DS2 HW #3 theoretical

① ⓐ $T(n) = 2T(\frac{n}{2}) + 1$

ⓑ $a=2,\ b=2,\ f(n)=1,\ \log_b a = 1,\ n^{\log_b a} = n$

$f(n) = O(n^{1-\varepsilon}) \Rightarrow T(n) = \Theta(n)$

② ⓐ $T(n) = T(n-1) + n$

ⓑ claim: $T(n) = O(n^2)$

$\therefore T(n) \leq cn^2$

Proof by induction: $\forall c \geq 1,\ T(1) = 1 \leq c * 1^2$

assume $T(n-1) \leq c(n-1)^2$

$$\begin{aligned} T(n) &= T(n-1) + n \\ &\leq c(n-1)^2 + n \\ &= c(n^2 - 2n + 1) + n \\ &= cn^2 - 2cn + c + n \\ &\leq cn^2 \text{ for any such that } 2cn + c + n \geq 0 \end{aligned}$$

$2cn + c \geq n$

$c(2n+1) \geq n$

$c \geq \frac{n}{(2n+1)} \approx \frac{1}{2}$

③ ⓐ $a=2,\ b=2,\ f(n)=n^3,\ \log_b a = 1,\ n^{\log_b a} = n$

$f(n) = \Omega(n^{1+\varepsilon}) \Rightarrow T(n) = \Theta(n^3)$

ⓑ $a=1,\ b=3,\ f(n) = 2\log(n),\ \log_b a = 0,\ n^{\log_b a} = 1$

$f(n) = \Omega(n^{0+\varepsilon}) \Rightarrow T(n) = \Theta(2\log(n)) = \Theta(\log(n))$

ⓒ $a=1,\ b=3,\ f(n) = n\log(n),\ \log_b a = 0,\ n^{\log_b a} = 1$

$f(n) = \Omega(n^{0+\varepsilon}) \Rightarrow T(n) = \Theta(n\log(n))$

ⓓ $a=2,\ b=3,\ f(n) = n^{\frac{3}{2}},\ \log_b a = 0.631,\ n^{\log_b a} = n^{0.631}$

$f(n) = O(n^{0.631-\varepsilon}) \Rightarrow T(n) = \Theta(n^{0.631})$

ⓔ $a=3,\ b=2,\ f(n) = n^2\log(n),\ \log_b a = 1.585,\ n^{\log_b a} = n^{1.585}$

$f(n) = \Omega(n^{1.585+\varepsilon}) \Rightarrow T(n) = \Theta(n^2\log(n))$

ⓕ $a=2,\ b=4,\ f(n) = \sqrt{n},\ \log_b a = \frac{1}{2},\ n^{\log_b a} = \sqrt{n}$

$f(n) = \Theta(\sqrt{n}) \Rightarrow T(n) = \Theta(\sqrt{n}\log(n))$

g) claim: $T(n) = O(\log(n))$

$\therefore T(n) \leq c \cdot \log(n)$

Proof by induction:

$\forall c, T(1) \leq c \cdot \log(1)$

assume $T(\sqrt{n}) \leq c \cdot \log(\sqrt{n})$

$T(n) = T(\sqrt{n}) + 2\log(n)$

$\leq c \cdot \log(\sqrt{n}) + 2\log(n)$

$= \frac{c}{2} \cdot \log(n) + 2\log(n)$

$= (\frac{c}{2} + 2) \cdot \log(n)$

chage $c = 4$

h) $a = 4, b = 4$, $f(n) = \frac{n}{\lg(n)}$, $\log_b a = 1$, $n^{\log_b a} = n$

$f(n) = O(n^{1-\varepsilon}) \Rightarrow T(n) = \Theta(n)$