Deep Learning for Computer Vision

Linear Filtering, Correlation and Convolution

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Review

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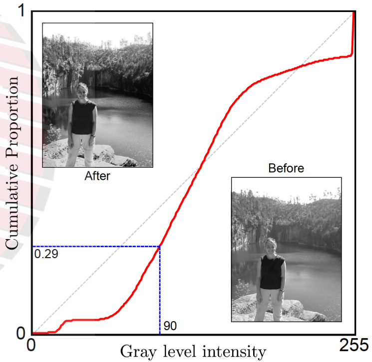
- Integrate $h(I)$ to obtain the cumulative distribution $c(I)$, whose each value $c_k = \frac{1}{M \times N} \sum_{l=1}^{k} h(l)$
- The transformed image $\hat{I}(i,j) = I_{MAX} \times c_{p_{ij}}$
- E.g., in figure, value 90 will be mapped to $I_{MAX} \times 0.29$ (rounded off)



Credit: Simon Prince, Computer Vision: Models, Learning, and Inference, Cambridge University Press

Image Filters: Linear Filter

- **Image Filter:** Modify image pixels based on some function of a local neighbourhood of each pixel

10	5	3
4	5	1
1	1	6

Some function →

	4	

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- **Linear Filter:** Replace each pixel by linear combination (a weighted sum) of neighbours
- Linear combination called kernel, mask or filter

10	5	3
4	5	1
1	1	6

Local image data

0	0	0
0	0.5	0
0	1	0.5

Kernel

	6.5	

Modified image data

Linear Filter: Cross-Correlation

Given a kernel of size $(2k+1) \times (2k+1)$:

- **Correlation** defined as:

$$G(i,j) = \underbrace{\frac{1}{(2k+1)^2}}_{\substack{\text{Uniform weight} \\ \text{to each pixel}}} \underbrace{\sum_{u=-k}^{k} \sum_{v=-k}^{k} I(i+u, j+v)}_{\substack{\text{Loop over pixels in considered} \\ \text{neighbourhood around } I(i,j)}}$$

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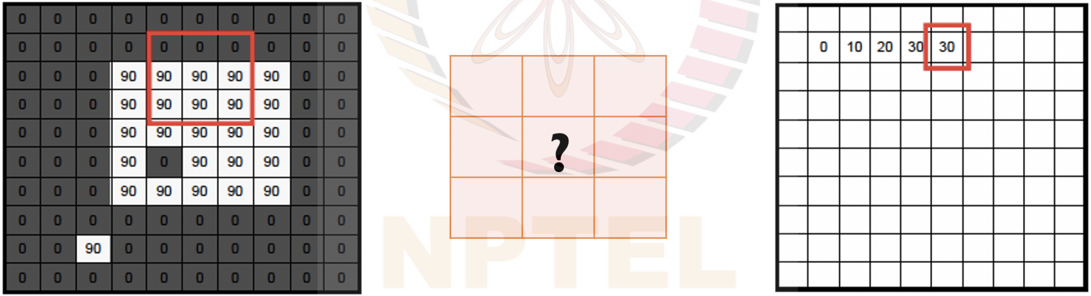
- Cross-correlation denoted by $G = H \otimes I$
- Can be viewed as "dot product" between local neighbourhood and kernel for each pixel
- Entries of kernel or mask $H(u,v)$ called filter co-efficients

Moving Average: Linear Filter

What values belong in the kernel H for the moving average example we saw earlier?

$$I(i,j) \otimes H(u,v) = G(i,j)$$



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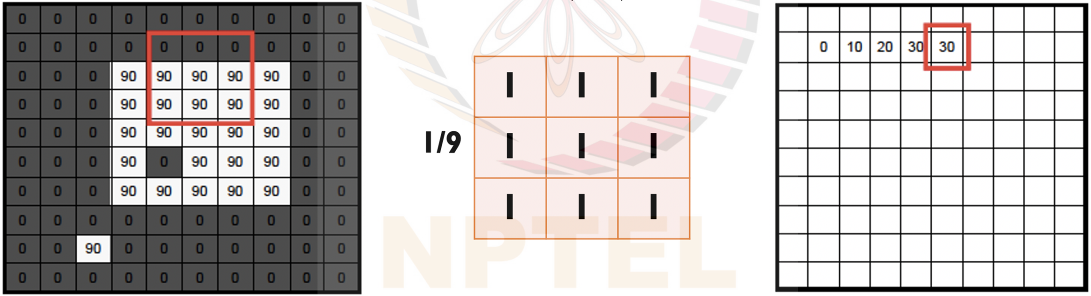

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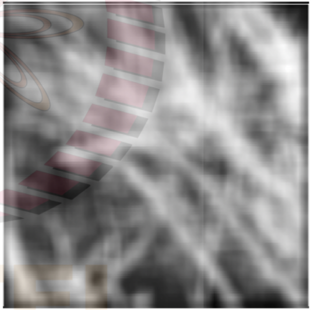
Moving Average Filter: Example

Effect of moving average filter (also known as **box filter**):



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Gaussian Average Filter

What if we want nearest neighbouring pixels to have the most influence on the output?

$I(i,j)$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
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$\otimes \quad H(u,v)$

	?	

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0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

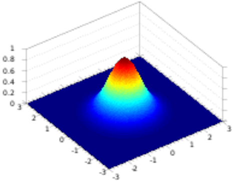
$\otimes \quad H(u,v)$

$\frac{1}{16}$

1	2	1
2	4	2
1	2	1

This kernel is an approximation of a 2D Gaussian function:

$$h(u,v) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{u^2+v^2}{\sigma^2}}$$



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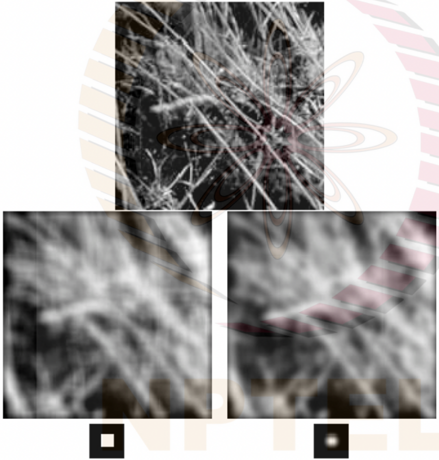
Averaging Filters: A Comparison



Box filter

Gaussian filter

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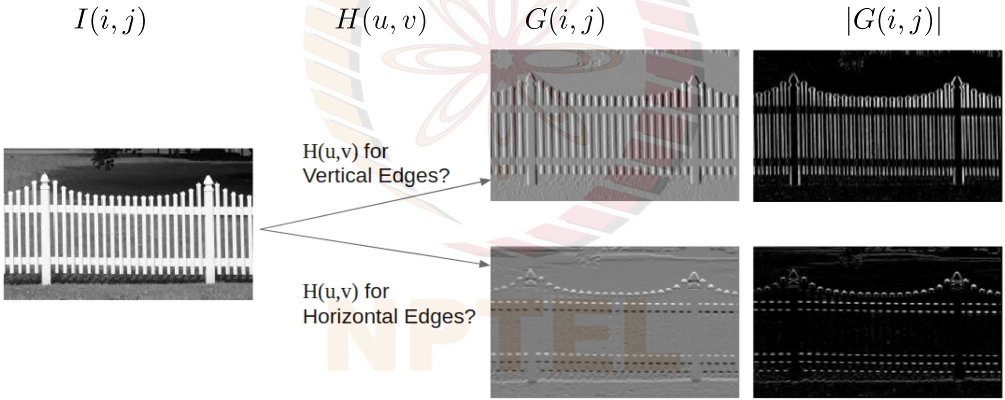
Other Filters: The Edge Filter

What should H look like to find the edges in a given image?



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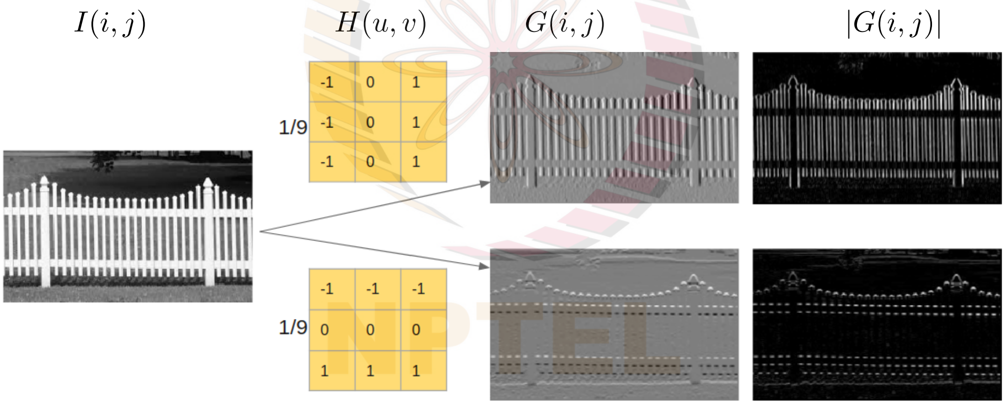

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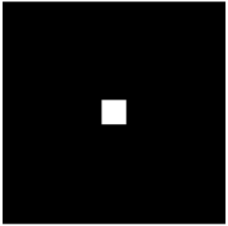
Beyond Correlation

What is the result of filtering the impulse signal (image) I with the arbitrary kernel H?

$I(i,j)$



$\otimes \quad H(u,v)$

a	b	c
d	e	f
g	h	i

$G(i,j)$

?

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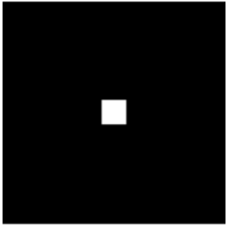
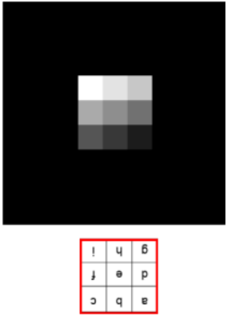


$\otimes \quad H(u,v)$

$G(i,j)$



Introducing Convolution

Given a kernel of size $(2k+1) \times (2k+1)$:

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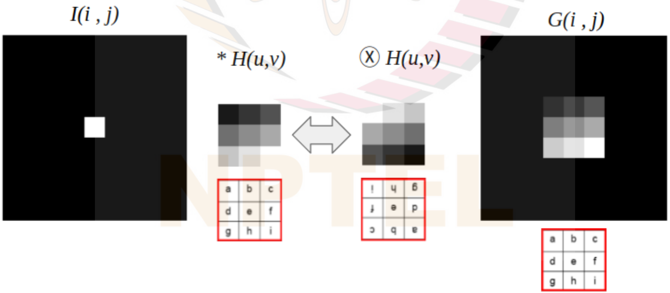
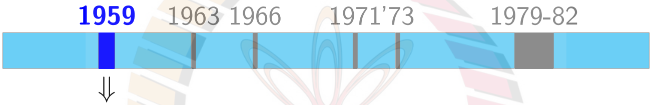
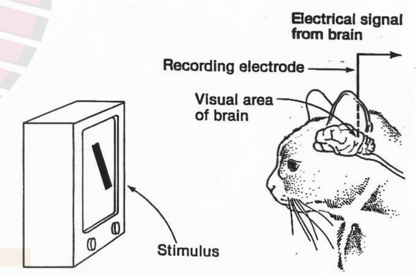


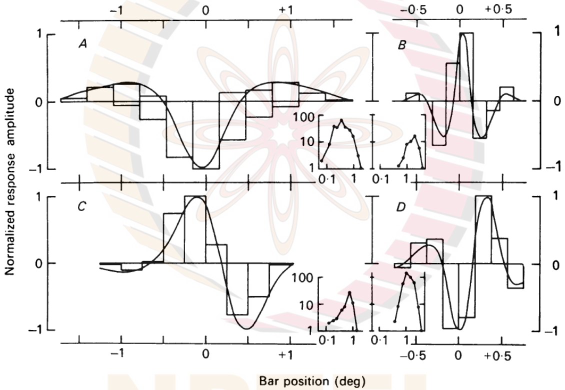
Recall: Early History



- David Hubel and Torsten Wiesel publish their work "*Receptive fields of single neurons in the cat's striate cortex*"
- Placed electrodes into primary visual cortex area of an anesthetized cat's brain
- Showed that simple and complex neurons exist, and that visual processing starts with simple structures such as oriented edges



Linear Summation in the Visual Cortex



Simple cells perform linear spatial summation over their receptive fields[1]

[1] *Movshon, Thompson and Tolhurst, Spatial Summation in the Receptive Fields of Simple Cells in the Cat's Striate Cortex, JP 1978*

Linear Shift-Invariant Operators

- Both correlation and convolution are Linear Shift-Invariant operators, which obey:
  - Linearity (or Superposition principle):

$$I \circ (h_0 + h_1) = I \circ h_0 + I \circ h_1$$

  - Shift-Invariance: shifting (or translating) a signal commutes with applying the operator

$$g(i,j) = h(i+k, j+l) \iff (f \circ g)(i,j) = (f \circ h)(i+k, j+l)$$

- Equivalent to saying that the effect of the operator is the same everywhere. Why do we need this in computer vision?

Source: Raquel Urtasun, Univ of Toronto

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Properties of Convolution

- **Commutative**: $a * b = b * a$
  - Conceptually no difference between filter and signal
- **Associative**: $a * (b * c) = (a * b) * c$
  - We often apply filters one after the other: (((a * b1) * b2) * b3)
  - This is equivalent to applying one cumulative filter: a * (b1 * b2 * b3)
- **Distributive over addition**: $a * (b + c) = (a * b) + (a * c)$
  - We can combine the responses of a signal over two or more filters by combining the filters
- **Scalars factor out**: $ka * b = a * kb = k(a * b)$
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Separable Convolution

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- Visual inspection

Separable Convolution

How can we tell if a given kernel K is separable?

- Visual inspection
- Analytically, look at the Singular Value Decomposition (SVD), and if only one singular value is non-zero, then it is separable.

$$K = U\Sigma V^T = \sum_i \sigma_i u_i v_i^T$$

where $\Sigma = \text{diag}(\sigma_i)$

$\sqrt{\sigma_1}u_1$and $\sqrt{\sigma_1}v_1$are the vertical and horizontal kernels

Source: Raquel Urtasun, Univ of Toronto

Practical Issues

Ideal size for the filter?

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Practical Issues

Ideal size for the filter?

The bigger the mask:

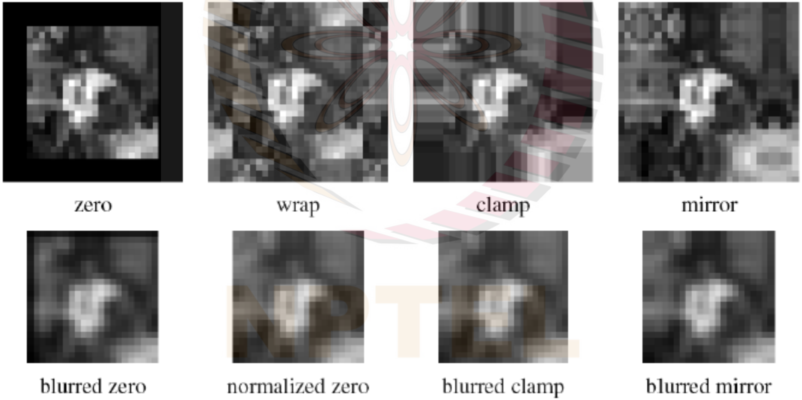
- more neighbours contribute
- smaller noise variance of output
- bigger noise spread
- more blurring
- more expensive to compute

What about the boundaries? Do we lose information?

- Without padding, we lose out on information at the boundaries.
- We can use a variety of strategies such as zero padding, wrapping around, copy the edge

Practical Issues

Different padding strategies:



Questions to Think About

- Do we then need (cross)-correlation at all?
- Are all filters always linear?

Homework

Readings

- Chapter 3 (§3.1-3.3), Szeliski, *Computer Vision: Algorithms and Applications*, 2010 draft
- Chapter 7 (§7.1-7.2), Forsyth and Ponce, *Computer Vision: A Modern Approach*, 2003 edition