Homework 3

Due: 10/24/2022

1. How many multiplies and additions are needed for each of the following computations?

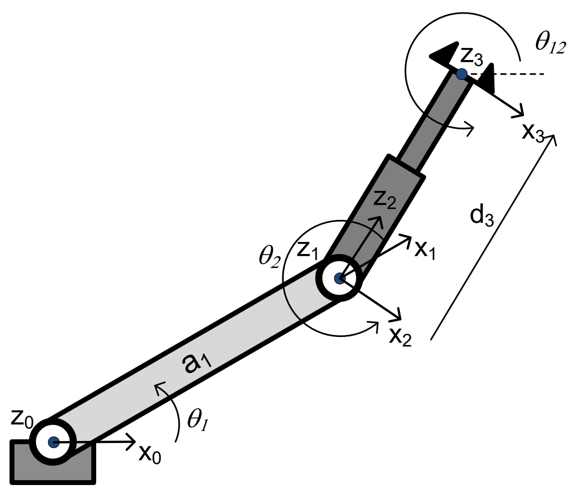
   (a) composition of rotations via rotation matrices, $C_1^2 C_0^1$

   (b) composition of rotations via quaternions, $\bar{q}_1^2 \otimes \bar{q}_0^1$

   (c) recoordinatization of a vector via rotation matrix, $C_1^2 \vec{r}^{\,1}$

   (d) recoordinatization of a vector via quaternion, $\bar{q}_1^2 \otimes \breve{r}^1 \otimes (\bar{q}_1^2)^{-1}$

2. Consider the three-link, planar robot shown below for which four coordinate frames have been assigned. Frame $\{0\}$ is fixed, frame $\{1\}$ rotates with angle θ_1 relative to frame $\{0\}$, frame $\{2\}$ rotates with angle θ_2 relative to frame $\{1\}$, and frame $\{3\}$ translates with distance d_3 relative to frame $\{2\}$.



The rotation matrices and displacements between frames and show below

$$C_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{r}_{01}^{\,0} = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix}, \quad C_2^1 = \begin{bmatrix} c_2 & 0 & -s_2 \\ s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{bmatrix}, \quad \vec{r}_{12}^{\,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_3^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \vec{r}_{23}^{\,2} = \begin{bmatrix} 0 \\ 0 \\ d_3 \end{bmatrix}$$

Note the notation $c_1 = \cos(\theta_1)$ and $s_1 = \sin(\theta_1)$.

(a) Determine the rotation matrix C_3^0.

(b) Determine the translation vector $\vec{r}^{\,0}_{03}$.

(c) Determine the following angular velocities as skew-symmetric matrices Ω and vectors $\vec{\omega}$. Note θ_1, θ_2 and d_3 can vary with time.

i. Ω^0_{01}, $\vec{\omega}^{\,0}_{01}$

ii. Ω^1_{12}, $\vec{\omega}^{\,1}_{12}$

iii. Ω^2_{23}, $\vec{\omega}^{\,2}_{23}$

iv. Ω^0_{03}, $\vec{\omega}^{\,0}_{03}$

3. Consider the time-varying coordinate transformation matrix C^n_b given below that describes the orientation of the body frame as it rotates with respect to the navigation frame.

$$C^n_b = \begin{bmatrix} \cos(t) & \sin(t)\sin(t^2) & \sin(t)\cos(t^2) \\ 0 & \cos(t^2) & -\sin(t^2) \\ -\sin(t) & \cos(t)\sin(t^2) & \cos(t)\cos(t^2) \end{bmatrix}$$

(a) Compute expression f or ψ, θ, and ϕ based on fixed-axis definition of roll,pitch,yaw assuming a 1,2,3 series of rotations (i.e. roll, pitch, yaw)

(b) Use Matlab to pl ot ψ, θ, and ϕ versus time

4. Consider the time-varying coordinate transformation matrix C^n_b given below that describes the orientation of the body frame as it rotates with respect to the navigation frame.

$$C^n_b = \begin{bmatrix} \cos(t) & \sin(t)\sin(t^2) & \sin(t)\cos(t^2) \\ 0 & \cos(t^2) & -\sin(t^2) \\ -\sin(t) & \cos(t)\sin(t^2) & \cos(t)\cos(t^2) \end{bmatrix}$$

(a) Determine the analytic form of the time-derivative of C^n_b (i.e., $\dot{C}^n_b = \frac{dC^n_b}{dt}$) via a term-by-term differentiation.

(b) Develop MATLAB functions which accept "t" (i.e., time) as a numerical input and return C^n_b and $\dot{C}^n_b$, respectively, as numerical outputs.

(c) Using the C^n_b and $\dot{C}^n_b$ functions from above, compute the angular velocity vector $\vec{\omega}^{\,n}_{nb}$ at time $t = 0$ sec (**Hint**: you might want to compute Ω^n_{nb} first).

i. What is the magnitude (i.e., $\dot{\theta}$, angular speed) of the angular velocity?

ii. About what unit vector ($\vec{k}^{\,n}_{nb}$) has the instantaneous rotation occurred?

(d) Using the C_b^n and $\dot{C}_b^n$ functions from above, compute the angular velocity vector $\vec{\omega}_{nb}^n$ at time $t = 0.5$ sec.

i. What is the magnitude (i.e., $\dot{\theta}$, angular speed) of the angular velocity?

ii. About what unit vector ($\vec{k}_{nb}^n$) has the instantaneous rotation occurred?

(e) Using C_b^n and $\dot{C}_b^n$ functions from above, compute the angular velocity vector $\vec{\omega}_{nb}^n$ at time $t = 1$ sec.

i. What is the magnitude (i.e., $\dot{\theta}$, angular speed) of the angular velocity?

ii. About what unit vector ($\vec{k}_{nb}^n$) has the instantaneous rotation occurred?

(f) In practice, direct measurement of the angular velocity vector $\vec{\omega}_{nb}^n$ can prove challenging, so a finite-difference approach may be taken given two sequential orientations represented by $C_b^n(t)$ and $C_b^n(t+\Delta t)$ a small time Δt apart. Consider the approximate value of the angular velocity vector $\vec{\omega}_{nb}^n$ derived by using the finite difference

$$\dot{C}_b^n(t) \approx \frac{C_b^n(t+\Delta t) - C_b^n(t)}{\Delta t}$$

at times $t = 0, 0.5$, and 1 sec. Compare the "analytic" values for $\dot{\theta}$ and $\vec{k}_{nb}^n$ (found in parts b, c and d) with your approximations from the finite difference using $\Delta t = 0.1$ sec. How large are the errors?

5. Given the geodetic coordinates of the peak of Mt. Everest as Latitude (L_b) 27deg 59min 16sec N, Longitude (λ_b) 86deg 56min 40sec E, and height (h_b) 8850 meters (derived by GPS in 1999):

(a) Develop a MATLAB

```
function [r_e__e_b]=llh2xyz(L_b,lambda_b,h_b)
```

to convert from geodetic curvilinear lat, lon, and height to ECEF rectangular x, y, and z coordinates (Please use SI units). Attach a printout of your function.

i. Test your llh2xyz function using coordinates of the peak of Mt Everest. What is $\vec{r}_{eb}^{\,e}$ at the peak?

(b) Develop a MATLAB function

```
function [L_b,lambda_b,h_b] = xyz2llh(r_e__e_b)
```

to convert ECEF x, y, and z coordinates to lat, lon, and height (use SI units). HINT: This should be an iterative transformation (i.e., not closed form).

(c) What is the acceleration due to gravity at the ellipsoid (i.e., at the ellipsoid $h_b = 0$. HINT: This should only be a function of lat—see **attached pages**)?

(d) What is the magnitude of the centrifugal acceleration $(-\Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb})$ at the ellipsoid and at the peak?

(e) What is the magnitude of the gravitational attraction at the ellipsoid and at the peak? HINT: See **attached pages** to compute $\vec{\gamma}^{\,e}_{ib} = \vec{\gamma}^{\,i}_{ib}.\big|_{\vec{r}^{\,i}_{ib} = \vec{r}^{\,e}_{eb}}$

The period of rotation of the Earth with respect to space is known as the sidereal day and is about 23 hours, 56 minutes, 4 seconds. This differs from the 24-hour mean solar day as the Earth's orbital motion causes the Earth-Sun direction with respect to space to vary, resulting in one more rotation than solar day each year (note that 1/365 of a day is about 4 minutes). The rate of rotation is not constant and the sidereal day can vary by several milliseconds from day to day. There are random changes due to wind and seasonal changes as ice forming and melting alters the Earth's moment of inertia. There is also a long-term reduction of the Earth rotation rate due to tidal friction [13].

For navigation purposes, a constant rotation rate is assumed, based on the mean sidereal day. The WGS 84 value of the Earth's angular rate is $\omega_{ie} = 7.292115 \times 10^{-5}$ rad s^{-1} [9].

2.3.5 Specific Force, Gravitation, and Gravity

Specific force is the nongravitational force per unit mass on a body, sensed with respect to an inertial frame. It has no meaning with respect to any other frame, though it can be expressed in any axes. *Gravitation* is the fundamental mass attraction force; it does not incorporate any centripetal components.[1]

Specific force is what people and instruments sense. Gravitation is not sensed because it acts equally on all points, causing them to move together. Other forces are sensed as they are transmitted from point to point. The sensation of weight is caused by the forces opposing gravity. There is no sensation of weight during freefall, where the specific force is zero. Conversely, under zero acceleration, the reaction to gravitation is sensed, and the specific force is equal and opposite to the acceleration due to gravitation. Figure 2.17 illustrates this for a mass on a spring. In both cases, the gravitational force on the mass is the same. However, in the stationary case, the spring exerts an opposite force.

A further example is provided by the upward motion of an elevator, illustrated in Figure 2.18. As the elevator accelerates upward, the specific force is higher and the occupants appear to weigh more. As the elevator decelerates, the specific force is lower than normal and the occupants feel lighter. In a windowless elevator, this

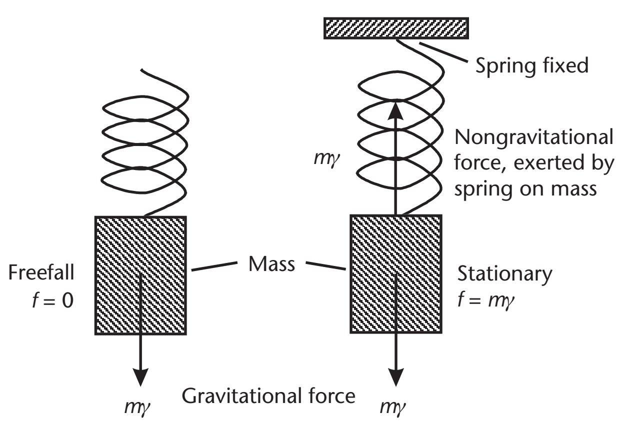


Figure 2.17 Forces on a mass on a spring.

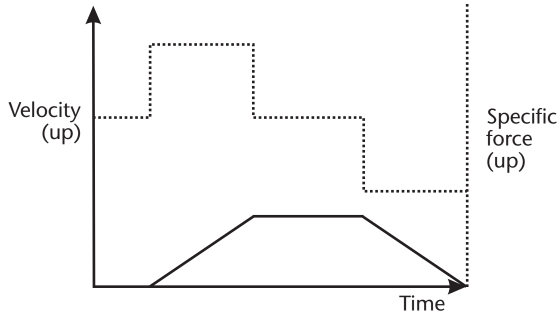


Figure 2.18 Velocity and specific force of an elevator moving up. (*From:* [4]. © 2002 QinetiQ Ltd. Reprinted with permission.)

can create the illusion that the elevator has overshot the destination floor and is dropping down to correct for it.

Thus, specific force, $\mathbf{f}$, varies with acceleration, $\mathbf{a}$, and the acceleration due to the gravitational force, $\boldsymbol{\gamma}$, as

$$\mathbf{f}_{ib}^{\gamma} = \mathbf{a}_{ib}^{\gamma} - \boldsymbol{\gamma}_{ib}^{\gamma} \tag{2.76}$$

Specific force is the quantity measured by accelerometers. The measurements are made in the body frame of the accelerometer triad; thus, the sensed specific force is $\mathbf{f}_{ib}^{b}$.

Before defining gravity, it is useful to consider an object that is stationary with respect to a rotating frame, such as the ECEF frame. This has the properties

$$\mathbf{v}_{eb}^{e} = 0 \quad \mathbf{a}_{eb}^{e} = 0 \tag{2.77}$$

From (2.35) and (2.44), and applying (2.34),

$$\dot{\mathbf{r}}_{ib}^{e} = \dot{\mathbf{r}}_{eb}^{e} = 0 \quad \ddot{\mathbf{r}}_{ib}^{e} = \ddot{\mathbf{r}}_{eb}^{e} = 0 \tag{2.78}$$

The inertially referenced acceleration in ECEF frame axes is given by (2.49), noting that $\dot{\boldsymbol{\Omega}}_{ie}^{e} = 0$ as the Earth rate is assumed constant:

$$\mathbf{a}_{ib}^{e} = \boldsymbol{\Omega}_{ie}^{e}\boldsymbol{\Omega}_{ie}^{e}\mathbf{r}_{ib}^{e} + 2\boldsymbol{\Omega}_{ie}^{e}\dot{\mathbf{r}}_{ib}^{e} + \ddot{\mathbf{r}}_{ib}^{e} \tag{2.79}$$

Applying (2.78),

$$\mathbf{a}_{ib}^{e} = \boldsymbol{\Omega}_{ie}^{e}\boldsymbol{\Omega}_{ie}^{e}\mathbf{r}_{eb}^{e} \tag{2.80}$$

Substituting this into the specific force definition, (2.76), gives

$$\mathbf{f}_{ib}^{e} = \boldsymbol{\Omega}_{ie}^{e}\boldsymbol{\Omega}_{ie}^{e}\mathbf{r}_{eb}^{e} - \boldsymbol{\gamma}_{ib}^{e} \tag{2.81}$$

The specific force sensed when stationary with respect to the Earth frame is the reaction to what is known as the acceleration due to *gravity*, which is thus defined by[2]

$$\mathbf{g}_b^{\gamma} = -\mathbf{f}_{ib}^{\gamma}\Big|_{\mathbf{a}_{eb}^{\gamma}=0,\ \mathbf{v}_{eb}^{\gamma}=0} \tag{2.82}$$

Therefore, from (2.81), the acceleration due to gravity is

$$\mathbf{g}_b^{\gamma} = \boldsymbol{\gamma}_{ib}^{\gamma} - \boldsymbol{\Omega}_{ie}^{\gamma}\boldsymbol{\Omega}_{ie}^{\gamma}\mathbf{r}_{eb}^{\gamma} \tag{2.83}$$

noting from (2.74) and (2.75) that

$$\mathbf{g}_b^{e} = \boldsymbol{\gamma}_{ib}^{e} + \omega_{ie}^{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{r}_{eb}^{e} \tag{2.84}$$

$$\mathbf{g}_b^{n} = \boldsymbol{\gamma}_{ib}^{n} + \omega_{ie}^{2}\begin{pmatrix} \sin^2 L_b & 0 & \cos L_b \sin L_b \\ 0 & 1 & 0 \\ \cos L_b \sin L_b & 0 & \cos^2 L_b \end{pmatrix}\mathbf{r}_{eb}^{n}$$

The first term in (2.83) and (2.84) is the gravitational acceleration. The second term is the outward centrifugal acceleration due to the Earth's rotation, noting that this is a virtual force arising from the use of rotating resolving axes (see Section 2.2.8). From the inertial frame perspective, a centripetal acceleration, (2.80), is applied to maintain an object stationary with respect to a rotating frame. It is important not to confuse gravity, $\mathbf{g}$, with gravitation, $\boldsymbol{\gamma}$. At the Earth's surface, the total acceleration due to gravity is about 9.8 m s^{-2}, with the centrifugal component contributing up to 0.034 m s^{-2}. In orbit, the gravitational component is smaller and the centrifugal component larger.

The centrifugal component of gravity can be calculated exactly at all locations, but calculation of the gravitational component is more complex. For air applications, it is standard practice to use an empirical model of the surface gravity, $\mathbf{g}_0$, and apply a simple scaling law to calculate the variation with height.[3]

The WGS 84 datum [9] provides a simple model of the acceleration due to gravity at the ellipsoid as a function of latitude:

$$g_0(L) \approx 9.7803253359\,\frac{(1 + 0.001931853\sin^2 L)}{\sqrt{1 - e^2\sin^2 L}}\ \text{m s}^{-2} \tag{2.85}$$

This is known as the Somigliana model. Note that it is a gravity field model, not a gravitational field model. The geoid (Section 2.3.3) defines a surface of constant gravity potential. However, the acceleration due to gravity is obtained from the gradient of the gravity potential, so is not constant across the geoid. Although the true gravity vector is perpendicular to the geoid (not the terrain), it is a reasonable approximation for most navigation applications to treat it as perpendicular to the ellipsoid. Thus,

$$\mathbf{g}_0^\gamma(L) \approx g_0(L)\mathbf{u}_{nD}^\gamma \tag{2.86}$$

where $\mathbf{u}_{nD}^\gamma$ is the down unit vector of the local navigation frame.

The gravitational acceleration at the ellipsoid can be obtained from the acceleration due to gravity by subtracting the centrifugal acceleration. Thus,

$$\boldsymbol{\gamma}_0^\gamma(L) = \mathbf{g}_0^\gamma(L) + \boldsymbol{\Omega}_{ie}^\gamma \boldsymbol{\Omega}_{ie}^\gamma \mathbf{r}_{eS}^\gamma(L) \tag{2.87}$$

where, from (2.65), the geocentric radius at the surface is given by

$$\mathbf{r}_{eS}^e(L) = R_E(L)\sqrt{\cos^2 L + (1 - e^2)^2 \sin^2 L} \tag{2.88}$$

The gravitational field varies roughly as that for a point mass, so gravitational acceleration can be scaled with height as

$$\boldsymbol{\gamma}_{ib}^\gamma \approx \frac{\left(r_{eS}^e(L_b)\right)^2}{\left(r_{eS}^e(L_b) + h_b\right)^2} \boldsymbol{\gamma}_0^\gamma(L_b) \tag{2.89}$$

For heights less than about 10 km, the scaling can be further approximated to $\left(1 - 2h_b / r_{eS}^e(L_b)\right)$. The acceleration due to gravity, $\mathbf{g}$, may then be recombined using (2.83). As the centrifugal component of gravity is small, it is reasonable to apply the height scaling to g where the height is small and/or poor quality accelerometers are used. Alternatively, a more accurate set of formulae for calculating gravity as a function of latitude and height is given in [9]. An approximation for the variation of the down component with height is

$$g_{b,D}^n(L_b, h_b) \approx g_0(L_b)\left[1 - \frac{2}{R_0}\left(1 + f + \frac{\omega_{ie}^2 R_0^2 R_P}{\mu}\right)h_b + \frac{3}{R_0^2}h_b^2\right] \tag{2.90}$$

where μ is the Earth's gravitational constant and its WGS 84 value [9] is $3.986004418 \times 10^{14}$ m^{3}s^{-2}.

When working in an inertial reference frame, only the gravitational acceleration is required. This can be calculated directly at varying height using [14]

$$\boldsymbol{\gamma}_{ib}^i = -\frac{\mu}{|\mathbf{r}_{ib}^i|^3}\left\{\mathbf{r}_{ib}^i + \frac{3}{2}J_2\frac{R_0^2}{|\mathbf{r}_{ib}^i|^2}\left\{\begin{array}{c}\left[1 - 5\left(r_{ib,z}^i \big/ |\mathbf{r}_{ib}^i|\right)^2\right] r_{ib,x}^i \\ \left[1 - 5\left(r_{ib,z}^i \big/ |\mathbf{r}_{ib}^i|\right)^2\right] r_{ib,y}^i \\ \left[3 - 5\left(r_{ib,z}^i \big/ |\mathbf{r}_{ib}^i|\right)^2\right] r_{ib,z}^i\end{array}\right\}\right\} \tag{2.91}$$

where J_2 is the Earth's second gravitational constant and takes the value 1.082627 $\times 10^{-3}$ [9].

Much higher precision may be obtained using a spherical harmonic model, such as the 360^2 coefficient EGM 96 gravity model [11]. Further precision is given by a gravity anomaly database, which comprises the difference between the measured and modeled gravity fields over a grid of locations. Gravity anomalies tend to be largest over major mountain ranges and ocean trenches.

2.4 Frame Transformations

An essential feature of navigation mathematics is the capability to transform kinematics between coordinate frames. This section summarizes the equations for expressing the attitude of one frame with respect to another and transforming Cartesian position, velocity, acceleration, and angular rate between references to the inertial, Earth, and local navigation frames. The section concludes with the equations for transposing a navigation solution from one object to another.[1]

Cartesian position, velocity, acceleration, and angular rate referenced to the same frame transform between resolving axes simply by applying the coordinate transformation matrix (2.7):

$$\mathbf{x}_{\beta\alpha}^{\gamma} = \mathbf{C}_{\delta}^{\gamma}\mathbf{x}_{\beta\alpha}^{\delta} \quad \mathbf{x} \in \mathbf{r}, \mathbf{v}, \mathbf{a}, \boldsymbol{\omega} \quad \gamma, \delta \in i, e, n, b \tag{2.92}$$

Therefore, these transforms are not presented explicitly for each pair of frames.[2] The coordinate transformation matrices involving the body frame—that is,

$$\mathbf{C}_{b}^{\beta}, \mathbf{C}_{\beta}^{b} \quad \beta \in i, e, n$$

—describe the attitude of that body with respect to a reference frame. The body attitude with respect to a new reference frame may be obtained simply by multiplying by the coordinate transformation matrix between the two reference frames:

$$\mathbf{C}_{b}^{\delta} = \mathbf{C}_{\beta}^{\delta}\mathbf{C}_{b}^{\beta} \quad \mathbf{C}_{\delta}^{b} = \mathbf{C}_{\beta}^{b}\mathbf{C}_{\delta}^{\beta} \quad \beta, \delta \in i, e, n \tag{2.93}$$

Transforming Euler, quaternion, or rotation vector attitude to a new reference frame is more complex. One solution is to convert to the coordinate transformation matrix representation, transform the reference, and then convert back.

2.4.1 Inertial and Earth Frames

The center and z-axes of the ECI and ECEF coordinate frames are coincident. The x- and y-axes are coincident at time t_0, and the frames rotate about the z axes at ω_{ie} (see Section 2.3.4). Thus,[1]