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③ ③ $\int \frac{1}{x^2-3x+4}dx = \int \frac{1}{(x-\frac{3}{2})^2+\frac{7}{4}}dx = \int \frac{\frac{4}{7}}{\frac{4}{7}\left((x-\frac{3}{2})^2+\frac{7}{4}\right)}dx$

$\underbrace{x^2-3x+4}_{}= (x+a)^2+b = x^2+2ax+a^2+b$

$-3=2a \qquad 4=(\frac{-3}{2})^2+b$

$a=-1.5 \qquad 4-\frac{9}{4}=b = \frac{7}{4}$

$= \frac{4}{7}\int \frac{1}{\frac{4}{7}(x-\frac{3}{2})^2+1}dx = \frac{4}{7}\int \frac{1}{(\frac{2}{\sqrt{7}})^2(x-\frac{3}{2})^2+1}dx$

$= \frac{4}{7}\int \frac{1}{(\frac{2}{\sqrt{7}}x-\frac{3}{\sqrt{7}})^2+1}dx = \Bigg/ \begin{array}{l} u=\frac{2}{\sqrt{7}}x-\frac{3}{\sqrt{7}} \\ du=\frac{2}{\sqrt{7}}dx \\ dx=\frac{\sqrt{7}}{2}du \end{array} \quad \frac{4}{7}\cdot\frac{\sqrt{7}}{2}\int \frac{1}{u^2+1}du =$

$\frac{2}{\sqrt{7}}\arctan u + C = \frac{2}{\sqrt{7}}\arctan(\frac{2}{\sqrt{7}}x-\frac{3}{\sqrt{7}}+C$

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תרגיל M - חדו"א 1

שאלה 1:

נסמן: $\int f(x) = F(x)$

① $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

$\int_b^c f(x)dx = [F(x)]_b^c = F(c) - F(b)$

נקבל ש:

$\int_a^b f(x)dx + \int_b^c f(x)dx = F(b) - F(a) + F(c) - F(b)$

נקבל:

$F(c) - F(a) = [F(x)]_a^c = \int_a^c f(x)dx$

② $\int_{-1}^{1} |x|dx = \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx = -\int_{-1}^{0} x\,dx + \int_0^1 x\,dx =$

$= -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 = -(0 - \frac{1}{2}) + (\frac{1}{2} - 0) = 1$

③ $\int_0^4 (1+|x-3|)^2 dx = \int_0^3 (1-(x-3))^2 dx + \int_3^4 (1-x-3)^2 =$

$\int_0^3 (16 - 8x + x^2)dx = \left[16x - 4x^2 + \frac{x^3}{3}\right]_0^3 = (16\cdot 3 - 4\cdot 9 + 9) = 23\frac{1}{3}$

$\int_3^4 (x^2 + 4x + 4)dx = \left[\frac{x^3}{3} + 2x^2 + 4x\right]_3^4$

שאלה 2

① $\int_a^b f(-x)dx = \bigg/ \begin{matrix} u = -x \\ du = -dx \end{matrix} = \int_{-a}^{-b} f(u) \cdot -du = -\int_{-a}^{-b} f(u)du =$

$\int_{-b}^{-a} f(u)du = [F(u)]_{-b}^{-a} = F(-a) - F(-b) = -F(-b) + F(-a)$

② $\int_{-a}^{a} f(x)dx = \int_a^0 f(x)dx + \int_0^a f(x)dx = \int_{-a}^{0} f(-x)dx + \int_0^a f(x)dx$

מהסעיף הקודם נקבל ש: (נסמן F היא פונ' הקדומה של f)

$\int_{-a}^{0} f(-x) = F(--a) - F(-0) = F(a) - F(0) = [F(x)]_0^a = \int_0^a f(x)dx$

(כי זוגית:

$\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_a^a f(x)dx = 2\int_0^a f(x)dx$

③ (i) $\int_{-1}^{1} |x|dx = 2\int_0^1 x\,dx = 2\left[\frac{x^2}{2}\right]_0^1 = 2(\frac{1}{2} - 0) = 1$

(ii) $\int_{-2}^{2} x^2 dx = 2\int_0^2 x^2 dx = 2\left[\frac{x^3}{3}\right]_0^2 = 2(\frac{8}{3} - 0) = \frac{16}{3}$

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(4) $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$= \int_{0}^{-a} -f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{-a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$

נחשב לפי סעיף 1: נסמן $\int F(x) = F$

$\int_{0}^{-a} f(-x) = -F(-\,-a) + F(-0) = F(0) - F(a) = [F(x)]_{a}^{0} = \int_{a}^{0} f(x)\,dx$

נציב בלי מינוס

$\int_{-a}^{a} f(x)\,dx = \int_{a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

לפי הנוסחה בסעיף 1

$\int_{a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{a}^{a} f(x)\,dx = 0$

(5) $\int_{-12}^{12} x^3\,dx =$

x^3 הוא זוגי אי זוגי נבדוק:

$\int_{-12}^{12} x^3\,dx = 0$

הוכחה שהפונ׳ אי זוגית

$f(-x) = (-x)^3 = -x^3 = -f(x)$

שאלה 3:

(1) $\int \frac{x^2+1}{x^2-1}\,dx$

חלוקת פולינום:

$$\begin{array}{r|l} 1 & \\ \hline x^2+1 & x^2-1 \\ x^2-1 & \\ \hline =2 & \end{array}$$

קיבלנו:

$1 + \frac{2}{x^2-1}$

$\int \frac{x^2+1}{x^2-1}\,dx = \int 1\,dx + 2\int \frac{1}{x^2-1}\,dx$

נחשב באמצעות פירוק לשברים חלקיים:

$\frac{1}{x^2-1} = \frac{1}{x^2-1^2} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$

נשווה מונה למונה

$1 = A(x+1)B(x-1)$

נציב $0, -1, 1$

$x=-1: \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$

$x=1: \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

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כעת:

נציב נוסח:

$$\frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

נציב חזרה ואינטגרל:

$$\int \frac{x^2+1}{x^2-1} = \int 1\,dx + \int \frac{1}{2(x-1)}\,dx - \int \frac{1}{2(x+1)}$$

$$= x + \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$\rightarrow \int \frac{1}{x-1}\,dx = \Big/ \begin{matrix} u = x-1 \\ du = dx \end{matrix} = \int \frac{1}{u}\,du = \ln|u| = \ln(x-1) + c$$

$$\rightarrow \int \frac{1}{x+1}\,dx = \Big/ \begin{matrix} u = x+1 \\ du = dx \end{matrix} = \int \frac{1}{u}\,du = \ln|u| = \ln(x+1) + c$$

נציב:

$$\int \frac{x^2+1}{x^2-1}\,dx = x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$$

② $\int \frac{2x-4}{x^2-x}\,dx$

פירוק לשברים חלקיים:

$$\frac{2x-4}{x^2-x} = \frac{2x-4}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$$

$$= \frac{Ax - A + Bx}{x(x-1)}$$

נשווה מונה למונה:

$$2x-4 = Ax - A + Bx$$

השוואת מקדמים:

$$2x-4 = (A+B)x - A$$

$$\begin{cases} 2 = A+B \Rightarrow 2 = 4+B \Rightarrow B = -2 \\ -4 = -A \Rightarrow A = 4 \end{cases}$$

$$\frac{2x-4}{x^2-x} = \frac{4}{x} - \frac{2}{x-1}$$

ולכן

$$\int \frac{2x-4}{x^2-x} = \int \frac{4}{x}\,dx - \int \frac{2}{x}\,dx$$

$$= 4\int \frac{1}{x}\,dx - 2\int \frac{1}{x-1}\,dx$$

$$\rightarrow \int \frac{1}{x}\,dx = \ln|x| + c$$

$$\rightarrow \int \frac{1}{x-1}\,dx = \Big/ \begin{matrix} u = x-1 \\ du = dx \end{matrix} = \int \frac{1}{u}\,du = \ln|u| = \ln|x-1| + c$$

וקיבלנו!

$$\int \frac{2x-4}{x^2-x} = 4\ln|x| - 2\ln|x-1| + c$$

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בס"ד

(4) $\int \frac{1}{x^3+x}\,dx =$

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$$= \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$$

נשווה מונה למונה:

$1 = Ax^2+A+Bx^2+Cx$

השוואת מקדמים:

$1 = (A+B)x^2+Cx+A$

$$\begin{cases} 0 = A+B \Rightarrow A=-B \\ 0 = C \\ 1 = A \Rightarrow B=-1 \end{cases}$$

קיבלנו:

$$\frac{1}{x^3+x} = \frac{1}{x} - \frac{x}{x^2+1}$$

כעת נחשב אינטגרל:

$$\int \frac{1}{x^3+x}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx$$

$\rightarrow \int \frac{1}{x}\,dx = \ln|x|+c$

$\rightarrow \int \frac{x}{x^2+1}\,dx = \Big/ \begin{matrix} u=x^2 \\ du=2x\,dx \end{matrix} = \int \frac{x}{u+1}\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u+1}\,du = \frac{1}{2}\ln|u+1|+c$

קיבלנו!

$$\int \frac{1}{x^3+x} = \ln|x| - \frac{1}{2}\ln|u+1| + c$$

(5) $\int \frac{1}{x^3-x}\,dx$

פירוק לשברים חלקיים

$$\frac{1}{x^3-x} = \frac{1}{x(x^2-1)} = \frac{A}{x} + \frac{Bx+C}{x^2-1} = \frac{A(x^2-1)+(Bx+C)x}{x(x^2-1)}$$

$$= \frac{Ax^2-A+Bx^2+Cx}{x(x^2-1)}$$

נשווה מונה למונה!

השוואת מקדמים:

Ax^2-A+Bx^2+Cx-1

$$\begin{cases} 0 = A+B \Rightarrow 0=1+B \Rightarrow B=-1 \\ 0 = C \\ -1 = A \end{cases}$$

קיבלנו!

$$\frac{1}{x^3-x} = \frac{1}{x} - \frac{1}{x^2-1}$$

חמן!

$$\int \frac{1}{x^3-x} = \int \frac{1}{x}\,dx - \int \frac{1}{x^2-1}\,dx$$

$\rightarrow \int \frac{1}{x}\,dx = \ln|x|$

$\rightarrow \int \frac{1}{x^2-1}\,dx$

פירוק לשברים חלקיים!

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רסק

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} =$$

$$\frac{Ax+A+Bx+B}{(x-1)(x+1)}$$

השוואת מקדמים:

$$\begin{cases} 0 = A+B \Rightarrow A=-B \\ 1 = A-B \Rightarrow 1=-2B \Rightarrow \boxed{B=-\frac{1}{2},\ A=\frac{1}{2}} \end{cases}$$

קיבלנו:

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{2(x-1)} - \frac{1}{2(x-1)}\,dx =$$

$$\int \frac{1}{2(x-1)}\,dx - \int \frac{1}{2(x+1)}\,dx$$

$$\rightarrow \int \frac{1}{2(x-1)}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx = \Big/ \begin{matrix} u = x-1 \\ du = dx \end{matrix} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| =$$

$$\frac{1}{2}\ln|x-1| + c$$

$$\rightarrow \int \frac{1}{2(x+1)}\,dx = \frac{1}{2}\ln|x+1| + c$$

נציב הכל חזרה!

$$\int \frac{1}{x^3-x}\,dx = \ln|x| - \left[\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right] + C$$

$$⑥ \int \frac{3x^2-x+1}{x^3-x^2}\,dx$$

פרוק לשברים חלקיים:

$$\frac{3x^2-x+1}{x^3-x^2} = \frac{3x^2-x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax\cdot(x-1)+B(x-1)+Cx^2}{x^2(x-1)}$$

$$= \frac{Ax^2-Ax+Bx-B+Cx^2}{x^2(x-1)}$$

נשווה מונה למונה:

$$3x^2-x+1 = Ax^2-Ax+Bx-B+Cx^2$$

השוואת מקדמים:

$$\begin{cases} 3 = A+C \Rightarrow -A = 3-C & A=0 \\ -1 = -A+B & \\ 1 = -B \Rightarrow B=-1 & C=3 \end{cases}$$

קיבלנו:

$$\frac{3x^2-x+1}{x^3-x^2} = \frac{-1}{x^2} + \frac{3}{x-1}$$

ולכן

$$\int \frac{3x^2-x+1}{x^3-x^2}\,dx = -\int \frac{1}{x^2}\,dx + 3\int \frac{1}{x-1}\,dx$$

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$\rightarrow \int \frac{1}{x^2}dx = \int x^{-2}dx = \frac{-1}{x} + c$ כפול

$\rightarrow \int \frac{1}{x-1}dx = \Big/ \begin{matrix} u = x-1 \\ du = dx \end{matrix} = \int \frac{1}{u}du = \ln|u| = \ln|x-1| + c$

נציב:

$\int \frac{3x^2 - x + 1}{x^3 - x^2}dx = \frac{1}{x} + 3\ln|x-1| + c$

⑦ $\int \frac{x^3 - 2x^2 + 2x - 2}{x^2 + 1}$

חלוקת פולינום:

$$\begin{array}{r|l} x^3 - 2x^2 + 2x - 2 & x^2 + 1 \\ \hline x^3 + x & x - 2 \\ -2x^2 + x - 2 & \\ -2x^2 - 2 & \\ \hline = x & \end{array}$$

נקבל:

$\frac{x^3 - 2x^2 + 2x - 2}{x^2 + 1} = x - 2 + \frac{x}{x^2 + 1}$

$\int \frac{x^3 - 2x^2 + 2x - 2}{x^2 + 1}dx = \int x\,dx - \int 2\,dx + \int \frac{x}{x^2+1}dx$

$= \frac{x^2}{2} - 2x + \int \frac{x}{x^2 + 1}dx$

נקבל:

$\int \frac{x}{x^2+1}dx = \Big/ \begin{matrix} u = x^2 + 1 \\ du = 2x\,dx \end{matrix} = \int \frac{x}{u}\frac{du}{2x}$

$= \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|x^2 + 1| + c$

נקבל:

$\int \frac{x^3 - 2x^2 + 2x - 2}{x^2 + 1}dx = \frac{x^2}{2} - 2x + \frac{1}{2}\ln|x^2 + 1| + c$

⑧ $\int \frac{x^3 + 5x^2 + 11x + 7}{x^2 + 4x + 5}dx =$

חלוקת פולינום:

$$\begin{array}{r|l} x^3 + 5x^2 + 11x + 7 & x^2 + 4x + 5 \\ \hline x^3 + 4x^2 + 5x & x + 1 \\ \hline = x^2 + 6x + 7 & \\ x^2 + 4x + 5 & \\ \hline 2x + 2 & \end{array}$$

נקבל ש:

$\frac{x^3 + 5x^2 + 11x + 7}{x^2 + 4x + 5} = x + 1 + \frac{2x - 2}{x^2 + 4x + 5}$

$\int \frac{x^3 + 5x^2 + 11x + 7}{x^2 + 4x + 5}dx = \int x + 1\,dx + \int \frac{2x + 2}{x^2 + 4x + 5}$

$\int x + 1\,dx = \frac{x^2}{2} + x + c$

$\int \frac{2x + 2}{x^2 + 4x + 5}dx$

נחפש להביא את המונה לצורת הנגזרת $2x + 4$ ש ימחקו:

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$$\int \frac{2x+2}{x^2+4x+5}dx = \int \frac{2x+4}{x^2+4x+5}dx - \int \frac{2}{x^2+4x+5}dx$$

$$\rightarrow \int \frac{2x+4}{x^2+4x+5}dx = \Big/ \begin{matrix} u = x^2+4x+5 \\ du = 2x+4\,dx \end{matrix} = \int \frac{1}{u}du = \ln|u| =$$

$$\ln|x^2+4x+5| + c$$

$$\rightarrow \int \frac{2}{x^2+4x+5}dx = 2\int \frac{1}{x^2+4x+5}dx = 2\ln|x^2+4x+5| + c$$

קיבלנו:

$$\int \frac{2x+2}{x^2+4x+5}dx = \ln|x^2+4x+5| - 2\ln|x^2+4x+5| + c$$

לכן:

$$\int \frac{x^3+5x^2+11x+7}{x^2+4x+5}dx = \frac{x^2}{2} + x + \ln|x^2+4x+5| - 2\ln|x^2+4x+5| + c$$

שאלה 4:

$$① \ \frac{1}{e^x+1}dx = \int \frac{e^x}{e^{2x}+e^x}dx = \Big/ \begin{matrix} u = e^x \\ du = e^x dx \end{matrix} = \int \frac{u^1}{u^2+u}du$$

נפרק לשברים חלקיים:

$$\frac{1}{u^2+u} = \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1} = \frac{A(u+1)+Bu}{u(u+1)} =$$

השוואת מקדמים:

$$1 = Au + A + Bu$$

$$\begin{cases} 0 = A+B \Rightarrow A = -B \\ 1 = A, \ B = -1 \end{cases}$$

$$\frac{1}{u^2+u} = \frac{1}{u} - \frac{1}{u+1}$$

קיבל:

$$\int \frac{1}{u^2+u}du = \int \frac{1}{u}du - \int \frac{1}{u+1}du = \ln|u| - \ln|u+1| + c =$$

קיבלנו:

$$\frac{1}{e^x+1}dx = \ln|e^x| + \ln|e^x+1| + c$$

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$$② \int_0^1 \frac{1}{e^{-x}+1}dx = \Big/ \begin{matrix} u = -x \\ du = -dx \end{matrix} = \int_0^{-1} \frac{1}{e^{u}+1} \, -du = -\int_0^{-1} \frac{1}{e^{u}+1} du$$

$$= \int_{-1}^{0} \frac{1}{e^{u}+1} du = \int_{-1}^{0} \frac{1 \cdot e^{u}}{(e^{u}+1)e^{u}} du = \int_{-1}^{0} \frac{e^{u}}{e^{2u}+e^{u}} du = \Big/ \begin{matrix} t = e^{u} \\ dt = e^{u}du \end{matrix}$$

$$= \int_{\frac{1}{e}}^{1} \frac{1}{t^2+t} dt$$

פירוק לשברים חלקיים:

$$\frac{1}{t^2+t} = \frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{A(t+1)+Bt}{t(t+1)} = \frac{At+A+Bt}{t(t+1)}$$

השוואת מונים:

$$1 = At + A + Bt$$

השוואת מקדמים:

$$\begin{cases} 0 = A+B \Rightarrow A = -B \\ 1 = A \Rightarrow B = -1 \end{cases}$$

קיבלנו:

$$\frac{1}{t^2+t} = \frac{1}{t} - \frac{1}{t+1}$$

כלומר:

$$\int_{\frac{1}{e}}^{1} \frac{1}{t^2+t} dt = \int_{\frac{1}{e}}^{1} \frac{1}{t} - \int_{\frac{1}{e}}^{1} \frac{1}{t+1} = \Big[\ln|t|\Big]_{\frac{1}{e}}^{1} - \Big[\ln|t+1|\Big]_{\frac{1}{e}}^{1}$$

$$(0-(-1)) - (\ln(2) - \ln(|\tfrac{1}{e}+1|))) = 1 - \ln(2) + \ln|e^{-1}+1|) = 0.62011$$