ODE hw #5

1 (a) $y''+y=u(t-3)$, $y(0)=0$, $y'(0)=1$

$\mathcal{L}(y''+y)=\mathcal{L}(u(t-3)) \Rightarrow \mathcal{L}(y'')+\mathcal{L}(y)=\mathcal{L}(u(t-3))$

$\Rightarrow s^2\mathcal{L}(y)-sy(0)-y'(0)+\mathcal{L}(y)=\frac{e^{-3s}}{s}$

$\Rightarrow (s^2+1)\mathcal{L}(y)=\frac{e^{-3s}+s}{s} \Rightarrow \mathcal{L}(y)=\frac{e^{-3s}+s}{s(s^2+1)}$

$\frac{1}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}=\frac{(A+B)s^2+Cs+A}{s(s^2+1)}=\frac{1}{s}-\frac{s}{s^2+1} \Rightarrow \frac{e^{-3s}+s}{s(s^2+1)}=\frac{e^{-3s}}{s}-\frac{e^{-3s}s}{s^2+1}+\frac{1}{s^2+1}$

$y(t)=\mathcal{L}^{-1}(\mathcal{L}(y))=\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}-\frac{e^{-3s}s}{s^2+1}+\frac{1}{s^2+1}\right)$

$=\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}\right)-\mathcal{L}^{-1}\left(\frac{e^{-3s}s}{s^2+1}\right)+\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)=u(t-3)-\mathcal{L}^{-1}\left(e^{-3s}\mathcal{L}(\cos t)\right)+\sin t$

$=u(t-3)-u(t-3)\cdot\cos(t-3)+\sin(t)$

$\Rightarrow \boxed{y(t)=u(t-3)(1-\cos(t-3))+\sin(t)}$

(c) $y''+y'+\frac{5}{4}y=u(t)\sin t-u(t-\pi)\sin(t-\pi)$, $y(0)=y'(0)=0$

$\mathcal{L}(y'')+\mathcal{L}(y')+\frac{5}{4}\mathcal{L}(y)=s^2\mathcal{L}(y)-sy(0)-y'(0)+s\mathcal{L}(y)-y(0)+\frac{5}{4}\mathcal{L}(y)$

$=(s^2+s+\frac{5}{4})\mathcal{L}(y)=\mathcal{L}(u(t)\sin t)-\mathcal{L}(u(t-\pi)\sin(t-\pi))=\mathcal{L}(\sin t)-e^{-\pi s}\mathcal{L}(\sin t)$

$=(1-e^{-\pi s})\cdot\frac{1}{s^2+1} \Rightarrow \mathcal{L}(y)=\frac{1-e^{-\pi s}}{(s^2+1)(s^2+s+\frac{5}{4})}$

$\frac{1}{(s^2+1)(s^2+s+\frac{5}{4})}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+s+\frac{5}{4}}=\frac{(A+C)s^3+(A+B+D)s^2+(\frac{5}{4}A+B+C)s+(\frac{5}{4}B+D)}{(s^2+1)(s^2+s+\frac{5}{4})}$

$\Rightarrow \begin{matrix} A=-C \\ D=1-\frac{5}{4}B \end{matrix} \Rightarrow \begin{matrix} B=4A+4 \\ D=-5A-4 \end{matrix} \Rightarrow \begin{matrix} A=-\frac{16}{17} & C=\frac{16}{17} \\ B=\frac{4}{17} & D=\frac{12}{17} \end{matrix} \Rightarrow \mathcal{L}(y)=\frac{1-e^{-\pi s}}{17}\cdot\left(\frac{4-16s}{s^2+1}+\frac{16s+12}{s^2+s+\frac{5}{4}}\right)$

$=\frac{4}{17}\cdot\frac{1}{s^2+1}-\frac{4}{17}\cdot\frac{e^{-\pi s}}{s^2+1}-\frac{16}{17}\cdot\frac{s}{s^2+1}+\frac{16}{17}\cdot\frac{e^{-\pi s}s}{s^2+1}+\frac{16}{17}\cdot\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+1}-\frac{16}{17}\frac{e^{-\pi s}(s+\frac{1}{2})}{(s+\frac{1}{2})^2+1}-\frac{14}{17}\cdot\frac{e^{-\pi s}}{(s+\frac{1}{2})^2+1}$

$\Rightarrow \mathcal{L}^{-1}(\mathcal{L}(y))=\frac{4}{17}\sin t-\frac{4}{17}u(t-\pi)\sin(t-\pi)-\frac{16}{17}\cos t+\frac{16}{17}u(t-\pi)\cos(t-\pi)+\frac{16}{17}\cdot e^{-\frac{1}{2}t}\cos t$

$-\frac{16}{17}\cdot e^{-\frac{1}{2}(t-\pi)}\cdot\cos(t-\pi)\cdot u(t-\pi)+\frac{4}{17}\cdot e^{-\frac{1}{2}t}\cdot\sin t-\frac{4}{17}e^{-\frac{1}{2}(t-\pi)}\cdot\sin(t-\pi)\cdot u(t-\pi)$

$\Rightarrow y(t)=\frac{4}{17}\left(\sin t-4\cos t+e^{-\frac{1}{2}t}(\sin t+4\cos t)+u(t-\pi)\left(\sin t-4\cos t+e^{-\frac{1}{2}(t-\pi)}(\sin t+4\cos t)\right)\right)$

$=\frac{4}{17}\left((\sin t-4\cos t)(1+u(t-\pi))+e^{-\frac{t}{2}}(\sin t+4\cos t)(1+e^{\frac{\pi}{2}}u(t-\pi))\right)$