Principles of Navigation
Homework 3

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MECH 6970

1. How many multiplications and additions are needed for each of the following computations?

   (a) Composition of rotations via rotation matrices, $C_1^2 C_0^1$

   Solution: The following depicts the 27 multiplications and 18 additions needed to calculate $C_1^2 C_0^1$.

$$C_1^2 C_0^1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

   (b) Composition of rotations via quaternions, $\bar{q}_1^{\,2} \otimes \bar{q}_0^{\,1}$

   Solution: The following depicts the 16 multiplications and 12 additions needed to calculate $\bar{q}_1^{\,2} \otimes \bar{q}_0^{\,1}$. a_{w-z} and b_{w-z} represent the elements of $\bar{q}_1^{\,2}$ and $\bar{q}_0^{\,1}$, respectively.

$$\bar{q}_1^{\,2} \otimes \bar{q}_0^{\,1} = \begin{bmatrix} a_w & -a_x & -a_y & -a_z \\ a_x & a_w & -a_z & a_y \\ a_y & a_z & a_w & -a_x \\ a_z & -a_y & a_x & a_w \end{bmatrix} \begin{bmatrix} b_w \\ b_x \\ b_y \\ b_z \end{bmatrix}$$
$$= \begin{bmatrix} a_w b_w + (-a_x b_x) + (-a_y b_y) + (-a_z b_z) \\ a_x b_w + a_w b_x + (-a_z b_y) + a_y b_z \\ a_y b_w + a_z b_x + a_w b_y + (-a_x b_z) \\ a_z b_w + (-a_y b_x) + a_x b_y + a_w b_z \end{bmatrix}$$

   (c) Recoordinatization of a vector via rotation matrix, $C_1^2 \vec{r}^{\,1}$

   Solution: The following depicts the 9 multiplications and 6 additions needed to calculate $C_1^2 \vec{r}^{\,1}$. a_{ij} and b_i represent the elements of C_1^2 and $\vec{r}^{\,1}$, respectively.

$$C_1^2\vec{r}^{\,1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{bmatrix}$$

(d) Recoordinatization of a vector via quaternion, $\bar{q}_1^{\,2} \otimes \breve{r}^1 \otimes \left(\bar{q}_1^{\,2}\right)^{-1}$

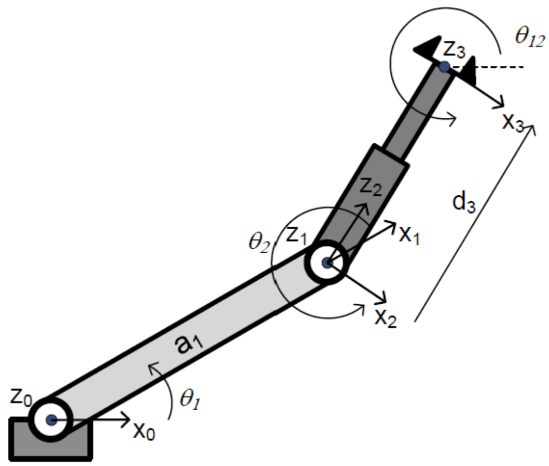
The following depicts the multiplications and additions needed to calculate $\bar{q}_1^{\,2} \otimes \breve{r}^1 \otimes \left(\bar{q}_1^{\,2}\right)^{-1}$. a_{w-z}, b_i, and c_{w-z} represent the elements of $\bar{q}_1^{\,2}$, $\breve{r}^1$, and $\bar{q}_1^{\,2}$, respectively.

$$\bar{q}_1^{\,2} \otimes \breve{r}^1 = \begin{bmatrix} a_w & -a_x & -a_y & -a_z \\ a_x & a_w & -a_z & a_y \\ a_y & a_z & a_w & -a_x \\ a_z & -a_y & a_x & a_w \end{bmatrix} \begin{bmatrix} 0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$
$$= \begin{bmatrix} (-a_xb_1) + (-a_yb_2) + (-a_zb_3) \\ a_wb_1 + (-a_zb_2) + a_yb_3 \\ a_zb_1 + a_wb_2 + (-a_xb_3) \\ (-a_yb_1) + a_xb_2 + a_wb_3 \end{bmatrix} = \begin{bmatrix} ab_w \\ ab_x \\ ab_y \\ ab_z \end{bmatrix}$$

As in b, the number of multiplications and additions up to this point is 16 and 12, respectively. Simply doubling these values gives the number of multiplications and additions for two quaternion multiplications. Therefore, the number of multiplications needed is 32 and the number of additions needed is 24. The final quaternion product is shown below.

$$\bar{q}_1^{\,2} \otimes \breve{r}^1 \otimes \left(\bar{q}_1^{\,2}\right)^{-1} = \begin{bmatrix} ab_w & -ab_x & -ab_y & -ab_z \\ ab_x & ab_w & -ab_z & ab_y \\ ab_y & ab_z & ab_w & -ab_x \\ ab_z & -ab_y & ab_x & ab_w \end{bmatrix} \begin{bmatrix} c_w \\ c_x \\ c_y \\ c_z \end{bmatrix}$$
$$= \begin{bmatrix} ab_wc_w + (-ab_xc_x) + (-ab_yc_y) + (-ab_zc_z) \\ ab_xc_w + ab_wc_x + (-ab_zc_y) + ab_yc_z \\ ab_yc_w + ab_zc_x + ab_wbc_y + (-ab_xc_z) \\ ab_zc_w + (-ab_yc_x) + ab_xc_y + ab_wc_z \end{bmatrix}$$

2. Consider the three-link, planar robot shown below for which four coordinate frames have been assigned. Frame $\{0\}$ is fixed, frame $\{1\}$ rotates with angle θ_1 relative to Frame $\{0\}$, frame $\{2\}$ rotates with θ_2 relative to frame $\{1\}$, and frame $\{3\}$ translates with distance d_3 relative to frame $\{2\}$.



The rotation matrices and displacements between frames are shown below.

$$C_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \vec{r}_{01}^{\,0} = \begin{bmatrix} a_1\cos(\theta_1) \\ a_1\sin(\theta_1) \\ 0 \end{bmatrix}, C_2^1 = \begin{bmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) \\ \sin(\theta_2) & 0 & \cos(\theta_2) \\ 0 & -1 & 0 \end{bmatrix}$$

$$\vec{r}_{12}^{\,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, C_3^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \vec{r}_{23}^{\,2} = \begin{bmatrix} 0 \\ 0 \\ d_3 \end{bmatrix}$$

(a) Determine the rotation matrix C_3^0.

Solution: C_3^0 can be determined by the following series of matrix multiplications.

$$C_3^0 = C_1^0 C_2^1 C_3^2 \tag{1}$$

$$C_3^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) \\ \sin(\theta_2) & 0 & \cos(\theta_2) \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C_3^0 = \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\sin(\theta_2) - \cos(\theta_2)\sin(\theta_1) & 0 \\ \cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1) & \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Determine the translation vector $\vec{r}_{03}^{\,0}$

Solution: $\vec{r}_{03}^{\,0}$ by the following summation of transformed relative frame vectors.

$$\vec{r}_{03}^{\,0} = \vec{r}_{01}^{\,0} + C_1^0 \vec{r}_{12}^{\,1} + C_2^0 \vec{r}_{23}^{\,2} \tag{2}$$

The second term in this summation is eliminated given $\vec{r}_{12}^{\,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. In addition to this, C_2^0 is defined as

$$\begin{aligned} C_2^0 &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) \\ \sin(\theta_2) & 0 & \cos(\theta_2) \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & 0 & -\cos(\theta_1)\sin(\theta_2) - \cos(\theta_2)\sin(\theta_1) \\ \cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1) & 0 & \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ 0 & -1 & 0 \end{bmatrix} \end{aligned}$$

The summation becomes the following

$$\begin{aligned} \vec{r}_{03}^{\,0} &= \begin{bmatrix} a_1\cos(\theta_1) \\ a_1\sin(\theta_1) \\ 0 \end{bmatrix} + C_2^0 \begin{bmatrix} 0 \\ 0 \\ d_3 \end{bmatrix} \\ &= \begin{bmatrix} a_1\cos(\theta_1) - d_3(\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)) \\ a_1\sin(\theta_1) - d_3(\sin(\theta_1)\sin(\theta_2) - \cos(\theta_1)\cos(\theta_2)) \\ 0 \end{bmatrix} \end{aligned}$$

(c) Determine the following angular velocities as skew-symmetric matrices Ω and vectors $\vec{\omega}$. Note θ_1, θ_2, and d_3 can vary with time.

Solution: The following relationship was used to solve these problems:

$$\dot{C}_b^a = \Omega_{ab}^{\,a} C_b^a \tag{3}$$

This relationship yields the following:

$$\Omega_{ab}^{\,a} = \dot{C}_b^a (C_b^a)^{-1} \tag{4}$$

i. Ω^0_{01}, $\vec{\omega}^0_{01}$
Solution: $\dot{C}^a_b$ and subsequent $\dot{C}$ were calculated using `diff()` in MATLAB. The following depicts the corresponding solution:

$$\begin{aligned}
\Omega^0_{01} &= \dot{C}^0_1 (C^0_1)^{-1} \\
&= \begin{bmatrix} -\dot{\theta}_1 \sin\theta_1 & -\dot{\theta}_1 \cos\theta_1 & 0 \\ \dot{\theta}_1 \cos\theta_1 & -\dot{\theta}_1 \sin\theta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\vec{\omega}^0_{01} &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}
\end{aligned}$$

ii. Ω^1_{12}, $\vec{\omega}^1_{12}$
Solution:

$$\begin{aligned}
\Omega^1_{12} &= \dot{C}^1_2 (C^1_2)^{-1} \\
&= \begin{bmatrix} -\dot{\theta}_2 \sin\theta_2 & 0 & -\dot{\theta}_2 \cos\theta_2 \\ \dot{\theta}_2 \cos\theta_2 & 0 & -\dot{\theta}_2 \sin\theta_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) \\ \sin(\theta_2) & 0 & \cos(\theta_2) \\ 0 & -1 & 0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & -\dot{\theta}_2 & 0 \\ \dot{\theta}_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\vec{\omega}^0_{01} &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix}
\end{aligned}$$

iii. Ω^2_{23}, $\vec{\omega}^2_{23}$

Solution:

$$\begin{aligned}
\Omega_{23}^{2} &= \dot{C}_3^2 (C_3^2)^{-1} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\vec{\omega}_{01}^{0} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

iv. Ω_{03}^{0}, $\vec{\omega}_{03}^{0}$

Solution: Like position vectors in b, angular velocities in different frames can be transformed and summed to find other relative angular velocities.

$$\begin{aligned}
\vec{\omega}_{03}^{0} &= \vec{\omega}_{01}^{0} + C_1^0 \vec{\omega}_{12}^{1} + C_2^0 \vec{\omega}_{23}^{2} \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + 0 \\
&= \begin{bmatrix} 0 \\ 0 \\ \theta_1 + \theta_2 \end{bmatrix}
\end{aligned}$$

Therefore,

$$\Omega_{03}^{0} = \begin{bmatrix} 0 & -(\theta_1 + \theta_2) & 0 \\ \theta_1 + \theta_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Consider the time-varying coordinate transformation matrix C_b^n given below that describes the orientation of the body frame as it rotates with respect to the navigation frame.

$$C_b^n = \begin{bmatrix} cos(t) & sin(t)sin(t^2) & sin(t)cos(t^2) \\ 0 & cos(t^2) & -sin(t^2) \\ -sin(t) & cos(t)sin(t^2) & cos(t)cos(t^2) \end{bmatrix}$$

   (a) Compute expressions for ψ,θ, and ϕ based on the fixed-axis definition of roll, pitch, yaw assuming a 1,2,3 series of rotations (roll, pitch, yaw).

   Solution: Euler angles from a 1-2-3 rotation can be extracted using the following:

$$\phi = tan^{-1}\left(\frac{-C_{23}}{C_{33}}\right)$$
$$\theta = sin^{-1}(-C_{13})$$
$$\psi = tan^{-1}\left(\frac{-C_{12}}{C_{11}}\right)$$

   (b) Use MATLAB to plot ψ, θ, and ϕ as a function of time.

   A simulation of the rotations was conducted for 10 seconds at 100 Hz. Figure 1 depicts the resulting Euler angles.

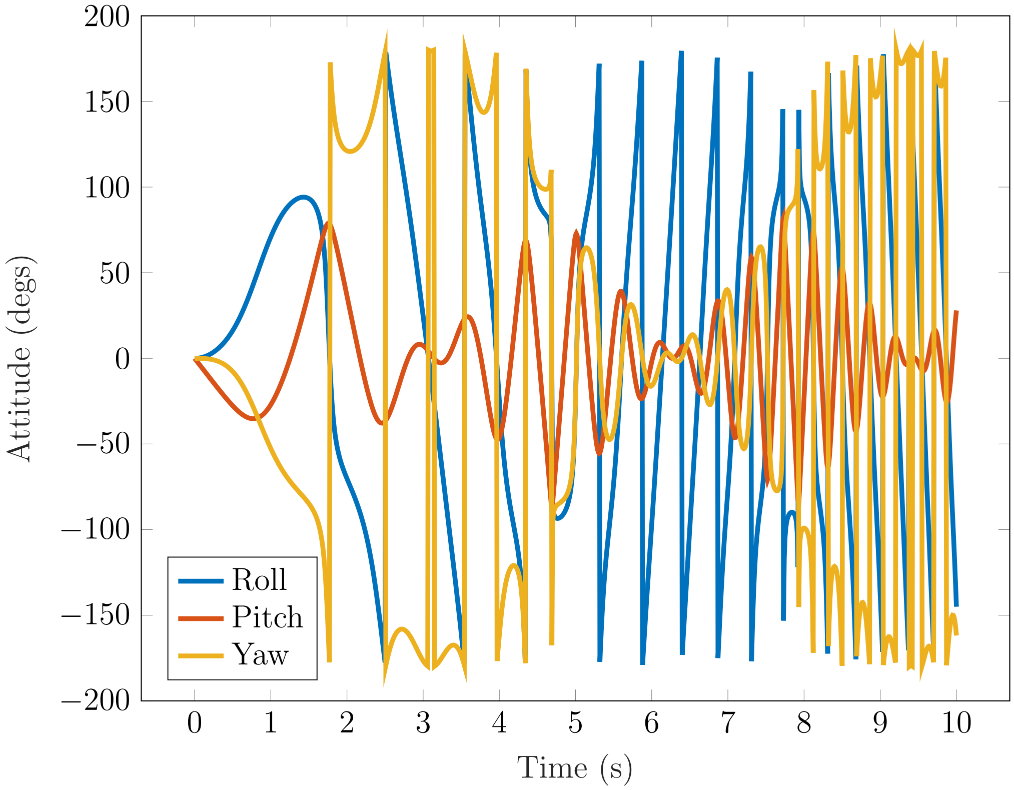


Figure 1: Euler Angles vs. Time

4. Consider the time-varying coordinate transformation matrix C_b^n given below that describes the orientation of the body as it rotates with respect to the navigation frame.

$$C_b^n = \begin{bmatrix} cos(t) & sin(t)sin(t^2) & sin(t)cos(t^2) \\ 0 & cos(t^2) & -sin(t^2) \\ -sin(t) & cos(t)sin(t^2) & cos(t)cos(t^2) \end{bmatrix}$$

(a) Determine the analytic form of the time-derivative of C_b^n (i.e. $\dot{C}_b^n = \frac{dC_b^n}{dt}$) via a term-by-term differentiation.

Solution: The following derivative was determined using the `diff()` function in MATLAB.

$$\dot{C}_b^n = \begin{bmatrix} -\sin(t) & \sin(t^2)\cos(t) + 2t\cos(t^2)\sin(t) & \cos(t^2)\cos(t) - 2\sin(t^2)\sin(t) \\ 0 & -2t\sin(t^2) & -2t\cos(t^2) \\ -\cos(t) & 2t\cos(t^2)\cos(t) - \sin(t^2)\sin(t) & -\cos(t^2)\sin(t) - 2t\sin(t^2)\cos(t) \end{bmatrix}$$

(b) Develop MATLAB functions which accept t (i.e time) as a numerical input and return C_b^n and $\dot{C}_b^n$, respectively, as numerical outputs.

Solution: The functions `timeRotation()` and `timeRotationDot()` are appended to this document.

(c) Using the C_b^n and $\dot{C}_b^n$ functions from above, compute the angular velocity vector $\vec{\omega}_{nb}^{\,n}$ at time $t = 0$ seconds. (HINT: You might want to compute Ω_{nb}^n)

Solution: The skew-symmetric matrices in c-e were calculated using the relationship described by Equation 4. The corresponding unit vectors of instantaneous rotation were determined with the following:

$$\vec{k}_{nb}^{\,n} = \frac{1}{2\sin(\theta)} \begin{bmatrix} \mathbf{C}_{b_{(3,2)}}^n - \mathbf{C}_{b_{(2,3)}}^n \\ \mathbf{C}_{b_{(1,3)}}^n - \mathbf{C}_{b_{(3,1)}}^n \\ \mathbf{C}_{b_{(2,1)}}^n - \mathbf{C}_{b_{(1,2)}}^n \end{bmatrix} \tag{5}$$

θ in Equation 5 is determined by the following:

$$\theta = \cos^{-1}\left(\frac{trace(\mathbf{C}_b^n) - 1}{2}\right) \tag{6}$$

i. What is the magnitude (i.e. $\dot{\theta}$, angular speed) of the angular velocity?

Solution: Given,

$$\Omega_{nb}^{n} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

the angular velocity vector $\vec{\omega}_{nb}^{\,n}$ is the following:

$$\vec{\omega}_{nb}^{\,n} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, the magnitude can be calculated using `vecnorm()` in MATLAB as

$$\dot{\theta} = 1 \text{ rad/s}$$

ii. About what unit vector ($\vec{k}_{nb}^{\,n}$) has the instantaneous rotation occurred?
Solution: Equation 5 is unable to yield a unit vector $\vec{k}_{nb}^{\,n}$ given $\theta = 0$. This is because no rotation has occurred yet at $t = 0$ s.

(d) Using the C_b^n and $\dot{C}_b^n$ functions from above, compute the angular velocity vector $\vec{\omega}_{nb}^{\,n}$ at time $t = 0.5$ seconds.

i. What is the magnitude (i.e. $\dot{\theta}$, angular speed) of the angular velocity?
Solution: Given,

$$\Omega_{nb}^{n} = \begin{bmatrix} 0 & 0.48 & 1 \\ -0.48 & 0 & 0.88 \\ -1 & -0.88 & 0 \end{bmatrix}$$

the angular velocity vector $\vec{\omega}_{nb}^{\,n}$ is the following:

$$\vec{\omega}_{nb}^{\,n} = \begin{bmatrix} 0.88 \\ 1 \\ -0.48 \end{bmatrix}$$

Therefore, the magnitude can be calculated using `vecnorm()` in MATLAB as

$$\dot{\theta} = 1.41 \text{ rad/s}$$

ii. About what unit vector ($\vec{k}_{nb}^{\,n}$) has the instantaneous rotation occurred?
Solution: Equation 5 yields the following unit vector $\vec{k}_{nb}^{\,n}$ given $\theta = 0.5578$ rad from Equation 6:

$$\vec{k}_{nb}^{\,n} = \begin{bmatrix} 0.439 \\ 0.892 \\ -0.112 \end{bmatrix}$$

(e) Using the C_b^n and $\dot{C}_b^n$ functions from above, compute the angular velocity vector $\vec{\omega}_{nb}^{\,n}$ at time $t = 1$ seconds.

i. What is the magnitude (i.e. $\dot{\theta}$, angular speed) of the angular velocity?
Solution: Given,

$$\Omega_{nb}^n = \begin{bmatrix} 0 & 1.68 & 1 \\ -1.68 & 0 & -1.08 \\ -1 & 1.08 & 0 \end{bmatrix}$$

the angular velocity vector $\vec{\omega}_{nb}^{\,n}$ is the following:

$$\vec{\omega}_{nb}^{\,n} = \begin{bmatrix} 1.08 \\ 1 \\ -1.68 \end{bmatrix}$$

Therefore, the magnitude can be calculated using `vecnorm()` in MATLAB as

$$\dot{\theta} = 2.24 \text{ rad/s}$$

ii. About what unit vector ($\vec{k}_{nb}^{\,n}$) has the instantaneous rotation occurred?
Solution: Equation 5 yields the following unit vector $\vec{k}_{nb}^{\,n}$ given $\theta = 1.3834$ rad from Equation 6:

$$\vec{k}_{nb}^{\,n} = \begin{bmatrix} 0.660 \\ 0.660 \\ -0.360 \end{bmatrix}$$

(f) In practice, direct measurement of the angular velocity vector $\vec{\omega}_{nb}^{\,n}$ can prove challenging, so a finite-difference approach may be taken given two sequential orientations represented by $C_b^n(t)$ and $C_b^n(t + \Delta t)$ a small time Δt apart. Consider the approximate value of the angular velocity vector $\vec{\omega}_{nb}^{\,n}$ derived by using the finite difference

$$\dot{C}_b^n(t) \approx \frac{C_b^n(t + \Delta t) - C_b^n(t)}{\Delta t}$$

at times $t = 0$,0.5, and 1 second. Compare the "analytic" values for $\dot{\theta}$ and $\vec{k}_{nb}^{\,n}$ (found in parts b, c, and d) with your approximations from the finite difference using $\Delta t = 0.1$ seconds. How large are the errors?

Solution: There are no errors between the unit vectors because they are calculated in the same manner. However, the angular velocity magnitudes are slightly different. The errors are depicted in Table 1.

Table 1: Angular Velocity Errors

Time (s)	0.0	0.5	1.0
Analytical $\dot{\theta}$	1.0	1.4142	2.2361
Finite Difference $\dot{\theta}$	0.0	1.3768	2.2491
$error_{\dot{\theta}}$	1.0	0.0375	-0.0130

The errors between the analytical and finite difference angular velocities are on the order of 0.01 rad/s.

5. Given the geodetic coordinate of the peak of Mt. Everest as Latitude (L_b) 27^o 59′ 16” N, Longitude (λ_b) 86^o 56′ 40” E, and height (h_b) 8850 meters (derived by GPS in 1999):

   (a) Develop a MATLAB function

   $$\text{function} \;\; r^e_{eb} = \text{llh2xyz}(L_b, \lambda_b, h_b)$$

   to convert from geodetic curvilinear Latitude, Longitude, and height to ECEF rectangular x,y, and z coordinates (Please use SI units). Attach a printout of your function.

   Solution: The function `llh2xyz()` is appended to this document.

   i. Test your llh2xyz function using coordinates of the peak of Mt. Everest. What is $\vec{r}^e_{eb}$?

      Solution: $\vec{r}^e_{eb}$ was determined to be the following:

      $$\vec{r}^e_{eb} = \begin{bmatrix} 300858.16 \\ 5636146.41 \\ 2979462.45 \end{bmatrix} \text{ m}$$

   (b) Develop a MATLAB function

   $$\text{function} \;\; [L_b, \lambda_b, h_b] = \text{xyz2llh}(r^e_{eb})$$

   to convert from ECEF rectangular x,y, and z coordinates to geodetic curvilinear Latitude, Longitude, and height (Please use SI units). Attach a printout of your function. HINT: This should be an interactive transformation (i.e. not closed form).

   Solution: The function `xyz2llh()` is appended to this document. This function was validated using the results of `llh2xyz()` to solve for the initial latitude, longitude, and height inputs.

   (c) What is the acceleration due to gravity at the ellipsoid (i.e. at the ellipsoid $h_b = 0$. HINT: This should only be a function of Latitude – see attached Pages)?

   Solution: Using the Somigliana model given in Groves, the acceleration due to gravity at Mt. Everest's latitude can be approximated as so:

   $$g_0(L) \approx 9.7803253359 \frac{(1 + 0.001931853 \sin^2(L))}{\sqrt{1 - e^2 \sin^2(L)}} \tag{7}$$

   $$\begin{aligned} g_0(27.99) &\approx 9.7803253359 \frac{(1 + 0.001931853 \sin^2(27.99))}{\sqrt{1 - e^2 \sin^2(27.99)}} \\ g_0(27.99) &\approx 9.7917 \text{ m/s}^2 \end{aligned}$$

(d) What is the magnitude of the centrifugal acceleration $(-\Omega^e_{ie}\,\Omega^e_{ie}\,\vec{r}^{\,e}_{eb})$ at the ellipsoid and at the peak?
Solution: Given,

$$\omega_{ie} = 7.2992115 \times 10^{-5} \text{ rad/s}$$

The centrifugal acceleration can be calculated as so:

$$\begin{aligned}
(\Omega^e_{ie}\,\Omega^e_{ie}\,\vec{r}^{\,e}_{eb})_{peak} &= \begin{bmatrix} 0 & -\omega_{ie} & 0 \\ \omega_{ie} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_{ie} & 0 \\ \omega_{ie} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 300858.16 \\ 5636146.41 \\ 2979462.45 \end{bmatrix} \\
&= \omega_{ie}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 300858.16 \\ 5636146.41 \\ 2979462.45 \end{bmatrix} \\
&= \begin{bmatrix} 0.00160 \\ 0.0300 \\ 0.0 \end{bmatrix} \text{ m/s}^2
\end{aligned}$$

$$\begin{aligned}
(\Omega^e_{ie}\,\Omega^e_{ie}\,\vec{r}^{\,e}_{eb})_{ellipsoid} &= \begin{bmatrix} 0 & -\omega_{ie} & 0 \\ \omega_{ie} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_{ie} & 0 \\ \omega_{ie} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 300441.60 \\ 5628342.55 \\ 2975309.29 \end{bmatrix} \\
&= \omega_{ie}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 300441.60 \\ 5628342.55 \\ 2975309.29 \end{bmatrix} \\
&= \begin{bmatrix} 0.00160 \\ 0.0299 \\ 0.0 \end{bmatrix} \text{ m/s}^2
\end{aligned}$$

The magnitudes of the centrifugal acceleration vectors are the following:

$$\begin{aligned}
\|\Omega^e_{ie}\,\Omega^e_{ie}\,\vec{r}^{\,e}_{eb}\|_{peak} &= 0.0301 \text{ m/s}^2 \\
\|\Omega^e_{ie}\,\Omega^e_{ie}\,\vec{r}^{\,e}_{eb}\|_{ellipsoid} &= 0.0300 \text{ m/s}^2
\end{aligned}$$

(e) What is the magnitude of the gravitational attraction at the ellipsoid and at the peak? HINT: See **attached pages** to compute $\vec{\gamma}^{\,e}_{ib} = \vec{\gamma}^{\,i}_{eb}|_{\vec{r}^{\,i}_{ib}=\vec{r}^{\,e}_{eb}}$
Solution: The equation for gravitational attraction is given in Equation 8.

$$\gamma_{eb}^{e} = -\frac{\mu}{|\mathbf{r}_{eb}^{e}|^3}\left\{\mathbf{r}_{eb}^{e} + \frac{3}{2}J_2\frac{R_0^2}{|\mathbf{r}_{eb}^{e}|^2}\begin{bmatrix}\left(1-5\left(\frac{\mathbf{r}_{eb_z}^{e}}{|\mathbf{r}_{eb}^{e}|^2}\right)^2\right)\mathbf{r}_{eb_x}^{e}\\ \left(1-5\left(\frac{\mathbf{r}_{eb_z}^{e}}{|\mathbf{r}_{eb}^{e}|^2}\right)^2\right)\mathbf{r}_{eb_y}^{e}\\ \left(3-5\left(\frac{\mathbf{r}_{eb_z}^{e}}{|\mathbf{r}_{eb}^{e}|^2}\right)^2\right)\mathbf{r}_{eb_z}^{e}\end{bmatrix}\right\} \tag{8}$$

The magnitude of the gravitational attractions at the ellipsoid and peak were calculated using Equation 8 in MATLAB.

$$\begin{aligned}\|\gamma_{ib}^{e}\|_{peak} &= 9.791\ \text{m/s}^2\\ \|\gamma_{ib}^{e}\|_{ellipsoid} &= 9.818\ \text{m/s}^2\end{aligned} \tag{9}$$

```
function Cbn = timeRotation(t)

Cbn = [cos(t), sin(t) * sin(t^2), sin(t) * cos(t^2); ...
        0, cos(t^2), -sin(t^2); ...
        -sin(t), cos(t) * sin(t^2), cos(t) * cos(t^2)];

end
```

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```
function CbnDot = timeRotationDot(t)

CbnDot = [-sin(t), sin(t^2)*cos(t) + 2*t*cos(t^2)*sin(t),   cos(t^2)*cos(t) -
 2*t*sin(t^2)*sin(t);
       0,                          -2*t*sin(t^2),
 -2*t*cos(t^2);
-cos(t), 2*t*cos(t^2)*cos(t) - sin(t^2)*sin(t), - cos(t^2)*sin(t) -
 2*t*sin(t^2)*cos(t)];

end
```

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```
function [Rbe_e] = llh2xyz(lat, lon, h)

    % Constants
    R_0 = 6378137.0; % Equatorial Radius (m)
    e = 0.0818191908425; % Eccentricity

    % Radius of Curvature
    R_E = R_0 / (sqrt(1-e^2*sin(lat)^2)); % Transverse Radius of Curvature (m)

    % LLH to ECEF
    x = (R_E + h)*cos(lat)*cos(lon);
    y = (R_E + h)*cos(lat)*sin(lon);
    z = ((1-e^2)*R_E + h)*sin(lat);

    Rbe_e = [x y z];

end
```

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```
function llh = xyz2llh(x, y, z)

    % Constants
    R_0 = 6378137; % Equatorial Radius (m)
    e = 0.0818191908425; % Eccentricity

    % Initial Latitude
    lat = atan2(z, (1 - e^2)*vecnorm([x y]));
    lastLat = 0;
    diffLat = lat - lastLat;

    while abs(diffLat) > 1e-6

        % Radius of Curvature
        R_E = R_0 / sqrt(1-e^2*sin(lat)^2); % Transverse Radius of Curvature
 (m)

        % Height
        h = (vecnorm([x y])/cos(lat))- R_E;

        % Latitude
        lastLat = lat;
        lat = atan2(z, (1 - e^2*(R_E/(h+R_E)))*vecnorm([x y]));
        diffLat = lat - lastLat;

    end

    % Longitude
    lon = atan2(y,x);
    llh = [lat lon h];

end
```

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