Calc 2 hw#9

(A) ① $u: \mathbb{R}^3 \to \mathbb{R}$ the function may potential to be a scalar Function

② True

③ False

(B) ③ $F(x,y) = (6xy - y^3)\hat{i} + (4y + 3x^2 - 3xy^2)\hat{j}$

$\frac{\partial P}{\partial y} = 6x - 3y^2$, $\frac{\partial Q}{\partial x} = 6x - 3y^2 = \frac{\partial P}{\partial y}$

$\frac{\partial \varphi}{\partial x} = P = 6xy - y^3 \Rightarrow \varphi(x,y) = 3x^2y - y^3x + c(y) = \frac{\partial \varphi}{\partial y} = 3x^2 - 3y^2x + c'(y)$

$\frac{\partial \varphi}{\partial y} = Q = 4y + 3x^2 - 3xy^2 \Rightarrow c'(y) = 4y \Rightarrow c(y) = 2y^2 \Rightarrow \boxed{\varphi(x,y) = 3x^2y - y^3x + 2y^2 + c}$

④ $F(x,y,z) = yz\hat{i} + xz\hat{j} + (xy + 3z^2)\hat{k}$

$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

$\frac{\partial R}{\partial y} = x = \frac{\partial Q}{\partial z}$, $\frac{\partial P}{\partial z} = y = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial x} = z = \frac{\partial P}{\partial y}$

$\frac{\partial \varphi}{\partial x} = P = yz \Rightarrow \varphi = yzx + c(y,z) \Rightarrow \frac{\partial \varphi}{\partial y} = zx + \frac{\partial c}{\partial y} = Q = xz$

$\Rightarrow \frac{\partial c}{\partial y} = 0 \Rightarrow c(y,z) = D(z) \Rightarrow \varphi(x,y,z) = xyz + D(z) \Rightarrow \frac{\partial \varphi}{\partial z} = xy + D'(z)$

$\frac{\partial \varphi}{\partial z} = R = xy + 3z^2 \Rightarrow D'(z) = 3z^2 \Rightarrow D(z) = z^3 + E \Rightarrow \varphi(x,y,z) = xyz + z^3 + E$

(C) ② $F = (y\cos x + y^2, \sin x + 2xy - 2y)$, $r(t) = (-t, \sin t)$, $\frac{\pi}{2} \le t \le \pi$

$r(\frac{\pi}{2}) = A(-\frac{\pi}{2}, 1)$, $r(\pi) = B(-\pi, 0)$

$\int_C \vec{F}\,ds = \int_A^B \vec{F} \cdot dr = \varphi(B) - \varphi(A)$

$\frac{\partial \varphi}{\partial x} = y\cos x + y^2 \Rightarrow \varphi(x,y) = y\sin x + xy^2 + c(y) \Rightarrow \frac{\partial \varphi}{\partial y} = \sin x + 2xy + c'(y)$

$= \sin x + 2xy - 2y \Rightarrow c'(y) = -2y \Rightarrow c(y) = -y^2 + D \Rightarrow \varphi(x,y) = y\sin x + xy^2 - y^2 + D$

$\Rightarrow \int_A^B \vec{F} \cdot ds = \varphi(-\pi, 0) - \varphi(-\frac{\pi}{2}, 1) = \frac{\pi}{2} + 2$

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(5) $r(t) = \frac{t^2+1}{t^2-1}\hat{i} + \cos\pi t\,\hat{j} + 2t\sin\pi t\,\vec{k}, \quad 0 \le t \le \frac{1}{2}$

$r(0) = A(-1,1,0),\ r(\frac{1}{2}) = B(-\frac{5}{3},0,1)$

$\int_C y\,dx + (x+z)\,dy + y\,dz = \varphi(B) - \varphi(A)$

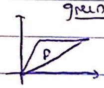
$\frac{\partial\varphi}{\partial x} = y \Rightarrow \varphi(x,y,z) = xy + c(y,z) \Rightarrow \frac{\partial\varphi}{\partial y} = x + \frac{\partial c}{\partial y} = x+z$

$\Rightarrow \frac{\partial c}{\partial y} = z \Rightarrow c(y,z) = zy + D(z) \Rightarrow \frac{\partial\varphi}{\partial z} = y + D'(z) = y$

$\Rightarrow D'(z) = 0 \Rightarrow D(z) = \epsilon \Rightarrow \varphi(x,y,z) = xy + zy + \epsilon$

$\Rightarrow \int_C y\,dx + (x+z)\,dy + y\,dz = \varphi(-\frac{5}{3},0,1) - \varphi(-1,1,0) = 1$

D) (3) green theorem:

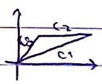


$\oint_C xy^2\,dx + (x^2+2y^3)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$= \iint_D (2x - 2xy)\,dx\,dy = \int_0^1\int_y^{3y} 2x - 2xy\,dx\,dy = \int_0^1 [x^2 - x^2y]_y^{3y}\,dy$

$= \int_0^1 9y^2 - 9y^3 - y^2 + y^3\,dy = \int_0^1 8y^2 - 8y^3\,dy = 8\left[\frac{1}{3}y^3 - \frac{1}{4}y^4\right]_0^1 = 8\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{2}{3}$

without green's theorem:



$\oint_C xy^2\,dx + (x^2+2y^3)\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy + \oint_{C_3} P\,dx + Q\,dy$

$\oint_{C_1} xy^2\,dx + (x^2+2y^3)\,dy \underset{\substack{x(t)=3t\\y(t)=t}}{=} \int_0^1 3t^3\cdot 3 + 9t^2 + 2t^3\,dt = \int_0^1 11t^3 + 9t^2\,dt = \left(\frac{11}{4}t^4 + 3t^3\right)_0^1 = 5\frac{3}{4}$

$\oint_{C_2} xy^2\,dx + (x^2+2y^3)\,dy \underset{\substack{x(t)=t\\y(t)=1}}{=} \int_3^1 t + (t^2+2)\cdot 0\,dt = \left[\frac{1}{2}t^2\right]_3^1 = \frac{1}{2} - 4\frac{1}{2} = -4$

$\oint_{C_3} xy^2\,dx + (x^2+2y^3)\,dy \underset{\substack{x(t)=t\\y(t)=t}}{=} \int_1^0 t^3 + t^2 + 2t^3\,dt = \int_1^0 3t^3 + t^2\,dt = \left[\frac{3}{4}t^4 + \frac{1}{3}t^3\right]_1^0 = -\frac{3}{4} - \frac{1}{3}$

$\Rightarrow \oint_C xy^2\,dx + (x^2+2y^3)\,dy = 5\frac{3}{4} - 4 - \frac{3}{4} - \frac{1}{3} = \frac{2}{3}$

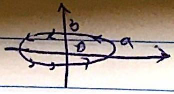
(7)

$\int_C yx^2\,dx + xy^2\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\underset{\substack{x=r\cos\theta\\y=r\sin\theta}}{=} \int_0^\pi\int_1^2 (r^3\sin^2\theta - r^3\cos^2\theta)\,r\,dr\,d\theta = \int_0^\pi\int_1^2 -r^3\cos 2\theta\,dr\,d\theta = \int_0^\pi \cos 2\theta\,d\theta \int_1^2 r^3\,dr$

$= \left[\frac{1}{2}\sin 2\theta\right]_\pi^0 \cdot \left[\frac{1}{4}r^4\right]_1^2 = 0$

⑥ ②



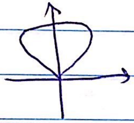
$P(x,y) = 0 \qquad Q(x,y) = x$

$$S = \oint_c x\,dy = \oint_c x(t)\frac{dy}{dt}\,dt = \oint_c x(t)\,y'(t)\,dt$$

$$\begin{aligned} x(t) &= a\cos t \\ y(t) &= b\sin t \end{aligned} \Rightarrow \begin{aligned} x'(t) &= a\sin t \\ y'(t) &= b\cos t \end{aligned} \Rightarrow S = \int_0^{2\pi} a\cos(t)\, b\cos(t)\,dt$$

$$= ab\int_0^{2\pi} \cos^2 t\,dt = ab\int_0^{2\pi} \frac{\cos 2t + 1}{2}\,dt = \frac{ab}{2}\left[\frac{1}{2}\sin 2t + t\right]_0^{2\pi} = ab\pi$$

③



$c(t) = (\sin 2t, \sin t), \quad 0 \le t \le \pi$

$Q(x,y) = x \qquad P(x,y) = 0$

$$S = \oint_c x(t)\,y'(t)\,dt \Rightarrow S = \int_0^{\pi} \sin 2t\,\cos t\,dt = \int_0^{\pi} 2\sin t\cos^2 t\,dt$$

$$= \left[-\frac{2}{3}\cos^3 t\right]_0^{\pi} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

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