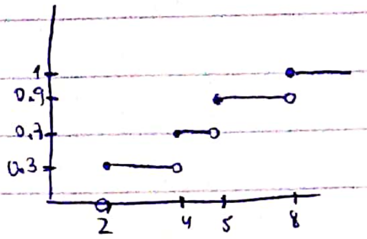
Probability HW #8

① $F(x) = \begin{cases} 0 & : x<2 \\ 0.3 & : 2 \leq x < 4 \\ 0.7 & : 4 \leq x < 5 \\ 0.9 & : 5 \leq x < 8 \\ 1 & : 8 \leq x \end{cases}$



④ $f_X(x) = \begin{cases} \alpha x^3(1-x) & : 0<x<1 \\ 0 & : \text{otherwise} \end{cases}$

ⓐ $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 cx^3(1-x)\,dx = c\left(\frac{x^4}{4} - \frac{x^5}{5}\right|_0^1 = c\left(\frac{1}{4}-\frac{1}{5}\right) = c\cdot\frac{1}{20} = 1 \Rightarrow c = 20$

ⓑ $\int_{-\infty}^{x} f(t)\,dt = \int_0^x 20t^3(1-t)\,dt = 20\left(\frac{t^4}{4} - \frac{t^5}{5}\right|_0^x = 20\left(\frac{x^4}{4} - \frac{x^5}{5}\right) = 5x^4 - 4x^5$

$F(x) = \begin{cases} 0 & ; x<0 \\ 5x^4 - 4x^5 & ; 0<x<1 \\ 1 & ; x>1 \end{cases}$

ⓒ $P(X<\frac{2}{3}) = F(\frac{2}{3}) = 5(\frac{2}{3})^4 - 4(\frac{2}{3})^5 = 5(\frac{16}{81}) - 4(\frac{32}{243}) = \frac{240-128}{243} = \frac{112}{243}$

ⓓ let Y = profit of a gallon

$P(Y=5) = F(\frac{2}{3}) - F(\frac{1}{3}) = \frac{112}{243} - \left(5(\frac{1}{3})^4 - 4(\frac{1}{3})^5\right) = \frac{112}{243} - \frac{11}{243} = \frac{101}{243}$

$P(Y=15) = 1 - P(Y=5) = \frac{243-101}{243} = \frac{142}{243}$

k	$P(Y=k)$
5	$\frac{101}{243}$
15	$\frac{142}{243}$

$E[Y] = 5\cdot\frac{101}{243} + 15\cdot\frac{142}{243} = \frac{2635}{243}$ $\quad E[Y^2] = 25\cdot\frac{101}{243} + 225\cdot\frac{142}{243} = \frac{34475}{243}$

$V[Y] = \frac{34435}{243} - \left(\frac{2635}{243}\right)^2 = 24.288$

⑦ ⓐ Area $= \frac{1}{2}bh = \frac{3}{2}h = 1 \Rightarrow h = \frac{2}{3}$. Find lines equations: $y - 0 = \frac{\frac{2}{3}-0}{1-0}(x-0) \to y = \frac{2}{3}x$

$y - 0 = \frac{\frac{2}{3}-0}{1-3}(x-3) \to y = 1 - \frac{x}{3}$

$f(x) = \begin{cases} \frac{2}{3}x & ; 0 \leq x < 1 \\ 1 - \frac{x}{3} & ; 1 \leq x \leq 3 \\ 0 & ; \text{otherwise} \end{cases}$ $\qquad F(x) = \begin{cases} 0 & ; x<0 \\ \frac{x^2}{3} & ; 0 \leq x < 1 \\ x - \frac{x^2}{6} - \frac{1}{2} & ; 1 \leq x \leq 3 \\ 1 & ; x>3 \end{cases}$

$\int_{-\infty}^{x} \frac{2}{3}t\,dt = \int_0^x \frac{2}{3}t\,dt = \left(\frac{2}{3}\frac{t^2}{2}\right|_0^x = \frac{x^2}{3}$

$\int_{-\infty}^{x} 1 - \frac{t}{3}\,dt = \int_1^x 1 - \frac{t}{3}\,dt = \left(t - \frac{t^2}{6}\right|_1^x = x - \frac{x^2}{6} - \left(1 - \frac{1}{6}\right) = x - \frac{x^2}{6} - \frac{5}{6}$

ⓑ $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(\frac{2}{3}x)\,dx + \int_1^3 x(1-\frac{x}{3})\,dx = \int_0^1 \frac{2}{3}x^2\,dx + \int_1^3 x - \frac{x^2}{3}\,dx$

$= \left(\frac{2x^3}{9}\right|_0^1 + \left(\frac{x^2}{2} - \frac{x^3}{9}\right|_1^3 = \frac{2}{9} + \left[\left(\frac{9}{2} - \frac{27}{9}\right) - \left(\frac{1}{2} - \frac{1}{9}\right)\right] = \frac{2}{9} + \left(\frac{8}{2} - \frac{7}{18}\right) = \frac{4}{3}$

ⓒ $P(0.25 < X < 1.5) = F(1.5) - F(0.25) = \frac{5}{8} - \frac{1}{48} = \frac{29}{48}$

⑨ $f(x) = \begin{cases} \frac{a}{x^3} & ; x \geq 1 \\ 0 & ; \text{otherwise} \end{cases}$

ⓐ $\int_1^{\infty} \frac{a}{x^3}\,dx = 1 \Rightarrow a \cdot \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx = a\left(\lim_{b\to\infty} \frac{-1}{2x^2}\right|_1^b = -\frac{a}{2}\left(\lim_{b\to\infty} \frac{1}{x^2}\right|_1^b = \frac{-a}{2}(0-1)$

$\frac{a}{2} = 1 \Rightarrow a = 2$

(b) $F(x) = \begin{cases} 1 - \frac{1}{x^2} & ; x \geq 1 \\ 0 & ; \text{otherwise} \end{cases}$

$\int_1^x \frac{2}{t^3}\, dt = \left(\frac{-1}{t^2}\right|_1^x = \frac{-1}{x^2} + 1$

(c) $M = \int_1^\infty x \cdot \frac{2}{x^3}\, dx = \int_1^\infty \frac{2}{x^2}\, dx = \left(\frac{-2}{x}\right|_1^\infty = \lim_{b\to\infty} -\frac{2}{b} + 2 = 2$

$E[x^2] = \int_1^\infty \frac{2}{x}\, dx = 2\left(\lim_{b\to\infty} \ln|x|\Big|_1^b = 2\left(\lim_{b\to\infty} b - 0\right) = \infty\right.$

$\sigma^2 = \infty - 2^2 = \infty$