202

# ODE hw #3

① ⓐ $xy = (1+2x)y'$ $\{k \in \mathbb{R}\}$ $y \equiv k$

$u(x) = y'(x)$

$xu'(x) = (1+2x)u(x) \Rightarrow x\frac{du}{dx} = (1+2x)u \Rightarrow \int \frac{1}{u}du = \int(\frac{1}{x}+2)dx + c$

$\Rightarrow \ln|u| = \ln|x| + 2x + c \Rightarrow u = y' = e^c \cdot e^{2x}|x| = De^{2x}\cdot|x| \mid \{D \in \mathbb{R}\}$

$\Rightarrow y = D\int e^{2x}\cdot|x|dx + C = \begin{cases} D\int e^{2x}\cdot x dx & : x>0 \\ -D\int e^{2x}\cdot x dx & : x<0 \end{cases} \Rightarrow y = \int Dxe^{2x}dx + c$

(the solution is defined $D \in \mathbb{R}$)

$\Rightarrow y = \frac{1}{2}Dxe^{2x} - \frac{1}{4}De^{2x} + c \Rightarrow y_1 = D_1e^{2x}(x-\frac{1}{2}) - D_2$ $\{D_1, D_2 \in \mathbb{R}\}$

ⓒ $y'' - 3y' = 9x$

$y'(x) = u(x)$

$u' - 3u = 9x \Rightarrow \mu(x) = e^{\int -3dx} = e^{-3x} \Rightarrow u'e^{-3x} - 3ue^{-3x} = 9xe^{-3x}$

$\Rightarrow ue^{-3x} = \int 9xe^{-3x}dx + c_1 = -3xe^{-3x} - e^{-3x} + c_1 \Rightarrow u = y' = c_1e^{3x} - 3x - 1$

$\Rightarrow y = \int c_1e^{3x} - 3x - 1\, dx = \frac{1}{3}c_1e^{3x} - \frac{3}{2}x^2 - x + c_2 \Rightarrow \boxed{y = D_1e^{3x} - \frac{3}{2}x^2 - x + D_2 \mid D_1, D_2 \in \mathbb{R}}$

② ⓒ $\begin{cases} y'' - 4y' + 4y = 0 \\ y(1) = y'(1) = 1 \end{cases}$

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)^2 = 0 \Rightarrow \lambda = 2$

general solution: $y(x) = c_1e^{2x} + c_2xe^{2x}$

specific solution:

$y(1) = 1 \Rightarrow c_1\cdot e^2 + c_2\cdot e^2 = 1 \Rightarrow c_2 = e^{-2} - c_1$

$y'(1) = 1 \Rightarrow 2c_1\cdot e^2 + c_2\cdot e^2 + 2c_2e^2 = 2c_1e^2 + 3c_2e^2 = 1 = 2c_1e^2 + 3(1 - c_1e^2)$

$= 3 - c_1e^2 \Rightarrow c_1 = \frac{2}{e^2} \Rightarrow c_2 = -\frac{1}{e^2} \Rightarrow \boxed{y(x) = 2e^{2x-2} - xe^{2x-2}}$ specific solution

ⓕ $\begin{cases} 3y'' + 2y' - 5y = 0 \\ y(0) = 11,\ y'(0) = -5 \end{cases} \Rightarrow$

$3\lambda^2 + 2\lambda - 5 = 0 \Rightarrow (3\lambda+5)(\lambda-1) = 0$

$\boxed{\lambda_1 = 1}$ $\boxed{\lambda_2 = -\frac{5}{3}}$

general solution: $y(x) = c_1e^x + c_2e^{\frac{5}{3}x}$

specific solution:

$y(0) = 11 \Rightarrow c_1 + c_2 = 11 \Rightarrow c_2 = 11 - c_1$

$y'(0) = -5 \Rightarrow c_1 - \frac{5}{3}c_2 = c_1 - 18\frac{1}{3} + \frac{5}{3}c_1 = \frac{8}{3}c_1 - 18\frac{1}{3} = -5 \Rightarrow 8c_1 - 55 = -15$

$\Rightarrow c_1 = 5 \Rightarrow c_2 = 6 \Rightarrow y(x) = 5e^x + 6e^{-\frac{5}{3}x}$

20p

3 a $\begin{cases} y''+3y'+2y=0 \\ y(0)=a,\ y'(0)=b \end{cases}$ $\lambda^2+3\lambda+2=0 \Rightarrow (\lambda+1)(\lambda+2)=0$ $\boxed{\lambda_1=-1}$ $\boxed{\lambda_2=-2}$

general solution: $\boxed{y(x)=C_1e^{-x}+C_2e^{-2x}}$

$y(0)=a \Rightarrow C_1+C_2=a \Rightarrow C_2=a-C_1$

$y'(0)=b \Rightarrow -C_1-2(a-C_1)=C_1-2a=b \Rightarrow C_1=2a+b \Rightarrow C_2=-a-b$

$\Rightarrow \boxed{y(x)=(2a+b)e^{-x}-(a+b)e^{-2x}}$

b $\lim_{x\to\infty}(y(x))=\lim_{x\to\infty}((2a+b)e^{-x}-(a+b)e^{-2x})=(2a+b)\cdot 0-(a+b)\cdot 0=0$

5 a $y''-2y''-8y=32x$

$\lambda^2-2\lambda-8=0 \Rightarrow (\lambda-4)(\lambda+2)=0 \Rightarrow \boxed{\lambda_1=4}$ $\boxed{\lambda_2=-2}$

$\Rightarrow y_h(x)=C_1e^{4x}+C_2e^{-2x}$

non homogenous solution: $y_p(x)=x^s\cdot b_1\cdot x$

$s=0$ since the algebraic plurality of $\alpha+\beta i$ as the root of the typical equation

therefore the specific solution is: $\boxed{y_p(x)=b_1x+b_0}$

$0-2b_1-8b_1x-8b_0=32x \Rightarrow -2b_1-8b_0=0$, $-8b_1=32$ $\Rightarrow b_1=-4$, $b_0=1$

$\Rightarrow \boxed{y_p(x)=-4x+1} \Rightarrow \boxed{y(x)=C_1e^{4x}+C_2e^{-2x}-4x+1}$

general solution.

c $y''-4y'+4y=e^x \Rightarrow \lambda^2-4\lambda+4=0 \Rightarrow (\lambda-2)^2=0 \Rightarrow \lambda=2$

$\Rightarrow y_h(x)=C_1e^{2x}+C_2xe^{2x}$

the algebraic plurality of $1=\alpha+\beta i$ $=0$ since 1 is not a root of the typical equation.

therefore, the specific solution is: $y(x)=be^x \Rightarrow y''-4y'+4y=be^x-4be^x+4be^x=$

$be^x=e^x \Rightarrow b=1 \Rightarrow \boxed{y_p(x)=e^x} \Rightarrow y(x)=C_1e^{2x}+C_2xe^{2x}+e^x$ general solution

e $y''-4y'+3y=12\sin 3x \Rightarrow \lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)=0 \Rightarrow \boxed{\lambda_1=1}\boxed{\lambda_2=3}$

$\Rightarrow \boxed{y_h(x)=C_1e^x+C_2e^{3x}}$

the right of the ODE $=12\sin 3x$, therefore, the solution thats left is:

$y_p(x)=x^s(c\cdot\cos 3x+d\cdot\sin 3x)$

the algebraic plurality of $\alpha+\beta i=3i$ is 0 therefore $s=0$

$-c\cdot 9\cos 3x-d\cdot 9\sin 3x-4(-c\cdot\sin 3x+d\cdot\cos 3x)+3c\cos 3x+3d\sin 3x=12\sin 3x$

$\Rightarrow -9c-12d+3c=0 \Rightarrow c=-2d$ $\quad -9d+12c+3d=12 \Rightarrow d=-\frac{2}{5} \Rightarrow c=\frac{4}{5}$

$\Rightarrow \boxed{y_p(x)=\frac{4}{5}\cos 3x-\frac{2}{5}\sin 3x} \Rightarrow \boxed{y(x)=C_1e^x+C_2e^{3x}+\frac{4}{5}\cos 3x-\frac{2}{5}\sin 3x}$ general solution

①  $y''' + y'' - 17y' + 15y = 3x \Rightarrow \lambda^3 + \lambda^2 - 17\lambda + 15 = 0$

$\Rightarrow (\lambda-3)(\lambda^2+4\lambda-5) = (\lambda-3)(\lambda+5)(\lambda-1) = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = -5, \lambda_3 = 1$

$\Rightarrow y_h(x) = c_1 e^{x} + c_2 e^{3x} + c_3 e^{-5x}$

$0 = \alpha + \beta i$ since the algebraic plurality of $0$ is a root of the typical equation who is 0.

therefore $s = 0$

$y_p(x) = \frac{1}{5}x + \frac{17}{75} \Rightarrow y(x) = c_1 e^{x} + c_2 e^{3x} + c_3 e^{-5x} + \frac{1}{5}x + \frac{17}{75}$

①  $y''' + 6y'' + 11y' + 6y = 20e^{3x} - xe^{-x} \Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\Rightarrow (\lambda+1)(\lambda^2+5\lambda+6) = (\lambda+1)(\lambda+2)(\lambda+3) = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$

$\Rightarrow y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

Find the solutions for $L(y(x)) = 20e^{3x}$ and $L(y(x)) = -xe^{-x}$ separately and add them together:

For $L(y(x)) = 20e^{3x}$ we'll get $s=0$, therefore: $g_1(x) = b_1 e^{3x}$

$27b_1 e^{3x} + 54b_1 e^{3x} + 33b_1 e^{3x} + 6b_1 e^{3x} = 20e^{3x}$

$\Rightarrow 120b_1 = 20 \Rightarrow b_1 = \frac{1}{6} \Rightarrow g_1(x) = \frac{1}{6}e^{3x}$

for $L(y(x)) = -xe^{-x}$ we'll get $s=1$, therefore

$g_2(x) = x(b_2 x + b_3)e^{-x} = b_2 x^2 e^{-x} + b_3 x e^{-x}$

$\Rightarrow g_2'(x) = -b_2 x^2 e^{-x} + (2b_2 - b_3)xe^{-x} + b_3 e^{-x} \Rightarrow g_2''(x) = b_2 x^2 e^{-x} - (2b_2 + 2b_2 - b_3)xe^{-x} + (2b_2 - 2b_3)e^{-x}$

$\Rightarrow g_2'''(x) = -b_2 x^2 e^{-x} + (6b_2 - b_3)xe^{-x} - (6b_2 - 3b_3)e^{-x}$

$\Rightarrow y''' + 6y'' + 11y' + 6y = (6b_2 - b_3 - 6(4b_2 - b_3) + 11(2b_2 - b_3) + 6b_3)xe^{-x} + (3b_3 - 6b_2 + 6(2b_2 - 2b_3) + 11b_3)e^{-x} = -xe^{-x}$

$\Rightarrow 2b_3 + 6b_2 = 0 \Rightarrow b_3 = -3b_2,\ 4b_2 = -1 \Rightarrow b_2 = -\frac{1}{4} \Rightarrow b_3 = \frac{3}{4}$

$\Rightarrow \boxed{g_2(x) = -\frac{1}{4}x^2 e^{-x} + \frac{3}{4}xe^{-x}} \Rightarrow y_p(x) = g_1(x) + g_2(x)$

$\Rightarrow \boxed{y(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} + \frac{1}{6}e^{3x} + \frac{3}{4}xe^{-x} - \frac{1}{4}x^2 e^{-x}}$ general solution

⑥ ⓒ $y'' - 4y = 8\cosh x \Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -2$

$\Rightarrow y_h(x) = c_1 e^{2x} + c_2 e^{-2x}$

~~therefore~~ we are looking for $g_1(x), g_2(x)$, therefore: $\begin{cases} g_1'(x)e^{2x} + g_2'(x)e^{-2x} = 0 \\ g_1'(x)\cdot 2e^{2x} - g_2'(x)2e^{-2x} = 8\cosh(x) \end{cases}$

$$\Rightarrow \begin{pmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{pmatrix} \begin{pmatrix} g_1'(x) \\ g_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 8\cosh x \end{pmatrix}$$

$= e^{x})$

therefore ~~po~~ according to the Kramer rule we'll get:

$$g_1'(x)=\frac{\begin{vmatrix}0 & e^{-2x}\\ 8\cosh x & -2e^{-2x}\end{vmatrix}}{\begin{vmatrix}e^{2x} & e^{-2x}\\ 2e^{2x} & -2e^{-2x}\end{vmatrix}}=\frac{-8\cosh(x)\,e^{-2x}}{-2-2}=2e^{-2x}\cdot\cosh(x)=e^{-x}+e^{-3x}$$

$$g_2'(x)=\frac{\begin{vmatrix}e^{2x} & 0\\ 2e^{2x} & 8\cosh x\end{vmatrix}}{\begin{vmatrix}e^{2x} & e^{-2x}\\ 2e^{2x} & -2e^{-2x}\end{vmatrix}}=\frac{8e^{2x}\cosh x-0}{-4}=-2e^{2x}\cdot\cosh(x)=e^{-3x}-e^{x}$$

$$\Rightarrow y_p(x)=e^{2x}\int e^{-x}+e^{-3x}\,dx-e^{-2x}\int e^{3x}\cdot e^{x}dx=e^{2x}\left(-e^{-x}-\frac{1}{3}e^{-3x}\right)-e^{-2x}\left(\frac{1}{3}e^{3x}-e^{x}\right)$$

$$=-e^{x}-\frac{1}{3}e^{-x}-\frac{1}{3}e^{x}-e^{-x}=-\frac{8}{3}\cosh x\Rightarrow y(x)=c_1e^{2x}+c_2e^{-2x}-\frac{8}{3}\cosh x$$

general solution.

(c) $y''+4y=\frac{1}{\cos 2x},\quad -\frac{\pi}{4}<x<\frac{\pi}{4}$

$\lambda^2+4=0\Rightarrow \lambda_1=2i\quad \lambda_2=-2i\Rightarrow y_h(x)=c_1\cos 2x+c_2\sin 2x$

$$\Rightarrow g_1'(x)=\frac{\begin{vmatrix}0 & \sin 2x\\ \sec 2x & 2\cos 2x\end{vmatrix}}{\begin{vmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{vmatrix}}=\frac{-\tan 2x}{2}:\ g_1(x)=\int-\frac{\tan 2x}{2}\,dx=\frac{1}{4}\ln(\cos 2x)$$

$$g_2'(x)=\frac{\begin{vmatrix}\cos 2x & 0\\ -2\sin 2x & \sec 2x\end{vmatrix}}{\begin{vmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{vmatrix}}=\frac{1}{2}\Rightarrow g_2(x)=\int\frac{1}{2}dx=\frac{x}{2}$$

$$\Rightarrow y_p(x)=\frac{1}{4}\cos 2x\ln(\cos 2x)+\sin(2x)\cdot\frac{x}{2}$$

$$\Rightarrow y(x)=c_1\cos(2x)+c_2\sin(2x)+\frac{1}{4}\cos(2x)\ln(\cos 2x)+\frac{x}{2}\cdot\sin(2x)$$

general solution.

(d) $y''-4y'+4y=\frac{e^{2x}}{x^2},\quad x>0$

$\lambda^2-4\lambda+4=0\Rightarrow(\lambda-2)^2=0\Rightarrow\lambda=2\Rightarrow y_h(x)=c_1e^{2x}+c_2xe^{2x}$

$$g_1'(x)=\frac{\begin{vmatrix}0 & xe^{2x}\\ \frac{e^{2x}}{x^2} & e^{2x}+2xe^{2x}\end{vmatrix}}{\begin{vmatrix}e^{2x} & xe^{2x}\\ 2e^{2x} & e^{2x}(2x+1)\end{vmatrix}}=\frac{-\frac{e^{4x}}{x}}{e^{4x}(2x+1)-2xe^{4x}}=-\frac{1}{x}\Rightarrow g_1(x)=\ln(x)$$

$$g_2'(x)=\frac{\begin{vmatrix}e^{2x} & 0\\ 2e^{2x} & \frac{e^{2x}}{x^2}\end{vmatrix}}{\begin{vmatrix}e^{2x} & xe^{2x}\\ 2e^{2x} & e^{2x}(2x+1)\end{vmatrix}}=\frac{\frac{e^{4x}}{x^2}}{e^{4x}}=\frac{1}{x^2}\Rightarrow g(x)=-\frac{1}{x}$$

$y_p(x) = -\ln(x)e^{2x} - e^{2x} \Rightarrow y(x) = c_1 e^{2x} + c_2 x e^{2x} - \ln(x) e^{2x} - e^{2x}$

$\Rightarrow y(x) = e^{2x}(c_1 + c_2 x - \ln(x))$ general solution

ⓘ $y'' - 2y' + y = e^x \ln x$ , $y(1) = y'(1) = 0$

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1 \Rightarrow y_h(x) = c_1 e^x + c_2 x e^x$

$$g_1'(x) = \frac{\begin{vmatrix} 0 & xe^x \\ e^x \ln x & (x+1)e^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}} = \frac{-xe^{2x}\ln x}{e^{2x}(x+1-x)} = -x\ln x \Rightarrow x\ln x \Rightarrow g_1(x) = \int -x\ln x\,dx = -\ln(x)\frac{x^2}{2} + \int \frac{x}{2}dx$$

$$g_1(x) = \frac{x^2}{2}\left(\frac{1}{2} - \ln(x)\right)$$

$$g_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x \ln x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}} = \frac{e^{2x}\ln x}{e^{2x}\cdot 1} = \ln x \Rightarrow g_2(x) = \int \ln x\,dx = x\ln x - x$$

$\Rightarrow y(x) = c_1 e^x + c_2 x e^x + \frac{x^2}{2} e^x\left(\frac{1}{2} - \ln(x)\right) + x^2 e^x (\ln(x) - 1)$ general solution

$y(1) = 0 \Rightarrow c_1 e + c_2 e + \frac{e}{4} - e = 0 \Rightarrow c_1 = \frac{3}{4} - c_2$

$y'(1) = 0 \Rightarrow c_1 e + c_2(e + e) + (2e + e)\left(-\frac{3}{4}\right) + \frac{1}{2}e = 0 \Rightarrow c_2 = 1 \Rightarrow c_1 = -\frac{1}{4}$

$\Rightarrow y(x) = e^x\left(-\frac{1}{4} + x + x^2\left(\frac{1}{2}\ln(x) - \frac{3}{4}\right)\right)$ general solution