A Survey on Knee-Oriented Multiobjective Evolutionary Optimization

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***Abstract*—Conventional multiobjective optimization algorithms (MOEAs) with or without preferences are successful in solving multi- and many-objective optimization problems. However, a strong hypothesis underlying their performance is that MOEAs are able to find a representative solution set to cover the entire Pareto-optimal front (PF) and decision makers are able to conveniently and precisely articulate their preference, which is not always easy to fulfill in practice. Accordingly, it is suggested that representative solutions in the naturally interesting regions of the PF rather than the whole PF should be targeted. A large body of research has been proposed to search or identify the knees or knee regions over the past decades. Therefore, this article aims to provide a comprehensive survey of the research on knee-oriented optimization. We start with a discussion of the importance and basic concepts of the knees, followed by a summary of knee-oriented benchmarks and indicators. After that, knee-oriented frameworks and techniques, and real-world applications are presented. Finally, potential challenges are pointed out and a few promising future lines of research are suggested. The survey offers a new perspective to develop MOEAs for solving multi- and many-objective optimization problems.**

***Index Terms*—Knee, multiobjective optimization, preference.**

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I. Introduction

Many real-world optimization problems often involve multiple conflicting objectives, and they are commonly referred to as multiobjective optimization problems (MOPs) [1]. MOPs become increasingly challenging to be optimized by the traditional Pareto-dominance-based optimization methods when the number of objectives is equal to or larger than 4 [2], which are known as many-objective optimization problems (MaOPs) [3]. Without loss of generality, an MOP or MaOP can be formulated as follows:

$$\begin{cases} \text{minimize} & \mathcal{F}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))^T \\ \text{subject to} & \mathbf{x} \in \Omega \end{cases} \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is the decision vector. $\Omega \subseteq \mathbb{R}^n$ is the decision space, and n is the number of decision variables. $\mathcal{F} : \Omega \to \mathbb{Y}$ consists of m objectives. $\mathbb{Y} \subseteq \mathbb{R}^m$ is the objective space. Given two candidate solutions $\mathbf{x_1}, \mathbf{x_2} \in \Omega$, $\mathbf{x_1}$ is said to Pareto dominate $\mathbf{x_2}$ (denoted by $\mathbf{x_1} \prec \mathbf{x_2}$), iif $\forall i \in \{1, \ldots, m\}$, $f_i(\mathbf{x_1}) \leq f_i(\mathbf{x_2})$ and $\exists j \in \{1, \ldots, m\}$, $f_j(\mathbf{x_1}) < f_j(\mathbf{x_2})$. A solution $\mathbf{x} \in \Omega$ is called Pareto optimal if no solution dominates $\mathbf{x}$. All Pareto-optimal solutions form the Pareto-optimal set (PS), i.e., $\text{PS} = \{\mathbf{x} \in \Omega | \nexists \mathbf{y} \in \Omega, \mathbf{y} \prec \mathbf{x}\}$. The mapping of the PS in the objective space is called the Pareto-optimal front (PF), i.e., $\text{PF} = \{\mathcal{F}(\mathbf{x}) | \mathbf{x} \in \text{PS}\}$.

In recent decades, much progress has been made in developing multiobjective evolutionary algorithms (MOEAs) for solving MOPs and MaOPs, which can be generally classified into four categories, i.e., dominance relationship-based methods, decomposition-based methods, indicator-based methods, and secondary criterion-based methods [4]–[6], as shown in Fig. 1.

The Pareto dominance relationship-based methods show competitive performance in solving MOPs [7]–[9], while many modified dominance relationships are developed to improve the search efficiency so as to solve MaOPs [10]–[12]. However, most modifications of dominance relationships need to tune parameters, which may be arduous to adapt them to different shapes of PFs [13]. The decomposition-based methods commonly use predefined weights or reference vectors [14], [15] to decompose an MOP or MaOP into a set of single- or multi-objective problems [16]–[18] and then optimize them simultaneously. Similarly, other reference information, such as reference points [19] or reference vectors [20], is commonly used to guide the search toward the corresponding subregions of the PF [21], [22]. Interested readers are referred to the comprehensive reviews [13], [15], [23].

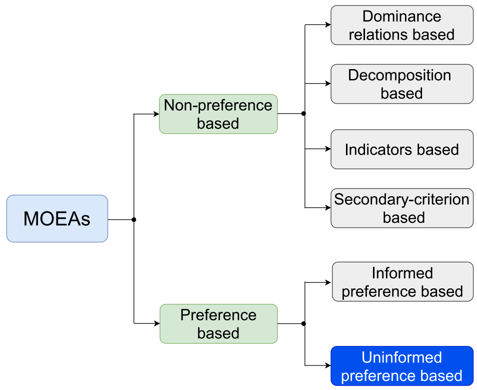

Fig. 1. Taxonomy of the classification of MOEAs.

The decomposition-based methods may improve the efficiency in solving the MaOPs but usually require fine-tuning of the weight vectors or references in the optimization. Performance indicators-based MOEAs [24] are commonly used in many-objective optimization [25]–[27] to tackle the "curse of dimensionality" [28]. They show competitive performance in getting nondominated solutions in terms of the performance indicators [29]–[31]. But their computational costs are commonly high because it is computationally expensive to compute the indicator values of the whole population at each generation during the optimization [24]. To address the weakness of the dominance-based methods in solving MaOPs, secondary criterion-based methods are developed by considering the criterion of convergence performance [32], [33], diversity performance [34], [35], or both of them [36]–[38] to distinguish solutions in high-dimensional objective spaces. However, additional criteria used in the optimization may decrease the efficiency of the optimization.

Usually, the nondominated solutions obtained by an MOEA will be selected by the decision makers (DMs) according to her/his *a priori* knowledge of the optimization problem. In real-world applications, DMs are commonly interested in only specific regions of the PF, i.e., regions of interest (ROIs) [39], rather than the entire PF [40]–[42]. Accordingly, preference-driven MOEAs are popular and they aim to provide DMs with a small set of representative solutions in the ROIs, assuming that the DMs have *informed preferences*, i.e., the DMs have sufficient *a priori* knowledge to specify clear preferences. These methods can be classified into *a priori* [42], *interactive* [43], and *a posteriori* approaches [40], respectively, when the DM gives the preference before, during, or after the optimization. In preference-driven MOEAs, different DMs may have different ways of articulating preferences, such as weight vectors [44], [45], reference points [46]–[48], reference vectors [49]–[51], preference relations [52]–[54], fuzzy preferences [55], utility functions [56]–[58], preference ranking [59]–[61], goal attainment [62], and other preferences such as extreme points or nadir point [63], [64]. Interested readers are referred to the corresponding surveys [40]–[42]. Despite significant progress made in developing MOEAs with informed preferences [40]–[42], the DMs may not have enough *a priori* knowledge to articulate preferences [14] due to the complexity of the optimization problems, torturous or even intractable interactive process, as well as resource-intensive and time-consuming computation to get a large number of representative solutions for posteriori approaches.

To address the above issues, knee solutions as *uninformed preference* or naturally interesting solutions of the PF [65] have received increased research attention. Knee points are regions of the PF where the solutions "bulge out the most" and one needs to sacrifice a large degradation in at least one objective to achieve a small gain on other objectives [66], as shown in Fig. 3, where $\mathcal{F}(K)$ is the knee point of the PF. Besides, the knees usually achieve the highest performance, such as the utility performance [67]–[69] and hypervolume (HV) [70] in comparison with their adjacent solutions [66]. In [71], the relationship between knee solutions and preferred solutions is discussed. According to the above discussion, knees are solutions of special interest on the PF when no specific user preference is given [68]. Note that the true PF is challenging to get, especially in solving real-world problems. Therefore, the investigation of the knees is based on the approximated PF in most scenarios when the true PF is unknown. There is also research work in which knees in a population during the optimization are studied. For example, knee candidates in a population are selected with a higher priority than nonknee solutions in environmental selection [70] since the knee solutions contribute to a larger HV [26] than the nonknee solutions in many-objective optimization [70] and dynamic multiobjective optimization [72]–[75].

Since the "knee" was first proposed by Das [66], increasing attention has been paid to the knee-oriented optimization. In Fig. 2, we can see the timeline of the development of the knee-related optimization and applications. From 1999 to 2009, many knee-oriented methods had been proposed to solve MOPs. For example, Branke *et al.* [67] defined the knees by using an expected marginal utility (EMU). In [67], some angle-based methods, such as bend angle and reflex angle, were proposed to identify the knees on the PF. Rachmawati and Srinivasan [76] described knees by using the tradeoff information between the objectives. After 2009, more and more research focuses on finding the knees of MaOPs, such as the dominance-based method [77] and enhanced utility-based method [78]. From 2004 to 2020, knee-oriented benchmark problems and performance indicators were proposed to evaluate the knee-oriented MOEAs. For example, DO-DK and DEB-DK were two sets of test problems proposed in [67], and the PMOP test suite as well as knee-oriented performance indicators were designed by Yu *et al.* [79]. Interestingly, the visualization of knee regions tends to be prevalent after 2015. For example, Tušar and Filipič [80] applied prosections to visualize knee regions in low-dimensional objective space; and He *et al.* [81] visualized knees in high-dimensional objective spaces by using the proposed achievement scalarizing function-based domination range (ASF-DR). Due to their attractive features, knee points are widely introduced in many applications. As shown in Fig. 2, the number of knee-related applications has increased rapidly since 1999. Knee solutions are successfully exploited in enhancing the performance of MOEAs in many-objective

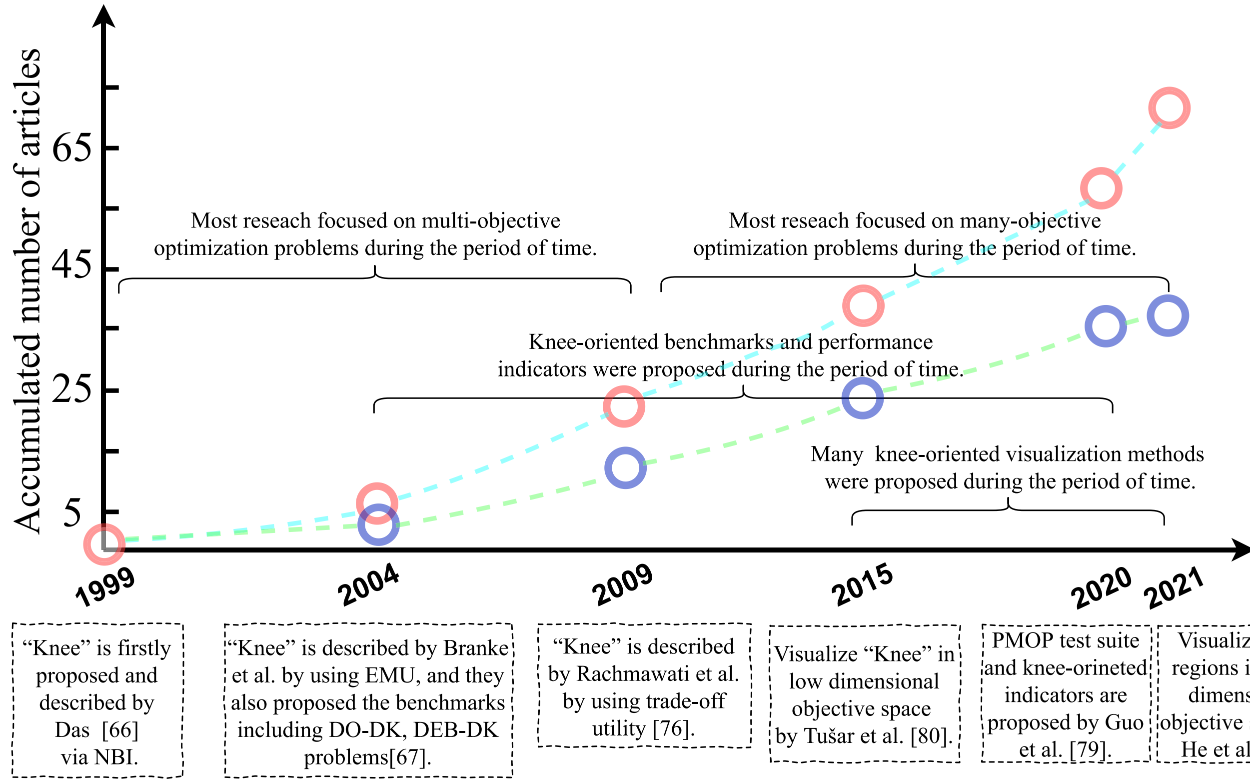

Fig. 2. Timeline of knee-related studies since 1999, where NBI is referred to normal boundary intersection [89]; DO-DK [67], DEB-DK [67], and PMOP test suites [79] are different kinds of knee-oriented benchmarks. ASF-DR is the achievement scalarizing function-based domination range [81]. The red and blue circles represent the accumulated number of knee-oriented articles and applications published since 1999 to the corresponding year, respectively.

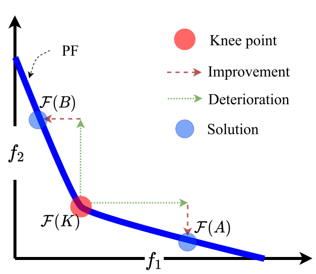

Fig. 3. Illustration example of a knee $\mathcal{F}(K)$ in the convex region of a PF.

optimization [70], [82] and dynamic optimization [72], [73], and applied to many real-world optimization problems, such as self-adaptive software [83], sparse reconstruction [84], driving strategy for electric vehicles [85], compressing deep neural networks [86], subspace learning [87], community detection [88], and hybrid electric vehicle management controller optimization [77].

In the preference-driven optimization, the preference-driven MOEAs are fully reviewed in [40]–[42] when the DM is able to specify informed preferences. Yu *et al.* [14] discussed the similarities and differences between reference-based and preference-based MOEAs. Li *et al.* [90] reviewed preference-driven MOEAs and gave a full investigation of the question "Does preference always help?", which demonstrates that the preference does not always lead to the ROIs if the given preference information is not well applied into the optimization, or invalid preference information. Deb and Gupta [65], [71] reviewed knee-oriented methodologies in solving bicriteria optimization problems and argued that there is a link between the knee solutions of bicriteria problems and the preferred methodologies used in problem-solving tasks. Li *et al.* [91] provided a short study on knee-related research from knee-oriented techniques to benchmarks. However, none of them fully reviewed the knee-related optimization, such as the consistency of the definitions of knees, knee-related methods, and applications. In contrast to the above existing surveys, this article comprehensively investigates the knee-related optimization from the following aspects.

1) The importance of knees.
2) The definitions of knees and their consistency in knee-oriented identification.
3) The knee-oriented benchmark problems and indicators.
4) The knee-oriented frameworks and techniques.
5) The knee-related real-world applications.
6) The potential lines of knee-oriented research.

The remainder of this article is organized as follows. Section II discusses the importance of the knee or advantages of knee-oriented MOEAs in comparison with the traditional MOEAs. Various definitions of the knee of the PF are given in Section III. The knee-oriented benchmarks and corresponding performance indicators are introduced in Section IV. Section V presents the basic frameworks of the knee-oriented MOEAs and typical knee-oriented techniques. The knee-related applications are summarized in Section VI. Promising lines of future research are suggested in Section VII. Finally, Section VIII concludes this article.

II. Importance of Knees

All the following discussions follow the taxonomy of MOEAs in Fig. 1. In this section, the limitations of the non-preference and informed preference-driven MOEAs are first introduced. Then, the unique features of knees and the benefits of knee-oriented MOEAs are discussed. Accordingly, the importance of knees comes from three aspects.

1) The nonpreference-based MOEAs to some extent fail to follow the hypothesis that the obtained solution set is able to represent the entire PF, i.e., the obtained solution set strikes a good balance between the convergence and diversity performances. The convergence indicates the distance from the solutions in the obtained solution set to the PF, while the diversity indicates whether the solution set can evenly cover the entire PF. However, the hypothesis may fail to hold due to the following reasons. First, it is unlikely to use a limited number of solutions (typically this is the population size) to represent the PF [92], since both the Pareto set and Pareto front of a continuous MaOP are piecewise continuous $(m-1)$-dimensional manifolds [93], according to the Karush–Kuhn–Tucker condition [94], where m is the number of objectives. In other words, using the same population size in solving both MOPs and MaOPs is hardly practical [14]. Second, it is usually computationally expensive to solve real-world applications [95], [96], making it hard to achieve a set of well-converged solutions toward the whole PF. Third, due to the complex manifold of the PF, different settings of the reference points on the PF may introduce strong bias on the evaluation, which is demonstrated in [97].
2) In informed preference-driven multiobjective optimization, an underlying assumption is that the DM is able to precisely elucidate his or her preferences. However, it does not hold true in many situations [90]. Foremost, few available observation data and high complexity of the MOPs (or MaOPs) make it hardly likely for the DMs to provide global preference information [43]. Second, articulation of the preferences is laborious, and sometimes intractable [14]. Besides, DM is not fully rational due to mental or physical factors [98]. As a result, they may provide different or even inconsistent preferences. Hence, reducing the DM's cognitive bias is another challenging issue. Additionally, selecting preferred solutions among a representative solution set in an *a posteriori* process becomes increasingly difficult when the number of objectives increases, because it is resource intensive and time consuming to obtain a representative solution set that covers the whole PF of an MaOP. Besides, preferences do not always lead to the ROIs, limited by the proper utilization of the preferences [14] and valid preferences [90]. Last but not least, how to fairly evaluate the performance of preference-driven MOEAs is an open issue, when there are different articulations of preference.
3) The unique features of the knees are particularly attractive when DMs cannot easily give informed preferences [79]. Above all, the knee solutions, which are regarded as uninformed preferences, do not require DMs to laboriously express their preferences [14]. Recall that knee solutions have significant advantages in some objectives but small degradation in other objectives [66]. An example is given in Fig. 3, where the knee candidate $\mathcal{F}(K)$ has a great decrease in f_2 but a slight increase in f_1 in comparison with the objectives of solution $\mathcal{F}(B)$. Similarly, it shows similar results when solution K is compared with solution $\mathcal{F}(A)$. In other words, the knees outperform their neighbors when the objectives are equally important. Besides, the knees usually achieve the highest performance, such as the utility performance [67]–[69] and HV [70]. Aiming to find the knee solutions enables the search algorithm to concentrate on a subset of solutions rather than covering the whole PF and makes it easier for the DM to select the preferred solutions [77]. As shown in Fig. 4, although the solutions in Fig. 4(a) seem to have better diversity, the solutions in Fig. 4(b) are better in terms of the degree of closeness to the knee regions.

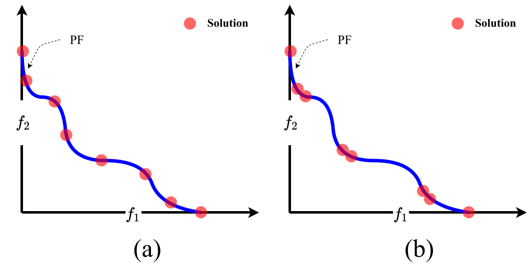

Fig. 4. Example for illustrating the aim of the knee-oriented search algorithms. Traditional MOEAs aim to find a solution set to represent the PF as shown in (a), while knee-oriented MOEAs focus only on the knee regions, as shown in (b).

From the above discussions, we can conclude that the knee points are good representative solutions on the PF and knee-oriented MOEAs are essential when DMs cannot clearly specify informed preferences.

III. Definitions of Knees

Knee points are commonly classified into three categories, i.e., knees in convex regions, knees in concave regions, and edge knees [79], which are hereafter called convex knees, concave knees, and edge knees, respectively. We first introduce the definitions for these terminologies, and then we examine whether the knee solutions identified based on different definitions are consistent with each other. Finally, the pros and cons of the definitions are summarized.

A. Definitions

Note that knee point identification is normally conducted in a normalized coordinate system using the transformation $\bar{f}_i = [(f_i - \min f_i)/(\max f_i - \min f_i)]$, where $\max f_i$ and $\min f_i$ are the maximum and minimum values of a Pareto set on the ith objective. The objective space normalization aims to help

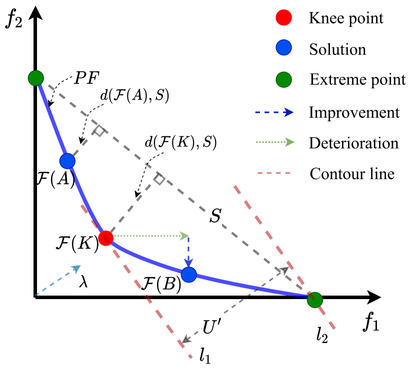

Fig. 5. Illustration of different definitions of the convex knee, where $d(\cdot)$ is to evaluate the distance from solutions to the CHIM (i.e., S) in Definition 1; $U^{'}$ is to evaluate the utility of a solution given the weight λ in Definition 2; and "Improvement" and "Deterioration" are used to calculate the tradeoff utility in Definition 3.

knee-oriented MOEAs obtain diverse solutions and reduce the negative effect of dominance resistant solutions (DRSs) [99].

Definition 1 [Convex Knee Based on Convex Hull of Individual Minima (CHIM)]: In [66], a convex knee solution of a PF is defined as the solution having the maximum distance from the PF to the CHIM which is constructed by the extreme points

$$\mathbf{x_k} = \arg\max_{\mathbf{x}} d(\mathcal{F}(\mathbf{x}), S) \tag{2}$$

where $d(\cdot)$ is the distance from solution $\mathbf{x}$ in the objective space to the CHIM denoted by S. The hyperplane S is defined as $f_1 + \cdots + f_m = 1$. In Fig. 5, $d(\mathcal{F}(K), S)$ and $d(\mathcal{F}(A), S)$ describe the distances from the solutions [$\mathcal{F}(K)$ and $\mathcal{F}(A)$] to the hyperplane S. The knee candidate $\mathcal{F}(K)$ has the largest distance to the CHIM.

Definition 2 (Convex Knee Based on EMU): In [68], a convex knee point is characterized as the solution with the highest EMU [67] ($\min U^{'}$), i.e.,

$$U^{'}\left(\mathbf{x}_i, \lambda^{'}\right) = \begin{cases} \min_{j\neq i}\left(U\left(\mathbf{x}_j, \lambda^{'}\right) - U\left(\mathbf{x}_i, \lambda^{'}\right)\right) & i = \arg\min U\left(\mathbf{x}_j, \lambda^{'}\right) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $U(\mathbf{x}, \lambda) = \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x})$, $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\sum_{i=1}^{m} \lambda_i = 1$. In Fig. 5, U' is the EMU of solution $\mathcal{F}(K)$ given the weight λ.

In the definition, a number of weight vectors are needed to calculate the EMU of solutions. Note that the number of reference vectors is critical in the identification of knees. Fewer knees will be found if a smaller number of reference vectors is given. On the contrary, more solutions around the knees will be regarded as the knee candidates if a larger number of reference vectors is utilized. Interested readers are referred to Section I of the supplementary material.

Definition 3 (Convex Knee Based on Tradeoff Utility): In [76], a knee in the convex region is characterized by maximizing the tradeoff utility (μ)

$$\begin{aligned} p = & \arg\max_{p_i}(\mu(p_i, P)) \\ \text{s.t.} \quad & \mu(p_i, P) = \min_{i\neq j, p_i \nprec p_j, p_j \nprec p_i} \tau\left(p_i, p_j\right) \\ & \tau\left(p_i, p_j\right) = \frac{\sum_{1<\iota<m} \max\left[0, f_\iota\left(p_j\right) - f_\iota\left(p_i\right)\right]}{\sum_{1<\iota<m} \max\left[0, f_\iota\left(p_i\right) - f_\iota\left(p_j\right)\right]} \end{aligned} \tag{4}$$

where p_i and p_j are two solutions from nondominated solution set P, which is an approximation of PF, and m is the number of objectives. In $\tau(p_i, p_j)$, the numerator and denominator are the aggregation of the deterioration and improvement in the exchange of the objectives of solutions p_i and p_j. In Fig. 5, $\tau(\mathcal{F}(K), \mathcal{F}(B))$ indicates deterioration/improvement.

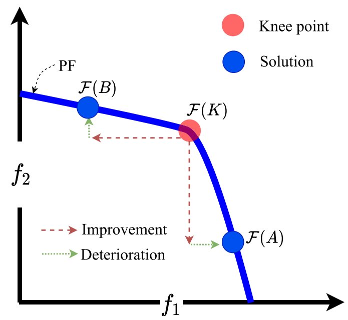

Fig. 6. Illustration of a concave knee of the PF, where $\mathcal{F}(K)$ is the concave knee. The improvement and deterioration, respectively, show the gain and loss when the concave knee moves to its adjacent nondominated solutions ($\mathcal{F}(A)$ and $\mathcal{F}(B)$) on the PF.

Definition 4 (Concave Knee): In [79], a knee in the concave region is characterized by minimizing the tradeoff utility (μ)

$$\begin{aligned} p = & \arg\min_{p_i}(\mu(p_i, P)) \\ \text{s.t.} \quad & \mu(p_i, P) = \max_{i\neq j, p_i \nprec p_j, p_j \nprec p_i} \tau\left(p_i, p_j\right) \\ & \tau\left(p_i, p_j\right) = \frac{\sum_{1<\iota<m} \max\left[0, f_\iota(p_j) - f_\iota(p_i)\right]}{\sum_{1<\iota<m} \max\left[0, f_\iota(p_i) - f_\iota(p_j)\right]} \end{aligned} \tag{5}$$

where m is the number of objectives, and P is an approximation of PF. In Fig. 6, a concave knee $\mathcal{F}(K)$ is shown.

Definition 5 (Edge Knee): The definition of edge knees was suggested by Deb and Gupta [65], which was intended to distinguish the edge knees from other extreme points. The edge knees are those extreme points for which a unit gain in one of the objectives requires at least an amount of γ sacrifice in the other objectives.

In Fig. 7, two examples are presented to show the edge knees, where the extreme points $\mathcal{F}(K)$ are the edge knees.

In summary, an example is given in Fig. 8 to show the different knee solutions that are identified according to different knee definitions. All definitions have found the same knee in region B, while the extreme points are only identified as the knee candidates by Definitions 2 and 3. However, the knee candidates obtained by three definitions are different in the knee regions (A and C). Interestingly, Definition 3 regards solutions a and b as the knee candidates because they also have higher tradeoff utility values than their adjacent solutions. More investigations are given in the supplementary material.

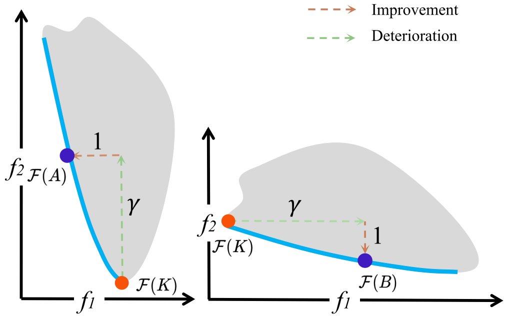

Fig. 7. Two examples of edge knees, where $\mathcal{F}(K)$ as the edge knee needs an amount of γ sacrifice on one objective to get a unit of gain in another objective.

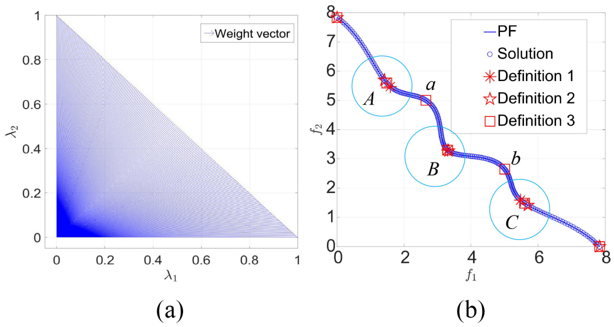

Fig. 8. Example to illustrate the different knee solutions resulting from different knee definitions, where (a) and (b) present 201 weight vectors used in Definition 2, and results obtained by different definitions on the DEB2DK problem [67], respectively.

B. Influence of the Definitions on Knee-Oriented MOEAs

In this section, we take the definitions of the convex knees as an example to investigate whether the knee candidates identified by different definitions are consistent with each other. DEB2DK [67], CKP [100], and PMOP9 problems [79] are selected as the test instances. In the experiment, DEB2DK and CKP problems are set to have three convex knee regions. Especially, the convex regions of DEB2DK are below the hyperplane constructed by the extreme points of the PF, while that of CKP above the hyperplane. PMOP9 problems are MaOPs with only one convex knee region. All experimental results are plotted in Figs. S1–S3 and discussed in Section I of the supplementary material. A summary of the experiments is presented as follows.

1) Investigation on the Identifications on DEB2DK With Respect to Different Distributions of Weight Vectors: According to the experiments on DEB2DK in Fig. S1 in the supplementary material, we can find that the knee candidates found by three definitions among the given representative solutions of the PFs are not always consistent. Definition 2 relies on the weight vectors and the number of knee candidates increases when the number of weight vectors increases. Definitions 1 and 3 are able to find knee candidates in the knee regions but the candidates are not all the same in the knee regions. Third, Definitions 2 and 3 find the same candidates in the convex knee regions and the same extreme points, when the number of weight vectors is large.

2) Investigation on the Identifications on CKP With Respect to Different Distributions of Weight Vectors: According to the experimental results on the CKP problem in Fig. S1 in the supplementary material, we find that Definition 2 fails to find the knee solutions in the convex regions of CKP with respect to different distributions of weight vectors, since the convex regions are above the hyperplane constructed by the extreme points. The reason is that the utility of the solutions in the knee regions is zero and therefore these solutions will not be selected. In contrast, Definitions 1 and 3 can find knee candidates, although Definition 1 only gets candidates in two out of three convex regions. Interestingly, Definition 3 is able to find candidates in all convex regions, however, it also gets a solution that is not in the convex region.

3) Investigation on the Identifications on DEB2DK and CKP With Respect to Different Distributions of Representative Solutions on the PFs: In this experiment, a large number of weight vectors and representative solutions of the PFs of DEB2DK and CKP problems are used in the investigation, and the visualization results are presented in Fig. S2 in the supplementary material. The results indicate that all definitions are able to find knee candidates in the knee regions of DEB2DK. Especially, Definitions 2 and 3 get the same knee candidates, while two knee candidates obtained by Definition 1 are different from those by Definitions 2 and 3. Definition 3 acquires two solutions that are not in the convex regions. Only Definitions 1 and 3 find the knee candidates in the convex knee regions of CKP and they are not completely the same. Interestingly, Definition 3 finds a candidate far away from the convex knee. Definition 2 fails to find knee candidates in the knee regions but only the extreme points.

4) Investigation on the Bias on the Knee Candidates of the Definitions: In this investigation, the PMOP9 problems with three and eight objectives are adopted and the results are shown in Fig. S3 in the supplementary material. The results show that all definitions have found knee candidates in the convex knee regions. Definition 1 favors the solutions close to the global knee regions having the largest distance value to the hyperplane. Definition 2 favors the solutions in the knee region and boundary regions. Definition 3 has a strong bias on the regions where the knee regions are deep [79].

From the above investigation, a conclusion can be made that the definitions are not always consistent with each other in the identification of knee candidates.

C. Summary

According to the results presented in Section I of the supplementary material, the pros and cons of the three knee definitions are summarized as follows.

1) *Definition 1:* The advantages are as follows.
   a) The definition is insensitive to the shapes of the PF [66], as it only relies on the distance to the hyperplane or CHIM constructed by the extreme points.
   b) The definition is irrespective of the weight vectors.
   c) The curve of the distance value contains the information of the shape of the PF [77], which

TABLE I
SUMMARY OF THE PROS AND CONS OF THE KNEE DEFINITIONS

Definition	Pros	Cons
Definition 1 [66]	• insensitive to the shapes of the PF. • irrespective of the weight vectors. • can observe the curvature of the PF. • can handle both MOPs and MaOPs • consistent with HV indicator.	• heavily rely on the distributions of the representative solutions of the PF. • inconsistent with the expected marginal utility. • heavily rely on the accuracy of the extreme points. • limited to the knee regions located in the "center" of the PF.
Definition 2 [68]	• irrespective of the extreme points. • easily find the best candidate. • can find the extreme points.	• difficult to handle MOPs with knee regions below the hyperplanes. • cannot reflect the curvature of the PF. • heavily dependent on the distribution of the weight vectors. • the knee candidates not always have the largest HV contribution in the knee regions.
Definition 3 [76]	• irrespective of the extreme points. • irrespective of the weight vectors. • can find the extreme points.	• favor exceptional solutions in non-convex regions. • the knee candidates not always have the largest HV contribution in the knee regions.

can be used for estimating the curvature of the PF.

d) The definition is applicable to MOPs with an arbitrary number of objectives [66].

e) In [70], the knees obtained by [66] are demonstrated to have a larger HV contribution [26] than their neighbors.

The disadvantages of this definition are as follows.

a) For the shallow knee region, the identification heavily relies on the distribution of the representative solutions. For example, the definition cannot find all knee solutions in the convex regions of CKP problems.

b) The obtained knee candidates may not have the largest utility value (the EMU or tradeoff utility) in their neighborhoods.

c) Apparently, the definition heavily relies on the extreme points used to construct the hyperplane.

d) Although Definition 1 is able to handle MOPs and MaOPs, the research findings in [101] indicate that the definition is limited to the knees located near to the "center" of the PF. For this reason, the study [101] slightly modified the definition by omitting the non-negativity condition of the CHIM and extended the Pareto explorer [102] to detect the knee candidates.

2) *Definition 2:* The advantages are summarized as follows.

a) The definition is irrespective of the extreme points.

b) It can easily acquire the best candidate in each knee region by recursively using the EMU measure [68], once the number of weight vectors is sufficient.

c) It can be adopted to find the extreme points that have higher EMU in comparison with their neighboring solutions.

The disadvantages are as follows.

a) The definition inherits the weakness of the weighted sum approach [1], i.e., it cannot handle MOPs whose knee regions are above the hyperplane, since the definition is based on the sum of the weighted objectives [16].

b) The utility of the nonknee solutions is assigned to 0. Consequently, it is difficult to differentiate them and the curve of the utility cannot reflect the curvature of the PF.

c) The number of knee candidates is critically dependent on the distribution of the weight vectors.

d) The definition is not always consistent with Definition 1, which means that the obtained knee candidate may not have the largest HV contribution in their neighborhoods.

3) *Definition 3:* The advantages mainly include the following.

a) The definition is irrespective of weight vectors and extreme points.

b) It describes the curvature of the PF and is able to locate the knees when the representative solutions in the knee regions are well distributed.

c) It can be adopted to find the extreme points which have a higher tradeoff utility than their neighbors.

The disadvantages are as follows.

a) The definition may result in solutions in nonconvex regions where the curvature of the solution changes.

b) Similar to Definition 2, the definition is not always consistent with Definition 1. Thus, the obtained knee candidates may not have the largest HV contribution in their neighborhoods.

In summary, these three definitions are not always consistent with each other in identifying knee candidates, and they have their unique advantages and disadvantages, as summarized in Table I. If the DM does not aim to get knee candidates with the highest EMU, Definition 1 is advisable when the extreme points are accurate. If the highest EMU is essential, then Definition 2 can be adopted when there is an adequate number of weight vectors. If the extreme points and weight vectors are unavailable, then Definition 3 is a good option. Especially, DM may use multiple methods to identify the most preferred solutions of interest (SOIs) [68] in the knee regions.

IV. KNEE-ORIENTED BENCHMARK PROBLEMS AND PERFORMANCE INDICATORS

In this section, we introduce the benchmarks problems dedicated to the design of knee regions on the PF for benchmarking the performance of knee-oriented MOEAs and corresponding performance indicators.

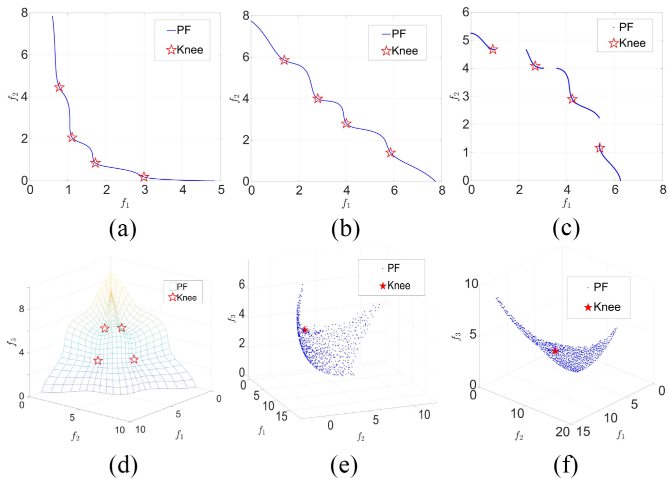

Fig. 9. PFs of DO2DK, DEB2DK, CKP, and DEB3DK have four knee regions, while the PFs of PEs have one knee region. (a) DO2DK. (b) DEB2DK. (c) CKP. (d) DEB3DK. (e) PE1. (f) PE2.

A. Benchmark Problems

Several benchmarks are tailored for knee-oriented MOEAs, including DO-DK [67], DEB-DK [67], [103], CKP [100], PEs[1] [101], and PMOP[2] [79] problems. DO-DK, DEB-DK, CKP, and PEs problems are mainly designed for multiobjective optimization, while PMOPs are for many-objective optimization.

1) Multiobjective Test Problems: The construction of the DO-DK, DEB-DK, and CKP problems is as follows:

$$\begin{cases} \text{minimize } f_i(\mathbf{x}) = g(\mathbf{x}_{II})r(\mathbf{x}_I)h_i(\mathbf{x}_I) \quad \forall i = 1:m \\ \text{where } \ \mathbf{x} = (\mathbf{x}_I, \mathbf{x}_{II}) \in \Omega \\ \mathbf{x}_I = (x_1, \ldots, x_{m-1}), \mathbf{x}_{II} = (x_{m-1}, \ldots, x_n) \end{cases} \tag{6}$$

where $g(\cdot)$, $r(\cdot)$, and $h(\cdot)$ are the landscape function, basic knee function, and shape function, respectively. The landscape function controls the difficulty level for MOEAs to converge to the PF. The basic knee function creates various knee regions on PF. The shape function determines the basic shape of the PF.

Fig. 9 presents the PFs of DO2DK, DEB2DK, CKP, and DEB3DK with four knee regions. The scales of the objectives of DO2DK and CKP are different. Especially, the PFs of DEB-DK are symmetrical, while those of the other two problems are asymmetrical. The knee regions of DO2DK and DEB-DK are below the hyperplane constructed by the extreme points, but those of CKP are above the hyperplane. Besides, the PF of CKP is disconnected. In summary, different scales of objectives and asymmetrical PF may create difficulties in the search of knee regions. The locations of the knee regions and discontinuity of the PF may increase different difficulty levels for the convergence of an MOEA toward knee regions.

PEs [101] are proposed to investigate the performance of CHIM-based methods, e.g., in [66]. The definitions of PE1 and PE2 are presented in (7) and (8), respectively

$$\begin{cases} \text{minimize } f_j(\mathbf{x}) = \sum_{i=1}^{n}(x_i - \alpha_i^j)^2 \quad \forall j = 1, 2, 3 \\ \text{where } x \in \Omega, \alpha^1 = (1, 1, 1)^T \\ \alpha^2 = (-1, -1, -1)^T, \alpha^3 = (1, -1, 1)^T \end{cases} \tag{7}$$

$$\begin{cases} \text{minimize } \widetilde{f}_i(\mathbf{x}) = f_i(x) + \frac{|f_1(x)+f_2(x)-12|}{2\sqrt{6}}||x + e_2|| \\ \text{minimize } \widetilde{f}_3(\mathbf{x}) = f_3(x) \\ \text{where } \quad \forall i = 1, 2, x \in \Omega, e_2 = (0, 1, 0)^T. \end{cases} \tag{8}$$

Note that f_i, $i = 1, 2, 3$, in (8) are the same as (7). The PFs and corresponding knee points of PEs are shown in Fig. 9. From plots (e) and (f), we can observe that the PFs of them are irregular [13] and the knee points are not in the right center of the PFs. Such features of PEs bring great difficulties to the CHIM-based methods [66] in the knee identification, since the method favors the knees near the center of the PF.

2) Many-Objective Test Problems: The PMOP test suite [79] extends the knee-oriented MOPs to MaOPs with more features of the knee regions, and problems become more challenging.

The mathematical construction of the PMOP test suite is as follows:

$$\begin{cases} \text{minimize } f_{i=1:m}(\mathbf{x}) = (1 + g(\ell(\mathbf{x}))) \cdot \eta(\mathbf{k}(\mathbf{x}_I)) \cdot h(\mathbf{x}_I) \\ \text{where } \mathbf{x} = (\mathbf{x}_I, \mathbf{x}_{II}) \in \Omega \\ \mathbf{x}_I = (x_1, \ldots, x_{m-1}), \mathbf{x}_{II} = (x_{m-1}, \ldots, x_n). \end{cases} \tag{9}$$

Differently, $\ell(\cdot)$ is the linkage function that can shift the location of the optima. $\eta(\mathbf{k}(\cdot))$ controls the knee regions on the PF, where $\eta(\cdot)$ and $\mathbf{k}(\cdot)$ are the stretching function and knee function, respectively. $\mathbf{k}(X_I) = \prod_{i=1}^{m-1} k_j(x_i)/(m-1)$, where k_j is the *basic* knee function and $j = \{1, \ldots, 6\}$. Six basic knee functions embedded into (9) create diverse features of knee regions and loads of difficulties for the knee-oriented search. Table II summarizes the features and roles of each function of (9). Specifically, $g(\cdot)$ is mainly designed for different difficulty levels for converging to the PF. $g(\cdot)$ considers not only the separability and multimodality of the landscape function but also diverse convergence rates to the PF when the objective functions have different landscape functions. $\ell(\cdot)$, $\eta(\cdot)$, $\mathbf{k}(\cdot)$, and $\mathbf{h}(\cdot)$ are for different types of difficulties to the identification of knee regions or knee points. $\ell(\cdot)$ builds linear or nonlinear dependencies between the decision variables

TABLE II
DIFFERENT ROLE FUNCTIONS TO TEST DIFFERENT ABILITIES OF KNEE-ORIENTED MOEAS

Functions	Features	Aims
$g(\cdot)$	g_1: None-separable, Uni-modal, g_2: Separable, Unimodal, g_3: Separable, Multi-modal, g_4: Separable, Multi-modal, g_5: None-separable, Multi-modal, g_6: Separable, Multi-modal, g_7: None-separable, Multi-modal, g_8: Separable, Multi-modal,	Optimization
$\ell(\cdot)$	ℓ_1: Linear shift, ℓ_2: Nonlinear shift,	Identification
$\eta(\cdot)$	Scalability of the PF	
$\mathbf{k}(\cdot)$	Symmetry and asymmetry, degeneration, Scalability of the number of knees, bias, differentiability, depth/location of the knee regions	
$\mathbf{h}(\cdot)$	h_1: Linear basic function and non-uniformity, h_2: Concave basic function and non-uniformity, h_3: Convex basic function and non-uniformity.	

[1]PEs are the short name of the predefined knee-oriented test problems proposed in [101] while the authors did not give names to them.
[2]https://github.com/GYResearch/PMOPs-Benchmark

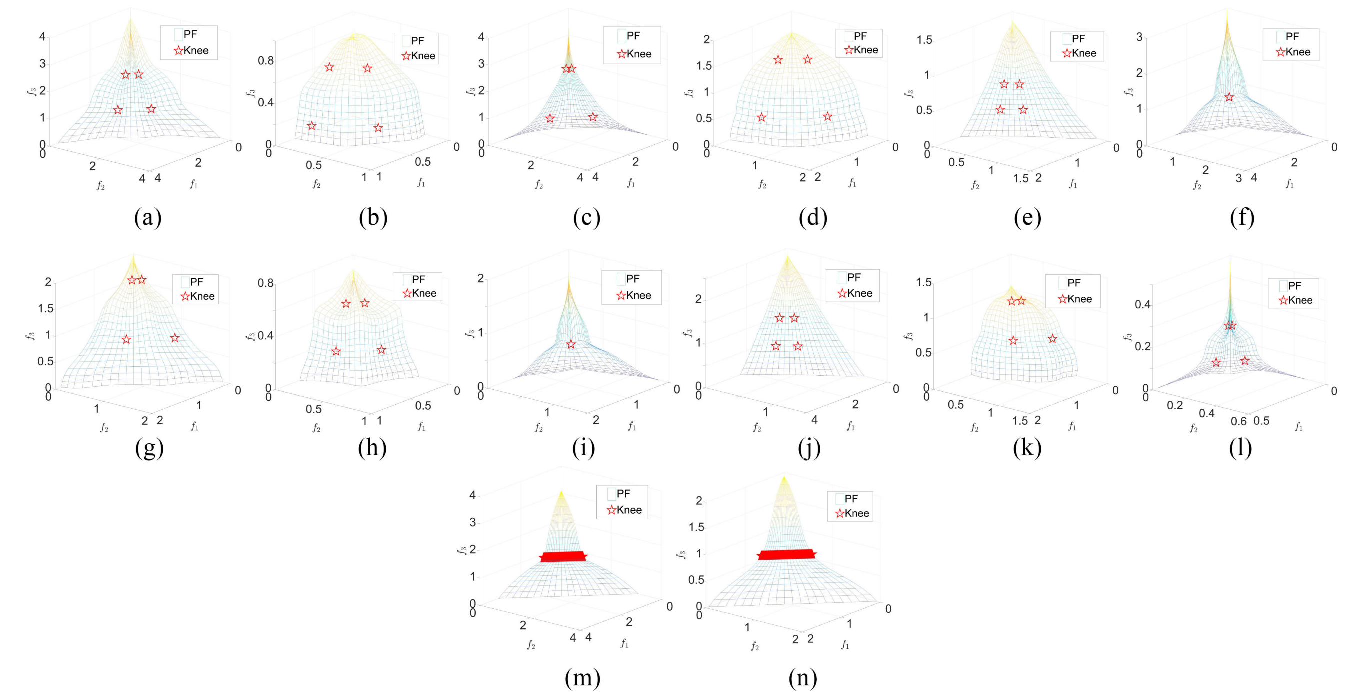

Fig. 10. PFs and the corresponding knees of PMOP test problems. (a) PMOP1. (b) PMOP2. (c) PMOP3. (d) PMOP4. (e) PMOP5. (f) PMOP6. (g) PMOP7. (h) PMOP8. (i) PMOP9. (j) PMOP10. (k) PMOP11. (l) PMOP12. (m) PMOP13. (n) PMOP14.

and the position of the optima. $\eta(\mathbf{k}(\cdot))$ creates diverse knee regions with different characteristics such as global or local knee regions with different depths/lengths defined in [79], differentiable or nondifferentiable knee regions, and degenerated knee regions. $\mathbf{h}(\cdot)$ determines the basic shape of the PF whether is linear, concave or convex, and the uniformity of the solutions on the PF. The PFs and corresponding knees of the PMOP test suite are shown in Fig. 10. Interested readers are referred to [79] for more details of the PMOP test suite.

In summary, DO-DK, DEB-DK, CKP, and PEs are widely used as the optimization problems for benchmarking knee-oriented MOEAs in the search of knee points or knee regions in the multiobjective space. In contrast, PMOPs are applied to the assessment of MOEAs' performance in accurately and effectively identifying knee regions of MOPs, in particular, in the high-dimensional objective space. In PMOPs, more diverse characteristics of knee regions are designed to meet different levels of challenges for knee-oriented identification. Besides, variable linkage and multimodality are taken into account to reflect other hardness in solving real-world problems. Overall, PMOPs are MaOPs and more flexible in controlling the difficulties of optimization or knee identification than the previously mentioned four test suites.

B. Performance Indicators

Performance indicators or metrics are important in the evaluation of different solution sets and comparison between different MOEAs. Inheriting from the generational distance (GD) [104] and the inverted GD (IGD) [105], three indicators are proposed in [79] for the evaluation of knee-oriented methods, namely, knee-driven GD (KGD), knee-driven inverted GD (KIGD), and knee-driven dissimilarity (KD).

The mathematical definitions of KGD, KIGD, and KD are defined as follows, given a set of well-distributed reference points Q in the knee regions, true knee points K on PF, and a set of obtained solution set P.

1) *KGD:*

$$\mathrm{KGD} = \frac{1}{|P|}\sum_{i=1}^{|P|} d(p_i, Q). \tag{10}$$

2) *KIGD:*

$$\mathrm{KIGD} = \frac{1}{|Q|}\sum_{i=1}^{|Q|} d(q_i, P). \tag{11}$$

3) *KD:*

$$\mathrm{KD} = \frac{1}{|K|}\sum_{i=1}^{|K|} d(k_i, P). \tag{12}$$

In (10)–(12), $d(\cdot)$ is the shortest Euclidean distance from a solution or point to a reference set. KGD reflects the convergence performance of a solution set approaching the knee regions of PF; KIGD shows the comprehensive performance (convergence and diversity) of a solution set covering the knee regions; and KD indicates the completeness of a solution set whether it can find at least one solution to the true knee point.

Note that a set of reference points must be generated for calculating the performance indicators. This is doable for evaluating the quality of solution sets obtained on benchmark problems since their PFs and the positions of the knees are known. For real-world applications, however, it becomes nontrivial to define a set of reference points that should evenly cover the knee regions. A possible way is to define the reference points based on the approximated PF consisting of the nondominated solutions obtained by different algorithms under comparison [106]. However, it remains challenging to figure out the number and location of the knee regions since different definitions may lead to different results.

V. Overview on Knee-Oriented MOEAs

In this section, the main frameworks of knee-oriented MOEAs are reviewed at first. Then, the techniques used for searching or identifying knee or knee regions are presented.

A. General Frameworks of Knee-Oriented Methods

Knee-oriented MOEAs have similar frameworks to the preference-driven MOEAs [40], [43], [107], including *a priori*, *interactive*, and *a posteriori* frameworks.

1) Priori Approaches: For knee-oriented *a priori* optimization, a preference model is constructed before the optimization is performed. The model can be any reference information that can be used to guide the optimization toward the knee regions of the PF. For example, the study in [77] uses a set of predefined reference vectors to partition the objective space into a number of subspaces, and the localized dominance relationship (localized α-dominance [108] is then adopted to drive the search process [77]) to search for the potential knee regions in the subspaces. These algorithms show competitive performance in the search of knees in many-objective optimization, although the number of knees they can identify relies on the number of reference vectors. In [109], P and Q are two predefined integers to determine the number of weight vectors so as to control the extent of the coverage and the focus on the knee regions. The extension study of [109] uses a parameter δ is preset to control the ratio of the nondominated solutions among the population [76]. Once the ratio is satisfied, the identification of knee candidates will be triggered. Another parameter k is predefined to determine the number of nearest neighbors, where the potential knee candidates are characterized from. Experimental results show that different settings of the parameters may result in different distributions of the knee regions. In [110], μ and η are predefined to trigger the identification of the knee candidates and control the spread of the knee regions, respectively. There are also other *a priori* strategies, such as knee-driven MOEA/D [111] and knee-driven NSGA-II [112]. Although a number of *a priori* approaches have been proposed to find knee candidates, the number of knees may rely on the predefined parameters, such as the number of reference vectors [108], [109]. In addition, little work has been done to investigate specific preferences or features between different knee regions such as the tradeoff between the interested objectives only during the optimization. In other words, only a small number of knee regions with specific features are explored instead of all knee regions.

2) Interactive Approaches: For knee-oriented interactive optimization, DMs are asked to iteratively intervene in the optimization. For each intervention, DMs are commonly provided with a small number of intermediate solutions and asked to rank them. The ranking is then incorporated into the subsequent search to find preferred solutions. In [81], the proposed ASF-DR provides the DM with the potential knee candidates. During the interactive stage, the DM selects the preferred solutions from the knee candidates by means of a visualization method or knee identification method. This interactive framework is generic and straightforward, albeit the interpretability of the knee-oriented visualization needs to be further investigated. Specifically, the gap between the visualization and informed preference needs to be bridged so that DMs can easily guide the optimization process. In [113], the potential knee candidates in SOIs are regarded as anchor points during the intervention. Then, the rest of the solutions are ranked according to their proximity to their closest anchor points. Accordingly, the DM can not only grasp enriched information of SOIs but also the information of a range of solutions in the vicinity of the SOIs. This method aims to provide the DMs with as much information as possible about the PF, which, however, may also make the interaction process intractable. Recently, a knee-oriented interactive framework is proposed that the implicit preference of the knees or knee candidates is transformed into reference vectors during the interactive optimization process [114], where the DMs are asked to choose their preferred knee candidates and meanwhile reference vectors are evolved in terms of the distribution of the potential knee regions. The method [114] is able to facilitate the DMs in preference articulation in the stage of decision making. Although this study makes a useful attempt to transform knees into reference vectors, more interpretable and convenient informed preferences are needed, because reference vectors are unable to quantify all different features between different knee points or regions.

3) Posteriori Approaches: In the decision-making stage, a set of solutions representing the PF is first achieved before the identification when the DM does not have enough *a priori* knowledge and cannot give specific preference information. Then, the knee-oriented *a posteriori* methods are adopted to find a small number of knee candidates among the given solution set so as to facilitate DMs in decision making. As introduced in Section V-B, there are five categories of knee-oriented methods, including the utility-based, angle-based, dominance-based, niching-based, and visualization-based methods, which are widely used for knee-oriented decision making, as shown in the summary of real applications in Section VI. However, many studies in solving real-world problems face an open challenge that it is resource intensive and time consuming to provide the DM with a large number of representative solutions covering the whole PF, especially when the number of objectives is large [14]. Besides, it is difficult to choose preferred solutions if there are too many or few knee candidates in a solution set.

B. Main Methods

Various techniques for the identification of knee candidates or knee regions have been proposed in recent decades, which can be summarized into utility-based, angle-based, dominance-based, niching-based, and visualization-based methods, as shown in Table III.

TABLE III
SUMMARY OF KNEE-ORIENTED TECHNIQUES

Categories	Methods	Features	Advantages	Disadvantages
Utility-based	Distance utility [66], [115]	Based on the distance to the CHIM constructed by extreme points	Easily extended to handle MaOPs	Bias central knees of the PF and sensitive to the degenerated PFs
	EMU [67], [68]	Rely on a number of well-distributed weight vectors	The marginal utility involves informed preference	Performance degradation with too many or few weight vectors
	MMD [69][78]	Combines the analyses of geometrical information and optimization performance	Able to discern the convex and concave knees	Estimation of the nadir point is a difficult task
	Modified ASF [81]	Replace reference points with ideal and nadir points in ASF	Flexible to integrate preference into the optimization	Sensitive to ideal points and nadir point
	Trade-off utility [116] [76], [65], [117][118]	Using the ratio between Improvement and Deterioration	Able to find global and local knees and robust to the shapes of PFs	Poor discrimination ability in many-objective optimization
	L_p scalarizing function [119] [120]	Using L_p to locate SOIs	Easily reflect local curvature of PF by tuning a parameter	Torturous tuning process in decision-making stage
Angle-based	Reflex angle [67]	Straight lines to its two adjacent neighboring points	Reflect the local information of the PF	Too much focus on local information reduces its versatility
	Bend angle [65]	Calculate bend angles with the assistance of the extreme points	Able to distinguish both convex and concave knees	Pool versatility on many-objective optimization problems
	Extended angle dominance [121], [122]	Calculating the extended angles by using an arctangent function	Able to get desirable extent of knee regions	Torturous setting of threshold angle and bias intensity
	Angle utility [123]	Scalarization function based on angles	Robust with the shape of PF and number of objectives	Suffer from the dominance resistant solutions
Dominance-based	Localized α-dominance [108]	Sort solutions in the sub-regions associated with different weight vectors	Fast convergence to convex knee regions and can eliminate DRSs	Need to predefine reference vectors and parameter α
	Localized knee-oriented dominance [77]	Different dominated areas for solutions in different locations	Can get rid of boundary solutions and solutions in convex regions	The performance relies on the number of reference vectors
	Extended angle dominance [121], [122]	Calculating the extended angles by using an arctangent function	Able to get desirable extent of knee regions	Torturous setting of threshold angle and bias intensity
	Cone-domination [124]	Different hyper-cones contribute different dominated area of a solution	Consistent with the HV contribution	Too many predefined hyper-cones reduce the versatility
	ϵ-dominance [125] [119]	Tuning ϵ to identify the knee candidates	Reflect the trade-off information between the objectives	Suffer from the torturous estimation of ϵ for different knee regions
Niching-based	Weighted sum niching approach [109]	Based on normal boundary intersection and weighted sum function	The extent and density of coverage of the knee regions are controllable	Performance degradation with the increase of the number of objectives
	Solution density [100]	Based on the radial coordinate values	Able to distinguish both convex and concave knees	Poor discrimination ability in many-objective optimization
	Niche strategy [112]	Mutual distance values in both decision space and objective space	Good convergence and diversity of population in the knee regions	Suffer high computational cost in solving MaOPs
	k-NSGA-II [110], K-MOEAD[110]	Modified crowding distance	Flexible to control the extent of the knee regions to be explored	Difficult to preset the trigger for knee-detection procedure
	Pareto explorer [101]	Explore Pareto landscape around optimal solutions	More robust than the CHIM-based method [66] on solving MaOPs	Heavily rely on the locations of the given optimal solutions
Visualization-based	Voronoi diagram [126]	Use Voronoi diagram to analyze the distribution of solutions	Able to reflect some geometry characteristics	Limited to 2-D Pareto front approximation
	ASF-DR [81]	Transform high-dimensional solutions into solutions in 2-D space	Able to visualize any dimension of knee regions	Break the Pareto relation
	Reeb Graph [127]	A graph-based approach	Able to extract knee points and geometry of the PF	The versatility on knee regions of MaOPs is unclear
	Level diagrams [128]	Plot distance value from a solution to ideal point on each objective	Easy to see the performance of solutions in subplots	Limited to 4-D Pareto front approximation
	Hyper-radial visualization [129]	Use the distance to ideal point as hyper-radius	Able to see the closeness to the ideal point	Limited to 4-D Pareto front approximation
	Prosections [80]	Only a section of the space is visualized at a time	Mostly preserves the dominance relations	Limited to 4-D and 5-D Pareto front approximation

1) Utility-Based Methods: Das [66] proposed a distance utility method based on CHIM as shown in (2), where the knee solutions are identified with the largest distances to the CHIM. Furthermore, the method [115] modifies the distance utility to find both convex and concave knees. The CHIM-based methods are easily extended in handling MaOPs. However, the CHIM-based methods tend to favor the knees in the center of the PF and the identification algorithm may be misled when the PF is degenerate [101]. In [67], the EMU is introduced to describe the knees among the tradeoff solution set, as shown in (3). The marginal utility is widely used in preference-driven MOEAs to find the best solution along the weight vector [130]. An improved version of [67] is proposed in [68] to recursively use the EMU to find the required number of SOI. Similarly, the study [69] identifies knee points by recursively using the minimum Manhattan distance (MMD) [78]. In comparison with the EMU approach, MMD does not need a set of uniformly distributed weight vectors, and the approach prefers the solution with the global minimum MMD value. It has been proved that the EMU approach cannot discern convex knee solutions from boundary solutions, which also have higher EMU values than their adjacent solutions [69]. In comparison with EMU-based methods, the MMD approach is more robust in handling MaOPs, although the algorithms proposed in [69] and [78] need boundary solutions and ideal points to assist the identification of knee candidates. Recently, a novel method [81] modifies the achievement scalarizing function (ASF) [131] with the nadir and ideal points to help the DM in finding the knee candidates. Note that the nadir points can be replaced by reference points when the DM has specific preferences. In this study, the differences between the ASF values of solutions in their neighborhood can help to locate the local knee regions, while the nadir point is difficult to estimate as the number of objectives increases [63]. In [76] and [116], the tradeoff utility [i.e., (4)] is applied to identify the knee candidates which have the maximum tradeoff utility values in comparison with their neighbors. Furthermore, the study [76] is extended by replacing the reference points in the preference-driven MOEA [132] with the knee-like points for the search of knee regions [117], where the knee-like solutions are identified

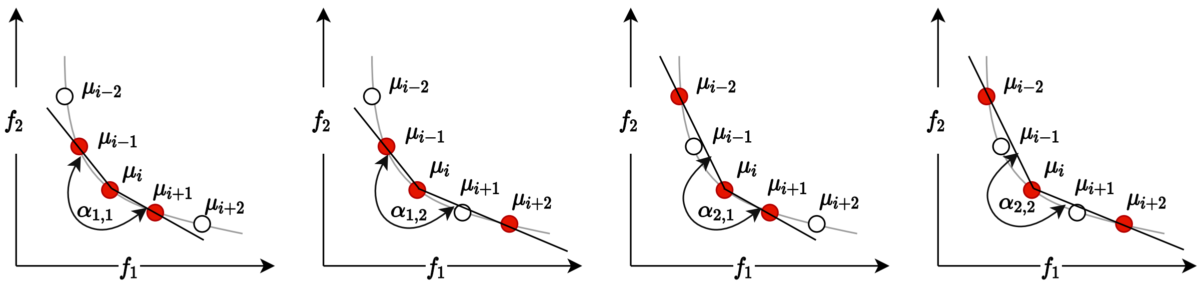

Fig. 11. Illustration of the reflex angle.

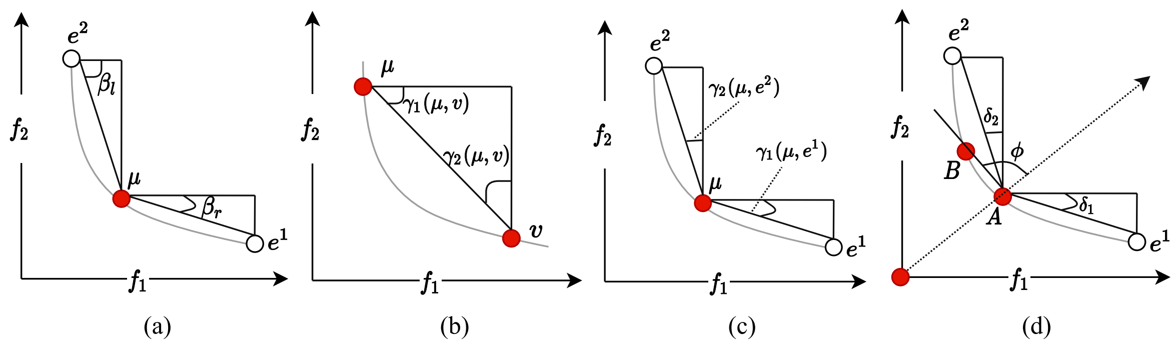

Fig. 12. Illustration of the (a) bend angle, (b) extended angle, (c) angle utility, and (d) knee-oriented dominance, respectively.

by [76]. A similar method based on mobile reference points is proposed in [133]. The experimental results show that the tradeoff utility-based methods are robust to different shapes of PFs, while suffering from high computational cost since each solution has to be compared with its neighbors in calculating the utility. Interestingly, the research in [118] aims to reduce the computation cost of the method in (4) and presents a GPU implementation to identify the knees in parallel. In [119], an approach based on local curvature is proposed, where different knees in different regions can be identified by adjusting the parameter α in L_p scalarizing function [120]. In fact, the proper design of the self-adjusting strategy is a challenging task when the shapes of the PF are complex.

2) Angle-Based Methods: In [67], the reflex angle is proposed to determine the desirability of a solution in convex regions. The reflex angle aims to reflect the local information of the PF. It is formed by connecting the solution by straight lines to its two adjacent neighboring points on the Pareto front. An illustration of the reflex angle is given in Fig. 11, and the largest angle among $\alpha_{1,1}$, $\alpha_{1,2}$, $\alpha_{2,1}$, and $\alpha_{2,2}$ is assigned to μ_i. However, concentration on the evaluation of the local information may reduce the robustness of the reflex angle in handling complex PFs. In [65], an extension of the reflex angle, i.e., the bend angle with the assistance of the extreme points is proposed to identify the global convex knees of the bicriterion problems. In Fig. 12(a), the subtraction of β_r from β_l is assigned to μ as the bend angle. The bend angle can be used for characterizing the convex knees that have the minimum negative bend angles. It can be concluded that such angle-based approaches are very simple to implement in solving MOPs but their accuracy strongly relies on the position or closeness of the solutions. On the basis of the bend angle, the extended angle dominance is proposed in [121] and [122]. In the identification, the solutions are first compared with the objectives using the Pareto dominance relationship. Then, the extended dominance angles of solutions will be compared with a predefined threshold. The extended angle dominance works well as a preference model to identify ROIs on the Pareto front [134]. An illustration of the extended angles is given in Fig. 12(b), where $\gamma_1(u, v)$ and $\gamma_2(u, v)$ are two extended angles between solutions μ and v. Particularly, the extended angle dominance performs robustly well in finding the knees on different shapes of the Pareto fronts once the predefined threshold is properly set. Instead of taking account of pairwise angles between all solutions on the PF in [121], the research in [123] proposes an angle utility to find the desired knee regions, where the solution with the maximum angle utility is regarded as the convex knee. In Fig. 12(c), e^1 and e^2 are two extreme points and the $max(\gamma_2(u, e^2), \gamma_1(u, e^1))$ is assigned to solution μ as its angle utility. The method is sensitive to the extreme points, especially when there are DRSs [99]. DRSs are the solutions that are extremely inferior to others in at least one objective and do not belong to the true PF. They commonly appear in the boundary regions such as the extreme regions and the solutions dominating them can hardly be found according to the Pareto dominance relationship. Accordingly, DRSs easily become the extreme points so that DRSs will decrease the accuracy of knee identification methods that rely on the extreme points.

3) Dominance-Based Methods: Modified dominance relationships may give the knee candidates a higher priority than their neighbors during the selection. In [99], the α-dominance is originally designed to handle DRSs [99]. However, the dominance possesses the tradeoff information between the objectives.

Consequently, the study [108] introduces α-dominance [99] in subregions to drive the optimization processes toward diverse knee regions of the subspaces and meanwhile the method can eliminate the impact from DRSs. However, the method needs to predefine a number of reference vectors and parameter α during the optimization. In order to relieve the influence, the research [77] proposes a knee-oriented dominance relationship[3] based on the framework of localized α-dominance [108] to assist the search of knee regions. The localized dominance relationship [77] can drive the search process toward potential knee regions, while eliminating the noninterested solutions, such as the DRSs [99] and boundary solutions. Especially, solutions at different locations in the objective space have different dominated areas. An illustration of the knee-oriented dominance relationship is given in Fig. 12(d). In comparing solutions A and B, $\delta_1 + \delta_2$ will be compared with $\tau \cdot \phi$, where τ is a self-adaptive parameter according to the location of a solution. It has been demonstrated [77], however, that the performance of localized dominance-based methods heavily relies on the distribution of the reference vectors which are generated by the NBI [89]. This may be attributed to the fact that the evenly distributed reference vectors generated by NBI [89] may not fit for irregular Pareto front shapes [13], [97], [135]. As suggested in [15] and [97], reference vectors originating from the nadir point [63] may be more capable of generating a set of evenly distributed solutions covering the PF. Therefore, self-adaptation of the reference vectors is essential to fit the distribution of knee regions on the PF. Different from the knee-oriented dominance, which is an outranking method aiming to find as many knee candidates as possible, angle utility [123] is a scalarizing method for finding the global knee. Another dominance-based method is the extended angle dominance [121], where a threshold angle is predefined to discard the solutions whose geometric angle is smaller than the threshold. Besides, a bias intensity of each objective is set in the pruning process to control the extent of the knee regions to be explored. Accordingly, the extended angle dominance suffers from the tiresome settings of the predefined parameters. In [124], a cone-domination is proposed, where the angles of every face of the hyper-cone region are predefined parameters to control the dominated area of a solution. The experimental results in [124] show that the method is able to provide DMs with a small number of knee candidates having the most of the HV contributions. Unfortunately, too many user-defined parameters may reduce the versatility in handling complex MaOPs. In [119], the study has investigated the ϵ-dominance [125] for knee identification since a solution with a higher ϵ value has higher tradeoff utility values than its neighbors. However, the ϵ-dominance is sensitive to the neighboring solutions. In other words, the estimation of ϵ values for different knee regions is a challenging task.

4) Niching-Based Methods: It is known that normal boundary intersection (NBI) [89] builds a connection between the weighted sum minimization and nonlinear programming [66]. On the basis of the connection, the weighted sum niching approach [109] with the assistance of parameters P and Q is proposed, where the parameters are used to estimate suitable weight vectors to control the exploration of the extent of the potential knee regions. Note that the accurate location of the knee regions is based on the percentage of the nondominated solutions in the current population during the optimization. A rate larger than 80% is suggested in [76]. In [100], it has been demonstrated that solutions in different knee regions with different curvatures possess different radial coordinate values [136]. Accordingly, the research [100] applies the difference of the radial coordinate values [136] for characterizing different knees, where the difference is denoted by density in [100]. When the density of a solution is smaller than a predefined threshold, then the solution is regarded as knee candidates. This method provides a new way of describing the knee regions, although the requirement to predefine the parameters reduces the versatility of the method in handling MaOPs. In [112], it has been proved that knee-driven search process assisted by a niche strategy [137] can improve the convergence and diversity of population toward the knee regions, where the niche strategy selects solutions in terms of their mutual distance values in both decision space and objective space is proposed. However, the method [112] is not competitive in recognizing the knee candidates and suffers high computational cost when the number of objectives increases. In [110], a knee preference-based selection is proposed, where a modified crowding distance is designed to improve the distribution of solutions in the niches of knee regions, and meanwhile the knee-region solutions have a higher priority to be selected. During the optimization, the method needs to preset a trigger for a knee detection procedure, where the trigger is 30%–50% of the maximum number of generations. Apparently, this rule may not be applicable to all MOPs. In [101], the Pareto explorer proposed in [102] is adopted to detect the knee candidates. First, several optimal solutions are required to be obtained. Then, an exploration of the Pareto landscape around the obtained optimal solutions is performed by moving along the Pareto set/front toward knee solutions or knee regions. It has been demonstrated in [101] that the method is less affected by the approximation quality of the CHIM than the method in [66], indicating that the continuation-like strategy is beneficial, in particular for handling sufficiently smooth MaOPs.

5) Visualization-Based Methods: In this section, we introduce the knee-oriented visualization methods. For example, a knee identification based on the Voronoi diagram (or Thiessen polygons) is proposed in [126]. The Voronoi diagram is to analyze the distribution information of the solutions on the PF, while the lines connected by the vertices indicate the locations of knee points. An example is given in Fig. 13(a) and (b), where intersections between the line and PF are the knee candidates. A visualization method based on the Voronoi diagram is proposed in [126] to find convex, concave, and edge knees. However, it is challenging to extend the method to MaOPs. In [81], a visualization method (ASF-DR) based on dimension reduction is proposed, where ASF-DR visualizes high-dimensional solutions in a 2-D space with respect to convergence distance (d^p) and divergence distance (d^d). The method is able to grasp the geometry information of

[3] https://github.com/GYResearch/LBD-MOEA

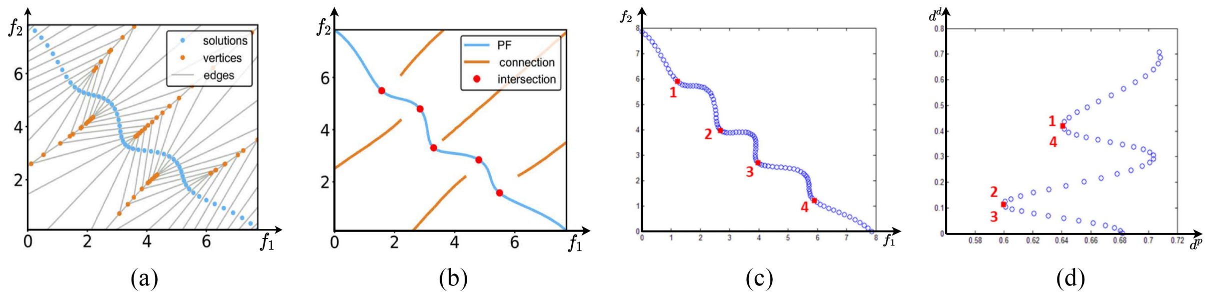

Fig. 13. Illustration of two representative visualization methods, i.e., Voronoi diagram [126] and ASF-DR [81]. (a) Voronoi diagram of given solutions. (b) Identification results. (c) Given solutions and knee points. (d) Transformed solutions.

PF, including the tradeoff between objectives, the location of extreme points, as well as the bend degrees of knee points. However, it is incompatible to the Pareto dominance relationship and different knees may be located at the same position in the 2-D visualization space. In Fig. 13(c) and (d), the knee candidates (1 and 4) are located at the same position in the visualization space and it also appears on knee candidates (2 and 3). Recently, a visualization method based on Reeb graph [127] is proposed. It is able to extract knee points and the connectivity and convexity of the Pareto front. However, the method does not fully exploit the topological structure of the Pareto front and the versatility remains unclear. In addition, several visualization methods are studied in [138] and it has been found that the level diagrams [128], hyper-radial visualization [129] and prosections [80] are able to visualize 4-D knee regions. In the level diagram, the *x*-coordinate is one of the objectives while the *y*-coordinate is the Euclidean distance from solutions to the ideal point. In hyper-radial visualization, the distances from solutions to the ideal point are regarded as hyper-radius in the visualization space. Prosections [80] are projections of only a section of the objective space at a time, and the method is able to visualize 4-D and 5-D convex knee regions. It should be mentioned that these three methods are verified only on optimization problems with one knee region which is located in the center of the PF and their performance on the problems with multiple knee regions is unclear.

In summary, the utility-based methods are able to find knee candidates by sorting solutions according to their utility values. However, they may also favor solutions in uninterested regions such as boundary regions. The angle-based methods are capable of reflecting the local curvatures, although more investigations are needed to understand the relationship between the sections of the angles and knee regions in high-dimensional objective spaces. The dominance-based methods are readily embedded into different knee-oriented frameworks. Unfortunately, they often also suffer from torturous settings of parameters. The niching-based and visualization-based methods are usually utilized in a posterior framework. The former is sensitive to the distribution of the given solutions, while the latter is prone to violating the Pareto dominance relationship because of dimensionality reduction methods.

VI. Applications

Due to the attractive properties of the knee points presented in Section II, knee-driven MOEAs have been successfully applied to many scientific and real-world problems, in particular in machine learning, community detection, and design optimization.

A. Pareto-Based Multiobjective Machine Learning

As indicated in [139], all machine learning problems are inherently MOPs, including regularization, clustering, image segmentation, and communication-efficient federated learning.

1) *Compressing Deep Neural Networks:* In [86], the minimum Manhattan distance [78] is introduced to guide the search toward the knee regions of the PF in simultaneously minimizing the complexity and maximizing the performance of deep neural networks. Experiments on LeNet[4] [140] and VGG-19 [141] have verified that the obtained knee points achieve a good tradeoff between the scale of parameters and performance on MNIST [142] and CIFAR-10 [143] datasets, respectively.
2) *Sparse Subspace Classification:* A Pareto-based sparse subspace learning algorithm is proposed [87] for high-dimensional data classification by considering two conflicting objectives, i.e., reconstruction error and sparsity. This article tests the method on real-world examples, including the UCI datasets[5] and gene expression data[6] and adopts the bend-angle measurement [67] to find the best candidates (knee points) on the PF. This article finds that the knee point can achieve good performance in classification with the lowest possible number of features.
3) *Sparse Subspace Classification:* A Pareto-based sparse subspace learning algorithm is proposed [87] for high-dimensional data classification by considering two conflicting objectives, i.e., reconstruction error and sparsity. This article tests the methods on real-world examples, including the UCI datasets[7] and gene expression data[8] and adopts the bend-angle measurement [67] to find the best candidates (knee points) on the PF. This article finds that the knee point can achieve good performance in classification with the lowest possible number of features.

[4] http://yann.lecun.com/exdb/lenet
[5] http://archive.ics.uci.edu/ml/index.php
[6] http://www.gems-system.org
[7] http://archive.ics.uci.edu/ml/index.php
[8] http://www.gems-system.org

4) *Image Segmentation:* In [144], the image segmentation problem is transformed into a biobjective optimization problem with two conflicting objectives, i.e., preserving the details and removing the noise of an image as much as possible. During the decision-making stage, a trade-off front of solutions is obtained when segmentation is conducted on natural images [145] such as flows, coins, and cameraman, and the knee candidates on the front are identified as the most suitable solution to the observed images because the authors do not have reference segmented images (or *a priori* knowledge for decision making).
5) *Federated Neural Architecture Search:* Federated learning is a distributed machine learning approach for privacy preservation [146]. A challenge is to address the high demand of communication and computational resources when a deep neural architecture search is carried. Accordingly, Zhu and Jin [147] proposed an evolutionary method to the real-time federated neural architecture search to balance the model performance and costs from the computation and communication. The experiments show that for 10 and 20 clients, the knee solutions have much lower model FLOPS[9] and better performance than the original ResNet [148]. Another finding is that the performance of the knee solution performs much more stable during the evolutionary search than that of the conventional offline evolutionary architecture search [149].

B. Design Optimization

Almost all real-world engineering design problems must optimize multiple conflicting objectives. In solving these problems, knee points have been used for automatically select preferred solutions.

1) *Hybrid Electric Vehicle Management Controller Design:* The controller design of hybrid electric vehicles (HEVs) aims to balance the driving comfort and environmental considerations by switching the power sources between the internal combustion engine and electric motor [150]. To alleviate the burden of DMs for incorporating preferences into the optimization, a localized dominance relationships-driven MOEA is proposed [77] to search for knee candidates of the HEV optimization problem. To facilitate DMs for informed decision making, another work [151] uses recursive EMU [68] to identify the knee candidates among a large number of Pareto-optimal solutions of the HEV optimization problem.
2) *Hybrid Direct-Current (DC) Link Capacitor Banks:* The capacitor is a basic electronic unit to store or release the electrical charges [152]. In real applications, there is few quantitative design in capacitor banks in terms of impedance characteristics, lifetime, power loss, cost, and volume. Therefore, a model-based design approach is proposed in [153] for the quantitative design of hybrid capacitor banks. In the experiment on a 5.5-kW photovoltaic application, the knee points on the optimized PF are chosen as the quantitative designs of the hybrid capacitor banks when no specified design target is given.
3) *Microgrid Dynamic Energy Management:* Commonly, model predictive control (MPC) is applied to diminish the adverse effect of the inaccurate data prediction in solving microgrid dynamic energy management problem. However, MPC technology needs DMs to frequently select solutions during the optimization. In order to relieve the burden of the DMs, Li *et al.* [118] introduces the knee point of the PF as the preferred solution. The experiments show that the proposed knee point-driven approach can effectively reduce the amount of computational consumption and obtain good convergence in solving microgrid dynamic energy management problems.

C. Self-Adaptive Software

The self-adaptive software [154], [155] is a special type of software that adaptively reconfigures itself during the runtime when the environment changes. In order to achieve adaptable software, the response time and energy consumption are regarded as two objectives in the optimization [83]. In the stage of decision making, the knee points are chosen because the knees achieve a good balance between the two objectives when the self-adaptive software (RUBis[10]) is optimized without human intervention [83].

D. Community Detection

In [88], two conflicting objectives are constructed for community detection. One is the modularity [156] to gather the densely connected nodes of a community network. The other one is the attribute similarity proposed in [88] to gather the nodes sharing the same attribute values. On account of the difficulty in providing DMs with the best solution, this article uses the angle-based measurement [67] to find the knee points as the best alternatives on the PF with respect to these two objectives, where each point is a partition of the network.

A summary of the knee-oriented optimization methods applied to various scientific and real-world applications is listed in Table S1 of the supplementary material. In these applications, knee points are commonly chosen since they have a good balance between different objectives and can relieve the burden of the DMs. However, most of the knee-oriented methods are *a posteriori* approaches which identify the knee candidates among a number of obtained solutions. Such a strategy may suffer from a large sacrifice of computing resources to get such a number of solutions. Hence, how to efficiently and effectively locate the knees beforehand is essential.

VII. Potential Research Directions

To address the remaining challenges in knee-oriented optimization, we suggest a number of promising lines of research in the following.

1) *Adopt Machine Learning Techniques to Assist Knee-Oriented Search:* A number of approaches in Section V

[9]FLOPS: floating-point operations per second.

[10]Rubis: Rice university bidding system, http://rubis.ow2.org/.

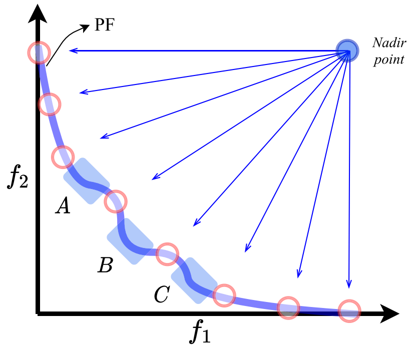

Fig. 14. Example of representative solution set of a PF, where A, B, and C are three knee regions. The red circles are the representative solutions. The arrows are reference vectors from the nadir point.

rely on the activation function or predefined parameters to control the approximation to the PF so as to balance the accuracy of the knee identification and sacrifice of the computational costs. Hence, self-learning strategies like reinforcement learning [157] may be helpful to adaptively control the activation function in the knee-oriented identification [91]. In addition, some methods [68], [76], [77], [108] are sensitive to the reference vectors. A possible way is to learn the distribution of the knee regions and tune the reference vectors, e.g., using the growing neural gas to tune the reference vectors [135].

2) *Choose Representative Knees or Other SOIs:* For a problem with many knee solutions, a critical issue is which knee candidates should be chosen by the DM. We recommend to consider other features to rank the knee candidates. For example, the HV contribution [70] or EMU [68] can be adopted to select a handful (7 ± 2) [68] of solutions according to the utility values. Besides, the difference between the decision spaces of the knees can be an alternative measure. In many real applications, the solutions finally chosen by the DM are required to be robust [158]. Consequently, the linkage between the decision space and objective space can be investigated to distinguish the knee candidates. For example, the evaluation of the robustness of the knees in the decision space becomes an important means in decision making for the DM. However, if a problem has no knee region, solutions having particular statistical meanings can be suggested. For example, the solutions whose objective values are close to the mean value, upper quartile, or lower quartile values of the corresponding objective values of the whole population may be of interest to the DM, since these solutions may reflect the distribution of the population and approximated PF.

3) *Actively Generate Reference Points in Knee Regions:* First, the reference points are essential for calculating the performance indicators. Usually, the way to generate reference points in knee regions is to use traditional MOEAs to fill the knee regions. However, the knee regions may not be covered. For example in Fig. 14, the solutions along the current reference vectors are not in the knee regions. Second, the solution set presented to the DM may contain only a small number of knee candidates in the knee regions. In order to improve the quality of the given reference set, we may actively produce knee candidates in the potential knee regions based on the current obtained solution set in Fig. 14. A possible way is to fit the complex PF and decision space by means of surrogates [95], [96], [159] or inverse models [160], [161] such that we may predict possible knee regions and generate knee candidates in the knee regions [162].

4) *Visualize Knees or Knee Regions:* In Section V, most visualization techniques [103], [126], [127] are only applicable in 2-D, 4-D, and 5-D spaces. Notably, the recently proposed ASF-DR [81] is able to visualize the shapes and locations of high-dimensional Pareto fronts. Unfortunately, different knee candidates may be mapped to the same location in the visualization space. Hence, ASF-DR can be improved to keep the consistency with the Pareto dominance by integrating extra information to distinguish the solutions located in the same position in the visualization space. Besides, it is essential to create a linkage between the visualization space and the preference space. Consequently, the visualization can provide DMs with an interface to interact with the optimization process and facilitate the preference articulation in terms of the links between preference and visualization [163].

5) *Investigate Knees in Constrained and Combinatorial Optimization Problems:* Although Section IV presents a number of knee-oriented benchmarks to comprehensively evaluate the knee-oriented methods with respect to the optimization and identification, the existing knee-oriented benchmarks do not consider some common features of the real applications, such as the constraints and mixed-integer decision variables, and little research has been conducted on knee-oriented investigations on such problems. The constraints may change the manifold of the PF and create new features of the knee regions. For example, some knee regions may be infeasible or discontinuous due to the constraints. Another interesting line of research is to investigate whether knee solutions exist in the combinatorial and mixed-variable optimization problems with multiple objectives [164], since the decision space is not discontinuous. In case knee regions exist, the identification of the structure of the neighborhood of the knees in the decision space remains unknown.

VIII. Conclusion

While traditional MOEAs aim to find a solution set to represent the PF, preference-driven MOEAs are required to find a solution set in the ROIs according to the preferences from the DMs. Unfortunately, it is nontrivial to find a preferred solution set in dealing with MaOPs, especially when the DM is unable to clearly express their preferences, and many questions regarding the importance of knee points, the influence of the definitions of knee solutions and their consistency on the search performance remain elusive. Therefore, this survey paper not only presents a categorization of methods for identifying knee solutions and discusses their advantages and disadvantages but also clarifies the importance of getting the knee solutions and the differences and inconsistencies between the existing definitions for knee solutions. In addition, we present the knee-oriented benchmarks and performance indicators dedicated to knee-oriented optimization. Finally, real-world applications in which knee-oriented optimization is performed are summarized, indicating that knee solutions have great practical significance.

Although knee-oriented multiobjective optimization has received much attention and many knee-oriented approaches have been proposed in recent years, a number of challenges remain to be tackled. For resolving the inconsistency of the definitions, it is a good idea to combine different definitions together to get robust performance in characterizing the knees. Machine learning techniques might be adopted to overcome the drawbacks of the knee-oriented methods such as the sensitivity of predefined parameters. In the decision-making stage, representative knee candidates or other SOIs can be studied when the number of knees is too many or too few. In addition, actively generating reference points in potential knee regions is worthy of further studies to improve the quality of a solution set. Visualization methods associated with references or preferences may be promising and meaningful for decision making, which can facilitate the DM in preference articulation. Last but not least, it is of interest to examine additional challenges when finding knee solutions in constrained, mixed-integer, and combinatorial optimization problems.

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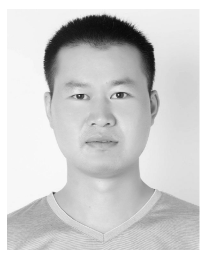

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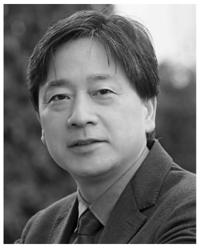

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