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Linking the TPR1, DPR1 and Arrow-Head Matrix Structures

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Abstract—Some recent polynomial root-finders rely on effective solution of the eigenproblem for special matrices such as DPR1 (that is, diagonal plus rank-one) and arrow-head matrices. We examine the correlation between these two classes and their links to the Frobenius companion matrix, and we show a Gauss similarity transform of a TPR1 (that is, triangular plus rank-one) matrix into DPR1 and arrow-head matrices. Theoretically, the known unitary similarity transforms of a general matrix into a block triangular matrix with TPR1 diagonal blocks enable the extension of the cited effective eigen-solvers from DPR1 and arrow-head matrices to general matrices. Practically, however, the numerical stability problems with these transforms may limit their value to some special classes of input matrices. © 2006 Elsevier Ltd. All rights reserved.

Keywords—The algebraic eigenproblem, Similarity transforms, DPR1, TPR1 and arrow-head matrices.

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1. INTRODUCTION

Similarity transforms of a matrix into a condensed form is a classical subject [1–3]. We revisit it motivated by the recent papers [4–7]. These papers propose effective algorithms for polynomial root-finding via eigen-solving for some highly structured generalized companion matrices. In particular this class includes DPR1 (that is, diagonal plus rank-one) and arrow-head matrices. There arises a natural further question on how much the cited progress can be extended to the eigenproblem for general matrices.

We relate this question to the recent unitary similarity transforms in [8,9] of an $n \times n$ general matrix into a TPR1 (that is, triangular plus rank-one) matrix or more precisely into a block triangular matrix with TPR1 diagonal blocks. The transforms use $(10/3)n^3 + O(n^2)$ arithmetic operations, that is, about as many as the customary unitary transforms into a Hessenberg matrix. We link these two groups of works by proposing non-unitary Gauss similarity transforms of an $n \times n$ TPR1 matrix into DPR1 and arrow-head matrices; the transforms are computed essentially by solving triangular and diagonally dominant linear systems of k equations for $k = 1, \dots, n-1$; this requires $(2/3)n^3 + O(n^2)$ arithmetic operations. We also link together DPR1, arrow-head and Frobenius companion matrices. Numerical stability of our nonunitary transforms require further study, but we demonstrate some inherent difficulties for constructing problems with unitary similarity transforms of TPR1 and Hessenberg matrices into arrow-head and DPR1 matrices.

The extension of our results from TPR1 to general matrices also requires some additional study because of numerical problems with the TPR1 representation of TPR1 matrices obtained in [8,9] from general matrices. We expect that these problems can be alleviated at least for some special matrix classes of interest, e.g., for banded or banded plus semiseparable matrices [4, 7–9]. Using the special compressed representation of TPR1 matrices in [4,6,7,10,11] could possibly help also in the case of general matrices.

Our transforms can be considered counterparts to the nonunitary similarity transform from a Hessenberg matrix into a block triangular matrix with Frobenius companion diagonal blocks presented in [1, pages 405–408]. The latter transform generally has numerical stability problems, and so have our transforms, although our case is a little more favorable because, unlike the Frobenius matrix defined by the coefficients of the characteristic polynomial of the input matrix, our DPR1 and arrow-head matrices can be defined by the scaled values of this polynomial on a fixed set of n points.

We organize our paper as follows. After some definitions in the next section, we specify Gauss similarity transforms in Section 3. By recursively applying these transforms, we cancel the off-diagonal entries of the triangular term of a TPR1 matrix to arrive at a DPR1 matrix in Section 4. In Section 5, we show correlation among DPR1, arrow-head and Frobenius companion matrices. In Section 6, we cover some modifications and extensions. Numerical tests have been performed by all authors jointly. Otherwise the paper is due to the first author.

2. DEFINITIONS

$\mathbb{R}^{n\times n}$ is the algebra of real $n \times n$ matrices. $M = (m_{i,j})_{i,j=1}^n$ is an $n \times n$ matrix, $\mathbf{v} = (v_i)_{i=1}^n$ is a column vector of dimension n, and $M^\top$ and $\mathbf{v}^\top$ are their transposes. I_k is the $k \times k$ identity matrix, $I = I_n$; $\mathbf{e}_j$ is the j-th column vector of I, so that $\mathbf{e}_1 = (1, 0, \dots, 0)^\top$, $\mathbf{e}_n = (0, 0, \dots, 1)^\top$. $\mathbf{0}$ is the null vector of a fixed dimension. $\mathbf{1} = (1, \dots, 1)^\top$. $R = (r_{i,j})_{i,j=1}^n$ is an $n \times n$ upper triangular matrix with $r_{i,j} = 0$ for $i > j$. $D = \operatorname{diag}(d_i)_{i=1}^n$ is the $n \times n$ diagonal matrix with the diagonal entries $d_1, \dots, d_n$. DPR1 and TPR1 denote the two classes of $n \times n$ matrices of the form $D + \mathbf{u}\mathbf{v}^\top$ and $R + \mathbf{u}\mathbf{v}^\top$, respectively. The NW and SE *arrow-head matrices* are the two classes of $n \times n$ matrices of the form $D + \mathbf{e}_h\mathbf{s}^\top + \mathbf{t}\mathbf{e}_h^\top$ for $h = 1$ and $h = n$, respectively. $C^{(k)} = (c_{i,j}^{(k)})$ for $k = 2, \dots, n$ is the $k \times k$ matrix of cyclic permutation such that $c_{i,j}^{(k)} = 1$ if $(i-j) \bmod k = 1$, $c_{i,j}^{(k)} = 0$ otherwise. (We have $C^{(k)}\mathbf{v} = (v_k, v_1, \dots, v_{k-1})^\top$ for $\mathbf{v} = (v_j)_{j=1}^k$ and $C^{(k)T}C^{(k)} = I_k$.)

$P_k = \mathrm{diag}(C^{(k)}, I_{n-k})$, $P^{(k)} = \mathrm{diag}(I_{n-k}, C^{(k)})$. (We have $P_k^\top P_k = P^{(k)T} P^{(k)} = I$, and the similarity transforms $M \longrightarrow P_k^T M P_k$ and $M \longrightarrow P^{(k)} M P^{(k)T}$ cyclically interchange the first k and the last k rows and columns of an $n \times n$ matrix M, respectively.) $P = (p_{i,j})_{i,j=1}^n$ is the reflection matrix such that $p_{i,j} = 1$ if $i = n+1-j$, $p_{i,j} = 0$ otherwise. We have $P^2 = I$; PAP is a NW arrow-head matrix if A is a SE arrow-head matrix and *vice versa*. Thus hereafter we only treat NW arrow-head matrices.

3. GAUSS SIMILARITY TRANSFORMS

Our basic tool is the *Gauss similarity transforms*

$$G_{\mathbf{x}} : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n},$$
$$G_{\mathbf{x}}(M) = \left(I + \mathbf{e}_1 \mathbf{x}^\top\right) M \left(I - \mathbf{e}_1 \mathbf{x}^\top\right),$$

defined by vectors $\mathbf{x} = (x_j)_{j=1}^n$ where $x_1 = 0$. We immediately observe the following property.

FACT 3.1. *For $x_1 = 0$, the Gauss transform $G_{\mathbf{x}}$ is a similarity transform, that is, $(I + \mathbf{e}_1\mathbf{x}^\top)(I - \mathbf{e}_1\mathbf{x}^\top) = I$.*

THEOREM 3.2. *A Gauss similarity transform $G_{\mathbf{x}}$ maps an upper triangular matrix R into an upper triangular matrix and maps a TPR1 matrix $T = R + \mathbf{u}\mathbf{v}^\top$ into a TPR1 matrix. More precisely,*

$$\begin{aligned} G_{\mathbf{x}}(R) &= R + \mathbf{e}_1\mathbf{x}^\top \left(R - r_{1,1} I\right), \\ G_{\mathbf{x}}\left(\mathbf{u}\mathbf{v}^\top\right) &= \left(\mathbf{u} + \mathbf{e}_1 \left(\mathbf{x}^\top \mathbf{u}\right)\right)\left(\mathbf{v}^\top - v_1 \mathbf{x}^\top\right), \\ G_{\mathbf{x}}(T) &= G_{\mathbf{x}}\left(R\right) + G_{\mathbf{x}}(\mathbf{u}\mathbf{v}^\top). \end{aligned} \tag{3.1}$$

By setting $a = u_1 + \mathbf{x}^\top\mathbf{u}$, we obtain that the first row of the matrix $G_{\mathbf{x}}(T)$ equals $\mathbf{w}^{(1)T} = \mathbf{e}_1^\top G_{\mathbf{x}}(T) = \mathbf{e}_1^\top R + \mathbf{x}^\top(R - r_{1,1}I) + a(\mathbf{v}^\top - v_1\mathbf{x}^\top) = \mathbf{e}_1^\top R + a\mathbf{v}^\top + \mathbf{x}^\top(R - (r_{1,1} + av_1)I)$, whereas the ith row for $i > 1$ equals $\mathbf{w}^{(i)T} = \mathbf{e}_i^\top G_{\mathbf{x}}(T) = \mathbf{e}_i^\top R + u_i(\mathbf{v}^\top - v_1\mathbf{x}^\top)$. In particular for $u_i = 0$ and $i > 1$, the ith rows of the matrices $G_{\mathbf{x}}(T)$ and T coincide with one another.

PROOF. We have the following equations, $G_{\mathbf{x}}(R) = (I + \mathbf{e}_1\mathbf{x}^\top)R(I - \mathbf{e}_1\mathbf{x}^\top) = (R + \mathbf{e}_1\mathbf{x}^\top R)(I - \mathbf{e}_1\mathbf{x}^\top) = R - R\mathbf{e}_1\mathbf{x}^\top + \mathbf{e}_1\mathbf{x}^\top R - \mathbf{e}_1\mathbf{x}^\top R\mathbf{e}_1\mathbf{x}^\top$. Substitute the equations $R\mathbf{e}_1 = r_{1,1}\mathbf{e}_1$ and $\mathbf{x}^\top R\mathbf{e}_1 = r_{1,1}\mathbf{x}^\top\mathbf{e}_1 = r_{1,1}x_1 = 0$ and obtain that $G_{\mathbf{x}}(R) = R + \mathbf{e}_1\mathbf{x}^\top(R - r_{1,1}I)$. Combine this equation with the following one, $G_{\mathbf{x}}(\mathbf{u}\mathbf{v}^\top) = (I + \mathbf{e}_1\mathbf{x}^\top)\mathbf{u}\mathbf{v}^\top(I - \mathbf{e}_1\mathbf{x}^\top) = (\mathbf{u} + \mathbf{e}_1(\mathbf{x}^\top\mathbf{u}))(\mathbf{v}^\top - v_1\mathbf{x}^\top)$ and deduce the theorem.

In virtue of Theorem 3.2, Gauss similarity transform maps a TPR1 matrix into a TPR1 matrix. Furthermore, we may represent both matrices in the same form $R + \mathbf{u}\mathbf{v}^\top$ where only the vector $\mathbf{v}$ and the first row $\mathbf{e}_1^\top R$ of the matrix R change.

4. SIMILARITY TRANSFORMS OF TPR1 INTO DPR1 MATRICES

A simple similarity transform of a TPR1 matrix into a DPR1 matrix is possible if the matrix R is nondefective, that is, has nonsingular matrices S and S^{-1} of its right and left eigenvectors, respectively. This holds, e.g., if all diagonal entries of the matrix R are distinct. Then we have $S^{-1}RS = D = \mathrm{diag}(r_{i,i})_{i=1}^n$, and therefore, $S^{-1}(R + \mathbf{u}\mathbf{v}^\top)S = D + \mathbf{s}\mathbf{t}^\top$ where $\mathbf{s} = S^{-1}\mathbf{u}$, $\mathbf{t}^\top = \mathbf{v}^\top S$. This gives us a similarity transform of a TPR1 matrix $R + \mathbf{u}\mathbf{v}^\top$ (with a nondefective matrix R) into a DPR1 matrix. The customary techniques [1–3] can be employed to compute the matrices S and S^{-1}.

Can we yield such a similarity transform where the matrix R is defective? Gauss transforms $G_{\mathbf{x}}$ enable us to carry this out recursively in $n - 1$ steps. Let us specify the basic step.

Apply the Gauss transform $G_{\mathbf{x}}$ to a triangular matrix R for the vector $\mathbf{x}$ that satisfies the following vector equation:

$$\left(\mathbf{e}_1^\top R + \mathbf{x}^\top \left(R - r_{1,1} I\right)\right) P_{\leftarrow}^\top = \mathbf{0}. \tag{4.1}$$

Here $P_{\leftarrow} = [\mathbf{0}\ I_{n-1}]$ denotes the projection matrix of the size $(n-1) \times n$ which shifts a vector of dimension n into its trailing subvector of dimension $n-1$. In addition to choosing $x_1 = 0$, we define the other $n-1$ components of the vector $\mathbf{x}$ by solving the above triangular linear system of $n-1$ equations.

This system is nonsingular if the $(1,1)$th entry $r_{1,1}$ of the matrix R is distinct from all other diagonal entries. We can ensure the latter assumption by changing the entry u_1 of the vector $\mathbf{u}$ because we can assume that $v_1 \neq 0$ (for otherwise we would deflate the matrix $T = R + \mathbf{u}\mathbf{v}^\top$). We can choose the values $|u_1|$ and $|r_{1,1}|$ large enough to make the linear system (4.1) diagonally dominant.

Updating the representation of the matrix T requires $2n$ arithmetic operations, and the solution of a linear system requires $2n^2$ arithmetic operations.

Now, having all off-diagonal entries in the first row of $G_{\mathbf{x}}(R)$ equal to zero, we interchange the first and the second last rows and columns of R. This reduces our original problem of the diagonalization of the matrix R to the case of $(n-1) \times (n-1)$ triangular matrix. By applying this process recursively (see Figure 1), in $n-1$ steps we arrive at a desired similarity transform of TPR1 matrix $R + \mathbf{u}\mathbf{v}^\top$ into a DPR1 matrix; this requires

$$2\sum_{k=1}^{n-1} k^2 = \frac{(n+1)(2n+1)n}{3}$$

arithmetic operations for solving $n-1$ triangular linear systems and at most $2\sum_{k=2}^{n} k = (n+1)n - 2$ arithmetic operations for possible updating the TPR1 representation of the matrix T.

$$\begin{pmatrix} X & X & X & X & X \\ 0 & X & X & X & X \\ 0 & 0 & X & X & X \\ 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & 0 & X \end{pmatrix} \Longrightarrow \begin{pmatrix} X & 0 & 0 & 0 & 0 \\ 0 & X & X & X & X \\ 0 & 0 & X & X & X \\ 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & 0 & X \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} X & X & X & X & X \\ 0 & X & X & X & X \\ 0 & 0 & X & X & X \\ 0 & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & X \end{pmatrix} \Longrightarrow \begin{pmatrix} X & 0 & 0 & 0 & 0 \\ 0 & X & X & X & X \\ 0 & 0 & X & X & X \\ 0 & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & X \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} X & X & X & X & X \\ 0 & X & X & X & X \\ 0 & 0 & X & 0 & 0 \\ 0 & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & X \end{pmatrix} \Longrightarrow \begin{pmatrix} X & 0 & 0 & 0 & 0 \\ 0 & X & X & X & X \\ 0 & 0 & X & 0 & 0 \\ 0 & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & X \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} X & X & X & X & X \\ 0 & X & 0 & 0 & 0 \\ 0 & 0 & X & 0 & 0 \\ 0 & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & X \end{pmatrix} \Longrightarrow \begin{pmatrix} X & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 \\ 0 & 0 & X & 0 & 0 \\ 0 & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & X \end{pmatrix}$$

Figure 1. Transform of a triangular term of a TPR1 matrix into a diagonal matrix (X marks nonzero terms).

5. CORRELATION AMONG THE DPR1, ARROW-HEAD AND FROBENIUS COMPANION MATRICES

The classes of TPR1 and arrow-head matrices are close to one another. Indeed, an arrow-head matrix $A = D + \mathbf{e}_1\mathbf{s}^\top + \mathbf{t}\mathbf{e}_1^\top$, $D = \mathrm{diag}(d_i)_i$, is a special case of a TPR1 matrix $R + \mathbf{u}\mathbf{v}^\top$ for $R = D + \mathbf{e}_1\mathbf{s}^\top + z\mathbf{e}_1\mathbf{e}_1^\top$, $\mathbf{u} = \mathbf{t} - z\mathbf{e}_1$, $\mathbf{v} = \mathbf{e}_1$, and any scalar z. The algorithm in our previous section transforms the matrix A into a DPR1 matrix. The Frobenius companion matrix is a special case of a TPR1 matrix, and so our transform turns it into a DPR1 matrix as well.

Now, conversely, consider a DPR1 matrix $M = D + \mathbf{u}\mathbf{v}^\top$ where $v_1 \neq 0$. (If $v_1 = 0$, we may deflate M.) Apply the Gauss transform $G_\mathbf{x}$ where $\mathbf{x} = (x_j)_{j=1}^n$, $x_1 = 0$, $x_j = v_j/v_1$ for $j = 2, \ldots, n$. By Theorem 3.2 we have $G_\mathbf{x}(M) = D + v_1(\mathbf{u} + (\mathbf{x}^\top\mathbf{u})\mathbf{e}_1)\mathbf{e}_1^\top + \mathbf{e}_1\mathbf{x}^\top(D - d_1 I)$, and so $G_\mathbf{x}(M)$ is an arrow-head matrix. By combining this transform with the one in the previous section we obtain a similarity transform of a TPR1 matrix into an arrow-head matrix. The known similarity transforms turn DPR1 and arrow-head matrices (via Hessenberg matrices) into block triangular matrices with Frobenius diagonal blocks [1, pages 405–408].

6. MODIFICATIONS AND EXTENSIONS

6.1. Arrow-Head and DPR1 Matrices with a Fixed Characteristic Polynomial

For the characteristic polynomial $c_A(\lambda)$ of an arrow-head matrix $A = D + \mathbf{e}_1\mathbf{s}^\top + \mathbf{t}\mathbf{e}_1^\top$, we have

$$c_A(\lambda) = \left(\lambda - w_1 - \sum_{k=2}^{n} w_k \Big/ (\lambda - d_k)\right) \prod_{i=2}^{n} (\lambda - d_i). \tag{6.1}$$

Here $D = \mathrm{diag}(d_i)_{i=1}^n$, $w_1 = d_1 + s_1 + t_1$; $w_k = s_k t_k$, $k = 2, \ldots, n$. The values $c_A(\lambda_i)$ for $i = 1, \ldots, n$ and any set $\lambda_1, \ldots, \lambda_n$ can be computed in $5n^2 + O(n)$ arithmetic operations. Based on (6.1) we can readily compute an arrow-head matrix A such that $c_T(\lambda) = c(\lambda)$ for a given polynomial $c(\lambda)$, and similarly for a DPR1 matrix A (see [5]).

6.2. Reduction of Eigen-Solving to the Cases of DPR1 and Arrow-Head Matrices

For a given $n \times n$ matrix A, one can immediately define a DPR1 or arrow-head matrix B with the same characteristic polynomial $c_A(x) = c_B(x)$ as soon as one computes the values $r_j = c_A(s_j)/q'(s_j)$, $j = 1, \ldots, n$, at n distinct points $s_1, \ldots, s_n$ where $q(x) = \prod_{j=1}^n (x - s_j)$. For some classes of matrices A this is a simple task, e.g., it takes linear time per point s_j for tridiagonal and, more generally, banded matrices A having a bandwidth in $O(1)$. In principle, this enables application of the efficient algorithms in [4–7] to approximating the eigenvalues of A, but the numerical properties of the transition between the matrices A and B require further study.

6.3. Unitary Versus Nonunitary Similarity Transforms of a Matrix into Condensed Forms

Since we apply nonunitary similarity transforms, further theoretical and experimental study of their numerical stability is in order. Can we yield unitary transforms of a general or TPR1 matrix into the DPR1 or arrow-head forms? No, if we restrict our study to the real case. Indeed, we have the following simple observation.

FACT 6.1. *The classes of Hermitian matrices as well as of rank-k matrices for any fixed k are closed under the unitary similarity transforms.*

The DPR1 and arrow-head matrices with real diagonals are subclasses of the HPR1 and HPR2 (that is, Hermitian plus rank-one and rank-two) matrices, respectively, and thus remain to be HPR1 and HPR2 matrices in unitary similarity transforms. Clearly, these classes do not cover the classes of TPR1 or Hessenberg matrices. Therefore, the similarity transform of a general Hessenberg or a general TPR1 matrix into an HPR1 or HPR2 matrix cannot be unitary.

In contrast, the customary unitary similarity transform maps an HPR1 matrix into a tridiagonal Hermitian plus rank-one matrix. At this point, we can apply the QR algorithm or other known effective algorithms to diagonalize the tridiagonal Hermitian term of this HPR1 matrix and thus to complete the mapping of the original HPR1 matrix into a DPR1 matrix with a real diagonal.

The above observations suggest the following research goals.

(1) Among the nonunitary similarity transforms of a matrix A into a condensed matrix, seek a reasonably fast ones which are sufficiently stable numerically.

(2) Devise unitary similarity transforms of a matrix A into non-Hermitian condensed matrices M. In particular if the matrix $M - \lambda I$ is readily invertible (e.g., is a complex unsymmetric tridiagonal plus small-rank matrix), then the inverse power iteration in [5] is still effective.

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