Calculas HW #5

30?

(1) (4) $f(x) = x^2 - 3|x|$

(a) all x

(b) $0 = x^2 - 3|x|$ $= \begin{cases} x<0 & x^2+3x \\ x \geq 0 & x^2-3x \end{cases}$

$0 = x^2 - 3x$

$0 = x(x-3)$

$x=0 \quad x=3$ $\quad (0,0) \; (3,0)$

$(-3,0)$

$0 = x^2+3x$

$0 = x(x+3)$

$x = -3$

Function is positive $x<-3$, $3<x$

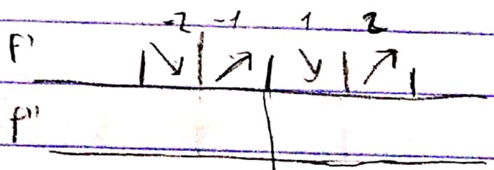
Function is negative $-3<x<0$, $0<x<3$

(c) $f'(x) = \begin{cases} x<0 & 2x+3 \\ x\geq 0 & 2x-3 \end{cases}$

$\lim_{x \to 0^+} x^2-3x = 0$

$\lim_{x \to 0^-} x^2+3x = 0$

they are equal so its continous



$2x+3 = 0 \Rightarrow -\frac{3}{2}$

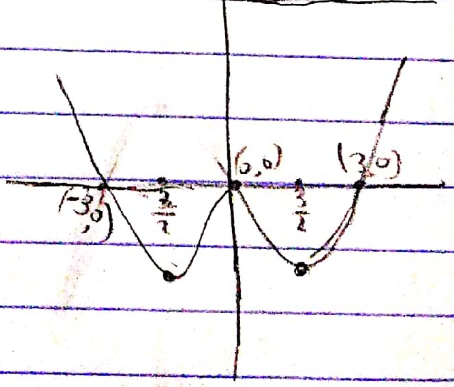
$2x-3 = 0 \Rightarrow +\frac{3}{2}$

$f(+\frac{3}{2}) = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}$

$f(-\frac{3}{2}) = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}$

min $(-1.5, -\frac{9}{4})$

min $(1.5, -\frac{9}{4})$



increasing $-\frac{3}{2}<x<0$, $\frac{3}{2}<x$

decreasing $0<x<\frac{3}{2}$, $x<-\frac{3}{2}$

(d) $f''(x) = \begin{cases} x<0 & 2 \\ x\geq 0 & 2 \end{cases}$ $\quad \begin{matrix} 2=0 \\ 2=0 \end{matrix}$ $\Big\}$ no inflection points

(e) no vertical asymptote because this function is continuous

horizontal:
$$\lim_{x\to+\infty} x^2-3x = x^2\left(1-\frac{3}{x}\right) = \infty$$
($x^2 \to +\infty$, $1 \to 1$, $\frac{3}{x} \to 0$)
$$\lim_{x\to-\infty} x^2+3x = x^2\left(1+\frac{3}{x}\right) = -\infty$$
($x^2 \to -\infty$, $1 \to 1$, $\frac{3}{x} = 0$)

no horizontal asymptote

slant?
$$\lim_{x\to+\infty} \frac{x^2-3x}{x} = \frac{x(x-3)}{x} = x-3 = \infty$$
$$\lim_{x\to-\infty} \frac{x^2+3x}{x} = \frac{x(x+3)}{x} = -\infty$$

no slant asymptote

(f) continuous:
$$\left.\begin{array}{l}\lim_{x\to0^+} x^2-3x = 0 \\ \lim_{x\to0^-} x^2+3x = 0\end{array}\right\}$$ equal therefore f is continuous on 0

f is continuous for all x

derivative:
$$f'(0) = \lim_{x\to0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0} \frac{x^2-3|x|}{x}$$
$$f'(0)^+ = \lim_{x\to0^+} \frac{x^2-3x}{x} = \lim_{x\to0^+} x-3 = -3$$
$$f'(0)^- = \lim_{x\to0} \frac{x^2+3x}{x} = \lim_{x\to0^-} x+3 = 3$$

(12) $f(x) = \dfrac{e^x-1}{e^x-2}$

(a) $e^x \neq 2$

$x \neq \ln(2)$

(b) $0 = \dfrac{e^x-1}{e^x-2} \to 0 = e^x-1$

$1 = e^x$

$\ln(1) = x$

$0 = x$

$(0,0) \to$ zero point

$x: \quad -1 \quad \overset{2\cdot 6\cdot e^x}{0} \quad 1$

$f(x) \quad 0.38 \mid 2.58$

pos pos

positive : $(-\infty,0) \cap (0,\infty)$

negative : none

ⓒ $f(x) = \dfrac{e^x - 1}{e^x - 2}$

$$f'(x) = \frac{(e^x-2)\cdot e^x - ((e^x-1)\cdot e^x)}{(e^x-2)^2} = \frac{e^{2x} - 2e^x - (e^{2x} - e^x)}{(e^x-2)^2}$$

$$\frac{e^{2x} - 2e^x - e^{2x} + e^x}{(e^x-2)^2} = \frac{-e^x}{(e^x-2)^2}$$

$$0 = \frac{-e^x}{(e^x-2)^2} \Rightarrow 0 = -e^x$$

$\Downarrow$

no solution

no critical points

for all x $\quad 0 > f'(x)$

ascent : for all x

descent : none

ⓓ $$f''(x) = \frac{-e^x\cdot(e^x-2)^2 - 2(e^x-2)\cdot e^x \cdot -e^x}{(e^x-2)^4} = \frac{-e^x(e^x-2)[e^x - 2 - 2e^x]}{(e^x-2)^4}$$

$$= \frac{-e^x(e^x-2)(-e^x-2)}{(e^x-2)^{4\,3}} = \frac{e^x(e^x+2)}{(e^x-2)^3}$$

convex : $(\ln 2, \infty)$

concave : $(-\infty, \ln 2)$

x	$-\infty$	$\ln 2$	∞
$f''(x)$	$-$		$+$

ⓔ vertical asymptotes : $x = \ln 2$

horizontal asymptote : $\lim\limits_{x\to\infty} \dfrac{e^x-1}{e^x-2} = \dfrac{e^x-2+1}{e^x-2} = 1 + \dfrac{1}{e^x-2} = 1 + \dfrac{1}{\infty} = 1$

$\boxed{y=1}$

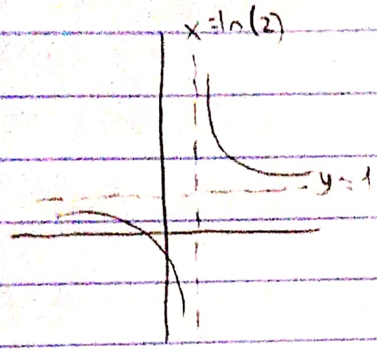
$\lim\limits_{x\to-\infty} \dfrac{e^x-1}{e^x-2} = \dfrac{e^x-2+1}{e^x-2} = 1 + \dfrac{1}{e^x-2} = 1 + \dfrac{1}{-\infty} = 1$

slant asymptote: none

(f) f is not continous at $x=\ln(2)$ because there is a vertical asymptote

differentiability: f is differentiable everywhere except at $x=\ln(2)$

even/odd: this function is odd $-f(x)=f(-x)$



(16) $f(x)=\ln\left|\frac{x-1}{x+1}\right|$ (a) Domain: all $\mathbb{R}-\{1,-1\}$

$\lim_{x\to1^-}\ln\left|\frac{x-1}{x+1}\right|=\ln\left|\frac{-0}{2}\right|=-\infty$

$\lim_{x\to1^+}\ln\left|\frac{x-1}{x+1}\right|=\ln\left|\frac{0}{2}\right|=-\infty$

vertical asymptote at $x=1$

$\lim_{x\to-1^-}\ln\left|\frac{x-1}{x+1}\right|=\ln\frac{-2}{-0}=\infty$

$\lim_{x\to-1^+}\ln\left|\frac{x-1}{x+1}\right|=\ln\left|\frac{-2}{0}\right|=-\infty$

vertical asymptote at $x=-1$ type II

$\lim_{x\to\pm\infty}\ln\left|\frac{x-1}{x+1}\right|=0$ so $y=0$ (x axis)

zeros: $x=0$

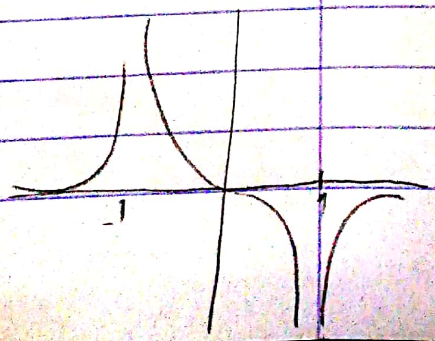
extreme pts: f(x) + -1 + 0 - 1 -

$f'(x)=\frac{1}{\frac{x-1}{x+1}}\cdot\frac{(x+1)-(x-1)}{(x+1)^2}=\frac{x+1}{x-1}\cdot\frac{2}{(x+1)^2}=\frac{2}{(x-1)(x+1)}$ no zero

ascent: $(-\infty,1)\cup(1,\infty)$

decend: $(-1,1)$

Convexity: $f''(x)=\frac{-2x\cdot2}{(x^2-1)^2}=\frac{-4x}{(x^2-1)^2}$



$\frac{-4x}{(x^2-1)^2}=0\Rightarrow-4x=0\Rightarrow x=0$

$x=0$ f(x) + -1 + 0 - 1 -

Convex: $(-\infty,0)$ concave: $(0,\infty)$

local discont. f is cont, for all x, when x = 1, -1 it is discont.

differentiability f is differentiable for all x except x = 1, -1

Odd $-f(x) \overset{?}{=} f(-x)$ ✓

$-\ln\left|\frac{x-1}{x+1}\right| = \ln\left|\frac{-x-1}{-x+1}\right|$ ✓

$-\log_e\left|\frac{x-1}{x+1}\right| = \log_e\left|\frac{x+1}{x-1}\right|$

$\log_e\left|\frac{x-1}{x+1}\right|^{-1} = \log_e\left|\frac{x+1}{x-1}\right|$

$\log_e\left|\frac{x+1}{x-1}\right| = \log_e\left|\frac{x+1}{x-1}\right|$

(18) $f(x) = x \cdot \sqrt{2x+1}$

a) $x \in [-\frac{1}{2}, \infty) \Leftarrow -\frac{1}{2} \le x \Leftarrow -1 \le 2x \Leftarrow 0 \le 2x+1$

b) $x \cdot \sqrt{2x+1} = 0$

$x = 0$ $\qquad \sqrt{2x+1} = 0 \Rightarrow 2x+1 = 0 \Rightarrow x = -\frac{1}{2}$

Zero pts $(-\frac{1}{2}, 0)$ $(0,0)$

$f(x)$	$-\frac{1}{2}$		0	
		$-$		$+$

pos $(0, \infty)$
neg $(-\frac{1}{2}, 0)$

c) $f'(x) = \sqrt{2x+1} + \frac{2}{2\sqrt{2x+1}} \cdot x = \sqrt{2x+1} + \frac{x}{2x+1}$

$\sqrt{2x+1} + \frac{x}{\sqrt{2x+1}} = 0 \Rightarrow \frac{x}{\sqrt{2x+1}} = -\sqrt{2x+1} \Rightarrow x = -(2x+1)$

$\Rightarrow x = -2x-1 \Rightarrow 3x = -1 \Rightarrow x = -\frac{1}{3}$

x	$-\frac{1}{2}$		$-\frac{1}{3}$	
$f'(x)$		$-$		$+$
$f(x)$		$\searrow$		$\nearrow$

ascend: $(-\frac{1}{3}, \infty)$
decend: $[-\frac{1}{2}, -\frac{1}{3}]$

$f(-\frac{1}{3}) = -\frac{\sqrt{3}}{9}$ $\qquad$ min $(-\frac{1}{3}, -\frac{\sqrt{3}}{9})$

d) $f''(x) = \frac{2}{2\sqrt{2x+1}} + \frac{\sqrt{2x+1} - \frac{2}{2\sqrt{2x+1}} \cdot x}{(\sqrt{2x+1})^2} = \frac{1}{\sqrt{2x+1}} + \frac{\sqrt{2x+1} - \frac{x}{\sqrt{2x+1}}}{2x+1}$

$= \frac{1}{\sqrt{2x+1}} + \frac{\frac{2x+1-x}{\sqrt{2x+1}}}{2x+1} = \frac{1}{\sqrt{2x+1}} + \frac{x+1}{(2x+1)\sqrt{2x+1}} = \frac{2x+1+x+1}{(2x+1)\sqrt{2x+1}}$

$= \frac{3x+2}{(2x+1)\sqrt{2x+1}}$

$\frac{3x+2}{(2x+1)\sqrt{2x+1}} = 0 \Rightarrow 3x+2 = 0 \Rightarrow 3x = -2 \Rightarrow x = -\frac{2}{3}$

x	$-\frac{1}{2}$	0
$f''(x)$		$+$
$f(x)$		$\cup$

inflection pt: none

convex: $[-\frac{1}{2}, \infty)$

concave: none

ⓔ asymptote vertical $\lim_{x \to \infty} x\sqrt{2x+1} = "\infty \cdot \infty" = \infty$ None

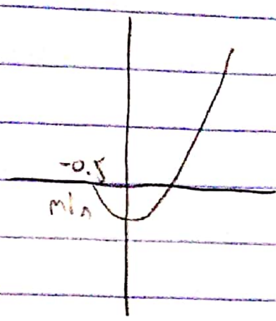
asymp horizontal $\lim_{x \to \infty} \frac{x\sqrt{2x+1}}{x} = \lim_{x \to \infty} \sqrt{2x+1} = \infty$ None

slant? none

ⓕ local discont: f is continuous for $x \in [-\frac{1}{2}, \infty)$

differentiability: f is diff for $x \in [-\frac{1}{2}, \infty)$

f is not odd or even



(23) $f(x) = x - \arctan(x)$

ⓐ Domain: all x

ⓑ $x - \arctan(x) = 0 \Rightarrow x = \arctan(x) \Rightarrow x = 0$

x	$-\infty$	-1	0	1	∞
f(x)		$-$	|	$+$	

Zero pt: (0,0)

pos $(0, \infty)$

neg $(-\infty, 0)$

ⓒ $f'(x) = 1 - \frac{1}{1+x^2}$

$1 - \frac{1}{1+x^2} = 0 \Rightarrow 1 = \frac{1}{1+x^2} \Rightarrow 1 = 1 + x^2 \Rightarrow x^2 = 0 \Rightarrow x = 0$

x	-1	0	1
f'(x)	$+$	|	$+$
f(x)	↗	|	↗

inflection pt: none

ascend: $(-\infty, 0) \cup (0, \infty)$

descend: none

ⓓ $f''(x) = \frac{-(-2x \cdot 1)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$ $\frac{2x}{(1+x^2)^2} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

x	-1	0	1
f''(x)	$-$	|	$+$
f(x)	∩	|	∪

inflection pt: $(0,0)$

convex: $(0,\infty)$

concave: $(-\infty,0)$

(e) asymp vertical: none

$$\lim_{x\to\infty} x-\arctan(x) = \infty - \frac{\pi}{2} = \infty$$

$$\lim_{x\to-\infty} x-\arctan(x) = -\infty-\left(-\frac{\pi}{2}\right) = -\infty$$

horizontal: none

$$\lim_{x\to\pm\infty} \frac{x-\arctan(x)}{x} = \lim_{x\to\pm\infty} \frac{x-\frac{\cos(x)}{\sin(x)}}{x} = \frac{x\sin(x)-\cos(x)}{x\sin(x)}$$

$$= 1-\frac{\cos(x)}{x\sin(x)} = \lim_{x\to\pm\infty} 1-\frac{1}{x}\cdot\arctan(x) = 1$$

$$\lim_{x\to\infty} x-\arctan(x)-x = \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} x-\arctan(x)-x = \lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$$

slant: $y = x+\frac{\pi}{2}$

$y = x-\frac{\pi}{2}$

(f) local diff: continuous for all x

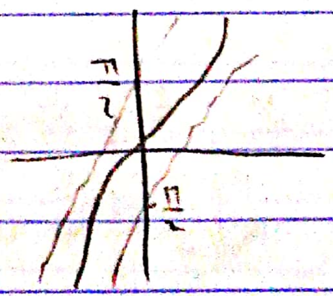
differentiability: diff for all x

odd $\quad -f(x) \stackrel{?}{=} f(-x)$

$$-(x-\arctan(x)) = -x-\arctan(-x)$$

$$-x+\arctan(x) = -x-\frac{\sin(-x)}{\cos(-x)}$$

$$-x+\arctan(x) = -x-\frac{-\sin(x)}{\cos(x)} \Rightarrow -x+\arctan(x) = -x+\arctan(x) \checkmark$$



② ⓑ $[0,3]$ $f(x) = 2x^3 - 9x^2 + 12x - 2$

D: all x

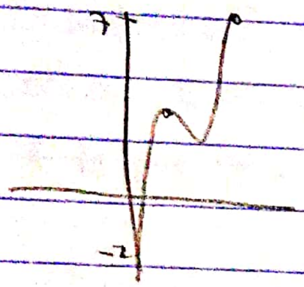
$f'(x) = 6x^2 - 18x + 12 = 0 \Rightarrow 6(x^2 - 3x + 2) = 0 \Rightarrow 6(x-2)(x-1) = 0$

$x = 1,\ x = 2$

x	0	1	$1\frac{1}{2}$	2	3
$f'(x)$	+		−		+
$f(x)$	↗		↘		↗

Max $(1,3)$

min $(2,2)$



$f(0) = 2$

$f(3) = 7$

abs max $(3,7)$

abs min $(0,-2)$

ⓓ (i) $f(x) = x - \ln x$

D: $x \in (0, \infty)$

$f'(x) = 1 - \frac{1}{x}$

$1 - \frac{1}{x} = 0 \Rightarrow 1 = \frac{1}{x} \Rightarrow x = 1$

$[0.5, 4]$

x	0	$\frac{1}{2}$	1	4
$f'(x)$		−		+
$f(x)$		↘		↗

Min $(1,1)$

ⓐ $f(\frac{1}{2}) = 1.19$ abs min $(1,1)$ $[\frac{1}{2}, 4]$

$f(4) = 2.6$ abs max $(4, 2.6)$

ⓑ $f(-2) =$ not in Domain

$f(4) = 2.6$

$[-2, 4]$

abs min $(1,1)$

abs max $(4, 2.6)$

(ii) $f(x) = x - \ln|x|$

D: $x \neq 0$

$f'(x) = 1 - \frac{1}{x}$

$1 - \frac{1}{x} = 0 \Rightarrow 1 = \frac{1}{x} \Rightarrow x = 1$

x	-2	0	$\frac{1}{2}$	1	4
$f'(x)$	+		−		+
$f(x)$	↗		↘		↗

min$(1,1)$

ⓐ $f(\frac{1}{2}) = 1.19$

$f(4) = 2.6$ abs min $(1,1)$ $[\frac{1}{2}, 4]$

abs max $(4, 2.6)$

ⓑ $f(-2) = -2.7$ abs min $(-2, -2.7)$

$f(4) = 2.6$ abs max $(4, 2.6)$

③ ⓐ $\arctan x - e^{-x} = 3$

$f(x) = \arctan x - e^{-x} - 3 = 0$ (Monotonic domain)

D: all x $f'(x) = \frac{1}{1+x^2} + e^{-x}$

f is pos for all x

$\frac{1}{1+x^2} + e^{-x} = 0 \Rightarrow$ no solution

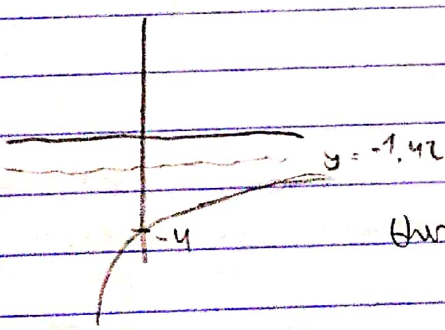
critical pt: none

asympt vertical: non

$\lim_{x \to \infty} \arctan x - e^{-x} - 3 = \frac{\pi}{2} - 3 = -1.42$

$\lim_{x \to -\infty} \arctan x - e^{-x} - 3 = -\infty$

$f(0) = -4$



there is no solution

ⓒ $1 + 8\ln x = 6x + x^2$

$f(x) = 1 + 8\ln x - 6x - x^2 = 0$

D: $x > 0$

$f'(x) = \frac{8}{x} - 6 - 2x$

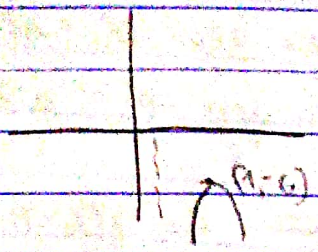
$\frac{8}{x} - 6 - 2x = 0 \Rightarrow 8 - 6x - 2x^2 = 0 \Rightarrow 2(x^2 + 3x - 4) = 0 \Rightarrow 2(x-1)(x+4)$

$\Rightarrow x = 1, x = -4$ ← not in domain.

x	0	$\frac{1}{2}$	1	2
f'(x)		+		−
f(x)		↗		↘

critical pt: $(1, -6)$

267



no solution to equation

(c) $1+8\ln|x| = 6x+x^2$

$f(x) = 1+8\ln|x|-6x-x^2=0$ $\quad$ D: $x\neq 0$

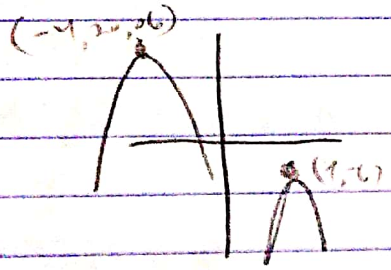
$f'(x) = \frac{8}{x}-6-2x$

$\frac{8}{x}-6-2x=0 \Rightarrow x=1 \quad x=-4$

x	$-\infty$	-4	0	1	
$f'(x)$	+	−	+	−	
$f(x)$	↗	↘	↗	↘	

critical pt: Max $(1,-6)$

Min $(-4,20,09)$



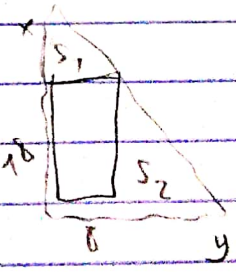
for all $x\in\mathbb{R}/\{0\}$

equation has 2 solution

(4) (b) $S_{\square} = 18\cdot 8 = 144$

$S_{\triangle} = \frac{(18+x)(8+y)}{2} = \frac{144+8x+18y+xy}{2}$

$= 72+4x+9y+\frac{xy}{2}$



$S_1 = \frac{8\cdot x}{2} = 4x$

$S_2 = \frac{18\cdot y}{2} = 9y$

$S\triangle = S\square + S_1 + S_2$

⇓

$72+4x+9y+\frac{xy}{2} = 144+4x+9y \Rightarrow \frac{xy}{2} = 72 \Rightarrow xy=144 \Rightarrow x=\frac{144}{y}$

D: $y\neq 0$

$S\triangle = 72+4\cdot\frac{144}{y} + \frac{144}{y}\cdot\frac{y}{2} + 9y$

⇓

$SA = 72 + \frac{576}{y} + 72 + 9y = 9y + \frac{576}{y} + 144$

$S'A = 9 - \frac{576}{y^2}$

$9 - \frac{576}{y^2} = 0 \Rightarrow 9y^2 - 576 = 0 \Rightarrow 9(y-8)(y+8) = 0$

$y = 8, y = -8$

$\Downarrow$

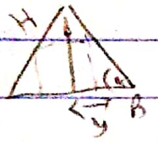
$y = 8$

$x = \frac{144}{8} = 18$

$18 + 18 = \boxed{36}$

$8 + 8 = \boxed{16}$

(d)



v = capacity

$V = x \cdot \pi y^2$

$\tan\alpha = \frac{H}{R} = \frac{x}{R-y} \Rightarrow x = \frac{H}{R} \cdot (R-y)$

$\Downarrow$

$V = \frac{H\pi}{R}(R-y) \cdot y^2$

constant

$f(y) = Ry^2 - y^3$

$f'(y) = 2Ry - 3y^2$

$2Ry - 3Ry^2 = 0 \Rightarrow y(2R - 3y)$

$y = B \quad y = \frac{2R}{3}$

$V_{max} = \frac{H}{R}(R-y) \cdot \pi \cdot \left(\frac{2R}{3}\right)^2$

(f) point of for tangent points are where the slopes are the same

$y = x^2 \quad y' = 2x \quad y = \frac{-3}{4} - 3 \quad$ - slope $= \frac{-3}{4}$

$\frac{-3}{4} = 2x \Rightarrow -3 = 8x \Rightarrow \frac{-3}{8} = x$ plug in $\left(\frac{-3}{8}, \frac{9}{64}\right)$

10. a) $f(x) = \sqrt{x}$

$\sqrt{65}$

$f(x_0+\Delta x) \approx \Delta x \cdot f'(x_0) + f(x_0)$

for Δx small $f(x) = \sqrt{x}$ $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$x_0 = 64$ $f(x_0) = 8$ $\Delta x = 1$ $f'(x_0) = \frac{1}{2}(64)^{-\frac{1}{2}} = \frac{1}{16}$

$f(65) \approx 8 + (1) \cdot \frac{1}{16} = 8.0625$

b) $\sqrt[3]{26}$

$f(x) = \sqrt[3]{x}$ $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$x_0 = 27$ $f(x_0) = 3$ $\Delta x = -1$ $f'(x_0) = \frac{1}{3}(27)^{-\frac{2}{3}} = \frac{1}{27}$

$f(26) \approx 3 + (-1) \cdot \frac{1}{27} = 2.962962$

9) d) $y = -2x+3$ asymp of $f(x) = \frac{ax^2+bx+c}{x+3}$

$y = 3x+7$ intersects $f(x)$ $x=-1$

$\lim_{x\to\infty} \frac{ax^2+bx+c}{x^2+3x} = -2 \Rightarrow a = -2$

$\lim_{x\to\infty} \frac{ax^2+bx+c}{x+3} + 2x = 3 \Rightarrow \lim_{x\to\infty} \frac{(b+6)x+c}{x+3} = 3 \Rightarrow b = -3$

$-3+7 = \frac{-2-3+c}{4} \Rightarrow -4 = -3+c \Rightarrow c = 1$

$$\boxed{\begin{aligned} a &= -2 \\ b &= -3 \\ c &= 1 \end{aligned}}$$

13) b) 2) $f(x) = \sin x$ $[\frac{\pi}{2}, \pi]$

$f'(c) = \frac{\sin(a) - \sin(b)}{a-b} = \frac{\sin(\pi) - \sin(\frac{\pi}{2})}{\pi - \frac{\pi}{2}}$ $\frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

$f'(c) = \cos c$

$\cos c = \frac{2}{\pi} \Rightarrow c = \cos^{-1}(\frac{2}{\pi}) \Rightarrow c = 0.88$

c) $f(x) = \ln x$ $[1, e]$

$\left.\begin{aligned} f'(c) &= \frac{\ln(1) - \ln(e)}{1-e} = \frac{-1}{1-e} \\ f'(c) &= \frac{1}{c} \end{aligned}\right\} \Rightarrow \frac{1}{c} = \frac{-1}{1-e} \Rightarrow 1-e = -c \Rightarrow c = e-1$

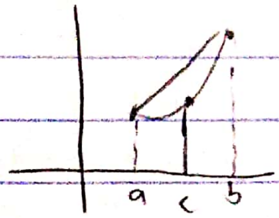
$c = 1.72$

14 b $|\arctan x - \arctan y| \le |x-y|$

$f'(c) = \frac{f(b)-f(a)}{b-a}$; $c \in [a,b]$ lagrange theorem

f is continuous $[a,b]$ and diff $c \in (a,b)$

$f'(c) = \frac{f(b)-f(a)}{b-a}$



$0 < \frac{1}{1+c^2} = f'(c) = \frac{\arctan x - \arctan y}{x-y} \le 1 \Rightarrow \frac{|\arctan x - \arctan y|}{|x-y|} \le 1$

$\Rightarrow |\arctan x - \arctan y| \le |x-y|$

d $\frac{x-y}{x} \le \ln\frac{x}{y} < \frac{x-y}{y}$

langrang theorem

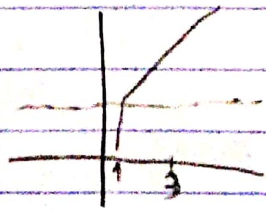
$\frac{1}{c} = \frac{\ln x - \ln y}{x-y}$

$\frac{1}{x} < \frac{\ln x - \ln y}{x-y} < \frac{1}{y}$

$\frac{x-y}{x} < \ln x - \ln y < \frac{x-y}{y}$

$\frac{x-y}{y} < \ln\frac{x}{y} < \frac{x-y}{y}$

15 a $f(x) = \begin{cases} x & 1 \le x < 3 \\ 1 & x = 3 \end{cases}$



$f(1) = f(3) = 1$

$f'(x) = 1$

e if f is defined in $[a,b]$, continous in $[a,b]$, diff (a,b), and $f(a) = f(b)$, so exist $c \in (a,b)$ such that $f'(c) = 0$ ← Ro theorem

langrange theorem

if f is defined in $[a,b]$ and continous at $[a,b]$, and diff in (a,b), so exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Homework sheet #5

(5) (a) $f(x) = \ln(x^2+1)$

Domain: $x \in \mathbb{R}$

$x=0$: $\ln 1 = 0$

$f(-x) = \ln((-x)^2+1)$

$= \ln(x^2+1) = f(x)$

Even

$y=0$: $0 = \ln(x^2+1)$

$1 = x^2+1$

$0 = x^2$

$f'(x) = \frac{1}{x^2+1} \cdot 2x = 0 = \frac{2x}{x^2+1}$

$x=0$ critical pt.

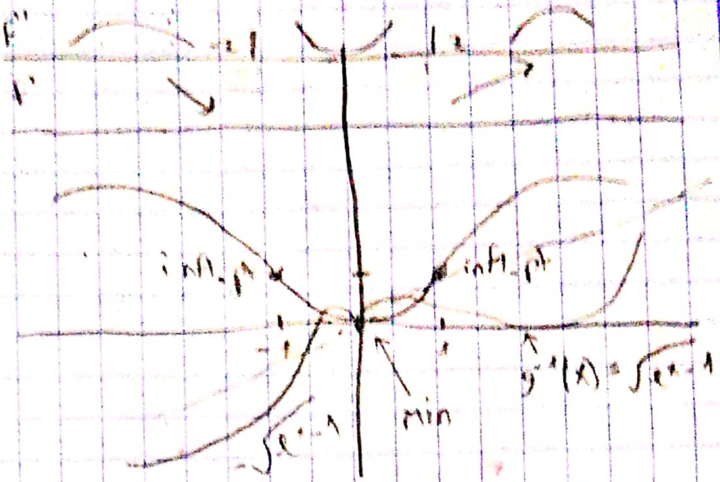
$f''(x) = \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2} = 0 \Rightarrow 2x^2 = 2$

$x^2 = 1 \Rightarrow x = \pm 1$ critical pt of f''

$f(1) = \ln 2$

$f(-1) = \ln 2$

horizontal asymptote: $\lim_{x \to \pm\infty} \ln(x^2+1) = \infty$



(b) $[0,\infty)$

invertible because it's (a) continuous

(b) monotonic (deriv is positive)

ⓒ $y = \ln(x^2-1)$

$x = \ln(y^2-1)$

$e^x = y^2-1$

$y^2 = e^x - 1$

$y = \pm\sqrt{e^x-1}$ $\boxed{y = \sqrt{e^x-1}}$

$y = \sqrt{x} \cdot 2\sqrt{x}$

⑦ $f(x) = \ln(x+\sqrt{x^2+k})$ $x \in R$

ⓐ Domain of f: $x+\sqrt{x^2+k} > 0 \Rightarrow \sqrt{x^2+k} > -x$ if x is positive or 0, its true

when x is negative, $x^2+k > x^2$ always true

ⓑ $f'(x) = \dfrac{1}{x+\sqrt{x^2+k}} \cdot \left(1+\dfrac{1}{2\sqrt{x^2+k}} \cdot 2x\right) = \dfrac{1+\dfrac{1}{\sqrt{x^2+k}}}{x+\sqrt{x^2+k}} \left(\dfrac{\sqrt{x^2+k}}{\sqrt{x^2+k}}\right) =$

$f'(x) = \dfrac{\sqrt{x^2+k}+1}{x\sqrt{x^2+k}+x^2+k}$

ⓒ invertible: ⓐ continuous ✓ (composition of $x+\sqrt{x^2+k}$ & $\ln x$ which is continuous in its domain)

ⓑ derivative is always positive

$x = \ln(y+\sqrt{y^2+k}) \Rightarrow e^x = y+\sqrt{y^2+k}$

$(e^x-y)^2 = (\sqrt{y^2+k})^2 \Rightarrow e^{2x} - 2ye^x + y^2 = y^2+k$

$2ye^x = e^{2x} - k \Rightarrow \boxed{y = \dfrac{e^{2x}-k}{2e^x}}$

ⓓ $k=1$

$\boxed{y' = \dfrac{e^{2x}-1}{2e^x}}$

$\sinh t = \frac{1}{2}(e^t - e^{-t}) = \dfrac{e^t}{2} - \dfrac{1}{2}e^t$

$f(x) = x^{\frac{2}{3}}$ $x \in R$

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \dfrac{2}{3x^{\frac{1}{3}}}$ $x \neq 0$ The domain in which f is differentiable is $x \neq 0$

(11) $f(x) = \tan x$

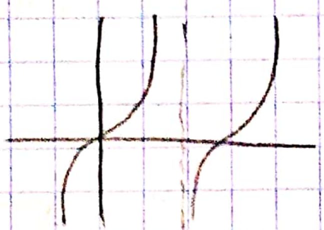
$f'(x) = \dfrac{1}{\cos^2 x} \neq 0 \rightarrow x \in (0, \pi)$

Does this contradict Rolle's Theorem?

<u>condition 1</u>: $f(a) = f(b) = 0$ $\quad \tan 0 = 0$ ✓

$\tan \pi = 0$

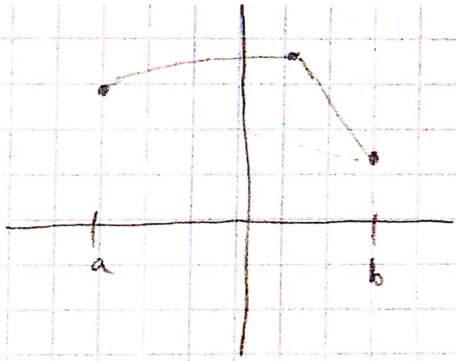
<u>condition 2</u>: tan x is continous on $[0, \pi]$ ✗



there is a discontinuity at $x = \frac{\pi}{2}$

So no contradiction to Rolle's Theorem because tan x doesn't satisfy the conditions in this domain.

(d) $f(x)$ is continous on $[a,b]$
(not differentiable on (a,b))



and there is no $c \in (a,b)$ such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$