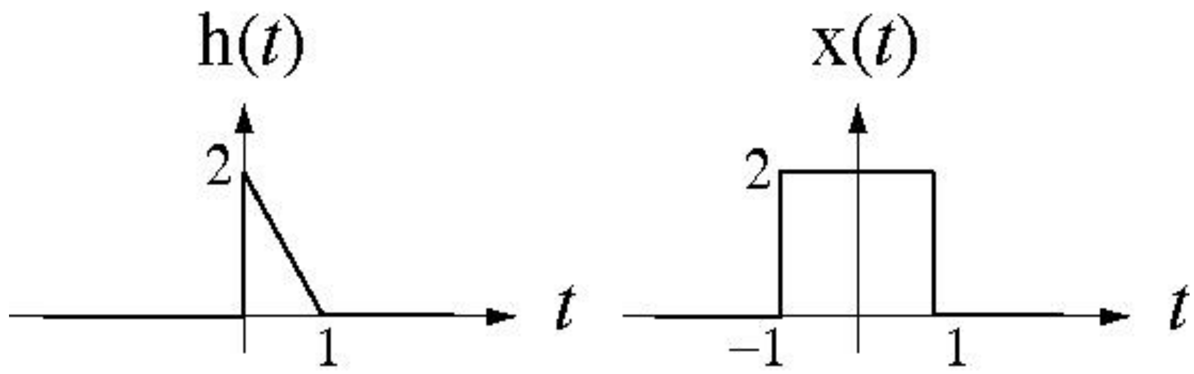
Key point: the system output at *t* is in response to all past and present values of the input signal, not just at time *t*

A Graphical Illustration of the Convolution Integral - 1

The convolution integral is defined by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

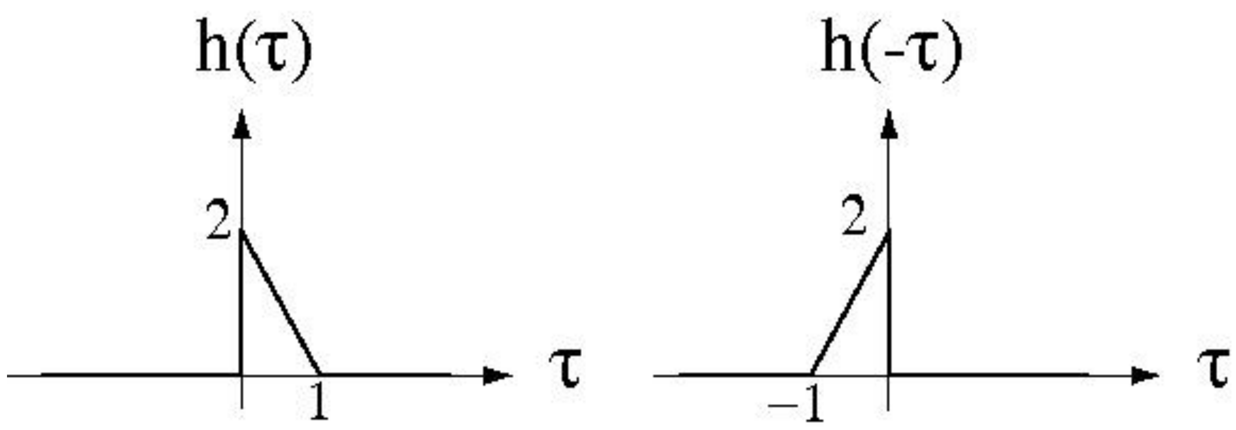
For illustration purpose, let the input, $x(t)$, and the impulse response, $h(t)$, be the two signals below.



A Graphical Illustration of the Convolution Integral - 2

In the convolution integral there is a factor, $h(t - \tau)$

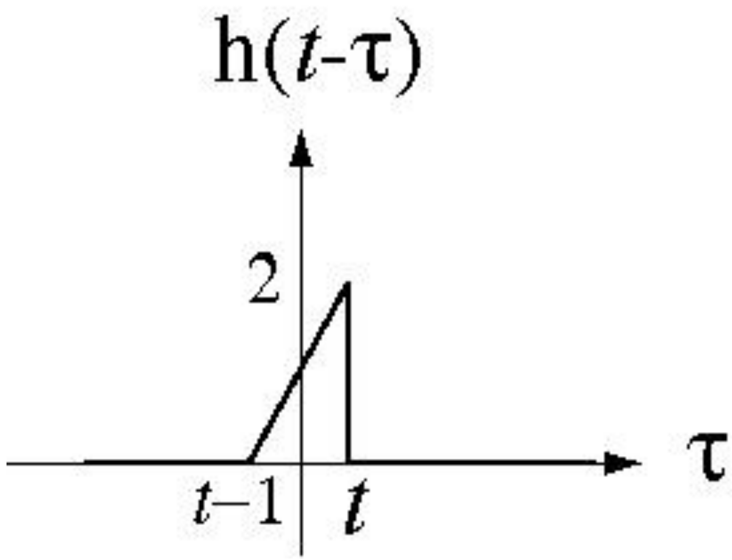
We can begin to visualize this quantity in the graphs below.



A Graphical Illustration of the Convolution Integral - 3

The functional transformation in going from $h(\tau)$ to $h(t - \tau)$ is

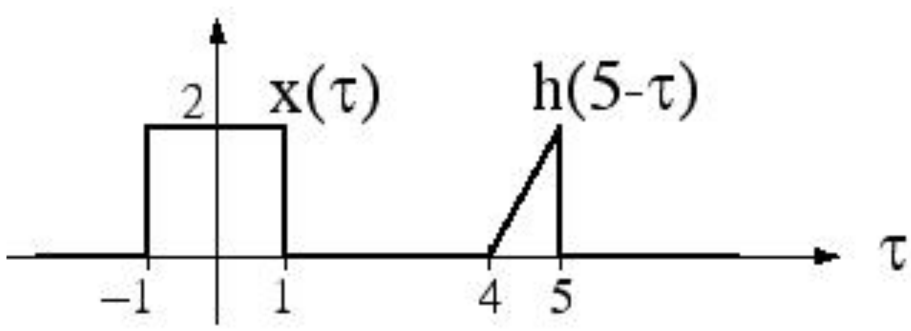
$$\mathrm{h}(\tau) \xrightarrow{\tau \to -\tau} \mathrm{h}(-\tau) \xrightarrow{\tau \to \tau - t} \mathrm{h}(-(\tau - t)) = \mathrm{h}(t - \tau)$$

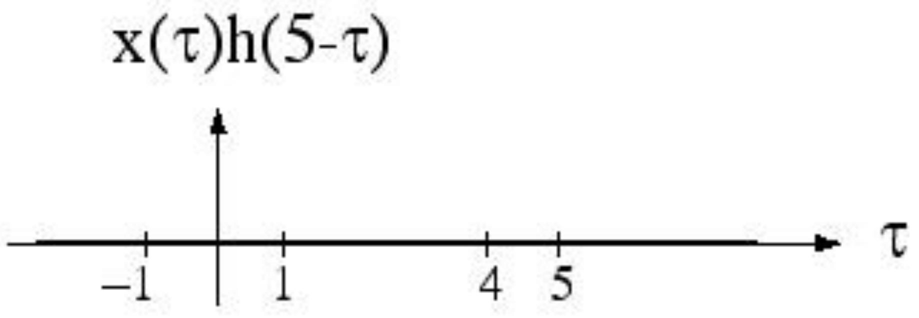


A Graphical Illustration of the Convolution Integral - 4

- The convolution value is the area under the product of $x(t)$ and $h(t - \tau)$. This area depends on what t is.
- For example, let $t = 5$, the area under the product is zero.
- If $y(t) = x(t) * h(t)$

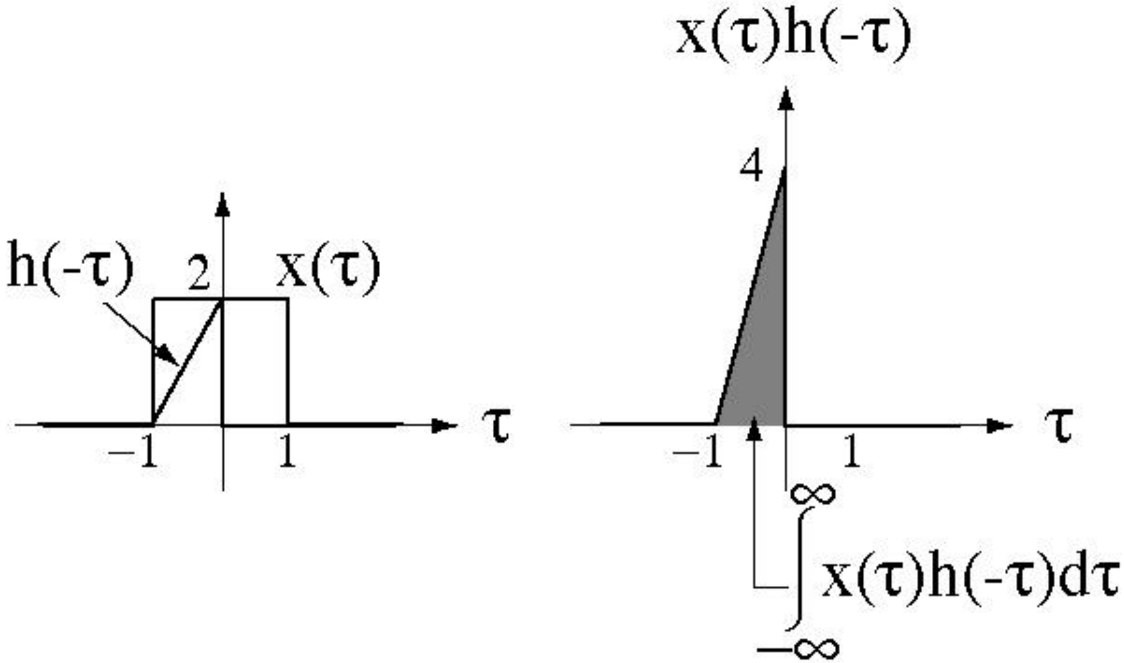
  then $y(5) = 0$.





A Graphical Illustration of the Convolution Integral - 5

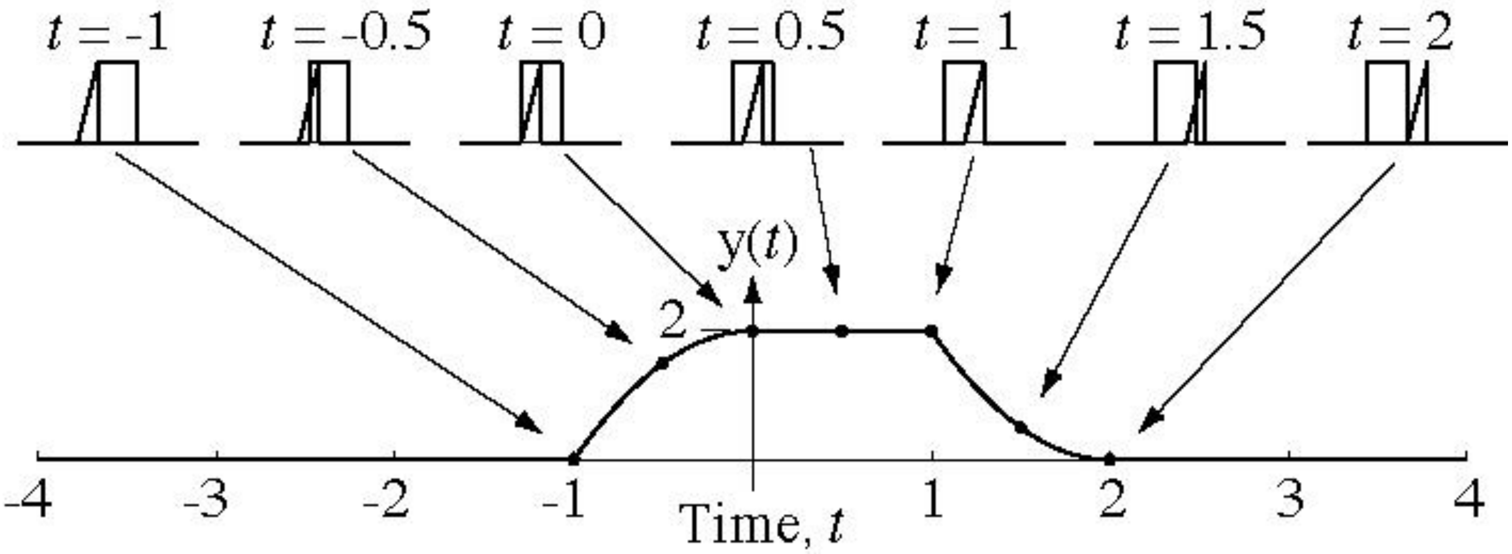
Now let $t = 0$.



Therefore $y(0) = 2$, the area under the product.

A Graphical Illustration of the Convolution Integral - 6

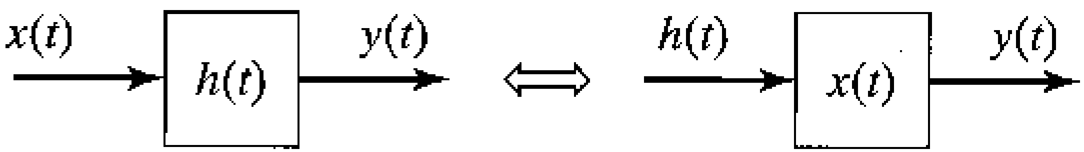
The process of convolving to find $y(t)$ is illustrated below.



Convolution Integral Properties - 1

Commutativity (可換性)

$$x(t)*y(t)=y(t)*x(t)$$



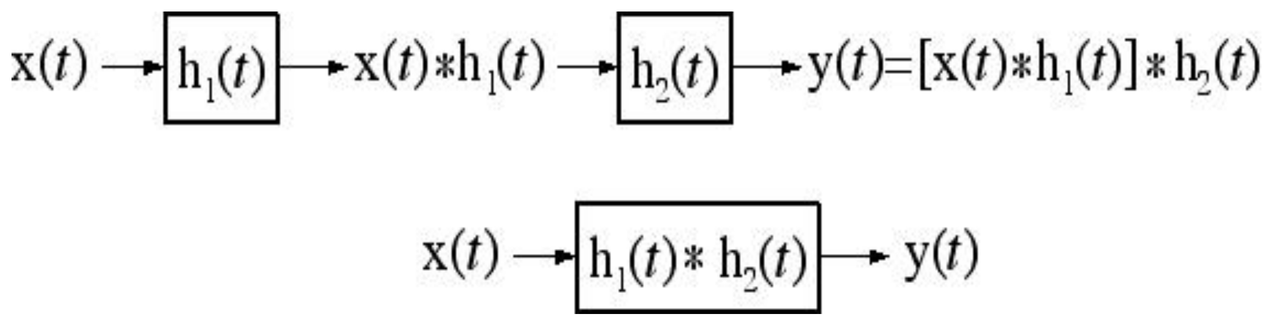
Convolution Integral Properties - 2

Associativity (結合性)

$$[x(t) * y(t)] * z(t) = x(t) * [y(t) * z(t)]$$

Cascade
Connection

カスケード接続

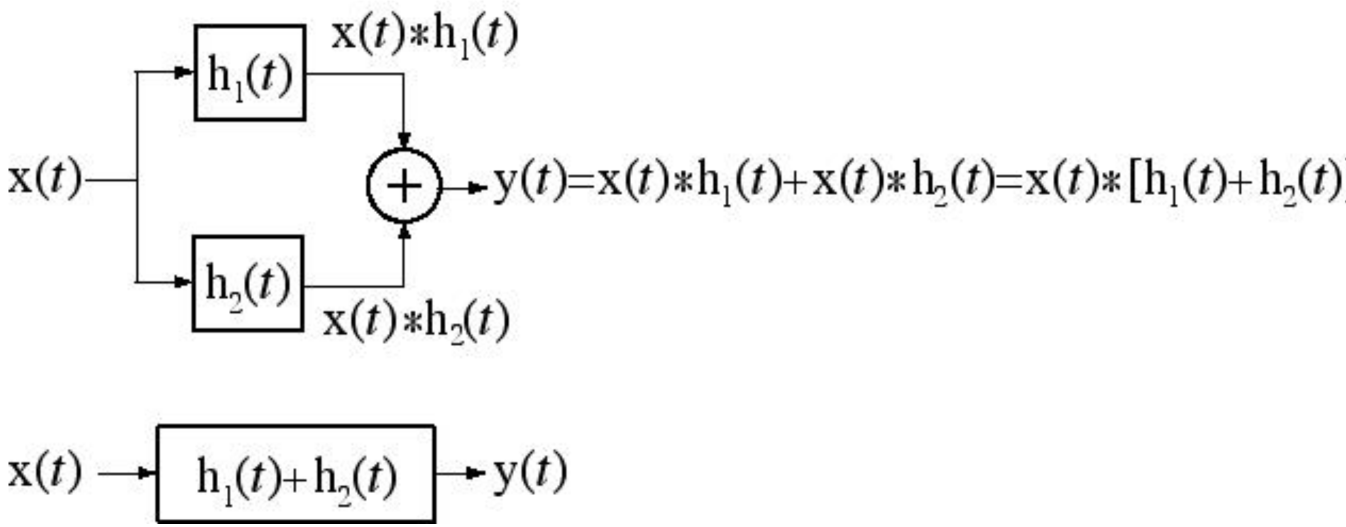


Convolution Integral Properties - 3

Distributivity (分配性)

$$[x(t)+y(t)]*z(t)=x(t)*z(t)+y(t)*z(t)$$

Parallel Connection



Stability and Impulse Response

A CT system is BIBO stable if its impulse response is absolutely integrable.

That is if $\int_{-\infty}^{\infty} |\mathrm{h}(t)| dt$ is finite.

For example, is this a stable system?

$$h(t) = e^{-3t}u(t)$$

$$y(t) = \int_{-\infty}^{\infty} |h(t)| dt < \infty$$

$$\rightarrow \int_{-\infty}^{\infty} |e^{-3t}| u(t) dt = \int_{0}^{\infty} e^{-3t} dt = 1/3$$

Unit Impulse Response and Unit Step Response

In a CT LTI system, let an input $x(t)$, the corresponding output $y(t)$. Then the input $\frac{d}{dt}(\mathrm{x}(t))$

will produce the response $\frac{d}{dt}(\mathrm{y}(t))$

For unit step response $s(t) = \mathbf{T}\{u(t)\}$

Then the unit impulse response is the first derivative of the unit step response

$$h(t) = \frac{ds(t)}{dt}$$

Remarks on Systems

- A system is represented in a *general manner*
  - Impulse response
  - Convolution
- Graphical illustration of the convolution integral
- Properties of convolution integral
- This Chapter: CT system
- Next Chapter: DT system