LA2 HW#6

① ⓐ $f(x)=2x^4,\ g(x)=x^3,\ V=C[0,1],\ \langle f,g\rangle=\int_0^1 f(x)g(x)\,dx$

$\langle f,g\rangle=\int_0^1 2x^4\cdot x^3\,dx=\left[\frac{1}{4}x^8\right]_0^1=\frac{1}{4}$

$\|f\|=\sqrt{\langle f,f\rangle}=\sqrt{\int_0^1 4x^8\,dx}=\sqrt{\left[\frac{4}{9}x^9\right]_0^1}=\sqrt{\frac{4}{9}}=\frac{2}{3}$

$\|g\|=\sqrt{\langle g,g\rangle}=\sqrt{\int_0^1 x^6\,dx}=\sqrt{\left[\frac{1}{7}x^7\right]_0^1}=\frac{1}{\sqrt{7}}$

$\|f-g\|=\sqrt{\langle f-g,f-g\rangle}=\sqrt{\int_0^1(2x^4-x^3)^2dx}=\sqrt{\int_0^1 4x^8-4x^7+x^6\,dx}=\sqrt{\left[\frac{4}{9}x^9-\frac{1}{2}x^8+\frac{1}{7}x^7\right]_0^1}$

$=\sqrt{\frac{4}{9}-\frac{1}{2}+\frac{1}{7}}=0.295$

$\cos(\alpha)=\frac{\langle f,g\rangle}{\|f\|\cdot\|g\|}=\frac{\frac{1}{4}}{\frac{2}{3}\cdot\frac{1}{\sqrt{7}}}=\frac{3\sqrt{7}}{8}\Rightarrow\alpha=\cos^{-1}\left(\frac{3\sqrt{7}}{8}\right)=0.125\text{ rad}=7.18\text{ deg}$

ⓒ $f(x)=-2x,\ g(x)=x^2,\ V=C[-1,1],\ \langle f,g\rangle=\int_{-1}^1 f(x)g(x)\,dx$

$\langle f,g\rangle=\int_{-1}^1 -2x^3\,dx=\left[-\frac{1}{2}x^4\right]_{-1}^1=-\frac{1}{2}+\frac{1}{2}=0$

$\|f\|=\sqrt{\langle f,f\rangle}=\sqrt{\int_{-1}^1 4x^2\,dx}=2\sqrt{\left[\frac{1}{3}x^3\right]_{-1}^1}=2\sqrt{\frac{2}{3}}$

$\|g\|=\sqrt{\langle g,g\rangle}=\sqrt{\int_{-1}^1 x^4\,dx}=\sqrt{\left[\frac{1}{5}x^5\right]_{-1}^1}=\sqrt{\frac{2}{5}}$

$\|f-g\|=\sqrt{\langle f-g,f-g\rangle}=\sqrt{\int_{-1}^1(-2x-x^2)^2dx}=\sqrt{\int_{-1}^1 4x^2+4x^3+x^4\,dx}=\sqrt{\left[\frac{4}{3}x^3+x^4+\frac{1}{5}x^5\right]_{-1}^1}$

$=\sqrt{\frac{4}{3}+1+\frac{1}{5}+\frac{4}{3}-1+\frac{1}{5}}=\sqrt{\frac{46}{15}}$

~~the inner product of f and g is 0, therefore they are vertical to one another~~
therefore the angle between is 90°

ⓔ $A=\begin{pmatrix}1&0\\3&0\end{pmatrix},\ B=\begin{pmatrix}0&0\\-1&0\end{pmatrix},\ \langle A,B\rangle=\mathrm{Trace}(AB^T)$

$\cancel{\langle A,B\rangle=\mathrm{Trace}\left(\begin{pmatrix}1&0\\3&0\end{pmatrix}\begin{pmatrix}0&-1\\0&0\end{pmatrix}\right)=\mathrm{Trace}\begin{pmatrix}0&-1\\0&-3\end{pmatrix}=-3}$

$\|A\|=\sqrt{\langle A,A\rangle}=\sqrt{\mathrm{Trace}(AA^T)}=\sqrt{\mathrm{Trace}\left(\begin{pmatrix}1&0\\3&0\end{pmatrix}\begin{pmatrix}1&3\\0&0\end{pmatrix}\right)}=\sqrt{\mathrm{Trace}\begin{pmatrix}1&3\\3&9\end{pmatrix}}=\sqrt{10}$

$\|B\|=\sqrt{\langle B,B\rangle}=\sqrt{\mathrm{Trace}(BB)^T}=\sqrt{\mathrm{Trace}\left(\begin{pmatrix}0&0\\-1&0\end{pmatrix}\begin{pmatrix}0&-1\\0&0\end{pmatrix}\right)}=\sqrt{\mathrm{Trace}\begin{pmatrix}0&0\\0&1\end{pmatrix}}=\sqrt{1}=1$

$\|A-B\|=\sqrt{\langle A-B,A-B\rangle}=\sqrt{\mathrm{Trace}\left(\begin{pmatrix}1&0\\4&0\end{pmatrix}\cdot\begin{pmatrix}1&4\\0&0\end{pmatrix}\right)}=\sqrt{\mathrm{Trace}\begin{pmatrix}1&4\\4&16\end{pmatrix}}=\sqrt{17}$

$\cancel{\cos(\alpha)=\frac{\langle A,B\rangle}{\|A\|\cdot\|B\|}}=\frac{-3}{\sqrt{10}\cdot 1}=-0.94\Rightarrow\alpha=\cos^{-1}(-0.94)=2.82\text{ rad}=\cancel{161.56\text{ deg}}$

② ⓐ $x=\begin{pmatrix}x_1\\x_2\end{pmatrix},\ y=\begin{pmatrix}y_1\\y_2\end{pmatrix},\ \langle x,y\rangle=x_1y_1+x_1y_2-x_2y_1+3x_2y_2$

in order for a function to define an inner product, it needs to be: ~~Symmetrical~~ symmetrical, linear homogenous, and positive

$\langle y,x\rangle=y_1x_1+y_1x_2-y_2x_1+3y_2x_2\neq\langle x,y\rangle$

in general the function is not symmetrical therefore it can't do the inner product exp: for $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $y = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ we'll get: $\langle x,y\rangle = 25$ $\langle y,x\rangle = 29$ inner product doesn't work

ⓒ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $\langle x,y\rangle = x_3^2 + x_3 y_3 + y_3^2$

the function isn't homogenous, therefore the inner product doesn't exist:

$\langle e_3, e_3\rangle = 1^2 + 1\cdot 1 + 1^2 = 3$, $\langle 2e_3, e_3\rangle = 4 + 2 + 1 = 7 \neq 2\langle e_3, e_3\rangle = 6$

ⓓ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $\langle x,y\rangle = x_1 y_1 + x_2 y_2$

the inner product must be able: $v = 0 \iff \langle v,v\rangle = 0$

but for $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ we'll get $\langle v,v\rangle = 0$ and v isn't a 0 vector therefore the function doesn't define inner product.

③ ⓐ $f(x) = 3x-1$, $V = C[-1,1]$, $\langle f,g\rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$

$g(x) = Ax + B$

$\langle f,g\rangle = 0 \Rightarrow \int_{-1}^{1} (3x-1)(Ax+B)\, dx = 0 \Rightarrow \int_{-1}^{1} 3Ax^2 + (3B-A)x - B \, dx = 0$

$\Rightarrow \left[Ax^3 + \frac{3B-A}{2} x^2 - Bx \right]_{-1}^{1} = 0 \Rightarrow A + \frac{3B-A}{2} - B + A - \frac{3B-A}{2} - B = 0 \Rightarrow A = B$

every function from the type $g(x) = Ax + A$ is orthogonal to $f(x)$.

④ ⓐ $-\langle 0,0\rangle = \langle -1\cdot 0, 0\rangle = \langle 0,0\rangle \Rightarrow 2\langle 0,0\rangle = 0 \Rightarrow \langle 0,0\rangle = 0$

ⓑ $\langle u,0\rangle = \langle 0,u\rangle = \langle -1\cdot 0, u\rangle = -\langle 0,u\rangle = -\langle u,0\rangle \Rightarrow 2\langle u,0\rangle = 0 \Rightarrow \langle u,0\rangle = 0$

⑤ ⓑ $\|u-v\|^2 = \langle u-v, u-v\rangle = \langle u, u-v\rangle - \langle v, u-v\rangle = \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle$

$= \|u\|^2 - \langle u,v\rangle - \overline{\langle u,v\rangle} + \|v\|^2 = \|u\|^2 - 2\mathrm{Re}\langle u,v\rangle + \|v\|^2$

ⓒ based on part a,b we'll get:

$\|u+v\|^2 + \|u-v\|^2 = \|u\|^2 + 2\mathrm{Re}\langle u,v\rangle + \|v\|^2 + \|u\|^2 - 2\mathrm{Re}\langle u,v\rangle + \|v\|^2 = 2(\|u\|^2 + \|v\|^2)$

⑥ ⓑ $v = (i,1)$, $u = (1,1)$

$\langle u,v\rangle = 1\cdot(-i) + 1\cdot 1 = 1 - i$

$\frac{1}{4}(\|u+v\|^2 - \|u-v\|^2) = \frac{1}{4}(\|(1+i, 2)\|^2 - \|(1-i, 0)\|^2) =$

$\frac{1}{4}((1+i)(1-i) + 2\cdot 2 - (1-i)(1+i)) = 1 \neq 1-i$

⑧ $\|u\| = 4$, $\|v\| = 5$, $u,v \in V$

$\langle u,v\rangle = 0$

$\|3u - 2v\| = \sqrt{\langle 3u-2v, 3u-2v\rangle} = \sqrt{9\langle u,u\rangle - 6\langle u,v\rangle - 6\langle v,u\rangle + 4\langle v,v\rangle}$

$= \sqrt{9\|u\|^2 + 4\|v\|^2} = \sqrt{9\cdot 16 + 4\cdot 25} = \sqrt{244} = 15.62$

(1b) **hermiticity:** $u=(z_1,z_2,z_3)$, $v=(w_1,w_2,w_3)$, $\langle u,v\rangle = z_1\bar{w}_1+z_2\bar{w}_2+z_3\bar{w}_3$ $\langle \theta \rangle$

$\overline{\langle v,u\rangle} = \overline{w_1\bar{z}_1+w_2\bar{z}_2+w_3\bar{z}_3} = \overline{w_1\bar{z}_1}+\overline{w_2\bar{z}_2}+\overline{w_3\bar{z}_3} = \bar{w}_1 z_1+\bar{w}_2 z_2+\bar{w}_3 z_3 = \langle u,v\rangle$

linear: $\langle u+x,v\rangle = (z_1+x_1)\bar{w}_1+(z_2+x_2)\bar{w}_2+(z_3+x_3)\bar{w}_3 = \langle u,v\rangle+\langle x,v\rangle$

homogeneous: $\langle \lambda u,v\rangle = \lambda z_1\cdot\bar{w}_1+\lambda z_2\cdot\bar{w}_2+\lambda z_3\cdot\bar{w}_3 = \lambda\langle u,v\rangle$

positive: $\langle u,u\rangle = z_1\bar{z}_1+z_2\bar{z}_2+z_3\bar{z}_3 = (a_1+b_1i)(a_1-b_1i)+(a_2+b_2i)(a_2-b_2i)+(a_3+b_3i)(a_3-b_3i)$

$= a_1^2+b_1^2+a_2^2+b_2^2+a_3^2+b_3^2 \geq 0 \qquad (z_j = a_j+b_ji)$

$\langle u,u\rangle = 0$ iff $a_1^2+b_1^2+a_2^2+b_2^2+a_3^2+b_3^2=0$

therefore $a_1=a_2=a_3=b_1=b_2=b_3=0$ meaning $u=0$

the function is an inner product $\mathbb{C}$