Statistics - homework 3

① ⓐ prove that the difference between this estimator's expected value and the true value is 0.

we must show that $E[\hat{\mu}_K] = \mu$

$$\hat{\mu}_K = \frac{\sum_{i=1}^{K} X_i}{K}$$

$$E[\hat{\mu}_K] = E\left[\frac{\sum_{i=1}^{K} X_i}{K}\right]$$

$$E[\hat{\mu}_K] = \frac{\sum_{i=1}^{K} E[X_i]}{K} \quad E[\hat{\mu}_K] = \frac{\sum_{i=1}^{K} \mu}{K} \quad E[\hat{\mu}_K] = \frac{K\mu}{K} \quad E[\hat{\mu}_K] = \mu$$

ⓑ $\hat{\mu}_M$ is a better estimator, since a higher value of K will result in lower variance

② ⓐ we must show that $E[\hat{\mu}] = \mu$

$$\hat{\mu} = a\bar{x}_1 + (1-a)\bar{x}_2$$

$$E(\hat{\mu}) = E[a\bar{x}_1 + (1-a)\bar{x}_2]$$

$$E[\hat{\mu}] = E[a\bar{x}_1] + E[(1-a)\bar{x}_2]$$

$$E[\hat{\mu}] = aE(\bar{x}_1] + (1-a)E[\bar{x}_2]$$

$$E[\hat{\mu}] = a\mu + (1-a)\mu$$

$$E[\hat{\mu}] = a\mu + \mu - a\mu$$

$$E[\hat{\mu}] = \mu$$

ⓑ $var(\hat{\mu}) = var(ax_1 + (1-a)x_2)$

$$var(\hat{\mu}) = a^2\sigma^2 + (1-a)^2\sigma^2$$

$$var(\hat{\mu}) = 2a^2\sigma^2 - 2a\sigma^2 + \sigma^2$$

The minimum variance occurs when the derivative with respect to a is 0

$$var(\hat{\mu})' = 4a\sigma^2 - 2\sigma^2 + 0 = 0$$

$$2a - 1 = 0$$

$$a = \frac{1}{2}$$

This is a minimum (not a maximum) if the second derivative is positive at $a = \frac{1}{2}$

$$var(\hat{\mu})'' = 4\sigma^2 \geq 0$$

④ ⓐ i. $\hat{\mu}_1$ unbiased.

$$\hat{\mu}_1 = \frac{\sum_{i=1}^{7} X_i}{7} \quad E[\hat{\mu}_1] = \frac{\sum_{X=1}^{7} E[X_i]}{7} \quad E[\hat{\mu}_1] = \frac{\sum_{i=1}^{7} \mu}{7}$$

$$E[\hat{\mu}_1] = \frac{7\mu}{7} = \mu$$

ii. $\hat{\mu}_2$ is unbiased

$\hat{\mu}_2 = \frac{2X_1 - X_6 + X_4}{2}$ $E[\hat{\mu}_2] = \frac{2E[X_1] - E[X_6] + E[X_4]}{2}$

$E[\hat{\mu}_2] = \frac{2\mu - \mu + \mu}{2}$ $E[\hat{\mu}_2] = \frac{2\mu}{2} = \mu$

(b) (i) $\hat{\mu}_1 = \frac{\sum_{i=1}^{7} X_i}{7}$ $var(\hat{\mu}_1) = var\left(\frac{\sum_{i=1}^{7} X_i}{7}\right)$

$var(\hat{\mu}_1) = \frac{\sum_{i=1}^{7} \sigma^2}{49}$ $v(\hat{\mu}_1) = \frac{\sigma^2}{7}$

(ii) $\hat{\mu}_2 = \frac{2X_1 - X_6 + X_4}{2}$ $var(\hat{\mu}_2) = var\left(\frac{2X_1 - X_6 + X_4}{2}\right)$

$var(\hat{\mu}_2) = \frac{4\sigma^2 - \sigma^2 + \sigma^2}{4}$ $var(\hat{\mu}_2) = \frac{4\sigma^2}{4} = \sigma^2$

$\hat{\mu}_1$ is a better estimator since its variance is lower.

(9) (a) $L(\theta) = \sum_{i=1}^{n} \log(f(x_i)) = \sum_{i=1}^{n} \log\left(\frac{2x_i}{a^2}\right) = \log\left(\sum_{i=1}^{n}(2x_i) - \sum_{i=1}^{n}(a^2)\right) = \log\left(2\sum_{i=1}^{n}(x_i) - na^2\right)$

Differentiate with respect to a and set equal to 0.

$L(\theta)' = \log\left(2\sum_{i=1}^{n}(x_i) - na^2\right)' = 2 + \frac{2}{a} = 0$

$a = -1$

this is a minimum since $L(\theta)''(1) = \frac{2}{a^2} \geq 0$

(10) $L(\theta) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{e^{-\lambda x_i} \lambda^m x_i^{m-1}}{\Gamma(m)} = \frac{\lambda^{mn} e^{-\lambda \sum_{i=1}^{n} x_i} \left(\prod_{i=1}^{n} x_i\right)^{m-1}}{\Gamma(m)^n}$

$\log(L(\theta)) = \log\left(\frac{\lambda^{mn} e^{-\lambda \sum_{i=1}^{n} x_i} \left(\prod_{i=1}^{n} x_i\right)^{m-1}}{\Gamma(m)^n}\right)$

$\log(L(\theta)) = \log\left(\lambda^{mn} e^{-\lambda \sum_{i=1}^{n} x_i} \left(\prod_{i=1}^{n} x_i\right)^{m-1}\right) - \log(\Gamma(m)^n)$

$\log(L(\theta)) = \log(\lambda^{mn}) + \log(e^{-\lambda \sum_{i=1}^{n} x_i}) + \log\left(\left(\prod_{i=1}^{n} x_i\right)^{m-1}\right) - n\log(\Gamma(m))$

$\log(L(\theta)) = mn\log(\lambda) - \lambda\sum_{i=1}^{n} x_i + (m-1)\log\left(\prod_{i=1}^{n} x_i\right) - n\log(\Gamma(m))$

differentiate with respect to lambda and set equal to 0

$\log(L(\theta))' = \frac{mn}{\lambda} - \sum_{i=1}^{n} x_i = 0$ $\lambda = \frac{mn}{\sum_{i=1}^{n} x_i}$

check the second derivative is negative $\log(L(\theta))'' = -\frac{mn}{\lambda^2} \leq 0$