
Riccati Differential Equation

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1 RICCATI DIFFERENTIAL EQUATION

1.1 Definition

The finite-horizon linear quadratic regular (LQR) optimal control problem is defined as

$$\begin{aligned} & \underset{\mathbf{u}(t)}{\text{minimize}} && \int_{t_0}^T (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} + 2\mathbf{x}^T \mathbf{S} \mathbf{u}) dt + \mathbf{x}(T)^T \mathbf{P}_T \mathbf{x}(T) \\ & \text{subject to} && \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} \\ & && \mathbf{P}(T) = \mathbf{P}_T \end{aligned} \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{Q} \in \mathbb{R}^{n \times n}$, $\mathbf{R} \in \mathbb{R}^{m \times m}$, $\mathbf{S} \in \mathbb{R}^{n \times m}$, $\mathbf{x}(T) \in \mathbb{R}^n$, and $\mathbf{P}_T \in \mathbb{R}^{n \times n}$. The solution to the finite-horizon LQR problem is

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t)$$

where

$$\mathbf{K}(t) = \mathbf{R}^{-1} [\mathbf{B}^T \mathbf{P}(t) + \mathbf{S}^T] \quad (2)$$

and where $\mathbf{K} \in \mathbb{R}^{m \times n}$ and $\mathbf{P} \in \mathbb{R}^{n \times n}$. The matrix function $\mathbf{P}(t)$ is found by solving the **Riccati differential equation backwards** in time (i.e. from $t = T$ to $t = t_0$) using the terminal condition $\mathbf{P}(T) = \mathbf{P}_T$. The Riccati differential equation is given by [5]

$$\dot{\mathbf{P}} = -[\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} - (\mathbf{P} \mathbf{B} + \mathbf{S}) \mathbf{R}^{-1} (\mathbf{B}^T \mathbf{P} + \mathbf{S}^T) + \mathbf{Q}] \quad (3)$$

1.2 Solving the ODE

The Riccati differential equation is a matrix-valued ODE of the form

$$\frac{d\mathbf{M}}{dt} = \mathbf{F}(t, \mathbf{M})$$

where $\mathbf{M} \in \mathbb{R}^{p \times q}$ (where p and q are arbitrary scalars). However, MATLAB's ODE solvers are only equipped to solve *vector*-valued ODEs of the form

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$$

where $\mathbf{y} \in \mathbb{R}^p$.

We can transform a matrix-valued ODE to a vector-valued ODE using the `odefun_mat2vec` and `odeIC_mat2vec` functions of the *ODE Solver Toolbox*, and then transform the results of the vector-valued ODE into the results of the corresponding matrix-valued ODE using the `odesol_vec2mat` function of the *ODE Solver Toolbox* [2, pp. 36–40][3].

1.3 Conditions for Existence and Uniqueness

Let's define the matrix \mathbf{M} as

$$\mathbf{M} = \begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix} \quad (4)$$

A unique solution to the Riccati differential equation exists if and only if the following conditions are satisfied [1][4, p. 35]:

1. \mathbf{M} is symmetric positive semidefinite ($\mathbf{M} \succeq 0$). If $\mathbf{S} = \mathbf{0}$, this condition reduces to the following two conditions:
 - (a) \mathbf{Q} is symmetric positive semidefinite ($\mathbf{Q} \succeq 0$).
 - (b) \mathbf{R} is symmetric positive definite ($\mathbf{R} \succ 0$).
2. \mathbf{P}_T is symmetric positive semidefinite ($\mathbf{P}_T \succeq 0$).
3. (\mathbf{A}, \mathbf{B}) stabilizable.
4. $(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{S}^T, \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T)$ detectable.
 - If $\mathbf{S} = \mathbf{0}$, this condition reduces to $(\mathbf{A}, \mathbf{Q}^{1/2})$ detectable.

REFERENCES

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