Codechef Learn, Episode 2
Lec-2 **SRD on Trees - Supernode Trees**

https://youtu.be/8VHWdNnP3h4

tanujkhattar@

SRD on Trees, Part-1 Recap

- Perform an ETT/HLD of the given tree to ensure P[x] < x in the linearized array.
- Do Array Square Decomposition on the linearized array s.t.
- Edges within the same block form a forest of Trees with at-most sqrt(N) nodes.
- Edges across blocks (x → y) start at root node (x) of a decomposed tree and end at some node (y) lying in another block to the left (BLOCK[y] < BLOCK[x])
- This structure can be used to support
  - Subtree updates → Reduce to lazy range updates [L, R] on the linearized ETT array.
  - Path Queries → Answered by spending O(B) time in blocks of x, y and LCA and O(1) time per block for every block in between.

Can we support Path Updates as well?

- Given a tree with N elements, support the following operations
  - Query U V X : Tell the minimum element >= X on the path from U to V
  - Update U V V: Add V to all elements on the path from U to V.
- https://www.codechef.com/problems/TRS

Can we support Path Updates as well?

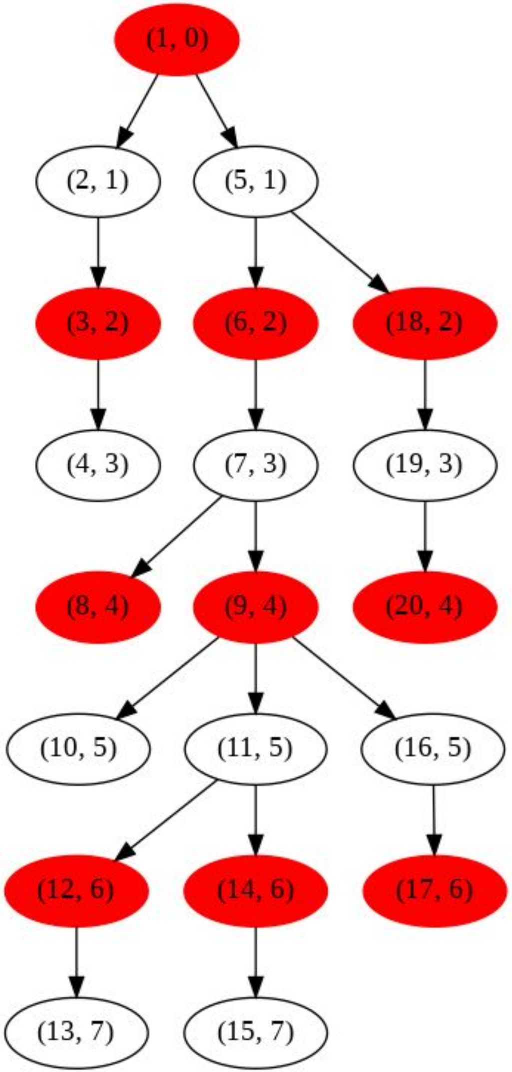
- We can support subtree updates because they reduce to a single range update in the ETT array.
  - We spend O(sqrt(N) * f(B)) time to process 1 range update [L, R].
  - f(B) is the time taken by the DS maintained at each node + lazy propagation.
- Updates on paths would require us to maintain another structure because ETT doesn't store path information by default.
  - Any ideas ?

Can we support Path Updates as well?

- We could do HLD and build a sqrt structure on top of it but then updates would take O(logN * sqrt(N) * f(B))
  - This is because HLD divides a path query into O(logN) different [L, R] ranges.
  - Hence, we would need to update O(logN) different [L, R] ranges via sqrt decomp on array.
  - Even though union of the [L, R] ranges <= N, processing O(logN) individual blocks in which L, R lie would be expensive if done naively.
- Another problem is that we need to process the decomposed trees s.t. we can efficiently query the path x → root[x] for any given x in the decomposed tree.
  - Answering “Min element >= V on the path from x → root[x]” efficiently via precomputation would again require additional complex processing like persistent segment tree ending at each node built using it’s parent node.
- We probably need something else for complex path queries & updates!

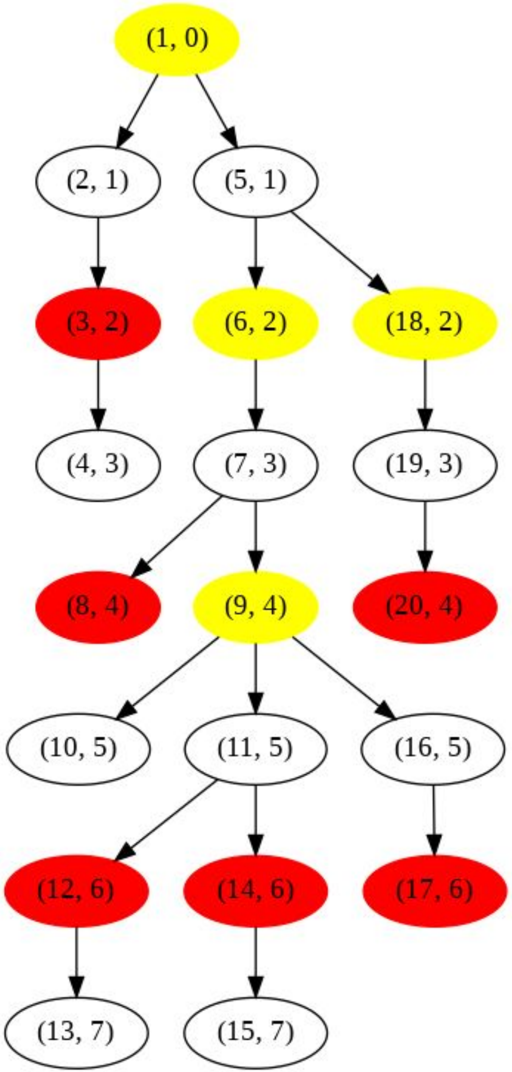
Supernode Trees Introduction

- Mark the nodes with level % B == 0 as special.
- How many special nodes can we have?
  - O(sqrt(N)) ?
  - O(N) ?



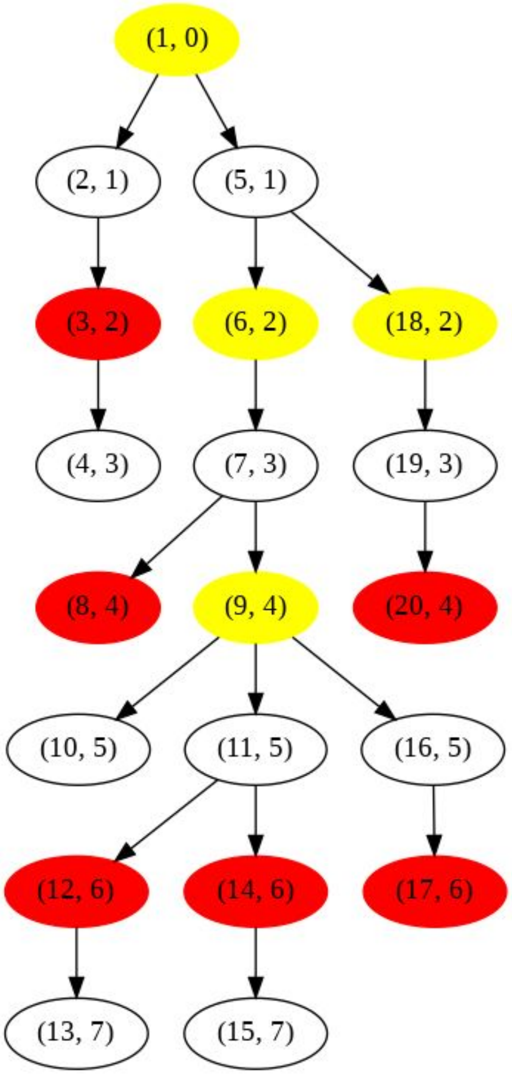
Supernode Trees Introduction

- What if we mark the nodes special when
  - level[N] % B == 0 and N is an “internal” node i.e.
  - N is not the bottomost marked node in the tree ?
- How many special nodes (yellow) can we now have?
  - O(sqrt(N)) ?
  - O(N) ?



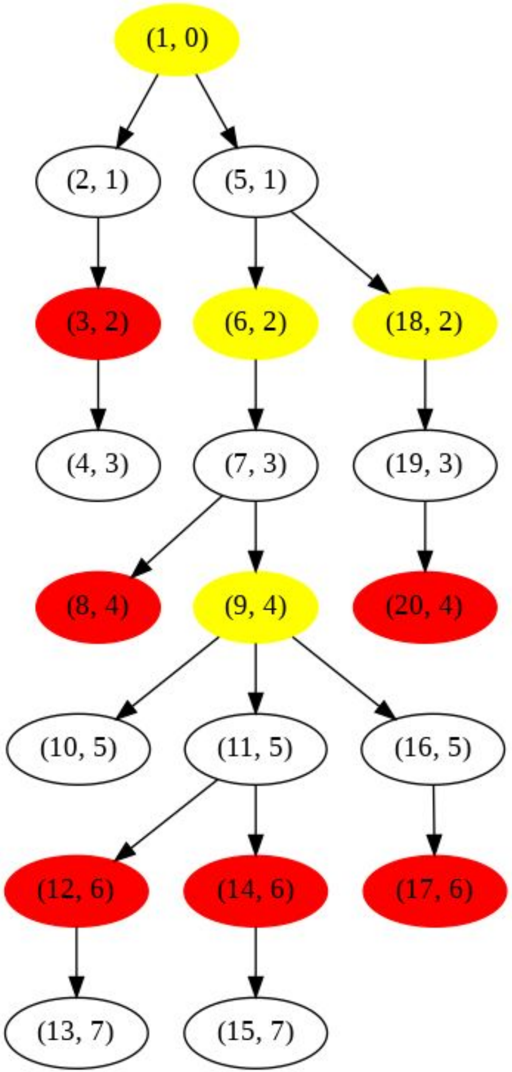
Supernode Trees Introduction

- What if we mark the nodes special when
  - level[N] % B == 0 and N is an “internal” node i.e.
  - N is not the bottomost marked node in the tree ?
- How many special nodes (yellow) can we now have?
  - Note that distance between two special nodes one below each other is O(B) .
  - Hence, each “internal” special node has at least O(B) non-special nodes below it.
  - Number of yellow “special” nodes would therefore be O(N / B).
  - If B ~ sqrt(N); we have ~O(sqrt(N)) marked nodes :)



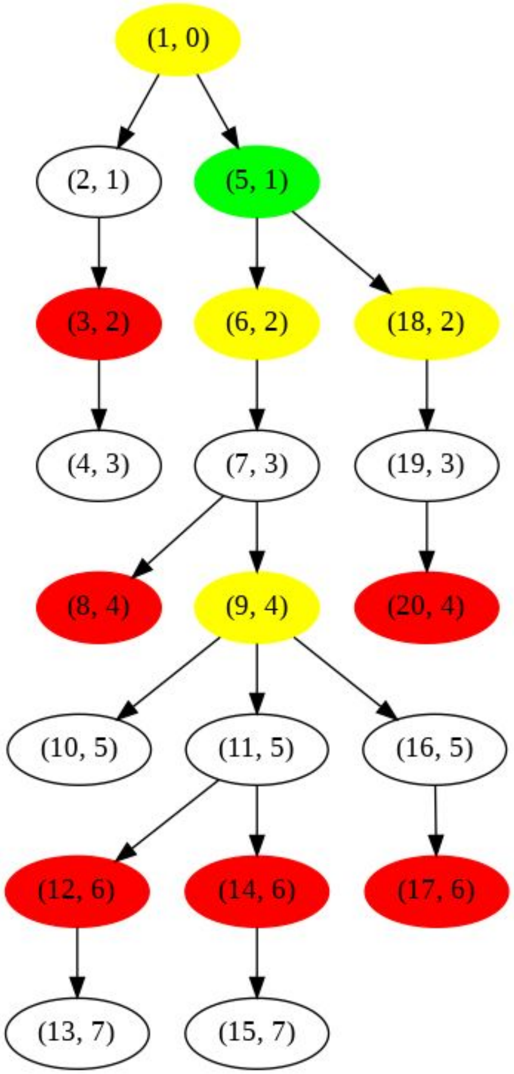
Supernode Trees Introduction

- What can we do with these (yellow) special marked nodes?
- Can we compress straight chains similar to HLD ?
  - 6 → 9 can be compressed to a single edge and processed together.
  - But, 1 → 6 and 1 → 18 have overlapping node 5.
- Can we do something to have only
  - O(sqrt(N)) marked nodes
  - Compress all chains into single edges between marked nodes, except edges below the bottomost marked node



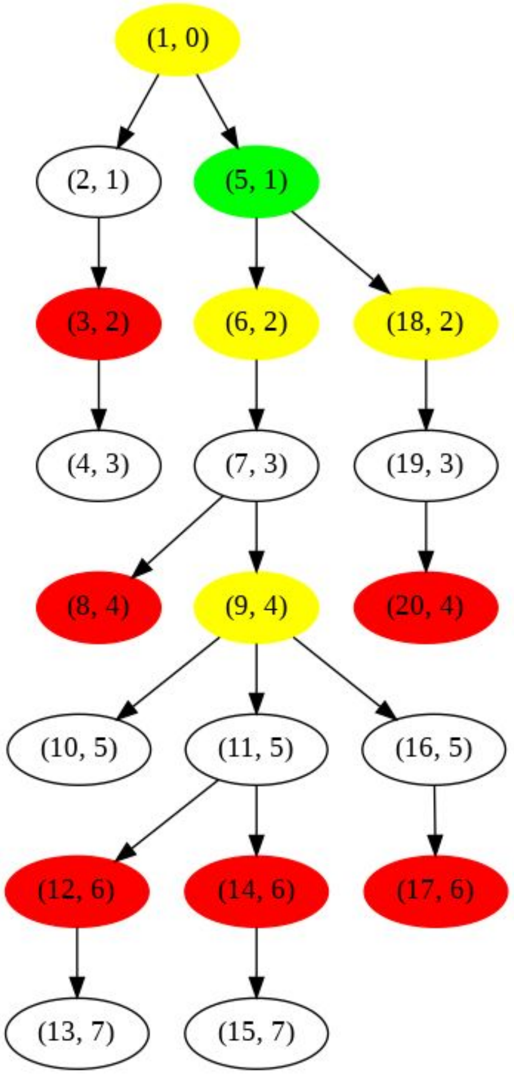
Supernode Trees Introduction

- What can we do with these (yellow) special marked nodes?
- Can we compress straight chains similar to HLD ?
  - 6 → 9 can be compressed to a single edge and processed together.
  - But, 1 → 6 and 1 → 18   have overlapping node 5.
- Can we do something to have only
  - O(sqrt(N)) marked nodes
  - Compress all chains into single edges between marked nodes, except edges below the bottomost marked node.
- Yes! Build an Auxiliary Tree of the K marked nodes
  - The idea is to add LCA of adjacent yellow nodes to the set of marked nodes.
  - It adds at-most K - 1 new nodes, thus total nodes ~O(sqrt(N))



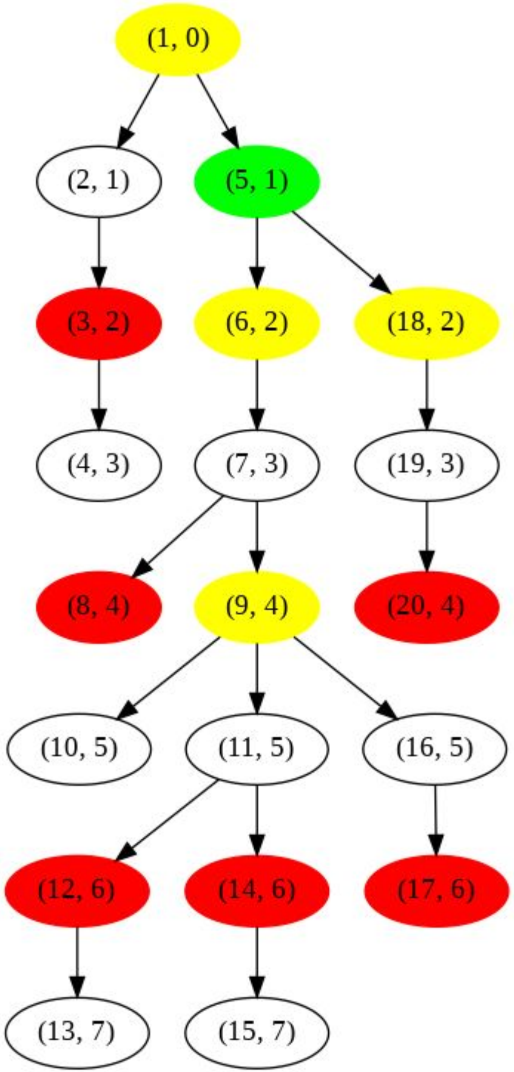
Supernode Trees Properties

- No. of “special” nodes in the supernode tree is O(sqrt(N))
  - Red Nodes: # of nodes O(N), not marked as “special”
    - level[x] % B == 0
  - Yellow Nodes: # of nodes O(N / B), marked as “special”
    - level[x] % B == 0 and
    - Has at least 1 red node in it’s subtree.
  - Green Nodes: # of nodes O(N / B), marked as “special”
    - Additional nodes marked to ensure that every top-down path between consecutive special nodes is a chain.
    - Can be found using Auxiliary Tree type processing.
- The “special” nodes are called “supernodes”



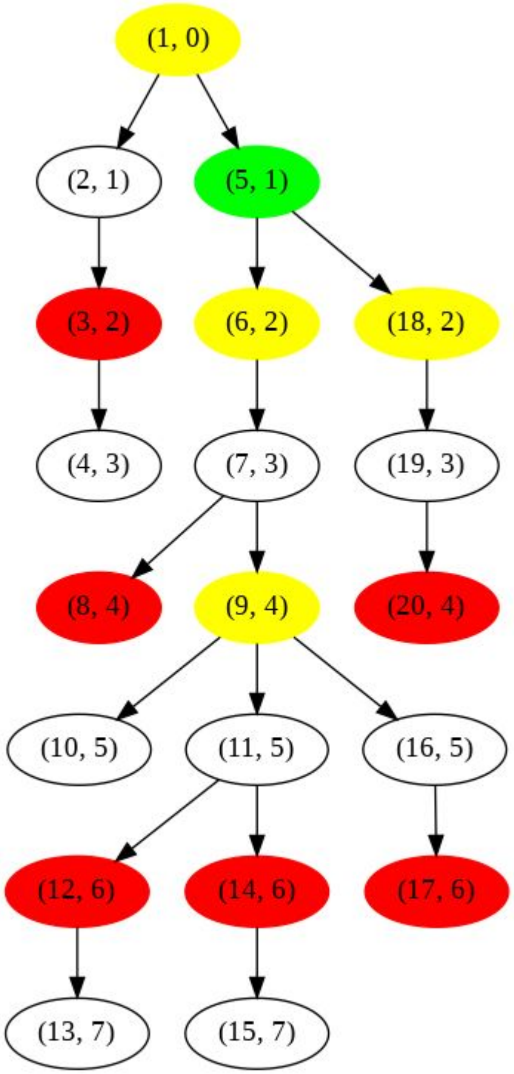
Supernode Trees Properties

- Any path x → y , where y is an ancestor of x, can be written as:
  - x → sp_x: at-most O(B) non-special nodes
  - sp_x → sp_y : at-most O(N / B) special nodes by traversing compressed edges
  - sp_y → y: at-most O(B) non-special nodes.
- This is similar to how a query / update [L, R] gets processed in Array Square Root Decomposition!
  - Note that paths starting below the bottom-most special nodes can have O(2 * B) non-special nodes because of ignored bottom red layer.



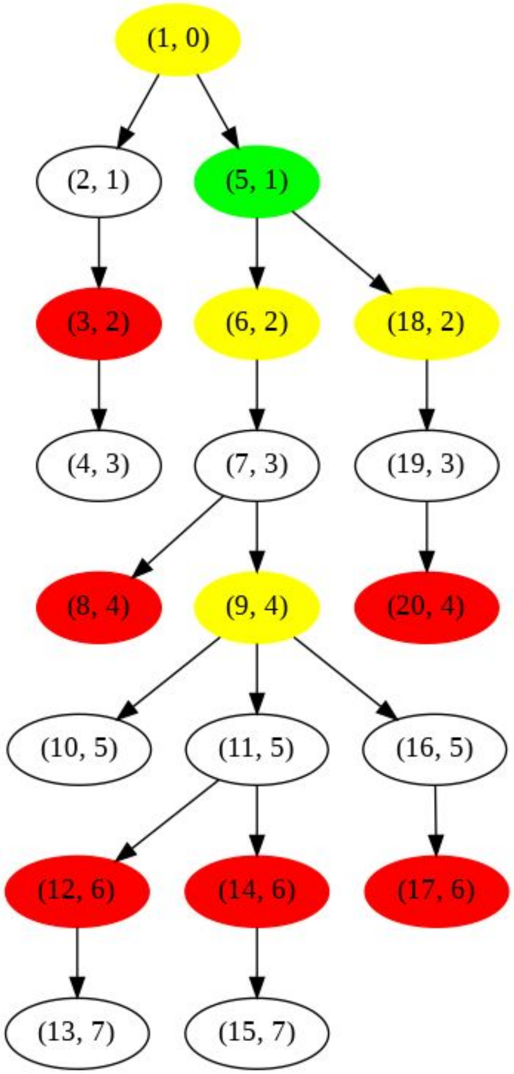
Supernode Trees Properties

- Thus, we can maintain a DS for each compressed chains to answer query / updates efficiently
  - This is similar maintaining a DS for each block in Array Sqrt Decomposition.
- To process a path query / update from x → y
  - x → sp_x: Brute force by traversing O(B) non-special nodes
  - sp_x → sp_y: Traverse O(N / B) special nodes by jumping over the compressed chains in O(f(B)) using the maintained DS.
  - sp_y → y: Brute force by traversing O(B) non-special nodes
- Total Complexity: O((B + N / B) * TimeTakenByDS)



Supernode Trees Implementation

```
int dfs(int x, int p = 0) {
  par[x] = head[x] = p;
  level[x] = level[p] + 1;
  bool seen_spcl = false, is_sqrt_node = !(level[x] % SQRT);
  is_spcl[x] = !p; // root is always special.
  int ret = 0;
  for (auto y : g[x]) {
    if (y == p) continue;
    int w = dfs(y, x);
    if (!w) continue;
    is_spcl[x] |= is_sqrt_node || (seen_spcl && is_spcl[w]);
    seen_spcl |= is_spcl[w];
    ret = w;
  }
  if (is_spcl[x]) head[x] = x;
  return (is_spcl[x] || is_sqrt_node) ? x : ret;
}
```

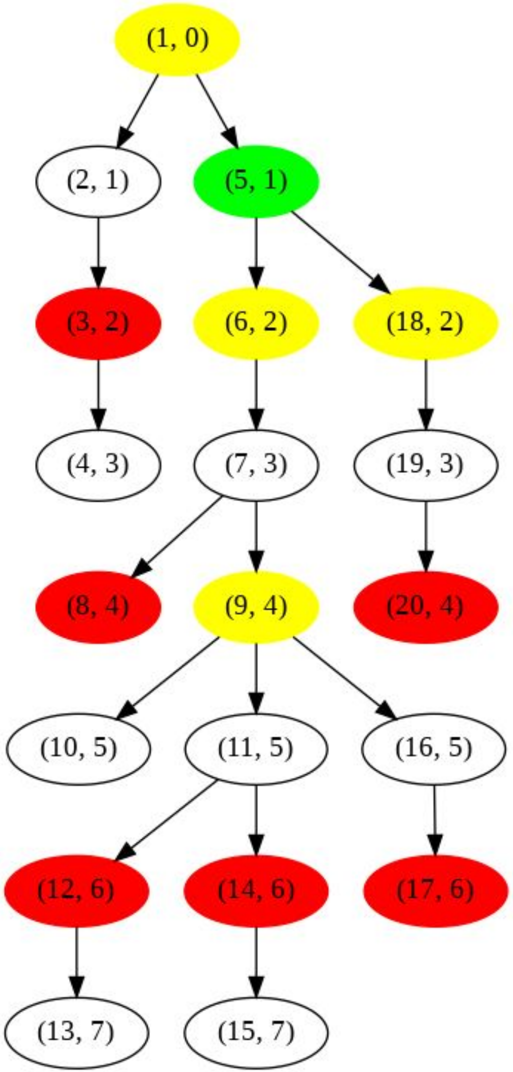


TRS using Supernode Trees

- Given a tree with N elements, support the following operations
  - Query   U V X : Tell the minimum element >= X on the path from U to V
  - Update U V V: Add V to all elements on the path from U to V.
- https://www.codechef.com/problems/TRS

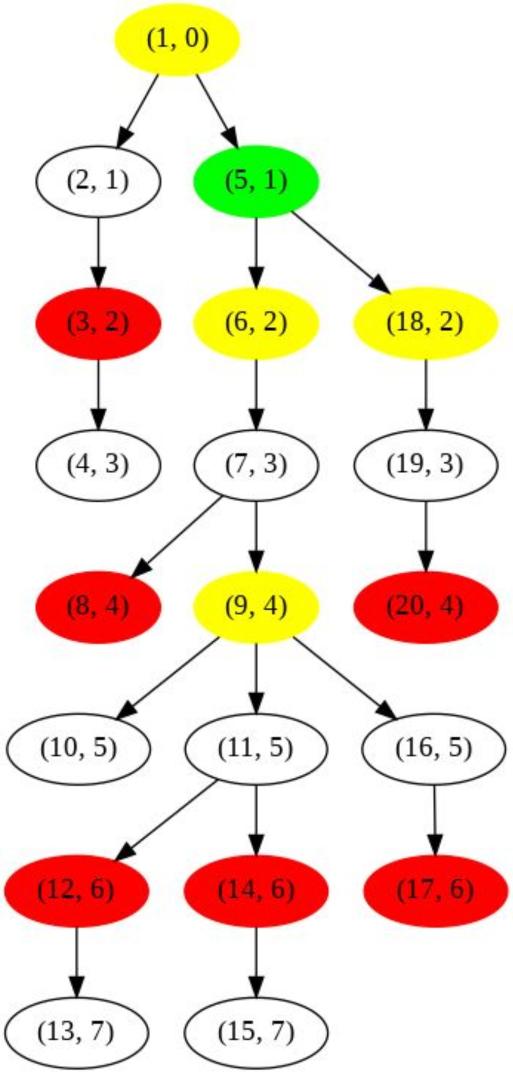
TRS using Supernode Trees

- Given a tree with N elements, support
  - Query   U V X : Tell minimum element >= X on the path from U to V
  - Update U V V: Add V to all elements on the path from U to V.
- Solution:
  - Construct the Supernode Tree of the given tree and maintain a multiset of all compressed edge values at each special node.
  - For update_up(U, P):
    - Brute force from U → sp_U in O(B * logB)
    - Lazy Update sp_U → sp_P in O(N / B)
    - Brute force from sp_P → P in O(B * logB)
  - For query_up(U, P)
    - Brute force from U → sp_U in O(B)
    - Lower Bound Query from sp_U → sp_P in O(N / B * logB)
    - Brute force from sp_P → P in O(B)



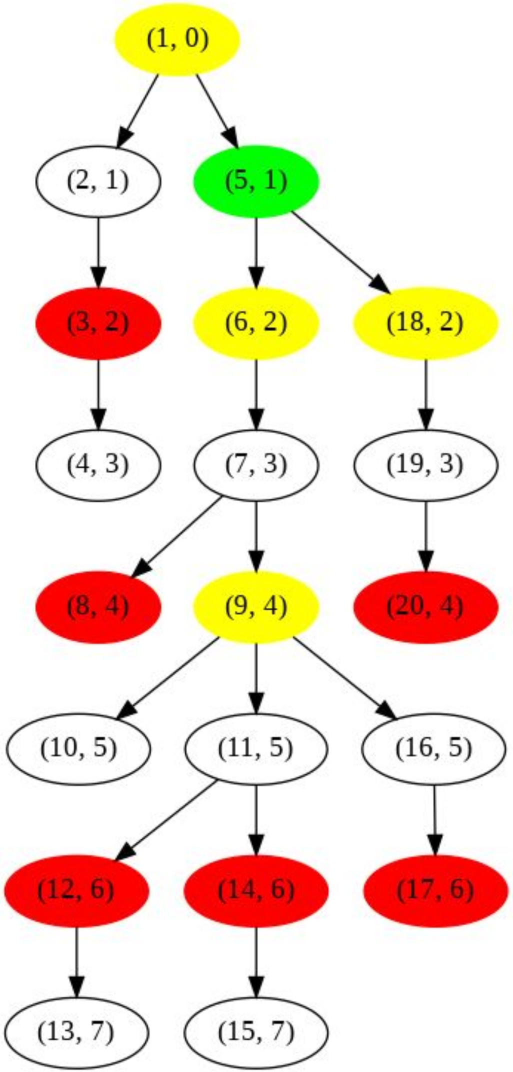
TRS using Supernode Trees

```
void update_up(int x, int p, int64_t add) {
  while (x != p) {
    int b = block[x];
    if (is_spcl[x] && level[head[par[x]]] >= level[p]) {
      block_add[b] += add;
      x = head[par[x]];
    } else {
      if (b) block_vals[b].erase(block_vals[b].find(val[x]));
      val[x] += add;
      if (b) block_vals[b].insert(val[x]);
      x = par[x];
    }
  }
}
```



TRS using Supernode Trees

```
int64_t query_up(int x, int p, int64_t min_val) {
  auto ans = INF;
  while (x != p) {
    int b = block[x];
    if (is_spcl[x] && level[head[par[x]]] >= level[p]) {
      auto it = block_vals[b].lower_bound(min_val - block_add[b]);
      if (it != block_vals[b].end()){
        ans = min(ans, *it + block_add[b]);
      }
      x = head[par[x]];
    } else {
      ans = min_op(ans, val[x] + block_add[b], min_val);
      x = par[x];
    }
  }
  return ans;
}
```



Implementation

- https://github.com/tanujkhattar/cp-teaching/blob/master/CWC/Ep02%20SRD%20on%20Trees/TRS.cpp

Supernode Tree vs HLD ?

- We do a similar thing in HLD, i.e. divide the tree into disjoint chains and process the chains so that we can “jump” over processed chains while traversing paths from x → y.
- However, one key difference here is that the size of each processed chain <= sqrt(N), hence we can store more complex data structures for each chain.
    - Eg: The multiset of all nodes in the chain!
    - This special property gives us the ability to extend Array Square Decomposition Ideas on Path Queries and hence solve some interesting hard problems!
    - Note that size of chains need to be bounded in order to support updates, as we end up iterating on elements partially inside the end chains.

Further Reading and Practice Problems

- https://arxiv.org/ftp/arxiv/papers/1303/1303.5481.pdf
- https://codeforces.com/blog/entry/46843
- https://codeforces.com/blog/entry/68231 , Problem - F editorial
- https://atcoder.jp/contests/abc133/tasks/abc133_f