Experimental Design and Linear Estimators

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Linear Model

- $a, b, c = -1$ or 1
- y : A function of a, b, c with Gaussian noise

A linear model : $\underbrace{y = \mu + \beta_a a + \beta_b b + \beta_c c}_{\text{unknown: } \mu,\ \beta_a,\ \beta_b,\ \beta_c} + \underbrace{\varepsilon}_{\text{noise}}$

Problem Estimate some of the unknown parameters from the experiments ⇩

There are $2^3 = 8$ types of experiments.

	1	2	3	4	5	6	7	8
(μ)	1	1	1	1	1	1	1	1
a	-1	1	-1	1	-1	1	-1	1
b	-1	-1	1	1	-1	-1	1	1
c	-1	-1	-1	-1	1	1	1	1
outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Each column corresponds to an experiment.

Linear Estimator

A linear model : $y = \mu + \beta_a a + \beta_b b + \beta_c c + \varepsilon$

$(a, b, c = -1$ or $1)$

unknown: $\mu, \beta_a, \beta_b, \beta_c$ — noise: ε

$$
\begin{array}{c|cccccccc}
 & ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \\
\hline
\begin{array}{c}(\mu)\\ a\\ b\\ c\end{array} &
\multicolumn{8}{c}{\left(\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{array}\right)} \\
\text{outputs} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8
\end{array}
$$

An example of linear (unbiased) estimator of β_a

ε : ignored

$$\hat{\beta}_a = \frac{1}{2}\{-y_1 + y_5\} = \frac{1}{2}\{-(\mu - \beta_a - \beta_b - \beta_c) + (\mu + \beta_a - \beta_b - \beta_c)\} = \beta_a$$

The linear estimator of β_a : $\hat{\beta}_a = \sum_{g=1}^{8} x_{ag} Y_g$

$(x_{a1}, \dots, x_{a8} \in \boldsymbol{R})$

Design of Experiments

One of the purpose of design of experiments is to construct the optimal experimental design.

	1	2	3	4	5	6	7	8
(μ)	1	1	1	1	1	1	1	1
a	-1	-1	-1	-1	1	1	1	1
b	-1	-1	1	1	-1	-1	1	1
c	-1	1	-1	1	-1	1	-1	1
outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Problem

Choose fewer experiments among 1, …, 8 with the minimum loss of information to estimate some of the unknown parameters.

Model selection, sparse modeling, …

Group Lasso

A linear model : $\boxed{y = \mu + \beta_a a + \beta_b b + \beta_c c + \varepsilon}$

Problem

- Obtain the good linear unbiased estimators.
- Choose fewer experiments.

⇩

Assume that we want to estimate β_a and β_b .

⇩

$$\underset{\boldsymbol{x}_a, \boldsymbol{x}_b \in \boldsymbol{R}^8}{\text{minimize}} \underbrace{\left\{\|\boldsymbol{x}_a\|^2 + \|\boldsymbol{x}_b\|^2\right\}}_{\text{Variances}} + \underbrace{\sum_{g=1}^{8} \lambda_g \left|\sqrt{x_{ag}^2 + x_{bg}^2}\right|}_{\text{Sparseness}}$$

Variances
(⇨ good estimators)

Sparseness
(⇨ fewer experiments)

subject to

$$\begin{pmatrix} \boldsymbol{M}\boldsymbol{x}_a \\ \boldsymbol{M}\boldsymbol{x}_b \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_a \\ \boldsymbol{e}_b \end{pmatrix}$$

Unbiasedness
(⇨ linear unbiased estimators)

Numerical Examples

(https://github.com/tanaken-basis/explasso)

$$\boxed{y = \mu + \beta_a a + \beta_b b + \beta_c c + \varepsilon}$$

Example 1) Assume that we want to estimate $\boxed{\beta_a, \beta_b}$

⇨ $(\lambda_1, \cdots, \lambda_8) = (10, 100, 20, 200, 40, 400, 80, 800)$

⇨ Solve the problem of **Group lasso**

⇨ The optimal solution :

	①	③	⑤
(μ)	1	1	1
a	−1	−1	1
b	−1	1	−1
c	−1	−1	−1

Example 2) Assume that we want to estimate $\boxed{\beta_a, \beta_b, \beta_c}$

⇨ $(\lambda_1, \cdots, \lambda_8) = (100, 0, 0, 100, 0, 100, 100, 0)$

⇨ **Group lasso** ⇨ **L4**

	⑧	⑤	③	②
(μ)	1	1	1	1
a	1	1	−1	−1
b	1	−1	1	−1
c	1	−1	−1	1

Patterns of the solution

The selection of the experiments strongly depends on the tuning parameters!

$$\underset{\boldsymbol{x}_a,\, \boldsymbol{x}_b \in \boldsymbol{R}^8}{\text{minimize}} \ \left\{\|\boldsymbol{x}_a\|^2 + \|\boldsymbol{x}_b\|^2\right\} + \sum_{g=1}^{8} \lambda_g \left|\sqrt{x_{ag}^2 + x_{bg}^2}\right|$$

$$\text{subject to} \ \begin{pmatrix} \boldsymbol{M}\boldsymbol{x}_a \\ \boldsymbol{M}\boldsymbol{x}_b \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_a \\ \boldsymbol{e}_b \end{pmatrix}$$

λ_g : unwantedness of the g-th experiment

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
	1	2	3	4	5	6	7	8
(μ)	1	1	1	1	1	1	1	1
a	-1	-1	-1	-1	1	1	1	1
b	-1	-1	1	1	-1	-1	1	1
c	-1	1	-1	1	-1	1	-1	1
outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Patterns of the solution

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
	1	2	3	4	5	6	7	8
(μ)	1	1	1	1	1	1	1	1
a	-1	-1	-1	-1	1	1	1	1
b	-1	-1	1	1	-1	-1	1	1
c	-1	1	-1	1	-1	1	-1	1
outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Example 3) Assume that we want to estimate $\boxed{\beta_a, \beta_b}$

$\Rightarrow$ $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8$ (highly symmetric)

$\Rightarrow$ **Group lasso** $\Rightarrow$

	1	2	3	4	5	6	7	8
(μ)	1	1	1	1	1	1	1	1
a	−1	−1	−1	−1	1	1	1	1
b	−1	−1	1	1	−1	−1	1	1
c	−1	1	−1	1	−1	1	−1	1

Sparse solutions (asymmetric solutions) are never obtained from …

Patterns of the solution

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
	1	2	3	4	5	6	7	8
(μ)	1	1	1	1	1	1	1	1
a	-1	-1	-1	-1	1	1	1	1
b	-1	-1	1	1	-1	-1	1	1
c	-1	1	-1	1	-1	1	-1	1
outputs	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Example 4) Assume that we want to estimate $\boxed{\beta_a, \beta_b}$

$$\Rightarrow \lambda_1, \lambda_3, \lambda_5 < \lambda_2, \lambda_4, \lambda_6, \lambda_7, \lambda_8$$

⇨ **Group lasso** ⇨

	1	3	5
(μ)	1	1	1
a	−1	−1	1
b	−1	1	−1
c	−1	−1	−1

or

	1	3	5	7
(μ)	1	1	1	1
a	−1	−1	1	1
b	−1	1	−1	−1
c	−1	−1	−1	1

or…

Patterns of the solution

Example 5) Assume that we want to estimate $\boxed{\beta_a, \beta_b}$

$\Rightarrow$ $\boxed{\lambda_1, \lambda_2, \lambda_5 < \lambda_4 < \lambda_3, \lambda_6, \lambda_7, \lambda_8}$

$\Rightarrow$ **Group lasso** $\Rightarrow$

	①	②	④	⑤
(μ)	1	1	1	1
a	−1	−1	−1	1
b	−1	−1	1	−1
c	−1	1	1	−1

or…

Example 6) Assume that we want to estimate $\boxed{\beta_a, \beta_b}$

$\Rightarrow$ $\boxed{\lambda_1, \lambda_2, \lambda_4 < \lambda_5 < \lambda_3, \lambda_6, \lambda_7, \lambda_8}$

$\Rightarrow$ **Group lasso** $\Rightarrow$

	①	②	④	⑤
(μ)	1	1	1	1
a	−1	−1	−1	1
b	−1	−1	1	−1
c	−1	1	1	−1

or…

The different orderings of $\lambda_1, \ldots, \lambda_8$ can give the same structure of the solution.

Question What makes the patterns of the solutions? $\Rightarrow$ **Gröbner basis**

Linear Unbiased Estimator

A linear model : $(a, b, c = -1 \text{ or } 1)$

$$y = \mu + \beta_a a + \beta_b b + \beta_c c + \varepsilon$$

unknown (μ, β_a, β_b, β_c); noise (ε)

$$\begin{array}{c|cccccccc}
 & ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \\
(\mu) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
a & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
b & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
c & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
\text{outputs} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8
\end{array}$$

An example of linear (unbiased) estimator of β_a

ε : ignored

$$\hat{\beta}_a = \frac{1}{2}\{-y_1 + y_5\} = \frac{1}{2}\{-(\mu - \beta_a - \beta_b - \beta_c) + (\mu + \beta_a - \beta_b - \beta_c)\} = \beta_a$$

$$= (\mu \quad \beta_a \quad \beta_b \quad \beta_c)\left\{-\frac{1}{2}\begin{pmatrix}1\\-1\\-1\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right\} = (\mu \quad \beta_a \quad \beta_b \quad \beta_c)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \beta_a$$

Linear Unbiased Estimator

A linear model : $y = \underbrace{\mu}_{\text{unknown}} + \underbrace{\beta_a}_{} a + \beta_b b + \beta_c c + \underbrace{\varepsilon}_{\text{noise}}$

$(a, b, c = -1 \text{ or } 1)$

unknown: $\mu, \beta_a, \beta_b, \beta_c$ — noise: ε

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \end{array} \\
\begin{array}{c} (\mu) \\ a \\ b \\ c \end{array} &
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{pmatrix} \\
\text{outputs} & \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \end{array}
\end{array}
$$

T : a subset of the columns ①,…,⑧

A linear combination of T is a linear unbiased estimator of β_a $\Leftrightarrow$ $(\text{A linear combination of } T) = \begin{pmatrix} 0 \\ k \\ 0 \\ 0 \end{pmatrix}$ $(k \neq 0)$

Linear Unbiased Estimator

$$
\begin{array}{c|cccccccc}
 & ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \\
\hline
(\mu) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
a & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
b & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
c & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{array}
= M
$$

$$
\begin{array}{c|cccccccc}
 & ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \\
\hline
(\mu) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\cancel{a} & \cancel{-1} & \cancel{-1} & \cancel{-1} & \cancel{-1} & \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} \\
b & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
c & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{array}
= M_{-a}
$$

Lemma

T : a subset of the columns ①,…,⑧

(A linear combination of T) $= \begin{pmatrix} 0 \\ k \\ 0 \\ 0 \end{pmatrix}$ $(k \neq 0)$ ⟺ $T \in \ker M_{-a} \,/\, \ker M$

Gröbner basis

$$\begin{array}{c|c} & \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \begin{array}{c} (\mu) \\ a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array} = M$$

$$\begin{array}{c|c} & \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \begin{array}{c} (\mu) \\ \cancel{a} \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cancel{-1} & \cancel{-1} & \cancel{-1} & \cancel{-1} & \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array} = M_{-a}$$

$$\left\{ \begin{array}{ll} \ker M: & \text{toric ideal} = \langle x_1x_4 - x_2x_3,\ x_1x_6 - x_2x_5,\ x_1x_8 - x_2x_7,\ \dots \rangle \\ \ker M_a: & \text{toric ideal} = \langle x_1 - x_5,\ x_2 - x_6,\ x_3 - x_7,\ x_4 - x_8,\ x_1x_4 - x_2x_3 \rangle \\ \ker M \,/\, \ker M_a: & \langle x_1 - x_5,\ x_2 - x_6,\ x_3 - x_7,\ x_4 - x_8 \rangle \end{array} \right.$$

Gröbner bases

$\Longrightarrow$ Any linear unbiased estimator of β_a is a linear combination of $y_1 - y_5,\ y_2 - y_6,\ y_3 - y_7,\ y_4 - y_8$.

Gröbner basis

$$\begin{array}{c|c} & \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \begin{array}{c} (\mu) \\ a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array} = M$$

$$\begin{array}{c|c} & \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \begin{array}{c} (\mu) \\ a \\ \cancel{b} \\ c \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ \cancel{-1} & \cancel{-1} & \cancel{1} & \cancel{1} & \cancel{-1} & \cancel{-1} & \cancel{1} & \cancel{1} \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array} = M_{-b}$$

$$\begin{cases} \ker M : & \text{toric ideal} = < x_1x_4 - x_2x_3,\ x_1x_6 - x_2x_5,\ x_1x_8 - x_2x_7,\ \ldots > \\ \ker M_b : & \text{toric ideal} = < x_1 - x_3,\ x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8,\ x_1x_6 - x_2x_5 > \\ \ker M \,/\, \ker M_b : & < x_1 - x_3,\ x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8 > \end{cases}$$

Gröbner bases

⇨ Any linear unbiased estimator of β_b is a linear combination of $y_1 - y_3,\ y_2 - y_4,\ y_5 - y_7,\ y_6 - y_8$.

Linear Unbiased Estimator

$$\begin{array}{c|c} & \begin{array}{cccccccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 \end{array} \\ \begin{array}{c} (\mu) \\ a \\ b \\ c \end{array} & \left(\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{array}\right) \\ \text{outputs} & \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \end{array} \end{array}$$

Assume that we want to estimate $\boxed{\beta_a, \beta_b}$

- Any linear unbiased estimator of β_a is a linear combination of $y_1 - y_5, y_2 - y_6, y_3 - y_7, y_4 - y_8$.
- Any linear unbiased estimator of β_b is a linear combination of $y_1 - y_3, y_2 - y_4, y_5 - y_7, y_6 - y_8$.

$\boxed{\lambda_1, \lambda_2, \lambda_5 < \lambda_4 < \lambda_3, \lambda_6, \lambda_7, \lambda_8}$ $\boxed{\lambda_1, \lambda_2, \lambda_4 < \lambda_5 < \lambda_3, \lambda_6, \lambda_7, \lambda_8}$

In the both cases, $y_1 - y_5$ and $y_2 - y_4$ are chosen.

Future Work / Conclusion

Lemma (in page 13)

T : a subset of the columns of design matrix.

A linear combination of $T = \begin{pmatrix} 0 \\ k \\ 0 \\ 0 \end{pmatrix}$ $(k \neq 0)$ $\Leftrightarrow$ $T \in \ker M_{-a} / \ker M$

- Any linear unbiased estimator is a linear combination of linear combinations of columns given as Gröbner basis.

Any least square estimator is a linear combination of linear combinations of columns given as Gröbner basis.

Given by Gröbner basis

Simple linear regression: $\hat{\beta} = \sum_{j<k} \tau_{jk} \left(\boxed{\frac{y_j - y_k}{x_j - x_k}} \right), \quad \tau_{jk} \propto (x_j - x_k)^2$
$(x_j - x_k \neq 0)$

- If we allow *errors* in the computation of $\ker M$ or $\ker M_{-a}$, the lemma ⇧ may also useful for high-dimensional $(n < p)$ data.

Future Work / Conclusion

$$\boxed{y = \mu + \beta_a a + \beta_b b + \beta_c c + \varepsilon}$$

$$\begin{array}{c|c} \begin{array}{c} (\mu) \\ a \\ b \\ c \\ \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \\ & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \end{array} = M$$

tolerance($error$)

$$\ker M\left(\frac{1}{3}\right) = \emptyset$$

$$\begin{array}{c|c} \begin{array}{c} (\mu) \\ b \\ c \\ \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \\ & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \end{array} = M_{-a}$$

$$\ker M_{-a}\left(\frac{1}{3}\right) = \left\{\frac{1}{3}(y_1 + y_2 - y_3)\right\}$$

$$\Rightarrow \frac{1}{3}(y_1 + y_2 - y_3) \in \ker M_{-a}\left(\frac{1}{3}\right) / \ker M\left(\frac{1}{3}\right)$$

Nearly unbiased estimator with tolerance of $1/3$

Future Work / Conclusion

Lemma (in page 13)

T : a subset of the columns of design matrix.

A linear combination of $T = \begin{pmatrix} 0 \\ k \\ 0 \\ 0 \end{pmatrix}$ $(k \neq 0)$ $\Leftrightarrow$ $T \in \ker M_{-a} \,/\, \ker M$

$$\ker M \subset \ker M_{-a} \subset \ker M_{-ab} \subset \ker M_{-abc} \subset \cdots$$

$$\operatorname{coker} M \supset \operatorname{coker} M_{-a} \supset \operatorname{coker} M_{-ab} \supset \operatorname{coker} M_{-abc} \supset \cdots$$

$$\ker M(\delta_1) \subset \ker M(\delta_2) \subset \ker M(\delta_3) \subset \ker M(\delta_4) \subset \cdots$$
$$(\delta_i\text{: tolerance}(error),\quad 0 < \delta_1 < \delta_2 < \delta_3 < \cdots)$$

- A connection with homology (matroid) ?
- A connection with persistent homology ?

Future Work / Conclusion

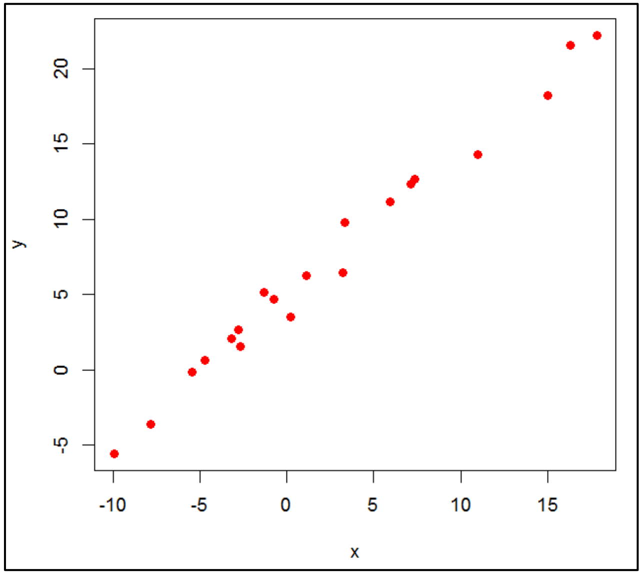
$$\boxed{\ker M \subset \ker M_{-a} \subset \ker M_{-ab} \subset \ker M_{-abc} \subset \ \cdots}$$

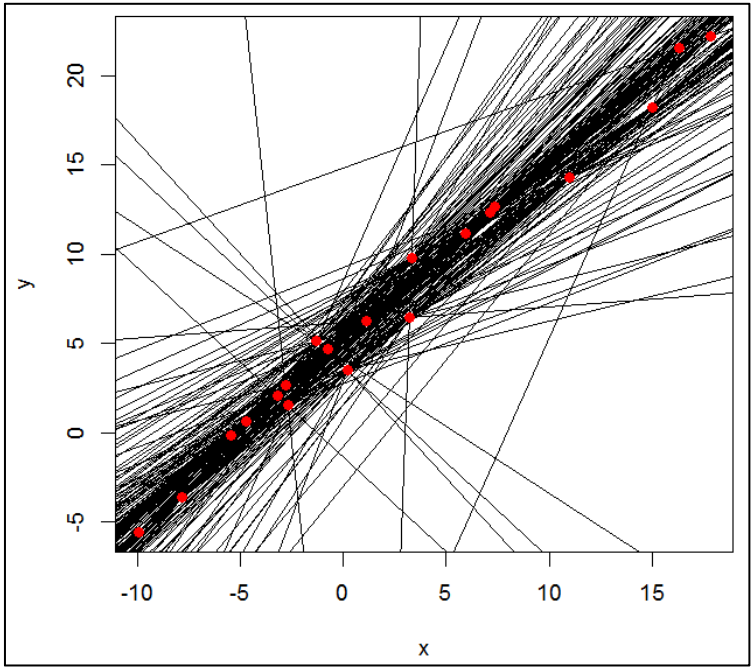
$$\begin{array}{c|c} (\mu) & \\ a & \\ b & \\ c & \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} = M_{+a}$$

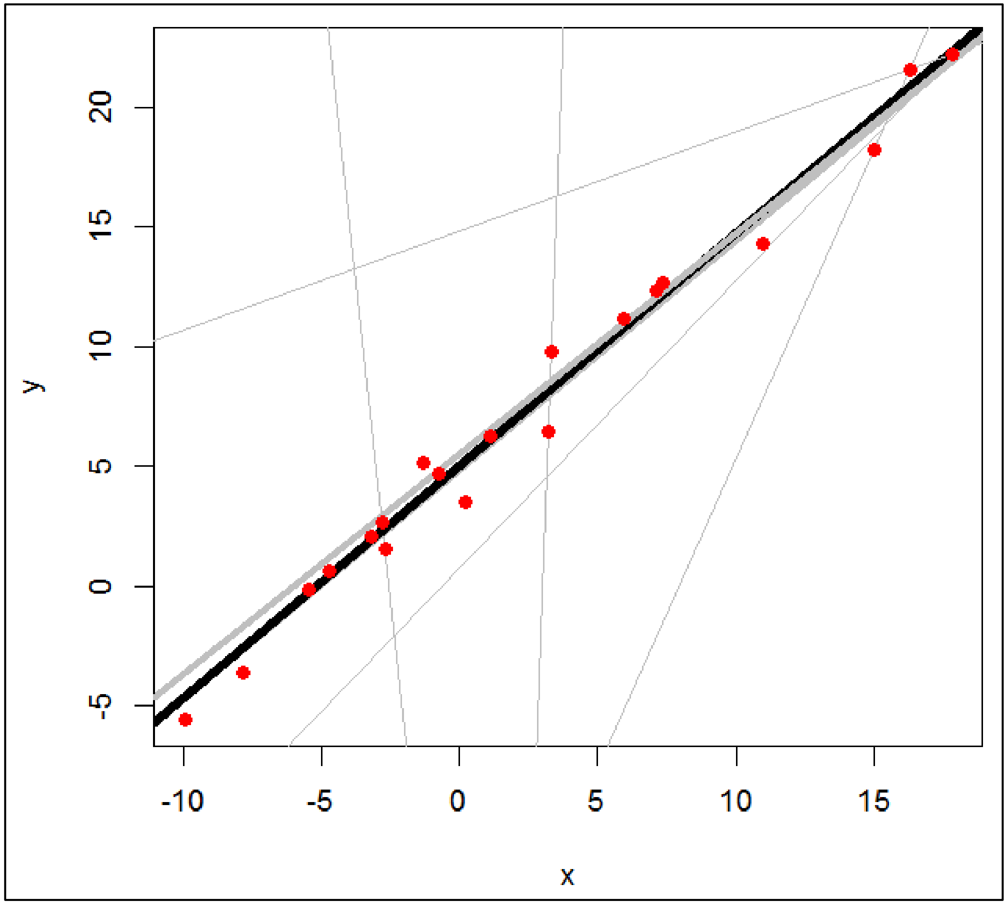
$$\begin{array}{c|c} & \\ (\mu) & \\ a & \\ b & \\ c & \end{array} \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array} = M_{-ab}$$

$$\boxed{\ker M_{-b} \,/\, \ker M = (\ker M_{-ab} \,/\, \ker M_a) \cap \ker M_{+a}}$$

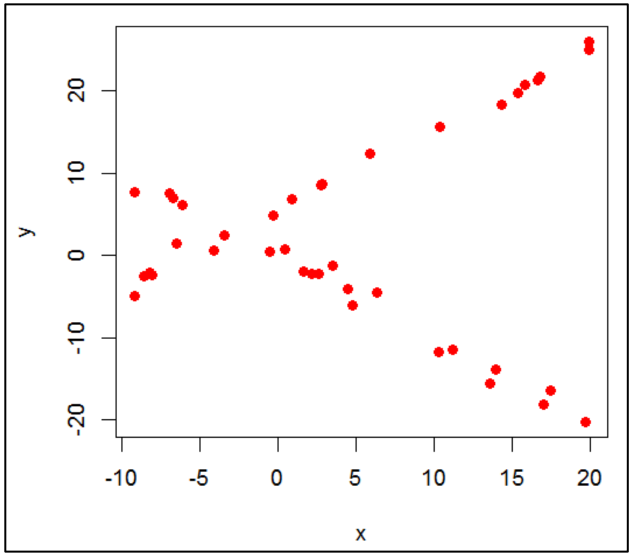
Linear regressions

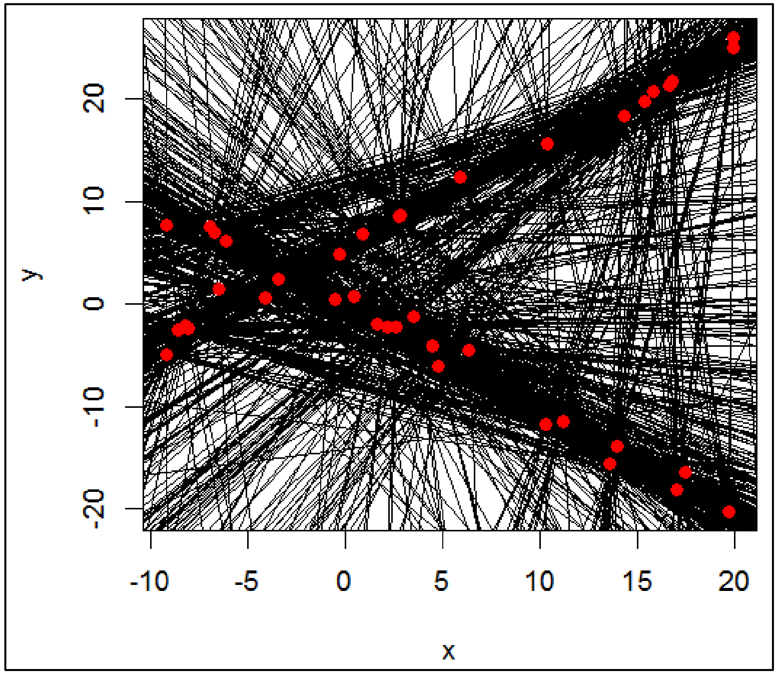


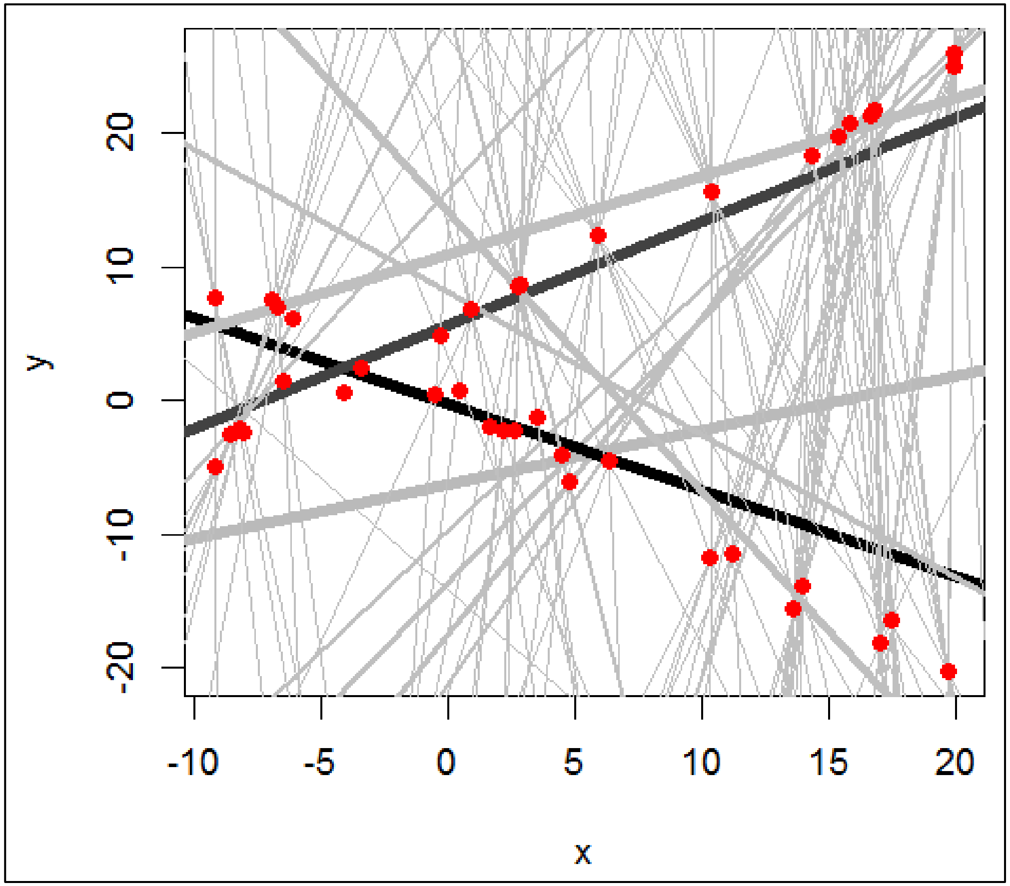




Linear regressions







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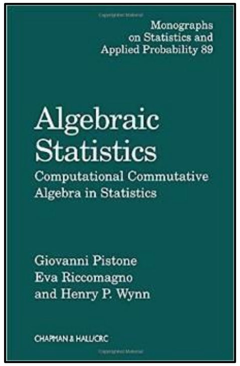
- 4ti2
- Algebraic Statistics: Computational Commutative Algebra in Statistics,
  C. Pistone, E. Riccomagno, H. Wynn, *Chapman & Hall*, (2000)



- Groebner do-jo
  (gu re bu na-a, do-o jo-o)