Calculus hw #5

(A) ② D is bounded, f is continuous in D

<u>False</u> since it's domain is open and not closed

③ f is continuous in D, D is closed

<u>false</u> since D is not bounded

④ f is not continuous in D

D is closed and bounded

<u>False</u> function is not continuous in D as $y \to \pm 1$.

(B) ① $f(x,y) = \sqrt{1-x^2-y^2}$, $x^2+y^2<1$

$$f'_x(x,y) = -\frac{x}{\sqrt{1-x^2-y^2}} \Rightarrow f'_x(x,y)=0 \Rightarrow x=0$$

$$\Rightarrow P_1 = (0,0) \in D$$

$$f'_y(x,y) = \frac{y}{\sqrt{1-x^2-y^2}} \Rightarrow f'_y(x,y)=0 \Rightarrow y=0$$

$$f'_{x^2}(x,y) = -\frac{\sqrt{1-x^2-y^2}+\frac{x^2}{\sqrt{1-x^2-y^2}}}{1-x^2-y^2} = \frac{y^2-1}{(1-x^2-y^2)^{1.5}}$$ $\Rightarrow$ switch between y to x and will get f'_{y^2}

$$\Rightarrow f'_{y^2}(x,y) = \frac{x^2-1}{(1-x^2-y^2)^{1.5}}$$

$$f'_{xy}(x,y) = \frac{d}{dy}\left(-x(1-x^2-y^2)^{-\frac{1}{2}}\right) = -xy(1-x^2-y^2)^{-1\frac{1}{2}} = \frac{-xy}{(1-x^2-y^2)^{1.5}}$$

$$H(x,y) = \begin{vmatrix} f_{x^2}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{y^2}(x,y) \end{vmatrix} \Rightarrow H(0,0) = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

max at $(0,0)$

④ $f(x,y) = y\sin x$

$$f'_x(x,y) = y\cos x, \quad f'_y(x,y) = \sin x \Rightarrow \begin{cases} y\cos x = 0 \\ \sin x = 0 \end{cases} \Rightarrow (x,y) = (\pi k, 0) \quad k \in \mathbb{Z}$$

$$f'_{x^2}(x,y) = -y\sin x, \quad f'_{y^2}(x,y) = 0, \quad f'_{xy}(x,y) = \cos x$$

$$\Rightarrow H(x,y) = \begin{vmatrix} -y\sin x & \cos x \\ \cos x & 0 \end{vmatrix} \Rightarrow H(\pi k, 0) = 0 - \cos^2(\pi k) = -(\pm 1)^2 = -1$$

$H(\pi k, 0) < 1$ for all K, therefore all the pts $(x,y) \in (\pi k, 0)$ we will get saddle pts

⑦

$$f(x,y) = e^{xy} \Rightarrow f'_x(x,y) = ye^{xy}, \quad f_y'(x,y) = xe^{xy}$$

$$f'_{xy}(x,y) = e^{xy} + xye^{xy} = e^{xy}(1+xy) \qquad \begin{cases} ye^{xy} = 0 \\ xe^{xy} = 0 \end{cases} \Rightarrow (x,y) = (0,0)$$

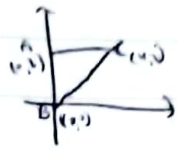
$$f'_{x^2}(x,y) = y^2e^{xy}, \quad f'_{y^2}(x,y) = x^2e^{xy}$$

(c)

$$\Rightarrow H(x,y) = \begin{vmatrix} y^2e^{xy} & e^{xy}(1+xy) \\ e^{xy}(1+xy) & x^2e^{xy} \end{vmatrix} \Rightarrow H(0,0) = -1 \Rightarrow$$

function has saddle pt $(0,0)$

(c) (2) $f(x,y) = x^2 - 2x + 2y^2 - 4y$



critical pts for f:

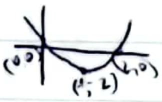
$$\begin{cases} f'_x(x,y) = 0 \\ f'_y(x,y) = 0 \end{cases} \Rightarrow \begin{cases} 2x-2 = 0 \\ 4y-4 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases} \Rightarrow \boxed{P_1 = (1,1) \in D}$$

min and max pts of D

AB: $\begin{cases} x(t) = 0 \\ y(t) = t \end{cases}$ $0 \le t \le 2 \Rightarrow \underset{(x,y)\in[A,B]}{f(x,y)} = 0^2 - 2\cdot 0 + 2t^2 - 4t = 2t^2 - 4t$

$t = 0, 2$

$4t - 4 = 0 \Rightarrow t = 1 \Rightarrow$



$\Rightarrow \boxed{P_2 = (0,0)} \Leftarrow t=0$, $\boxed{P_3 = (0,2)} \Leftarrow t=2$ max; $t=1$ $\boxed{P_4 = (0,1)}$ min

AC: $\begin{cases} x(t) = t \\ y(t) = 2 \end{cases}$ $0 \le t \le 4 \Rightarrow \underset{(x,y)\in[A,C]}{f(x,y)} = t^2 - 2t + 2\cdot 4 - 4\cdot 2 = t^2 - 2t \Rightarrow$

$\Rightarrow$ min : $t=1 \Rightarrow \boxed{P_5 = (1,2)}$, max: $t = 4 \Rightarrow \boxed{P_6 = (4,2)}$

BC: $\begin{cases} x(t) = 0 + t(4-0) \\ y(t) = 0 + t(2-0) \end{cases} \Rightarrow \begin{cases} x(t) = 4t \\ y(t) = 2t \end{cases}$ $0 \le t \le 1 \Rightarrow \underset{(x,y)\in[B,C]}{f(x,y)} = (4t)^2 - 2\cdot 4t + 2(2t)^2 - 4\cdot 2t$

$= 24t^2 - 16t \Rightarrow$ derivative equal to 0 $\Rightarrow 48t - 16 = 0 \Rightarrow t = \frac{1}{3} \Rightarrow$

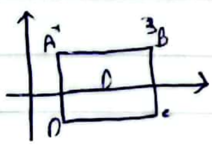
$\Rightarrow$ min: $t = \frac{1}{3} \Rightarrow \boxed{P_7 = \left(\frac{4}{3}, \frac{2}{3}\right)}$ max: $t=1 \Rightarrow \boxed{P_6 = (4,2)}$

absolute min and max:

(x,y)	$P_1=(1,1)$	$P_2=(0,0)$	$P_3=(0,2)$	$P_4=(0,1)$	$P_5=(1,2)$	$P_6=(4,2)$	$P_7=(\frac{4}{3},\frac{2}{3})$
$f(x,y)$	-3	0	0	-2	-1	8	$-2\frac{2}{3}$

absolute min $(1,1)$ absolute max $(4,2)$

(4) $f(x,y) = (4x - x^2)\cos y$



critical pts:

$$\begin{cases} f'_x(x,y) = 0 \\ f'_y(x,y) = 0 \end{cases} \Rightarrow \begin{cases} (4-2x)\cos y = 0 \\ (x^2 - 4x)\sin y = 0 \end{cases} \Rightarrow \begin{cases} 4 - 2x = 0 \Rightarrow x = 2, y = 0 \\ x^2 - 4x = 0 \Rightarrow x = 0 \; y = \frac{\pi}{2} \\ \Rightarrow x = 4 \; y = \frac{\pi}{2} \end{cases}$$

y doesn't exist in the area $\Rightarrow \boxed{P_1 = (2,0) \in D}$

where finding P:

AB: $\begin{cases} x(t)=t \\ y(t)=\frac{\pi}{4} \end{cases}$ $1\le t\le 3 \Rightarrow \underset{(x,y)\in[A,B]}{f(x,y)} = 2\sqrt{2}t - \frac{\sqrt{2}t^2}{2} \Rightarrow f'(x,y)=0 \Rightarrow -\sqrt{2}t+2\sqrt{2}=0 \Rightarrow t=2$

$\Rightarrow$  $\Rightarrow$ max: $t=2 \Rightarrow \boxed{P_2=(2,\frac{\pi}{4})}$, min: $t=1$ $\boxed{P_3=(1,\frac{\pi}{4})}$, $t=3$ $\boxed{P_4=(3,\frac{\pi}{4})}$

the function is even: $f(x,y)=f(x,-y)$

therefore for BC the extreme pts are: Max: $\boxed{P_5=(2,-\frac{\pi}{4})}$

Min: $\boxed{P_6=(1,-\frac{\pi}{4})}$ $\boxed{P_7=(3,-\frac{\pi}{4})}$

we see that P_3, P_4, P_6, P_7 are at the edges of AD, BC. therefore we need to check only internal pts:

AD: $\begin{cases} x(t)=1 \\ y(t)=t \end{cases}$ $-\frac{\pi}{4}\le t\le\frac{\pi}{4} \Rightarrow \underset{(x,y)\in[A,D]}{f(x,y)} = 3\cos t \Rightarrow f'(x,y) = -3\sin t$

$\Rightarrow -3\sin t=0 \Rightarrow t=0 \Rightarrow$ max: $\boxed{P_8=(1,0)}$

BC: $\begin{cases} x(t)=3 \\ y(t)=t \end{cases}$ $-\frac{\pi}{4}\le t\le\frac{\pi}{4} \Rightarrow \underset{(x,y)\in[B,C]}{f(x,y)} = 3\cos t \Rightarrow$ extreme pt when $t=0$ $\Rightarrow \boxed{P_9=(3,0)}$

absolute min and max:

(x,y)	$P_1=(2,0)$	$P_2=(2,\frac{\pi}{4})$	$P_3=(1,\frac{\pi}{4})$	$P_4=(3,\frac{\pi}{4})$	$P_5=(2,-\frac{\pi}{4})$	$P_6=(1,-\frac{\pi}{4})$	$P_7=(3,-\frac{\pi}{4})$	$P_8=(1,0)$	$P_9=(3,0)$
$f(x,y)$	4	$2\sqrt{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	$2\sqrt{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	3	3
	absolute max $(2,0)$		absolute min $(1,\frac{\pi}{4})$	absolute min $(3,\frac{\pi}{4})$		absolute min $(1,-\frac{\pi}{4})$	absolute min $(3,-\frac{\pi}{4})$		

(5) no, it speaks about the whole function being closed and bounded, and here the area isn't bounded

(D) (1) $f(x,y,z)=xyz$, $x+y+z=48 \Rightarrow z=48-x-y$

$\Rightarrow g(x,y) = f(x,y,48-x-y) = xy(48-x-y) = 48xy - x^2y - y^2x$

$g'_x(x,y)=0 \Rightarrow 48y-2xy-y^2=0 \Rightarrow 2xy = 48y-y^2$

$g'_y(x,y)=0 \Rightarrow 48x-2xy-x^2=0 \Rightarrow 48x-x^2-48y+y^2=0$

$\Rightarrow (y-24)^2-(x-24)^2=0 \Rightarrow (y-24)^2=(x-24)^2 \Rightarrow y-24=x-24 \Rightarrow \boxed{y=x}$

$\Rightarrow y-24=24-x \Rightarrow y=48-x$

one of them will become 0

$x = y$, therefore:

$h(x) = f(x, x, 48-2x)$

$h(x) = x^2(48-2x) = 48x^2 - 2x^3$

$h'(x) = 96x - 6x^2 \Rightarrow x(96-6x) = 0 \Rightarrow x = 0$

$\Rightarrow 96 - 6x = 0 \Rightarrow x = 16 \Rightarrow y = 16 \Rightarrow z = 16$

critical pt: $P = (16, 16, 16)$

16^3

(4) $f(x,y) = xy$, $x^2 + y^2 \le 1$

critical pt

$f'_x(x,y) = y \Rightarrow y = 0$

$f'_y(x,y) = x \Rightarrow x = 0$

$\Rightarrow P = (0,0)$

$x(t) = \cos(t)$

$y(t) = \sin(t)$

$\Rightarrow f(x,y) = \sin t \cos t = \frac{1}{2} \cdot \sin 2t$

$0 \le t \le 2\pi$

$f'(x(t), y(t)) = \cos 2t \Rightarrow \cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2}, \frac{3\pi}{2}$

4 solutions:

$\cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2} + \pi k \Rightarrow t = \frac{\pi}{4} + \frac{\pi}{2} k \Rightarrow t_1 = \frac{\pi}{4}, t_2 = \frac{3\pi}{4}$

$t_3 = \frac{5\pi}{4}, t_4 = \frac{7\pi}{4}$

(x,y)	$P_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$P_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$P_3 = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$P_4 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
$f(x,y)$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	absolute max		absolute max	

(E) (2) $f(x,y,z) = xyz$, $g(x,y,z) = 2xy + 2xz + 2yz - 10 = 0$

$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

$\nabla f(x,y,z) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$, $\nabla g(x,y,z) = \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix} = \begin{pmatrix} 2y + 2z \\ 2x + 2z \\ 2x + 2y \end{pmatrix}$

I $yz = \lambda(2y - 2z) \Rightarrow yz - 2\lambda z \Rightarrow z = \frac{2\lambda y}{y - 2\lambda}$

$\Rightarrow$ II $xz = \lambda(x + 2z) \Rightarrow \frac{2\lambda xy}{y - 2\lambda} = 2\lambda(x + \frac{2\lambda y}{y - 2\lambda}) \Rightarrow \frac{xy}{y - 2\lambda} = x + \frac{2\lambda y}{y - 2\lambda} \Rightarrow xy = xy - 2\lambda x + 2\lambda y \Rightarrow x = y$

III $xy = \lambda(2x + 2y) \Rightarrow y^2 = 4\lambda y \Rightarrow y = 4\lambda$ (y not 0)

IV. $2xy + 2xz + 2yz = 10 \Rightarrow 2 \cdot 4\lambda \cdot 4\lambda + 2 \cdot 4\lambda \cdot 4\lambda + 2 \cdot 4\lambda \cdot 4\lambda = 10 \Rightarrow 96\lambda^2 = 10 \Rightarrow \lambda = \frac{\sqrt{5}}{\sqrt{48}}$

$\Rightarrow x = y = z = 4\lambda = 4\frac{\sqrt{5}}{\sqrt{48}} = \frac{\sqrt{5}}{\sqrt{3}} \Rightarrow P = (\frac{\sqrt{5}}{\sqrt{3}}, \frac{\sqrt{5}}{\sqrt{3}}, \frac{\sqrt{5}}{\sqrt{3}})$

(3) $f(x,y,z) = xyz$, $g(x,y,z) = 2xy + 2xz + 2yz - 10 = 0$

$\nabla f(x,y,z) = \lambda \nabla g(x_0, y_0, z_0)$

$\nabla f(x,y,z) = \begin{pmatrix} 2x - 4 \\ 2y - 2 \\ 2z + 2 \end{pmatrix}$, $\nabla g(x,y,z) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Scanned with CamScanner

$\Rightarrow$ $2x-4 = \lambda \Rightarrow 2x-4 = 2y-2 \Rightarrow x = y+1$

$2y-2 = \lambda \Rightarrow z = -\frac{\lambda}{4}-1$

$2z+2 = \lambda \Rightarrow 2y+1+\frac{1}{2}\lambda+1 = 1 \Rightarrow 2y+\frac{1}{2}\lambda+1 = 0 \Rightarrow 2y+x-2+1 = 0$

$\Rightarrow 3y = 0 \Rightarrow y = 0 \Rightarrow x = 1 \Rightarrow \lambda = -2 \Rightarrow z = 0 \Rightarrow \boxed{P = (1,0,0)}$

5) 1) $f(x,y) = x^2 - y^2$ $\quad x^2+y^2 = 4$

$g(x,y) = x^2+y^2-4 = 0$

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$\nabla f(x,y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 2x \\ -2y \end{pmatrix}, \quad \nabla g(x,y) = \begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$\Rightarrow \begin{array}{l} 2x = \lambda 2x \\ -2y = \lambda 2y \end{array} \Rightarrow 2x(\lambda-1) = 0 \Rightarrow x = 0, y = \pm 2 \Rightarrow P_1 = (0,2) \quad P_2 = (0,-2)$

$\Rightarrow \lambda = 1 \Rightarrow y = 0, x = \pm 2 \Rightarrow P_3 = (2,0) \quad P_4 = (-2,0)$

find the absolute min and max:

(x,y)	$(0,2)$	$(0,-2)$	$(2,0)$	$(-2,0)$
f	-4	-4	4	4
	abs min	abs min	abs max	abs max

3) $f(x,y,z) = x+y+z$ $\quad x^2+4y^2+9z^2 = 36$

$g(x,y,z) = x^2+4y^2+9z^2-36 = 0$

$\nabla f(x,y,z) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \nabla g(x,y,z) = \begin{pmatrix} 2x \\ 8y \\ 18z \end{pmatrix}$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \Rightarrow \begin{array}{l} 1 = \lambda \cdot 2x \Rightarrow x = \frac{1}{2\lambda} \\ 1 = \lambda \cdot 8y \Rightarrow y = \frac{1}{8\lambda} \\ 1 = \lambda \cdot 18z \Rightarrow z = \frac{1}{18\lambda} \end{array} \Rightarrow \left(\frac{1}{2\lambda}\right)^2 + 4\left(\frac{1}{8\lambda}\right)^2 + 9\left(\frac{1}{18\lambda}\right)^2 = 36$

$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{16\lambda^2} + \frac{1}{36\lambda^2} = 36 \Rightarrow \lambda^2 = \frac{\frac{1}{4}+\frac{1}{16}+\frac{1}{36}}{36} = \frac{49}{72^2} \Rightarrow \lambda = \pm \frac{7}{72}$

(x,y)	$P_1 = \left(\frac{36}{7}, \frac{9}{7}, \frac{4}{7}\right)$	$P_2 = \left(-\frac{36}{7}, -\frac{9}{7}, -\frac{4}{7}\right)$
$f(x,y)$	7	-7

6) 2) $f(x,y,z) = 3x-y-3z$, $\quad g(x,y,z) = x+y-z = 0$, $\quad h(x,y,z) = x^2+2z^2-1 = 0$

$\nabla f(x,y,z) = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}, \nabla g(x,y,z) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \nabla h(x,y,z) = \begin{pmatrix} 2x \\ 0 \\ 4z \end{pmatrix}$

$\nabla f(x,y,z) = \lambda_1 \nabla g(x,y,z) + \lambda_2 \nabla h(x,y,z)$

$\Rightarrow \begin{array}{l} 3 = \lambda_1 + 2\lambda_2 x \\ -1 = \lambda_1 \\ -3 = -\lambda_1 + 4\lambda_2 z \end{array}$ $\quad \boxed{\lambda_1 = -1} \Rightarrow \boxed{\lambda_2 = -\frac{1}{2}} \Rightarrow \boxed{x = -2z}$

$x+y-z = 0 \Rightarrow y = 3z$

$x^2+2z^2-1 = 0 \Rightarrow 4z^2+2z^2 = 1 \Rightarrow z = \pm\frac{1}{\sqrt{6}}$

$\Rightarrow$	$P_1 = \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{3}{2}}, \frac{1}{\sqrt{6}}\right)$	$P_2 = \left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}\right)$
$P(x,y)$	$-2\sqrt{6}$ abs min	$2\sqrt{6}$ abs max

ii)

4) $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x + y + z - 1 = 0$, $h(x,y,z) = x + 2y + 3z - 6 = 0$

$\nabla f(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$, $\nabla g(x,y,z) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\nabla h(x,y,z) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\nabla f(x,y,z) = \lambda_1 \nabla g(x,y,z) + \lambda_2 \nabla h(x,y,z)$

$\Rightarrow$ $2x = \lambda_1 + \lambda_2$ $\quad$ $x + y + z - 1 = 0 \Rightarrow \frac{\lambda_1 + \lambda_2}{2} + \frac{\lambda_1 + 2\lambda_2}{2} + \frac{\lambda_1 + 3\lambda_2}{2} = 1 \Rightarrow \lambda_1 = \frac{2}{3} - 2\lambda_2$

$2y = \lambda_1 + 2\lambda_2$

$2z = \lambda_1 + 3\lambda_2$ $\quad$ $x + 2y + 3z = 6 \Rightarrow \frac{\lambda_1 + \lambda_2}{2} + \lambda_1 + 2\lambda_2 + \frac{3\lambda_1 + 9\lambda_2}{2} = 6 \Rightarrow 3\lambda_1 + 7\lambda_2 = 6$

$\Rightarrow 2 - 6\lambda_2 + 7\lambda_2 = 6 \Rightarrow \lambda_2 = 4 \Rightarrow \lambda_1 = -7\frac{1}{3} \Rightarrow x = -\frac{5}{3}, y = \frac{1}{3}, z = \frac{7}{3} \Rightarrow P = \left(-\frac{5}{3}, \frac{1}{3}, \frac{7}{3}\right)$

$f(-4,5,0) = 41 > f\left(-\frac{5}{3}, \frac{1}{3}, \frac{7}{3}\right) = 8\frac{1}{3}$ $\quad$ the pt P is abs min of the function

B) 1) $f(x,y) = x^2 + y^2$, $g(x,y) = (x-1)^3 - y^2 = 0$

$\nabla f(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$, $\nabla g(x,y) = \begin{pmatrix} 3(x-1)^2 \\ -2y \end{pmatrix}$

$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 3(x-1)^2 \\ -2y \end{pmatrix} \Rightarrow$ $2x = 3\lambda(x-1)^2$

$2y = -2\lambda y \Rightarrow 2y(1+\lambda) = 0 \Rightarrow y = 0 \Rightarrow (x-1)^3 - 0^2 = 0 \Rightarrow x = 1$