Math 778E Homework 3

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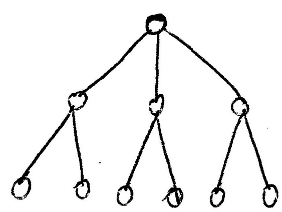
Problem 1

Proposition 1. *The Petersen graph is the unique Moore graph of degree 3 and diameter 2.*

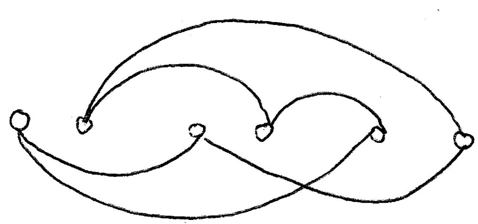
Proof. Let G be a 3-regular Moore graph of diameter 2 (and so G has girth 5). Desginating an arbitrary vertex as the root, we know that the following structure is forced by the fact that G has girth 5.



Since G has diameter 2, the remaining edges must be introduced between the leaves of this structure. Let the leaves in the previous image be labeled from left-to-right $v_1, \ldots, v_6$. We will refer to the subsets $\{v_1, v_2\}$, $\{v_3, v_4\}$, and $\{v_5, v_6\}$ as the three "cherries" of G. Now, when adding edges, we make use of the following observation.

- No vertex may have two neighbors beloning to the same cherry (else there is a cycle of length at most 4 in G).

Including edges under this constraint, we find that all remaining edges are forced (up to isomorphism).

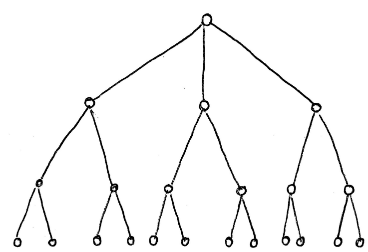


Thus, the resulting graph is unique (up to isomorphism) and is isomorphic to the Petersen graph, as desired. □

Problem 2

Proposition 2. *Every 3-regular graph with girth 7 has at least 24 vertices.*

Proof. Let G be a 3-regular graph with girth 7. Desginating an arbitrary vertex as the root, we know that the following structure is forced by the fact that G has girth 7.



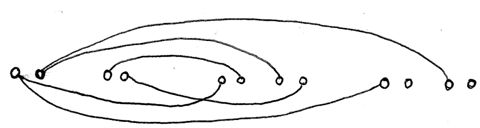
Currently, we have made use of 22 vertices. We proceed by trying to include the missing edges without introducing new vertices. We will see that in order to include the missing edges, we are forced to add an additional two vertices, thus bringing the lower bound to the desired 24 vertices.

Let the leaves in the previous image be labeled from left-to-right $v_1, \ldots, v_{12}$. We will refer to the subsets $\{v_1, \ldots, v_4\}$, $\{v_5, \ldots, v_8\}$, and $\{v_9, \ldots, v_{12}\}$ as the three "clusters" of G and the subsets $\{v_1, v_2\}$, $\{v_3, v_4\}, \ldots, \{v_{11}, v_{12}\}$ as the six "cherries" of G. Now, when adding edges, we make use of the following two observations.

- No vertex may have two neighbors beloning to the same cluster (else there is a cycle of length at most 6 in G).

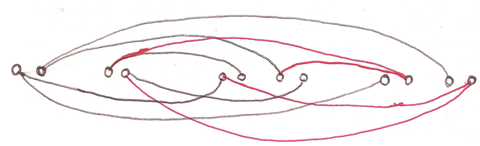
- No two vertices belonging to the same cherry may have non-ancestral neighbors belonging to a common cherry (else there is a cycle of length 6 in G).

Including edges under these constraints, we find that the following structure is forced (up to isomorphism) upon $\{v_1, \ldots, v_{12}\}$.



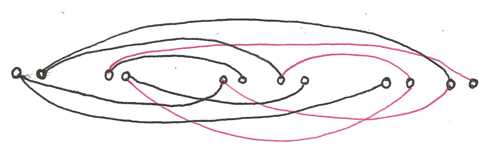
We consider now two cases: either $v_3 \sim v_{10}$ and $v_4 \sim v_{12}$ or $v_3 \sim v_{12}$ and $v_4 \sim v_{10}$.

Case 1 If $v_3 \sim v_{10}$ and $v_4 \sim v_{12}$, then the following additional structure is imposed on G.



Now, any edge leaving v_6 or v_8 induces a cycle of length less than seven, and so we must introduce a new vertex to accomodate these edges. In fact, we must introduce two new vertices, since joining v_6 and v_8 to a common new vertex will incude a cycle of length six. Thus, we have arrived at the desired minimum 24 vertices.

Case 2 If $v_3 \sim v_{12}$ and $v_4 \sim v_{10}$, then the following additional structure is imposed on G.



As before, any edge leaving v_6 or v_8 induces a cycle of length less than seven, and so we must introduce a two new vertices to accomodate these edges. Thus, we have arrived at the desired minimum 24 vertices. □

Problem 3

Proposition 3. *If G has girth at least g, $(u, v) \in E(G)$, and both x, y are distance at least $g-1$ from $\{u, v\}$, then $G-(u, v)+(x, u)+(y, v)$ has girth at least g.*

Proof. Denote $G-(u,v)+(x,u)+(y,v)$ by H and suppose, for contradiction, that $g(H) < g$. Let C be a cycle witnessing the girth of H. It must be that one of (x, u) or (y, v) belongs to C or both belong to C (otherwise, C belongs to G, contradicting the fact that $g(G) \geq g$).

Case 1 Suppose exactly one of (x, u) or (y, v) belongs to C. Without loss of generality, let it be (x, u). Since $|C| < g$, we see that there is a path of length less than $g-1$ from u to x in G, contradicting the fact that $d(x, y) \geq g-1$.

Case 2 Suppose both (x, u) and (y, v) belong to C. Observe that, beginning at the vertex u, one of two walks around C must encounter the vertices u, v, y, and x in that order. If not, then C contains a path from u to y missing both v and x. As this path is present in G, it follows that the path is length at least $g-1$, and so $|C| > g-1$, which is impossible. Now, let P be the path between u and v on C missing x and y. It follows that $uPvu$ is a cycle in G of length less than g, contradicting the fact that $g(G) \geq g$. □

Problem 4

Proposition 4. *Let the vertices of G be the lines and points on the hypersurface $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 = 0$ in the 4-dimensional projective plane over $GF(q)$ and connect a point to a line if they are incident. Prove that G is a Moore graph of degree $q+1$ and girth 8.*

Proof. Let $\mathcal{F}$ be the hypersurface $\{x \mid x^T x = 0\}$ in the 4-dimensional projective space over $GF(q)$.

Claim 1 $\mathcal{F}$ contains no plane.

Suppose π is a plane contained in $\mathcal{F}$ and let $u, v \in \pi$. It follows that $u^T u = 0$, $v^T v = 0$, and $(u+v)^T(u+v) = 0$. Hence, $u^T v = 0$. Letting V denote the linear subspace of the 5-dimensional space, we have $V \subset V^\perp$, implying that $\dim V^\perp \geq \dim V = 3$, which is a contradiction with the fact that $\dim V + \dim V^\perp = 5$.

Claim 2 There are exactly $q+1$ lines of $\mathcal{F}$ through each point p of $\mathcal{F}$. Let $\Sigma = \{x \mid p^T x = 0\}$ be the tangent hyperplane at p. Each line through p is contained in Σ, as any point v of such a line as in the previous claim satisfies $v^T p = 0$. Conversely, if $v \in \mathcal{F} \cap \Sigma$, then all points of the line space spanned by p and v belong to $\mathcal{F} \cap \Sigma$.

Let π be any plane in Σ avoiding p. It follows that $\pi \cap \mathcal{F}$ is a non-degenerate conic section, since it it contained a line, then this line together with p would give a plane contained in $\mathcal{F}$, contradicting the previous claim. Hence, $\pi \cap \mathcal{F}$ has $q + 1$ points. Now, the lines in $\mathcal{F}$ through b are exactly those lines connecting p to the points of $\pi \cap \mathcal{F}$, so there are $q + 1$ of them.

Observe also that the previous considerations together with the first claim imply that no three lines in $\mathcal{F}$ form a triangle.

Claim 3 $|\mathcal{F}| = q^3 + q^2 + q + 1$. Observe that, given any line Λ and point p on $\mathcal{F} - \Lambda$, there is a unique line on $\mathcal{F}$ containing p and meeting Λ. In fact, the tangent hyperplane Σ at p cannot contain Λ, so Σ intersects Λ in a single point a, and so the line ap is this unique line. Thus, if we consider any line Λ, any point of $\mathcal{F}$ is incident with exactly one of these lines. Hence, $|\mathcal{F} - \Lambda| = q^2(q + 1)$, proving the claim.

Since each line contains $q+1$ points and each point is incident with $q+1$ lines of $\mathcal{F}$, the number of lines of $\mathcal{F}$ is the same. let $\mathcal{L}$ denote the set of lines of $\mathcal{F}$.

Form a bipartite graph on $\mathcal{F} \cup \mathcal{L}$ by connecting $p \in \mathcal{F}$ to $\Lambda \in \mathcal{L}$ if and only if $p \in \Gamma$. The resulting bipartite graph is $(q+1)$-regular by claim 2 and has $2(q^3 + q^2 + q + 1)$ vertices by claim 3. It contains no cycle of length four or six, since a 4-cycle would correspond to two lines on $\mathcal{F}$ meeting in two points and a 6-cycle to 3 lines on $\mathcal{F}$ forming a triangle, both of which are impossible. $\square$

Problem 5

Proposition 5. *Let G be a graph with minimum degree at least 3 and girth $g \geq 3$. Prove that G contains $\frac{3}{8g} 2^{\frac{g}{2}}$ independent cycles.*

Proof. We show first that if G is a graph on n vertices such that $3 \leq d_G(v) \leq d$ for every vertex v, then we require at least $\frac{n+2}{d+1}$ vertices to represent all cycles of G. If Z represents all circuits, then $V(G) - Z$ spans a forest and hence, it spans at most $n - |Z| - 1$ edges. Since each vertex has degree at least 3, there are $3(n - |Z|)$ edges leaving the vertices of $V(G) - Z$. The edges spanned by this set are counted twice, but still we have at least

$$3(n - |Z|) - 2(n - |Z| - 1) = n - |Z| + 2$$

edges connected $V(G) - Z$ to Z. On the other hand, a vertex in Z is incident with at most d edges, hence the number of $(V(G) - Z, Z)$-edges is at most

$d|Z|$. Thus,

$$d|Z| \geq n - |Z| + 2,$$

or, equivalently,

$$|Z| \geq \frac{n+2}{d+1}.$$

We show next that, if G has girth g and $\delta(G) \geq 3$, then we need at least $\frac{3}{8}2^{\frac{g}{2}}$ vertices to represent all cycles of G. Let d denote the maximum degree of G. The same counting as in the previous argument gives

$$n \geq 1 + d(2^{\left\langle \frac{g-1}{2} \right\rangle} - 1).$$

Now, any set Z representing all cycles satisfies

$$\begin{aligned} |Z| &\geq \frac{n+2}{d+1} \\ &\geq \frac{3 + d(2^{\left\langle \frac{g-1}{2} \right\rangle} - 1}{d+1} \\ &\geq \frac{3 + 3(2^{\left\langle \frac{g-1}{2} \right\rangle} - 1}{3+1} \\ &\geq \frac{3}{8}2^{\frac{g}{2}}. \end{aligned}$$

Now, let $G_1 = G$ and define G_{i+1} be obtained from G_i by removing the vertices of a minimal cycle C_i in G_i, removing all vertices not in cycles of $G_i - V(C_i)$, and suppressing all vertices of degree 2. Now, assume the girth of each G_i is at most g (otherwise, consider G_i instead of G). Hence, $|V(C_1) \cup \cdots \cup V(G_\nu)| \leq \nu g$.

On the other hand, $V(C_1) \cup \cdots \cup V(C_\nu)$ represents all cycles of G. By the previous argument,

$$|V(C_1) \cup \cdots \cup V(C_\nu)| \geq \frac{3}{8}2^{\frac{g}{2}}.$$

Hence,

$$\nu \geq \frac{3}{8g}2^{\frac{g}{2}}.$$

□

Problem 6

Proposition 6. *Let G be a k-regular simple graph on $2k+1$ vertices. Prove that G has a Hamiltonian cycle.*

Proof. Let H be the graph obtained by adding a vertex v to G and connecting it to other vertices. Thus, H is a graph on $2k+2$ vertices with minimum degree $k+1$, and so possesses a Hamiltonian cycle by Dirac's Theorem. In G, therefore, there must be a Hamiltonian path $v_0v_1\cdots v_{2k}$. Now, it must be that, in G, if v_0 is adjacent to v_i, then v_{2k} is adjacent to v_{i-1}. As v_0 and v_{2k} are of degree k, we also have that if v_0 is not adjacent to v_i, then v_{2k} is not adjacent to v_{i-1}.

Suppose first that v_0 is adjacent to $v_1,\ldots,v_k$ and v_{2k} is adjacent to $v_k,\ldots,v_{2k-1}$. We know there is i with $1\le i\le k$ such that v_i is not adjacent to v_k. Thus, v_i is adjacent to v_j for some $k<j\le 2k-1$, as v_i has degree k in G. Hence, we have the Hamiltonian cycle $v_iv_{i-1}\cdots v_0v_{i+1}\cdots v_{j-1}v_{2k}\cdots v_jv_i$.

Now, let $1\le i\le 2k-1$ be such that v_{i+1} is adjacent to v_0 but v_i is not. By the above, v_{i-1} is adjacent to v_{2k}, and so G contains the cycle given by $v_{i-1}\cdots v_0v_{i+1}\cdots v_{2k}$ which has length $2k$.

Let now $C=u_1\cdots u_{2k}$ be any cycle of length $2k$ in G and let u_0 be the remaining vertex in G. Since C is maximal in G, u_0 cannot be adjacent to two neighboring vertices of C, so it is adjacent to every other point in C (say, $u_1,u_3,\ldots,u_{2k-1}$). Replacing u_{2i} with u_0, we get another maximal cycle, and so u_{2i} must also be adjacent to u_1, u_3, …, u_{2k-1}. It follows that u_1 is adjacent to u_0, u_2, …, u_{2k}, which is a contradiction with the fact that G is k-regular. □