① ⓐ $f(x) = \dfrac{x^3-3x^2+2}{x^2-4x+3} = \dfrac{x^3-3x^2+2}{(x-3)(x-1)}$ $x \neq 3$, $x \neq 1$

$x=3$:

① $f(3) = \dfrac{2}{0} = \infty$

② $\lim\limits_{x\to 3}$

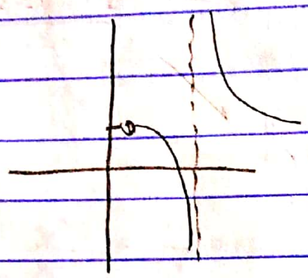
$\lim\limits_{x\to 3^+} \dfrac{x^3-3x^2+2}{(x-3)(x-1)} = \dfrac{+}{+} = +\infty$

$\lim\limits_{x\to 3^-} \dfrac{x^3-3x^2+2}{(x-3)(x-1)} = \dfrac{+}{-} = -\infty$

infinite discon

$x=1$:

① $f(1) = \dfrac{0}{0} = 0$, Removable Discont



ⓒ $f(x) = \dfrac{1}{1+5^{\frac{1}{x-1}}}$ $x \neq 1$

$x=1$: ① $f(1) = \dfrac{1}{1+5^{\frac{1}{0}}}$

$\lim\limits_{x\to 1^+} \dfrac{1}{1+5^{\frac{1}{x-1}}} = \dfrac{1}{1+5^{\frac{1}{0^+}}} = \dfrac{1}{1+5^{+\infty}} = \dfrac{1}{+\infty} = 0$

$\lim\limits_{x\to 1^-} \dfrac{1}{1+5^{\frac{1}{x-1}}} = \dfrac{1}{1+5^{\frac{1}{0^-}}} = \dfrac{1}{1+5^{-\infty}} = \dfrac{1}{1} = 1$

jump discont

ⓕ $f(x) = \begin{cases} \dfrac{1}{x+2} & x<-2 \\ x^2-3 & -2 \le x < 1 \\ x+1 & x \ge 1 \end{cases}$ Domain : all x

$x=-2$ $x=1$

will never be -2 in $\dfrac{1}{x+2}$

$x=-2$: $\lim\limits_{x\to -2^-} \dfrac{1}{x+2} = \dfrac{1}{0^-} = -\infty$

$\lim\limits_{x\to -2^+} x^2-3 = 1$

infinite Discont Type III

$\lim\limits_{x\to 1^+} x+1 = 2$

$\lim\limits_{x\to 1^-} x^2-3 = -2$

jump Discont Type II

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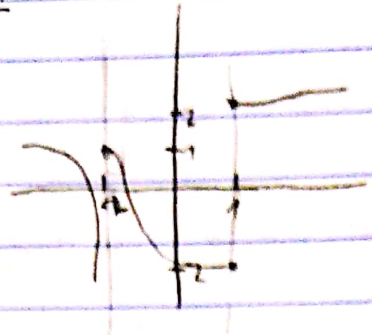
graph:



(9) $f(x) = \begin{cases} \frac{2x^3}{x-1} & x < 0 \\ \frac{\sin x}{x} & 0 < x \leq \pi \\ \frac{1}{x-4} & x > \pi \end{cases}$

domain:
$x \neq 4$
$x = 0$
$x = \pi$

$\underline{x=0}$: $\lim_{x \to 0^-} \frac{2x^3}{x-1} = \frac{0}{-1} = 0$

$\lim_{x \to 0^+} \frac{\sin x}{x} = 1$

Jump Discon

$\underline{x=\pi}$: $\lim_{x \to \pi^-} \frac{\sin x}{x} = 0$

$\lim_{x \to \pi^+} \frac{1}{x-4} = \frac{1}{-0.85}$

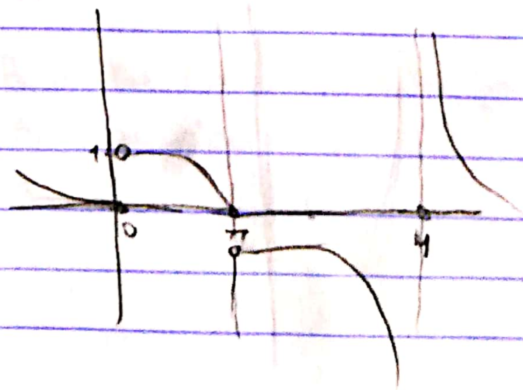
Jump Discon

$\underline{x=4}$: $\lim_{x \to 4^-} \frac{1}{x-4} = \frac{1}{4^- - 4} = -\infty$

$\lim_{x \to 4^+} \frac{1}{x-4} = \frac{1}{4^+ - 4} = +\infty$

Infinite Discon

graph:



0 little ...
?

(2) (b) $f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ A & x = 0 \end{cases}$

$\underline{x=0}$: $\lim_{x\to 0^-} \sin\frac{1}{x}$ = no limit, bounded

$\lim_{x\to 0^+} \sin\frac{1}{x}$ = no limit, bounded

$\lim_{x\to 0^-}$ bounded = $\lim_{x\to 0^+}$ bounded $\neq A$ there is a Infinite discont (Type II)

(d) $f(x) = \sin x \cdot \sin\frac{1}{x}$ Domain: $x \neq 0$ when $\frac{1}{x} = 0$ undefined

$\underline{x=0}$: $\left.\begin{aligned} &\lim_{x\to 0^+} \sin x \sin\frac{1}{x} = 0 \\ &\lim_{x\to 0^-} \sin x \sin\frac{1}{x} = 0 \end{aligned}\right\}$ Removable (Hole) Discont

(e) $f(x) = \begin{cases} e^{\frac{-1}{x^2}} & x \neq 0 \\ A & x = 0 \end{cases}$ $\rightarrow$ $-\frac{1}{x^2}$ $x = 0$ undefined

$\underline{x=0}$: $\lim_{x\to 0^+} e^{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{-1}{x^2} = \frac{-1}{0^+} = -\infty$

$\lim_{x\to 0^-} e^{\frac{-1}{x^2}} = \lim_{x\to 0^-} \frac{-1}{x^2} = \frac{-1}{0^-} = -\infty$

if $A=0$ the function would be continous, if $A \neq 0$ it would be a Removable Discont

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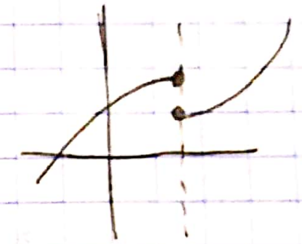
3) a) $f(x) = \begin{cases} x^3 & x < 2 \\ ax+b & 2 < x < 3 \\ x^2+2 & x^2+2 \quad x \geq 3 \end{cases}$

for which values of a & b is this continuous?

all elementary functions are continuous

$\boxed{x=2,\ x=3 \text{ may not be continuous?}}$



x=2: ① $f(2) = 2a+b$

② $\lim_{x \to 2^-} x^3 = 8$

$\lim_{x \to 2^+} ax+b = 2a+b$

$\boxed{2a+b = 8}$

x=3: ① $f(3) = 11 \quad (3^2+2)$

② $\lim_{x \to 3^-} ax+b = 3a+b$

$\lim_{x \to 3^+} x^2+2 = 11$

$\boxed{3a+b = 11}$

$$\begin{array}{r} 2a+b = 8 \\ 3a+b = 11 \\ \hline -a = -3 \end{array}$$

$\boxed{\begin{matrix} a = 3 \\ b = 2 \end{matrix}}$

Homework sheet #3

(3)(b) $f(x) = \begin{cases} ax^3 & x<2 \\ ax^2+bx & 2\leq x<3 \\ x^2+ax & x\geq 3 \end{cases}$

$x=2$: $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2)$

$\underbrace{8a = 4a+2b}$ $= 4a+2b$

$4a = 2b \Rightarrow \boxed{2a=b}$

$x=3$: $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = f(3)$

$9a+3b = 9+3a$ $= 9+3a$ $\to 6a+3b=9 \Rightarrow \boxed{2a+b=3}$

next pg continues

$2a + 2a = 3$

$4a = 3$

$\boxed{a = \frac{3}{4}}$

$\boxed{b = 2\left(\frac{3}{4}\right) = \frac{3}{2}}$

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3.07

(4) $f(x) = \begin{cases} \dfrac{3x+4}{x-2} & x \le 0 \\ \dfrac{\sin ax}{x} & 0 < x \le \pi \\ \dfrac{x^2-a}{x^2-3x-4} & x > \pi \end{cases}$ $\rightarrow (x-4)(x+1)$ Domain: $x \ne 4$

(a) $\underline{x=0}$: continous

$\underline{x=\pi}$: $\lim\limits_{x\to\pi^-} \dfrac{\sin 2x}{x} = \dfrac{0}{\pi} = 0$

$\lim\limits_{x\to\pi^+} \dfrac{x^2+2}{x^2-3x-4} = \dfrac{x^2+2}{-3.555} = -3.33$

$\underline{x=4}$: $\lim\limits_{x\to4^-} \dfrac{x^2-a}{x^2-3x-4} = \dfrac{16+2}{4^-} = -\infty$

$\lim\limits_{x\to4^+} \dfrac{x^2-a}{x^2-3x-4} = \dfrac{16+2}{4^+} = +\infty$

$\lim\limits_{x\to0^+} \dfrac{\sin ax}{x} \cdot \dfrac{a}{a} = a = -2$

$\lim\limits_{x\to0^-} \dfrac{3x+4}{x-2} = \dfrac{4}{-2} = -2$

2 discontinuity

(b) $\left.\begin{array}{l} \lim\limits_{x\to0^-} \dfrac{3x+4}{x-2} = \dfrac{4}{-2} = -2 \\ \lim\limits_{x\to0^+} \dfrac{\sin ax}{x} \cdot \dfrac{a}{a} = a \end{array}\right\} a \ne -2$ Removable Discont.

$\underline{x=4}$: $\left.\begin{array}{l} \lim\limits_{x\to4^-} \dfrac{x^2-a}{x^2-3x-4} = \dfrac{16-a}{4^-} = -\infty \\ \lim\limits_{x\to4^+} \dfrac{x^2-a}{x^2-3x-4} = \dfrac{16-a}{4^+} = \infty \end{array}\right\}$ Infinite discont.

$\underline{x=\pi}$: $\left.\begin{array}{l} \lim\limits_{x\to\pi^-} \dfrac{\sin ax}{x} = \dfrac{0}{\pi} = 0 \\ \lim\limits_{x\to\pi^+} \dfrac{x^2-a}{x^2-3x-4} = \dfrac{x^2-a}{-3.888} \end{array}\right\}$ Jump Discont

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(5) (b) $\lim_{x\to 0} \frac{\arcsin 3x}{\tan 5x} = \frac{0}{0}$ $/\cdot\frac{3x}{3x}$ $\frac{\arcsin 3x}{\tan 5x}\cdot\frac{3x}{3x} =$

$L=1$

$\frac{\frac{3x}{\frac{\sin 5x}{\cos 5x}}} = \frac{\cos 5x\cdot 3x}{\sin 5x}\cancel{\cdot\frac{5x}{5x}} = L=1 = \frac{\cos 5x\cdot 3x}{5x} = \frac{3\cos 5x}{5} = \frac{3\cdot 1}{5} = \frac{3}{5}$

(d) $\lim_{x\to +\infty}\left(5x\cdot\arctan\frac{3}{x}\right)$

$t = \arctan\frac{3}{x}$

$\tan(t) = \frac{3}{x}$

$x = \frac{3}{\tan(t)}$

$\lim_{t\to 0}\frac{5\cdot 3}{\frac{\sin(t)}{\cos(t)}}\cdot t = \frac{t}{\sin(t)}\cdot 5\cdot 3\cdot\cos(t) = 1\cdot 5\cdot 3\cdot\cos(t) = 5\cdot 3 = \boxed{15}$

$L=1$ $L=1$

(f) $\lim_{x\to -\infty}\arctan x = -\frac{\pi}{2}$

(6) (b) $\tan x - 1 + \log_2 x = 0$ $[0, 1.5]$

we want an accuracy of 0.01

1.5

$f(0.1) = 4.22$ $0+0.75$

$f(1.5) = 13.686$

$f(0.8) = 0.272$ $1.5 - 0.375$

$f(1.15) = -1.436$

$f(0.975) = -0.438$ $0.8+0.175$

$f(0.8875) = -0.0561$ $1.15 - 0.09375$

$f(0.87) = 0.0155$ $0.975 + 0.046875$

there is a root $0.87 < a < 0.8875$

using The Intermediate value Theorem:

$\textcircled{7}$ (a) $P(x) = x^5 - 4x^3 + 5x^2 + 3x + 7$

$$\lim_{x \to \infty} x^5 - 4x^3 + 5x^2 + 3x + 7 = x^5\left(1 - \frac{4}{x^2} + \frac{5}{x^3} + \frac{3}{x^4} + \frac{7}{x^5}\right) = \infty$$

$$\lim_{x \to -\infty} x^5 - 4x^3 + 5x^2 + 3x + 7 = x^5\left(1 - \frac{4}{x^2} + \frac{5}{x^3} + \frac{2}{x^4} + \frac{7}{x^5}\right) = -\infty$$

because all polynomials are continous and the polynomial approaches both ∞ & $-\infty$.

If a continous function has values of opposite signs inside an interval then it has a root in that interval

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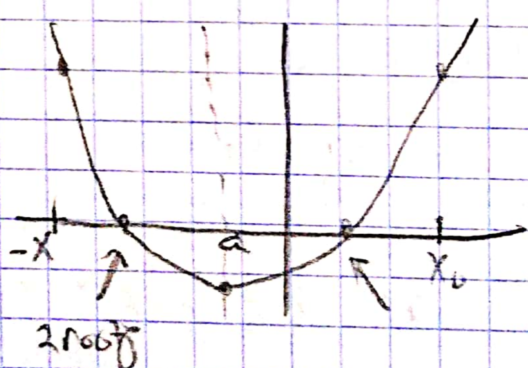
Homework Sheet #3:

7 c Prove: $Q(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0$

Even degree polynomial (n is even) & there exists $a \in \mathbb{R}$ such that $Q(a) < 0$ then the polynomial has at least 2 roots

$\lim_{x \to \infty} x^n + \dots + a_0 = \infty$

$\lim_{x \to -\infty} x^n + \dots + a_0 = +\infty$



For some large $x_0 \in \mathbb{R}$, $f(x_0) > 0$ because the $\lim_{x \to \infty} f(x) = \infty$

also $f(-x_0) > 0$ because $\lim_{x \to -\infty} f(x) = \infty$

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13 (a) Prove: If f is continuous at x_0 & $f(x_0)<0$ then there exists a neighborhood of x_0 in which $f(x_0)<0$ for all x

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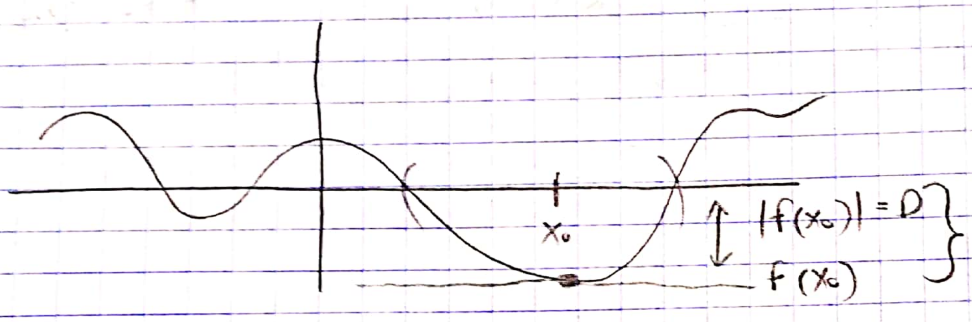

Proof: By the definition of continuity,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

By the definition of a limit, $|f(x) - f(x_0)| < \varepsilon$

whenever $|x - x_0| < \delta$ take $\varepsilon = \frac{D}{2}$

$$|f(x) - f(x_0)| < \frac{D}{2}$$

$$-\frac{D}{2} + f(x_0) < f(x) < \frac{D}{2} + f(x_0)$$

$$-\frac{3D}{2} < f(x) < \frac{D}{2}$$

There will be a neighborhood around $x = x_0$ where all the values are negative

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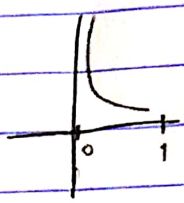
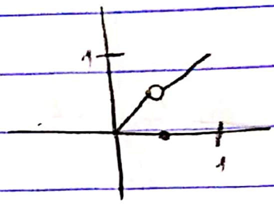
11 b $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



d $f(x) = \begin{cases} x & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases}$



12 a prove,

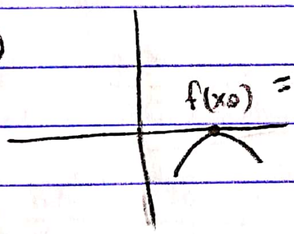
the function is continous from the right $f(x_0) = \lim\limits_{x \to x_0^+} f(x)$

from here we can understand that its not continous from the left, because if it was continous, it would be continous in x_0 because $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x)$, contradicts

b prove,

from here we can understand the $\lim\limits_{x \to x_0} f(x)$ does not exist because its given the f is not continous in x_0.

13 b



$-(x - x_0)^2 = f(x) < 0$

$f(x_0) = 0$

16 c $f(x) = \dfrac{2x^3 + x - 18}{3x^2 - 2x - 8}$

$(x-2)\left(x+\frac{4}{3}\right)3$ $\quad x = 2$

vertical: $(x-2)(3x+4)$ $\quad x = -\frac{4}{3}$

$x = 2$

$\lim\limits_{x \to 2} \dfrac{2x^3 + x - 18}{3x^2 - 2x - 8} = \dfrac{0}{0}$

$\dfrac{(\cancel{x-2})(2x^2+4x+9)}{(\cancel{x-2})(3x+4)}$

$2x^2 + 4x + 9$

$2x^3 + x - 18 \; | \; x - 2$

$2x^3 - 4x^2$

$4x^2 + x$

$4x^2 - 8x$

$9x - 18$

$9x - 18$

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$$\lim_{x\to 2} \frac{2x^2+4x+9}{3x+4} = \frac{25}{10} = 2.5$$

$$\lim_{x\to -\frac{4}{3}} \frac{2x^3+x-18}{3x^2-2x-18} = \frac{-24\frac{2}{27}}{0} = \infty$$

$$\lim_{x\to -\frac{4}{3}^+} \frac{2x^3+x-18}{3x^2-2x-18} = \frac{\cancel{(x-2)}(2x^2+4x+9)}{\cancel{(x-2)}(3x+4)} = \frac{7.22}{0^+} = \infty$$

$$\lim_{x\to -\frac{4}{3}^-} \frac{2x^2+4x+9}{3x+4} = \frac{7.22}{0^-} = -\infty$$

<u>horizontal</u> :

$$\lim_{x\to\infty} \frac{2x^3+x-18}{3x^2-2x-18} = \frac{\cancel{x^2}(2x+\underbrace{\frac{1}{x}}_{L=0}-\underbrace{\frac{18}{x^2}}_{L=0})}{\cancel{x^2}(3-\underbrace{\frac{2}{x}}_{L=0}-\underbrace{\frac{18}{x^2}}_{L=0})} = \frac{2x}{3} = \infty$$

$$\lim_{x\to-\infty} \frac{2x^3+x-18}{3x^2-2x-18} = \frac{2x}{3} = -\infty$$

no horizontal asymptote

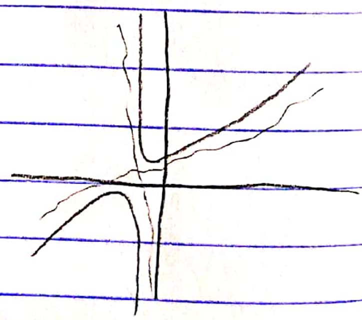
<u>slant</u> :

$$\lim_{x\to\infty} \frac{\frac{2x^3+x-18}{3x^2-2x-18}}{x} = \frac{2x^3+x-18}{3x^3-2x^2-18x} = \frac{\cancel{x^3}(2+\underbrace{\frac{1}{x^2}}_{L=0}-\underbrace{\frac{18}{x^3}}_{L=0})}{\cancel{x^3}(3-\underbrace{\frac{2}{x}}_{L=0}-\underbrace{\frac{18}{x^2}}_{L=0})} = \boxed{\frac{2}{3} = a}$$

$$\lim_{x\to\infty} \frac{2x^3+x-18}{3x^2-2x-18} - \frac{2}{3}x = \frac{2x^3+x-18-\frac{2}{3}x(3x^2-2x-18)}{3x^2-2x-18} =$$

$$\frac{\cancel{2x^3}+x-18-\cancel{2x^3}+\frac{4}{3}x^2+12x}{3x^2-2x-18} = \frac{\frac{4}{3}x^2+13x-18}{3x^2-2x-18} = \frac{\cancel{x^2}(\frac{4}{3}+\underbrace{\frac{13}{x}}_{L=0}-\underbrace{\frac{18}{x^2}}_{L=0})}{\cancel{x^2}(3-\underbrace{\frac{2}{x}}_{L=0}-\underbrace{\frac{18}{x^2}}_{L=0})} =$$

$$\frac{\frac{4}{3}}{3} = \boxed{\frac{4}{9} = b} \qquad \boxed{y = \frac{2}{3}x + \frac{4}{9}}$$



(e) $f(x) = 2x - \sqrt{4x^2 + 7}$

vertical: no vertical asymptote → Domain: all x

horizontal: $\lim_{x \to \infty} 2x - \sqrt{4x^2+7} = \infty$

$\lim_{x \to -\infty} 2x + \sqrt{4x^2+7} = -\infty$

slant: $\lim_{x \to \infty} \dfrac{2x + \sqrt{4x^2+7}}{x} = \dfrac{\frac{2x}{x} \pm \sqrt{\frac{4x^2}{x^2} + \frac{7}{x^2}}}{\frac{x}{x}} = \dfrac{2 + \sqrt{4 + \frac{7}{x^2}}}{1} =$

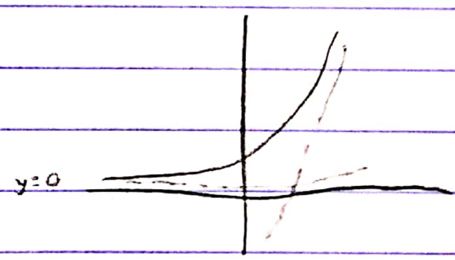
$2 + \sqrt{4} = \boxed{4} = a$

$\lim_{x \to \infty} 2x + \sqrt{4x^2 - 7} - 4x = \dfrac{\left(\sqrt{4x^2+7} - 2x\right)\left(\sqrt{4x^2+7} + 2x\right)}{\sqrt{4x^2+7} + 2x}$

$\dfrac{4x^2 + 7 - 4x^2}{\sqrt{4x^2+7} + 2x} = \dfrac{7}{\sqrt{4x^2+7} + 2x} = \dfrac{\frac{7}{x}}{\sqrt{4 + \frac{7}{x^2}} + 2} = \dfrac{7}{x\left(4 + \sqrt{\frac{7}{x^2}}\right)} = \dfrac{7}{\infty} = 0$

$\boxed{y = 4x}$

horizontal asymptote at $y = 0$



(9) $f(x) = \sqrt{4x^2 + x}$

vertical: Domain: $x = 0$, $x = -\frac{1}{4}$

$\lim_{x \to 0} \sqrt{4x^2 + x} = 0$

$\lim_{x \to 0} \sqrt{4x^2 + x} = 0$

slant: $\lim_{x \to \infty} \sqrt{4x^2 + x} = \infty$

$\lim_{x \to -\infty} \sqrt{4x^2 + x} = -\infty$

$$\lim_{x\to+\infty} \frac{\sqrt{4x^2+x}}{x} = \frac{\sqrt{x^2(4+\frac{1}{x})}}{x} = \frac{x\sqrt{4}}{x} = 2$$

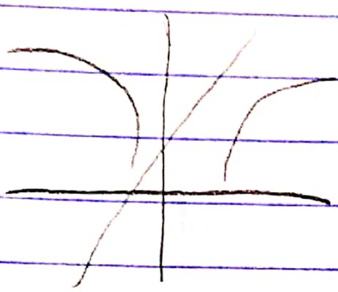
$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+x}}{x} = \frac{\sqrt{x^2(4+\frac{1}{x})}}{x} = \frac{-x\sqrt{4}}{x} = -2$$

$$\lim_{x\to+\infty} \sqrt{4x^2+x} - 2x = \frac{4x^2+x-4x^2}{\sqrt{4x^2+x}+2x} = \frac{x}{4x} = \frac{1}{4}$$

$$\lim_{x\to-\infty} \sqrt{4x^2+x} + 2x = \frac{4x^2+x-4x^2}{\sqrt{4x^2+x}-2x} = \frac{x}{-4x} = -\frac{1}{4}$$

$x \to +\infty \quad y = 2x + 0.25$

$x \to -\infty \quad y = -2x - 0.25$



(i) $f(x) = \frac{e^x}{e^x - 1}$ <u>Domain</u> : $x \neq 0$

$$\lim_{x\to 0^+} \frac{e^x}{e^x-1} = \frac{1}{0^+} = +\infty$$

$$\lim_{x\to 0^-} \frac{e^x}{e^x-1} = \frac{1}{0^-} = -\infty$$

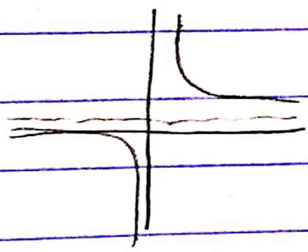
vertical asymptote $x=0$ $(y = \pm\infty)$

$$\lim_{x\to+\infty} \frac{e^x}{e^x-1} = \frac{e^x}{e^x(1-\frac{1}{e^x})} = 1$$

$$\lim_{x\to-\infty} \frac{e^x}{e^x-1} = \frac{0}{-1} = 0$$

horizontal asymptote $y=1$, $y=0$

not slant asymptote

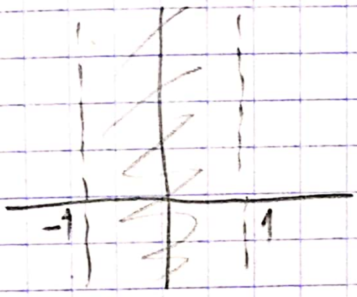


j) $f(x) = x + \arccos\frac{1}{x}$ $(x \neq 0)$

arccos x: Domain $-1 \leq x \leq 1$ $\qquad -1 \leq \cos x \leq 1$

$\arccos\frac{1}{x}$: $\qquad -1 \leq \frac{1}{x} \leq 1$

$\rightarrow x \geq 1$ or $x \leq -1$



<u>vertical asymptote</u>

x & arccos x are elementary functions.

$\frac{1}{x}$ has a discontinuity at $x = 0$

BUT $x = 0$ is not in the domain of this function

sums of elementary functions are continuous.

Scanned with CamScanner

$0 \leq \arccos X \leq \pi$

SLANT? a: $\lim\limits_{\substack{x \to \infty \\ x \to -\infty}} \dfrac{x + \arccos \frac{1}{x}}{x} = \lim\limits_{\substack{x \to \infty \\ x \to \infty}} \dfrac{1 + \arccos \frac{1}{x}}{x} =$

$1 + \underbrace{\frac{1}{x}\left(\arccos \frac{1}{x}\right)}_{0} = 1 = a$

Vanishing

bounded

b: $\lim\limits_{\substack{x \to \infty \\ x \to -\infty}} x + \arccos \frac{1}{x} - x = \lim\limits_{\substack{x \to \infty \\ x \to -\infty}} \arccos \frac{1}{x} = \arccos 0 = \frac{\pi}{2}$

$\boxed{y = x + \frac{\pi}{2}}$