DP Optimizations

1D-1D Optimizaton :

$$dp[i] = \min_{k<i}(dp[k] + cost[k][i])$$

If cost satifies quadrangle inequality, $Best[i] <= Best[i+1]$. Hence, for each i see for what all j>i, i could be the $Best[j]$ using binary search. Store the beginning of each segment in a vector. Whenever we consider a new value of i, perform the following to update the segments:

- While the new value of i is better than the value at the back of the vector, pop the back.
- Binary search in the current segment to find the turning point, and push this value of i together with the turning point onto the back of the vector.

Knuth's Optimization

$$dp[i][j] = \min_{i<k<j}(dp[i][k-1] + dp[k+1][j]) + cost[i][j]$$

If cost satisfies quadrangle inequality and it is monotonic, then $Best[i][j-1] <= Best[i][j] <= Best[i][j+1]$. So, build the dp over length (j=i+len) and for dp[i][j] traverse only from Best[i][j-1] to Best[i][j+1]. Reduces $O(N^3)$ to $O(N^2)$

Games

Grundy numbers. For a two-player, normal-play (last to move wins) game on a graph (V, E): $G(x) = \text{mex}(\{G(y) : (x, y) \in E\})$, where $\text{mex}(S) = \min\{n \geqslant 0 : n \notin S\}$. x is losing iff $G(x) = 0$.

Sums of games.

- *Player chooses a game and makes a move in it.* Grundy number of a position is xor of grundy numbers of positions in summed games.
- *Player chooses a non-empty subset of games (possibly, all) and makes moves in all of them.* A position is losing iff each game is in a losing position.
- *Player chooses a proper subset of games (not empty and not all), and makes moves in all chosen ones.* A position is losing iff grundy numbers of all games are equal.
- *Player must move in all games, and loses if can't move in some game.* A position is losing if any of the games is in a losing position.

Misère Nim. A position with pile sizes $a_1, a_2, \ldots, a_n \geqslant 1$, not all equal to 1, is losing iff $a_1 \oplus a_2 \oplus \cdots \oplus a_n = 0$ (like in normal nim.) A position with n piles of size 1 is losing iff n is *odd*.
Green HackenBush Game If two players are playing a game where move allowed is to cut an edge of a (rooted) tree then for each node,set value of the node to xor of (1 + value of it's child) for all children of the node in the tree. For leaves, value is 0. See value of root to determine the winner. For a (rooted) graph, where each player can remove an edge from the component connected to the root and one unable to remove loses. Create bridge tfaree and do same as tree except val[node] = orignalVal[node] $\oplus$ (numEdges[node] & 1)

Maths

Angular bisector of angle ABC is line BD, where $D = \frac{BA}{|BA|} + \frac{BC}{|BC|}$.

Center of incircle of triangle ABC is at the intersection of angular bisectors, and is $\frac{a}{a+b+c}A + \frac{b}{a+b+c}B + \frac{c}{a+b+c}C$, where a, b, c are lengths of sides, opposite to vertices A, B, C. Radius $= \frac{2\Delta}{a+b+c}$.

Sums

$(m+1) * \sum_{k=0}^{n} k^m = (n+1)^{m+1} - \left(\sum_{r=2}^{m+1}\left(\binom{m+1}{r}\sum_{k=0}^{n} k^{m+1-r}\right)\right)$

$S_n = \sum_{k=1}^{n}[a+(k-1)d]r^{k-1} = \frac{a-[a+(n-1)d]r^n}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2}$

Derangements. Number of permutations of $n = 0, 1, 2, \ldots$ elements without fixed points is $1, 0, 1, 2, 9, 44, 265, 1854, 14833, \ldots$ Recurrence: $D_n = (n-1)(D_{n-1} + D_{n-2}) = nD_{n-1} + (-1)^n$. Corollary: number of permutations with exactly k fixed points is $\binom{n}{k}D_{n-k}$.

Catalan Numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. DP Recurrence : $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ and $C_0 = 1$. It is the no of ways of arranging a bracket sequence of length n. Also equal to number of unlabelled binary trees of size n.

Stirling numbers of 1^{st} kind. $s_{n,k}$ is $(-1)^{n-k}$ times the number of permutations of n elements with exactly k permutation cycles. $|s_{n,k}| = |s_{n-1,k-1}| + (n-1)|s_{n-1,k}|$.

Stirling numbers of 2^{nd} kind. $S_{n,k}$ is the number of ways to partition a set of n elements into exactly k non-empty subsets. $S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$. $S_{n,1} = S_{n,n} = 1$.

Bell numbers. B_n is the number of partitions of n elements. $B_0, \ldots = 1, 1, 2, 5, 15, 52, 203, \ldots$ $B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_k = \sum_{k=1}^{n} S_{n,k}$. Bell triangle: $B_r = a_{r,1} = a_{r-1,r-1}$, $a_{r,c} = a_{r-1,c-1} + a_{r,c-1}$.

Linear diophantine equation. $ax + by = c$. Let $d = \gcd(a, b)$. A solution exists iff $d|c$. If (x_0, y_0) is any solution, then all solutions are given by $(x, y) = (x_0 + \frac{b}{d}t, y_0 - \frac{a}{d}t)$, $t \in \mathbb{Z}$. To find some solution (x_0, y_0), use extended GCD to solve $ax_0 + by_0 = d = \gcd(a, b)$, and multiply its solutions by $\frac{c}{d}$. Linear diophantine equation in n variables: $a_1x_1 + \cdots + a_nx_n = c$ has solutions iff $\gcd(a_1, \ldots, a_n)|c$. To find some solution, let $b = \gcd(a_2, \ldots, a_n)$, solve $a_1x_1 + by = c$, and iterate with $a_2x_2 + \cdots = y$. Multiplicative inverse of a modulo m: x in $ax + my = 1$, or $a^{\phi(m)-1} \pmod{m}$.

Chinese Remainder Theorem. System $x \equiv a_i \pmod{m_i}$ for $i = 1, \ldots, n$, with pairwise relatively-prime m_i has a unique solution modulo $M = m_1m_2\ldots m_n$: $x = a_1b_1\frac{M}{m_1} + \cdots + a_nb_n\frac{M}{m_n} \pmod{M}$, where b_i is modular inverse of $\frac{M}{m_i}$ modulo m_i.

System $x \equiv a \pmod{m}$, $x \equiv b \pmod{n}$ has solutions iff $a \equiv b \pmod{g}$, where $g = \gcd(m, n)$. The solution is unique modulo $L = \frac{mn}{g}$, and equals: $x \equiv a + T(b-a)m/g \equiv b + S(a-b)n/g \pmod{L}$, where S and T are integer solutions of $mT + nS = \gcd(m, n)$.

Euler's phi function. $\phi(n) = |\{m \in \mathbb{N}, m \leqslant n, \gcd(m, n) = 1\}|$.
$\phi(mn) = \frac{\phi(m)\phi(n)\gcd(m,n)}{\phi(\gcd(m,n))}$. $\phi(p^a) = p^{a-1}(p-1)$. $\sum_{d|n}\phi(d) = \sum_{d|n}\phi(\frac{n}{d}) = n$.

Euler's theorem. $a^{\phi(n)} \equiv 1 \pmod n$, if $\gcd(a, n) = 1$.

Wilson's theorem. p is prime iff $(p-1)! \equiv -1 \pmod p$.

Mobius function. $\mu(1) = 1$. $\mu(n) = 0$, if n is not squarefree. $\mu(n) = (-1)^s$, if n is the product of s distinct primes. Let f, F be functions on positive integers. If for all $n \in N$, $F(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} \mu(d)F(\frac{n}{d})$, and vice versa. $\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}$. $\sum_{d|n} \mu(d) = 1$.
If f is multiplicative, then $\sum_{d|n} \mu(d)f(d) = \prod_{p|n}(1 - f(p))$, $\sum_{d|n} \mu(d)^2 f(d) = \prod_{p|n}(1 + f(p))$.

Primitive roots. If the order of g modulo m (min $n > 0$: $g^n \equiv 1 \pmod m$) is $\phi(m)$, then g is called a primitive root. If Z_m has a primitive root, then it has $\phi(\phi(m))$ distinct primitive roots. Z_m has a primitive root iff m is one of 2, 4, p^k, $2p^k$, where p is an odd prime. If Z_m has a primitive root g, then for all a coprime to m, there exists unique integer $i = \text{ind}_g(a)$ modulo $\phi(m)$, such that $g^i \equiv a \pmod m$. $\text{ind}_g(a)$ has logarithm-like properties: $\text{ind}(1) = 0$, $\text{ind}(ab) = \text{ind}(a) + \text{ind}(b)$. If p is prime and a is not divisible by p, then congruence $x^n \equiv a \pmod p$ has $\gcd(n, p-1)$ solutions if $a^{(p-1)/\gcd(n,p-1)} \equiv 1 \pmod p$, and no solutions otherwise. (Proof sketch: let g be a primitive root, and $g^i \equiv a \pmod p$, $g^u \equiv x \pmod p$. $x^n \equiv a \pmod p$ iff $g^{nu} \equiv g^i \pmod p$ iff $nu \equiv i \pmod p$.)

Postage stamps/McNuggets problem. Let a, b be relatively-prime integers. There are exactly $\frac{1}{2}(a-1)(b-1)$ numbers *not* of form $ax+by$ ($x, y \geqslant 0$), and the largest is $(a-1)(b-1)-1 = ab-a-b$.

Ballot Theorem. If A receives p votes and B receives q votes with $p > q$ then probability that A is strictly ahead of B at all times is given by $\frac{p-q}{p+q}$. If ties allowed : $\frac{p+1-q}{p+1}$

Leibniz formula for Determinants $det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$. where $sgn(\sigma)$ is $+1/-1$ based on even/odd parity of number of inversions of permutation σ.

2D Recurrence using FFT For any 2D recurrence of the form $F_{n,p} = \sum_{i=0}^{k} a_i(n).F_{n-1,p-i}$. We can write it as follows use polynomial multiplication to compute the values of recurrence fast.

$$\sum_{i=0}^{kn} F_{n,i}.x^i = \prod_{i=1}^{n} \sum_{j=0}^{k} a_j(i).x^j$$

Trick for 2D FFT $\frac{P(1)+P(\delta)+...+P(\delta^{n-1})}{n}$ is sum of all indexes of polynomial P divisible by n where δ = n'th root of unity.

Fermat's two-squares theorem. Odd prime p can be represented as a sum of two squares iff $p \equiv 1 \pmod 4$. A product of two sums of two squares is a sum of two squares. Thus, n is a sum of two squares iff every prime of form $p = 4k + 3$ occurs an even number of times in n's factorization.

Counting Primes Fast To count number of primes lesser than big n. Use following recurrence. dp[n][j] = dp[n][$j+1$] + dp[n/p_j][j] where dp[i][j] stores count of numbers lesser than equal to i having all prime divisors greater than equal to p_j. Precompute this for all i less than some small k and for others use the recurrence to compute in small time.

Graphs

Mirsky's Theorem The height of a poset is the maximum cardinality of a chain, a totally ordered subset of the given partial order.Mirsky's theorem states that, for every partially ordered set, the height also equals the minimum number of antichains (subsets in which no pair of elements are ordered) into which the set may be partitioned. In such a partition, every two elements of the longest chain must go into two different antichains, so the number of antichains is always greater than or equal to the height.

Dilworth's Theorem An antichain in a partially ordered set is a set of elements no two of which are comparable to each other, and a chain is a set of elements every two of which are comparable. Dilworth's Theorem states that in any finite partially ordered set, the maximum number of elements in any antichain equals the minimum number of chains in any partition of the set into chains. Comparability graphs enjoy the nice property that the maximum independent set size is equal to the minimum number of cliques whose union covers all nodes (this is Dilworth's theorem). Furthermore, cliques in comparability graphs correspond to directed paths and vice-versa (once the edges are oriented as implied above). So, the problem is reduced to computing the minimum number of paths required in a directed graph to cover all nodes. This is solved via bipartite matching. Build a bipartite graph B with a copy of the nodes of the original graph on both sides. Call the original directed graph G and say it has n nodes. Add an edge from a node u on the "left" to a node v on the "right" in B if u-v is a directed edge in G. Then, the size of a minimum path cover in G (and, hence, a maximum independent set) is exactly n minus the maximum matching size in B. Minimum Disjoint Path Cover in a DAG is n - maximum matching in the corresponding bipartite graph with each vertex partitioned into 2 sets.

Konig's Thoerem In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover. Consider a bipartite graph where the vertices are partitioned into left (L) and right (R) sets. Suppose there is a maximum matching which partitions the edges into those used in the matching (E_m) and those not (E_0). Let T consist of all unmatched vertices from L, as well as all vertices reachable from those by going left-to-right along edges from E_0 and right-to-left along edges from E_m. This essentially means that for each unmatched vertex in L, we add into T all vertices that occur in a path alternating between edges from E_0 and E_m. Then $(L \setminus T) \cup (R \cap T)$ is a minimum vertex cover. Intuitively, vertices in T are added if they are in R and subtracted if they are in L to obtain the minimum vertex cover.

Matrix-tree theorem Let matrix $T = [t_{ij}]$, where t_{ij} is negative of the number of multiedges between i and j, for $i \neq j$, and $t_{ii} = \deg_i$. Number of spanning trees of a graph is equal to the determinant of a matrix obtained by deleting any k-th row and k-th column from T. If G is a multigraph and e is an edge of G, then the number $\tau(G)$ of spanning trees of G satisfies recurrence $\tau(G) = \tau(G - e) + \tau(G/e)$, when $G - e$ is the multigraph obtained by deleting e, and G/e is the contraction of G by e (multiple edges arising from the contraction are preserved.)

Cycle Spaces The (binary) cycle space of an undirected graph is the set of its Eulerian subgraphs. This set of subgraphs can be described algebraically as a vector space over the two-element finite field. One way of constructing a cycle basis is to form a spanning forest of the graph, and then for each edge e that does not belong to the forest, form a cycle C C_e consisting of e together with the path in the forest connecting the endpoints of e. The set of cycles C_e formed in this way are linearly independent (each one contains an edge e that does not belong to any of the other cycles) and has the correct size $mn + c$ to be a basis, so it necessarily is a basis. This is fundamental cycle basis.

Cut Spaces The family of all cut sets of an undirected graph is known as the cut space of the graph. It forms a vector space over the two-element finite field of arithmetic modulo two, with the symmetric difference of two cut sets as the vector addition operation, and is the orthogonal complement of the cycle space. To compute the basis vector for the cut space, consider any spanning tree of the graph. For every edge e in the spanning tree, remove the edge and consider the cut formed. Thus dimension of the basis vector for cut space is n-1.

Miscellaneous

- Number of perfect matchings of a bipartite graph is equal to the permanent of the adjacency matrix obtained. To check the parity of the number of perfect matchings, we can evaluate the permanent of the matrix in Z_2 which can be done easily coz in Z_2, Perm(A) = Deter(A).