Homework 1 Differential Equation

2.1?

① ①

$\frac{dy}{dx} = 3y \qquad y(x) = 4e^{3x}$

$y(x) = 4e^{3x} \Rightarrow y'(x) = 3 \cdot \underbrace{4e^{3x}}_{y(x)} \Rightarrow 3y$ ✓

③ $y(x) = \sin 4x \qquad \frac{d^2y}{dx^2} + 16y = 0 \Rightarrow \frac{d^2y}{dx^2} = -16y$

$y(x) = \sin 4x \Rightarrow y'(x) = 4\cos 4x \Rightarrow y''(x) = -16\sin 4x \Rightarrow -16y$ ✓

⑤ $y' + 2xy = 1 \qquad y(x) = e^{-x^2}\int_0^x e^{t^2}\,dt + ce^{-x^2}$

$y' = (-2xe^{-x^2})\left(\int_0^x e^{t^2}\,dt\right) + e^{-x^2}\left(e^{x^2}\right) - 2cxe^{-x^2}$

$= -2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1 - 2cxe^{-x^2}$

$y' + 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 2cxe^{-x^2} = 1$

$y' + 2x\left[e^{-x^2}\int_0^x e^{t^2}\,dt + ce^{-x^2}\right] = 1$

$y' + 2xy = 1$ ✓

⑦ ②

$y' = x+3 \qquad c = 0, 1, -6$

$y = \int (x+3)\,dx = \frac{x^2}{2} + 3x + 6$

Domain: all x

$c=0: \; y(x) = \frac{x^2}{2} + 3x = 0$

$x\left(\frac{x}{2} + 3\right) = 0$

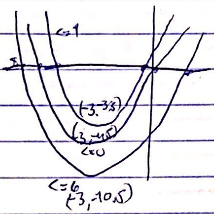
$\boxed{x=0} \qquad \frac{x}{2} = -3$

$\boxed{x=-6}$

$y' = x+3 = 0$

$x = -3$

$y(-3) = 4.5 - 9 = 4.5$



$c=1: \; y(x) = \frac{x^2}{2} + 3x + 1 = 0$

$x^2 + 6x + 2 = 0$

$x = \frac{-6 \pm \sqrt{36-8}}{2} = \frac{-6 \pm \sqrt{28}}{2}$

$= \frac{-6 \pm 2\sqrt{7}}{2} = \frac{2(-3 \pm \sqrt{7})}{2}$

$= -6$

$y(-3) = 4.5 - 9 + 1 = 3.5$

$= -3 \pm \sqrt{7} \approx -3 \pm 2.65$

$\approx -5.65, -0.35$

$c=6: \; y(x) = \frac{x^2}{2} + 3x - 6 = 0$

$x^2 + 6x - 12 = 0$

$x = \frac{-6 \pm \sqrt{36+48}}{2}$

$= \frac{-6 \pm \sqrt{84}}{2} = \frac{-6 \pm 2\sqrt{21}}{2}$

$y(-3) = 4.5 - 9 - 6 = 10.5$

$= -3 \pm \sqrt{21} \approx -3 \pm 4.58$

$\approx \boxed{-7.58, 1.58}$

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④ $y' = \frac{2}{x} + 3 \Rightarrow y = \int \left(\frac{2}{x} + 3\right) dx = 2\ln|x| + 3x + c$

$y_1(1) = 0$

$y_2(1) = 1$

$y_3(-2) = -6$

$\Rightarrow 0 = 2\ln|1| + 3(1) + c$

$0 = 3 + c$

$-3 = c$

$y_2(1) = 1 \Rightarrow 1 = 2\ln|1| + 3(1) + c$

$1 = 3 + c$

$-2 = c$

$y_3(-2) = -6 \Rightarrow -6 = 2\ln|-2| + 3(-2) + c$

$-6 = 1.34 - 6 + c$

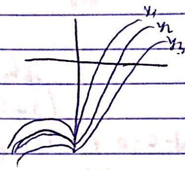
$-1.34 = c$

$y_1(x) = 2\ln|x| + 3x - 3$

$y_2(x) = 2\ln|x| + 3x - 2$

$y_3(x) = 2\ln|x| + 3x - 1.34$

Domain : $x \neq 0$



②① $y' = 3x^2+5 \Rightarrow y = \int (3x^2+5)\,dx = x^3+5x+C$

$y(1)=1$

$\boxed{y(x) = x^3+5x+C}$

$\boxed{y(x) = x^3+5x-5}$

$y(1)=1 \Rightarrow 1 = (1)^3+5(1)+C$

$1 = 1+5+C$

$1 = 6+C$

$-5 = C$

④ $y' = \ln|x-1| \Rightarrow y = \int \ln|x-1|\,dx$ $\quad u = \ln|x-1| \quad du = \frac{1}{x-1}$

$y(0)=1$ $\quad dv = 1 \quad v = x$

$= x\ln|x-1| - \int \frac{x}{x-1}\,dx$

$= x\ln|x-1| - \int \left(1+\frac{1}{x-1}\right)dx$

$= x\ln|x-1| - [x+\ln|x-1|+C]$

$\boxed{y = x\ln|x-1| - x - \ln|x-1| + C}$

$\boxed{y = x\ln|x-1| - x - \ln|x-1| + 1}$

$y(0)=1 \Rightarrow 1 = 0-0-\ln|-1|+C$

$1 = C$

③③ $(y-3)\,y' = \frac{4y}{x}$ $\quad y' = \frac{1}{x}\left(\frac{4y}{y-3}\right)$ $\quad \frac{4y}{y-3} = 0 \Rightarrow y = 0$

$\left(\frac{y-3}{4y}\right)\frac{dy}{dx} = \frac{1}{x}$ $\quad (y \neq 0, y \neq 3)$ $\quad \frac{y-3}{y}y' = 4x$

$\frac{y-3}{4y}dy = \frac{1}{x}dx$ $\quad y - 3\ln|y|$

$\int \frac{y-3}{4y}dy = \int \frac{1}{x}dx$

$\int \left(\frac{1}{4}-\frac{3}{4y}\right)dy = \ln x + C$ $\quad \boxed{\frac{1}{4}y - \frac{3}{4}\ln|y| = \ln x + C}$

⑥ $\cos^2 x \frac{dy}{dx} = y+3$ $\quad \frac{dy}{dx} = \sec^2 x(y+3)$

$y+3 = 0 \Rightarrow y(x) = 3$

$\frac{1}{y+3}\,dy = \sec^2 x\,dx$

$\int \frac{1}{y+3}\,dy = \int \sec^2 x\,dx$

$\ln|y+3| = \tan x + C$

$|y+3| = Ce^{\tan x}$ $\quad C = e^c,\ C>0$

$y+3 = Ce^{\tan x}$

$y = Ce^{\tan x} - 3$

⑨ $\frac{dy}{dx} = 3x^2(y+2) \Rightarrow \frac{1}{y+2}dy = 3x^2\,dx$

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$$\int \frac{1}{y+2}\,dy = \int 3x^2\,dx$$

$\ln|y+2| = x^3 + c$ $\quad e^{x^3+c} = e^{x^3}e^{c} = ce^{x^3}$ $\quad c>0$

$|y+2| = ce^{x^3}$ $\quad c>0$

$y+2 = ce^{x^3}$ $\quad \boxed{y = ce^{x^3} - 2}$

③ $y' + \left(\frac{1}{x}+1\right)y = e^{x}$ $\quad e^{\int\left(\frac{1}{x}+1\right)dx} = e^{\ln x + x}$

$\underbrace{e^{\ln x + x}y' + \left(\frac{1}{x}+1\right)e^{\ln x + x}y}_{} = e^{\ln x + 2x}$

$\left(e^{\ln x + x}y\right)' = xe^{2x}$ $\quad \int xe^{2x}dx \quad u = x \quad du = 1 \quad dv = e^{2x} \quad v = \frac{1}{2}e^{2x}$

$xe^{x}y = \int xe^{2x}dx$ $\quad = \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}dx$

$xe^{x}y = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c$ $\quad = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c$

$$\boxed{y = \frac{1}{2}e^{x} - \frac{1}{4x}e^{x} + \frac{c}{xe^{x}}}$$

⑤ $\sin x\, y' + y\cos x = \sin x\cos x$

$y' + y\frac{\cos x}{\sin x} = \cos x$ $\quad e^{\int\frac{\cos x}{\sin x}dx}$ $\quad u = \sin x \quad du = \cos x\,dx \Rightarrow e^{\int\frac{1}{u}du} = e^{\ln|u|}$

$= e^{\ln|\sin x|} = |\sin x|$

$\sin x\, y' + y\cos x = \sin x\cos x$

$y\sin x = \int \sin x\cos x\,dx$ $\quad u = \sin x \quad du = \cos x\,dx \Rightarrow \int u\,dv = \frac{u^2}{2} = \frac{\sin^2 x}{2}$

$y\sin x = \frac{1}{2}\sin^2 x + c$

$$\boxed{y = \frac{1}{2}\sin x + \frac{c}{\sin x}}$$

⑦ $(1-x^2)y' = xy + 2$

$(1-x^2)y' - xy = 2$

$y' - \frac{x}{1-x^2}y = \frac{2}{1-x^2}$ $\quad e^{\int\frac{x}{1-x^2}dx}$ $\quad u = 1-x^2 \quad du = -2x\,dx \quad -\frac{1}{2}du = x\,dx \Rightarrow e^{-\frac{1}{2}\int\frac{1}{u}du} = e^{-\frac{1}{2}\ln|u|}$

$= e^{-\frac{1}{2}\ln(|1-x^2|)}$

$\frac{1}{\sqrt{1-x^2}}y' - \frac{x}{(1-x^2)^{\frac{3}{2}}}y = \frac{2}{(1-x^2)^{\frac{3}{2}}}$ $\quad = \frac{1}{\sqrt{|1-x^2|}}$

$\frac{1}{\sqrt{1-x^2}}y = \int\frac{2}{(1-x^2)^{\frac{3}{2}}}dx$ $\quad x = \sin u \quad dx = \cos u\,du \quad u = \arcsin x$

$= 2\int\frac{\cos u}{(1-\sin^2 u)^{\frac{3}{2}}}du$ $\quad \sin^2 x + \cos^2 x = 1 \quad \cos^2 x = 1 - \sin^2 x$

$= 2\int\frac{\cos u}{(\cos^2 u)^{\frac{3}{2}}}du$

$= 2\int\frac{\cos u}{\cos^3 u}du$

$= 2\int\frac{1}{\cos^2 u}du$

$= 2\tan u + c$

$= 2\tan(\arcsin x) + c$

$= \frac{2x}{\sqrt{1-x^2}} + c$

$\frac{1}{\sqrt{1-x^2}}y = \frac{2x}{\sqrt{1-x^2}} + c$

$y = 2x + c\sqrt{1-x^2}$

check: $y' = 2 + c\left(\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right)$

$y' = 2 - cx(1-x^2)^{-\frac{1}{2}}$

$(1-x^2)y' = 2(1-x^2) - cx(1-x^2)^{\frac{1}{2}}$

$= 2 - 2x^2 - cx(1-x^2)^{\frac{1}{2}}$

$= 2 - x\left(2x + c(1-x^2)^{\frac{1}{2}}\right) = 2 - xy$

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1) 9) $(x-1)y' - y = (x-1)^4 \qquad y(4) = 9$

$y' - \left(\frac{1}{x-1}\right)y = (x-1)^3 \qquad e^{\int \frac{1}{x-1}dx} = e^{\ln|x-1|} = x-1$

$(x-1)y' - y = (x-1)^4$

$(x-1)y = \int (x-1)^4 dx \Rightarrow (x-1)y = \frac{1}{5}(x-1)^5 + c \Rightarrow y = \frac{1}{5}(x-1)^4 + \frac{c}{x-1}$

$9 = \frac{1}{5}(3)^4 + \frac{c}{3} \Rightarrow 9 = \frac{1}{5}(81) + \frac{c}{3} \Rightarrow 9 = 16.2 + \frac{c}{3} \Rightarrow c = 21.6 \Rightarrow y(x) = \frac{1}{5}(x-1)^4 + \frac{21.6}{x-1}$

1) 1) $y' = 1 + \frac{y}{x}$

$z = \frac{y}{x}$

$y = zx \Rightarrow y' = xz' + z$

$xz' + z = 1 + z$

$x\frac{dz}{dx} = 1$

$dz = \frac{1}{x}dx$

$z = \ln x + c$

$\boxed{y = x\ln x + c}$

$y' = \ln x + 1 + c$

$= 1 + \ln x + c$

$= 1 + \frac{x(\ln x + c)}{x}$

$= \frac{1 + x\ln x + cx}{x} = 1 + \frac{y}{x}$

2) $y' = 1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2 \qquad z = \frac{y}{x} \qquad y' = xz' + z$

$xz' + z = 1 - z + z^2$

$xz' = z^2 - 2z + 1 \qquad \frac{dz}{dx} = \left(\frac{1}{x}\right)(z^2 - 2z + 1) \qquad z^2 - 2z + 1 = 0$

$(z-1)(z-1) = 0$

$\frac{1}{(z-1)^2}dz = \frac{1}{x}dz \qquad z = 1$

$\int \frac{1}{(z-1)^2}dz = \int \frac{1}{x}dx \qquad u = z-1, \; du = dz \qquad \int \frac{1}{u^2}du = \ln x + c \Rightarrow -\frac{1}{u} = \ln x + c$

$\frac{-1}{z-1} = \ln x + c \qquad z - 1 = \frac{-1}{\ln x + c} \qquad z = \frac{-1}{\ln x + c} + 1 \qquad \boxed{y = \frac{-x}{\ln x + c} + x}$

5) $x(x+y)y' - y^2 = xy$

$x(x+y)y' = y^2 + xy$

$y' = \frac{y^2 + xy}{x^2 + xy} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \Rightarrow y' = \frac{\left(\frac{y}{x}\right)^2 + \frac{y}{x}}{1 + \frac{y}{x}} \qquad z = \frac{y}{x} \qquad y' = xz' + z$

$y = zx$

$xz' + z = \frac{z^2 + z}{1+z}$

$x\frac{dz}{dx} = \frac{z^2 + z}{1+z} - \frac{z(1+z)}{1+z} = \frac{z^2 + z}{1+z} - \frac{z^2 + z}{1+z} = 0 \qquad x\frac{dz}{dx} = 0$

7) $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$

$= \frac{1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2}{1} \qquad z = \frac{y}{x}$

$y = zx \qquad y' = xz' + z$

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$x z' + z = z^2 + z + 1$

$x z' = z^2 + 1 \Rightarrow \frac{1}{z^2+1}\, dz = \frac{1}{x}\, dx$

$\arctan z = \ln x + c \Rightarrow z = \tan(\ln x + c)$

$y = x \tan(\ln x + c$

③ ② $y' - xy = \frac{2x}{y}$ $\quad \alpha = -1 \quad z = y^2 \Rightarrow z' = 2yy'$

$yy' - xy^2 = 2x$

$\frac{1}{2} z' - xz = 2x$

$z' + 2xz = 4x \quad e^{\int 2x\,dx} = e^{x^2}$

$e^{x^2} z' + 2x e^{x^2} z = 4x e^{x^2}$

$z e^{x^2} = \int 4x e^{x^2}\, dx$

$z e^{x^2} = 2e^{x^2} + c$

$z = 2 + c e^{-x^2}$

$y = \sqrt{2 + ce^{-x^2}}$

④ $3y - 2y' = y^4 e^{3x}$

$2y' - 3y = -e^{3x} y^4 \quad \alpha = 4 \quad z = y^{1-\alpha} = y^{-3} \quad z' = -3y^{-4} y'$

$\frac{2}{y^4} y' - 3y^{-3} = -e^{3x} \quad z = \frac{1}{y^3} \quad -\frac{1}{3} z' = \frac{y'}{y^4}$

$-\frac{2}{3} z' - 3z = -e^{3x} \quad y = \sqrt[3]{\frac{1}{z}} \quad -\frac{2}{3} z' = \frac{2y'}{y^4}$

$z' + 2z = \frac{3}{2} e^{3x} \quad e^{\int 2\,dx} = e^{2x}$

$e^{2x} z' + 2e^{2x} z = \frac{3}{2} e^{5x}$

$e^{2x} z = \frac{3}{2} \int e^{5x}\, dx$

$e^{2x} z = \left(\frac{3}{2}\right)\left(\frac{1}{5}\right) e^{5x} + c \Rightarrow e^{2x} z = \frac{3}{10} e^{5x} + c \Rightarrow z = \frac{\frac{3}{10} e^{5x} + c}{e^{2x}} \Rightarrow \frac{1}{z} = \frac{e^{2x}}{\frac{3}{10} e^{3x} + c}$

$y = \sqrt[3]{\frac{e^{2x}}{\frac{3}{10} e^{3x} + c}}$

⑦ $\frac{y'}{\sqrt{y}} = e^{x^3} - \frac{4\sqrt{y}}{x}$

$y' = \sqrt{y}\, e^{x^3} - \frac{4y}{x}$

$y' + \frac{4}{x} y = y^{\frac{1}{2}} e^{x^3} \quad \alpha = \frac{1}{2} \quad z = y^{1-\alpha} = y^{1/2} \Rightarrow y = z^2$

$\frac{y'}{\sqrt{y}} + \frac{4}{x}\sqrt{y} = e^{x^3} \quad z' = \frac{y'}{2\sqrt{y}}$

$2z' = \frac{y'}{\sqrt{y}}$

$2z' + \frac{4}{x} z = e^{x^3}$ $\quad e^{\int \frac{2}{x} dx} = e^{2\ln|x|}$

$z' + \frac{2}{x} z = \frac{1}{2} e^{x^3}$

$e^{2\ln x} z' + \frac{2}{x} e^{2\ln x} z = \frac{1}{2} e^{x^3} e^{2\ln x}$

$z e^{2\ln x} = \frac{1}{2} \int e^{x^3} e^{2\ln x} dx$

$z x^2 = \frac{1}{2} \int e^{x^3} x^2 dx$

$z x^2 = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) e^{x^3} + C \Rightarrow z = \frac{1}{6} e^{x^3} x^{-2} + C x^{-2}$ $\quad y = \left(\frac{1}{6} e^{x^3} x^{-2} + C x^{-2}\right)^2$

① ② $\frac{dy}{dx} = -\frac{x}{y}$ $\quad y(3) = 4$ $\quad (0,5), (0,10), (-5,5), (-5,0)$

$y\,dy = -x\,dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + c \Rightarrow y^2 = -x^2 + c \Rightarrow y = \sqrt{c - x^2} \Rightarrow 4 = \sqrt{c-9} \Rightarrow c - 9 = 16$

$c = 25$

$y(x) = \sqrt{25 - x^2}$ $\quad (0,5)$ ✓ $\quad (-5,0)$ ✓

$(-5,5)$ ✓ $\quad (0,10)$ ✗

④ $2y' = y^3 \cos x$ $\quad y(0) = 1$ $\quad (-4,1)\ (-\infty, 0)\ (0, \infty)\ (-\infty, \infty)$

$\frac{dy}{dx} = \left(\frac{1}{2} \cos x\right)(y^3)$ $\quad y^3 = 0 \Rightarrow y = 0$

$y(x) = 0$

$y^{-3} dy = \frac{1}{2} \int \cos x\, dx$

$-\frac{1}{2} y^{-2} = \frac{1}{2} \sin x + c$ $\quad y(x) = \sqrt{\frac{1}{1 - \sin x}}$

$y^{-2} = c - \sin x$

$y^2 = \frac{1}{c - \sin x}$ $\quad \frac{1}{1 - \sin x} \to 1 - \sin x = 0$

$\sin x = 1$

$x = \frac{\pi}{2} + 2\pi k = -\frac{3\pi}{2}, \frac{\pi}{2} \ldots$

$y = \sqrt{\frac{1}{c - \sin x}}$ $\quad (-4,1)$ ✓

$1 = \sqrt{\frac{1}{c - 0}}$ $\quad (-\infty, 0)$ ✗

$(0, \infty)$ ✗

$1 = \sqrt{\frac{1}{c}} \Rightarrow c = 1$ $\quad (-\infty, \infty)$ ✗

⑥ ② $\frac{dy}{dx} = -\frac{x}{y}$ $\quad y(x) = \begin{cases} \sqrt{25 - x^2} & -5 \leq x < 0 \\ -\sqrt{25 - x^2} & 0 \leq x \leq 5 \end{cases}$ $\quad (-5, 5)$

$y\,dy = -x\,dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + c$ $\quad c - x^2 \geq 0$

$c \geq x^2$

$y^2 = -x^2 + c$ $\quad c = 25$ ✓

$y = \pm\sqrt{c - x^2}$ $\quad y(x) = \pm\sqrt{25 - x^2}$

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(3) $x^3 + y^3 = 3cxy$

$$\frac{dy}{dx} = \frac{cy - x^2}{y^2 - cx} = f(x,y) \Rightarrow f(kx, ky) = \frac{cky - k^2x^2}{k^2y^2 - ckx} = \frac{cy - kx^2}{ky^2 - cx} \times$$ this is not homogeneous

$x^3 + y^3 = 3cxy$

$3x^2 + 3y^2 y' = 3c(xy' + y)$

$x^2 + y^2 y' = cxy' + cy$

$y^2 y' - cxy' = cy - x^2$

$y'(y^2 - cx) = cy - x^2$

$y' = \frac{cy - x^2}{y^2 - cx}$ ✓

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