矢吹太朗『コンピュータでとく数学』（オーム社，2024）

1 実行環境

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
data = {{1, 7}, {3, 1}, {6, 6}, {10, 14}};
model = LinearModelFit[data, X1, X1]
model["BestFitParameters"]
```

Out[ ]=
```
FittedModel[ 2. + 1. X1 ]
```

Out[ ]=
```
{2., 1.}
```

2 数と変数

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
2 * (-3)
```
Out[ ]=
```
-6
```

In[ ]:=
```
2 (-3)
```
Out[ ]=
```
-6
```

In[ ]:=
```
(1 + 2) * 3
```
Out[ ]=
```
9
```

In[ ]:=
```
2^10
```
Out[ ]=
```
1024
```

In[ ]:=
```
-2 < -1
```
Out[ ]=
```
True
```

In[ ]:=
```
2 + 2 == 5
```
Out[ ]=
```
False
```

In[ ]:=
```
If[7 < 5, 10, 20]
```
Out[ ]=
```
20
```

In[ ]:=
```
x < 1
```
Out[ ]=
```
x < 1
```

In[ ]:=
```
x == y
```
Out[ ]=
```
x == y
```

In[ ]:=
```
x^2 - 1 == (x + 1) (x - 1) // Simplify
```
Out[ ]=
```
True
```

In[ ]:=
```
Not[0 < 1] (* 方法1 *)
! (0 < 1) (* 方法2 *)
```
Out[ ]=
```
False
```
Out[ ]=
```
False
```

In[ ]:=
```
Or[0 < 1, 2 > 3]  (* 方法1 *)
(0 < 1) || (2 > 3) (* 方法2 *)
```
Out[ ]=
```
True
```
Out[ ]=
```
True
```
In[ ]:=
```
And[0 < 1, 2 > 3]  (* 方法1 *)
(0 < 1) && (2 > 3) (* 方法2 *)
```
Out[ ]=
```
False
```
Out[ ]=
```
False
```
In[ ]:=
```
Not[10 < x]
```
Out[ ]=
```
10 ≥ x
```
In[ ]:=
```
Clear["Global`*"];
```
In[ ]:=
```
x = 5; x == 5
```
Out[ ]=
```
True
```
In[ ]:=
```
a = 1 + 2;
b = 9;
a (b + 1)
```
Out[ ]=
```
30
```
In[ ]:=
```
a = 1 + 2; b = 9; a * (b + 1)
```
Out[ ]=
```
30
```
In[ ]:=
```
a = 1 + 2
```
Out[ ]=
```
3
```
In[ ]:=
```
a = 3;
Clear[a]; (* 変数を記号にする. *)
Expand[(a + 1)^2]
```
Out[ ]=

$1 + 2 a + a^2$

In[ ]:=
```
x1 = 2; x2 = 3; x1 + x2
```
Out[ ]=
```
5
```
In[ ]:=
```
Subscript[x, 1] = 2; Subscript[x, 2] = 3; Subscript[x, 1] + Subscript[x, 2]
```
Out[ ]=
```
5
```

```
In[ ]:= x = 1; y = x + 1; x = 2; y
Out[ ]=
2

In[ ]:= x = 1;
y := x + 1; (* yは「2」ではなく「x + 1」になる. *)
x = 2;
y     (* 「x + 1」は「2 + 1」つまり3. *)
Out[ ]=
3

In[ ]:= Clear["Global`*"];

In[ ]:= f = 2 x + 3;
f /. x → 5
Out[ ]=
13

In[ ]:= g = a + b;
g /. {a → x, b → y}
Out[ ]=
x + y

In[ ]:= f = Function[x, 2 x + 3];
f[5]
Out[ ]=
13

In[ ]:= Clear[f];
f[x_] := 2 x + 3
f[5]
Out[ ]=
13

In[ ]:= Clear[f, a];
f = Function[x, 2 x + 3];
g = f[a];
{f[5], g /. a → 5}
Out[ ]=
{13, 13}

In[ ]:= Clear[f];
f[x_] := 1 / x
f[1]
Out[ ]=
1
```

In[ ]:=
```
f1[x_] := Piecewise[{{1/x, x ≠ 0}}, Undefined]

f2[0] = Undefined;
f2[x_] := 1/x

f3[0] = Undefined;
f3[x_ /; x ≠ 0] := 1/x

f4[x_] := If[x ≠ 0, 1/x, Undefined]

f5[x_] := Which[x ≠ 0, 1/x, True, Undefined]

{f1[1], f2[1], f3[1], f4[1], f5[1]} (* 全て1 *)
{f1[0], f2[0], f3[0], f4[0], f5[0]} (* 全てUndefined *)
```

Out[ ]=
```
{1, 1, 1, 1, 1}
```

Out[ ]=
```
{Undefined, Undefined, Undefined, Undefined, Undefined}
```

In[ ]:=
```
Function[x, 2 x + 3][5]
```

Out[ ]=
```
13
```

In[ ]:=
```
Clear[f];
f[x_, y_] := x + y
f[2, 3]
```

Out[ ]=
```
5
```

In[ ]:=
```
Clear[g];
g[x_] := x[[1]] + x[[2]]
x = {2, 3}; g[x]
```

Out[ ]=
```
5
```

In[ ]:=
```
g[{x1_, x2_}] := x1 + x2
g[x]
```

Out[ ]=
```
5
```

In[ ]:=
```
Apply[f, x]
```

Out[ ]=
```
5
```

In[ ]:=
```
g[{2, 3}]
```

Out[ ]=
```
5
```

In[ ]:=
```
Clear["Global`*"];
```

In[∘]:=
```
Expand[(x + 1)^2]
```
Out[∘]=

$1 + 2x + x^2$

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
N[Sqrt[2], 30]
```
Out[∘]=

1.41421356237309504880168872421

In[∘]:=
```
pi2 = FromDigits[RealDigits[N[Pi], 2], 2]
pi10 = FromDigits[RealDigits[N[Pi], 10], 10]
Abs[Pi - pi2] < Abs[Pi - pi10] (* True *)
```
Out[∘]=

$$\frac{884\,279\,719\,003\,555}{281\,474\,976\,710\,656}$$

Out[∘]=

$$\frac{3\,141\,592\,653\,589\,793}{1\,000\,000\,000\,000\,000}$$

Out[∘]=

True

In[∘]:=
```
0.1 + 0.2 == 0.3
```
Out[∘]=

True

In[∘]:=
```
Chop[0.1 + 0.2 - 0.3] == 0
```
Out[∘]=

True

In[∘]:=
```
1/10 + 2/10 == 3/10
```
Out[∘]=

True

In[∘]:=
```
(* 「`」はシングルクォートではなくバッククォート *)
Block[{Internal`$EqualTolerance = 0.}, 0.1 + 0.2 == 0.3] (* False *)
```
Out[∘]=

False

In[∘]:=
```
Chop[0.1 + 0.2 - 0.3] == 0 (* True *)
```
Out[∘]=

True

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
Clear[x];
Simplify[Sin[x]^2 + Cos[x]^2]
```
Out[∘]=

1

In[ ]:=
```
FullSimplify[Sqrt[5 + 2 Sqrt[6]]]
```
Out[ ]=

$\sqrt{2} + \sqrt{3}$

In[ ]:=
```
Simplify[Sqrt[(x - 1)^2], x - 1 ≥ 0]
```
Out[ ]=

$-1 + x$

In[ ]:=
```
Reduce[Sqrt[(x - 1)^2] == x - 1, x, Reals]
```
Out[ ]=

$x \geq 1$

In[ ]:=
```
Clear[a, b];
Reduce[Sqrt[a] Sqrt[b] == Sqrt[a b], Reals]
```
Out[ ]=

b ≥ 0 && a ≥ 0

3 データ構造

```
In[]:= Clear["Global`*"];

In[]:= v = {2, 3, 5}; Length[v]
Out[]= 3

In[]:= v[[3]] = 0.5; v
Out[]= {2, 3, 0.5}

In[]:= Range[5]
Out[]= {1, 2, 3, 4, 5}

In[]:= Range[0, 1, 0.1]
Out[]= {0., 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.}

In[]:= Subdivide[0, 100, 4]
Out[]= {0, 25, 50, 75, 100}

In[]:= v = {2, 3};
       1.1 v
Out[]= {2.2, 3.3}

In[]:= u = {10, 20}; v = {2, 3};
       u + v
Out[]= {12, 23}

In[]:= v + 1
Out[]= {3, 4}

In[]:= u = {10, 20}; v = {2, 3};
       u . v
Out[]= 80

In[]:= a = {2, 3, 4}; b = a; b[[3]] = 0.5; a
Out[]= {2, 3, 4}
```

```
In[ ]:= v = {2, -1, 3, -2};
        Cases[v, x_ /; x > 0]     (* パターンマッチングによる抽出 *)
        Select[v, Function[x, x > 0]] (* 関数による抽出 *)
        Select[v, Positive]     (* 組込み関数の利用 *)
Out[ ]=
        {2, 3}
Out[ ]=
        {2, 3}
Out[ ]=
        {2, 3}

In[ ]:= v = {2, -1, 3, -2};
        UnitStep[v]
Out[ ]=
        {1, 0, 1, 0}

In[ ]:= v = {2, -1, 3, -2};
        n = Length[v];   (* vのサイズ *)
        u = Table[Null, n]; (* Nullは「値がない」ということ. *)
        Do[u[[i]] = If[v[[i]] < 0, 0, 1], {i, 1, n}];

In[ ]:= Table[If[x < 0, 0, 1], {x, v}]
Out[ ]=
        {1, 0, 1, 0}

In[ ]:= v = {2, -1, 3, -2};
        f = Function[x, If[x < 0, 0, 1]];
        Map[f, v]
Out[ ]=
        {1, 0, 1, 0}

In[ ]:= v = {2, -1, 3, -2};
        f = Function[x, If[x < 0, 0, 1], Listable];
        f[v]
Out[ ]=
        {1, 0, 1, 0}

In[ ]:= u = {1, 7, 2, 9}; v = {2, 3, 5, 7};
        f = Function[{a, b}, If[a < b, -1, 1]];
        MapThread[f, {u, v}]
Out[ ]=
        {-1, 1, -1, 1}

In[ ]:= Clear["Global`*"];

In[ ]:= x = <|"apple" → "りんご", "orange" → "みかん"|>;
        x["orange"]
Out[ ]=
        みかん
```

In[ ]:=
```
AppendTo[x, "grape" → "ぶどう"];
x["grape"]
```

Out[ ]=

ぶどう

In[ ]:=
```
x["apple"] =.
KeyExistsQ[x, "apple"]
```

Out[ ]=

False

In[ ]:=
```
Clear[x];
x["apple"] = "りんご";
x["orange"] = "みかん";

x["orange"]      (* みかん *)

x["grape"] = "ぶどう";
x["grape"]      (* ぶどう *)

x["apple"] =.
Head[x["apple"]] =!= x (* False *)
```

Out[ ]=

みかん

Out[ ]=

ぶどう

Out[ ]=

False

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
df = Transpose[Dataset[<|"name" → {"A", "B", "C"},
          "english" → {60, 90, 70},
          "math" → {70, 80, 90},
          "gender" → {"f", "m", "m"}|>]]
```

Out[ ]=

name	english	math	gender
A	60	70	f
B	90	80	m
C	70	90	m

In[ ]:=
```
df = Dataset[{
  <|"name" → "A", "english" → 60, "math" → 70, "gender" → "f"|>,
  <|"name" → "B", "english" → 90, "math" → 80, "gender" → "m"|>,
  <|"name" → "C", "english" → 70, "math" → 90, "gender" → "m"|>}]
```

Out[ ]=

name	english	math	gender
A	60	70	f
B	90	80	m
C	70	90	m

In[ ]:=
```
df[All, {"english", "math"}]
```

Out[ ]=

english	math
60	70
90	80
70	90

In[ ]:=
```
Normal[df[[All, "english"]]]
```

Out[ ]=
```
{60, 90, 70}
```

In[ ]:=
```
m = Values[Normal[df[[All, {2, 3}]]]]
```

Out[ ]=
```
{{60, 70}, {90, 80}, {70, 90}}
```

In[ ]:=
```
{english, math} = Transpose[m]
```

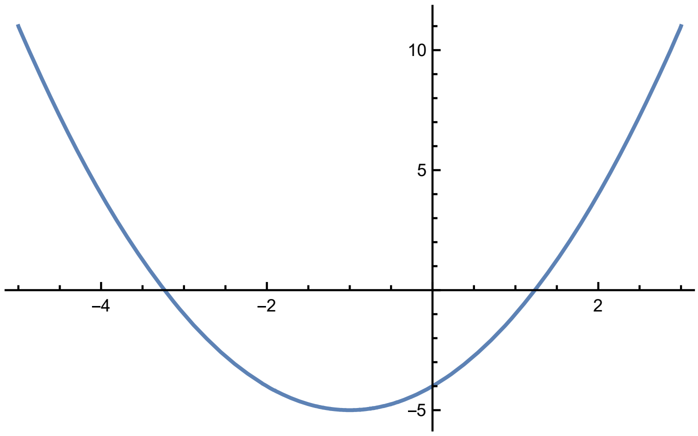
Out[ ]=
```
{{60, 90, 70}, {70, 80, 90}}
```

4 可視化と方程式

In[ ]:=
```
Clear["Global`*"];
```

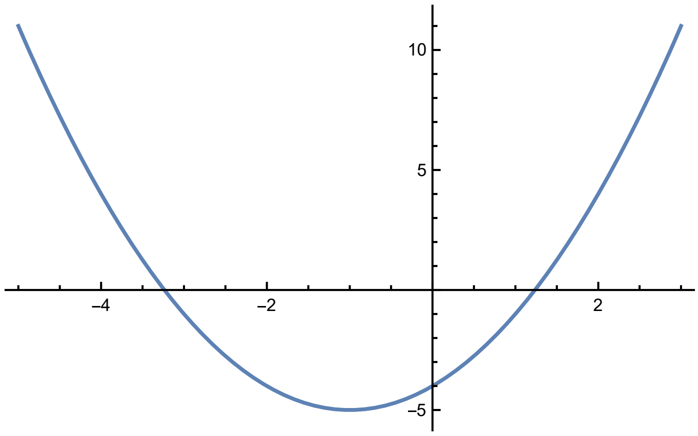
In[ ]:=
```
Plot[x^2 + 2 x - 4, {x, -5, 3}]
```

Out[ ]=



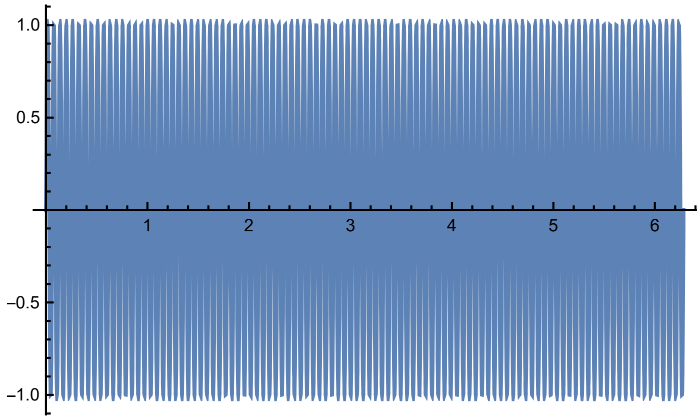
In[ ]:=
```
x = Subdivide[-5, 3, 100];
y = x^2 + 2 x - 4;
ListLinePlot[Transpose[{x, y}]]
```

Out[ ]=



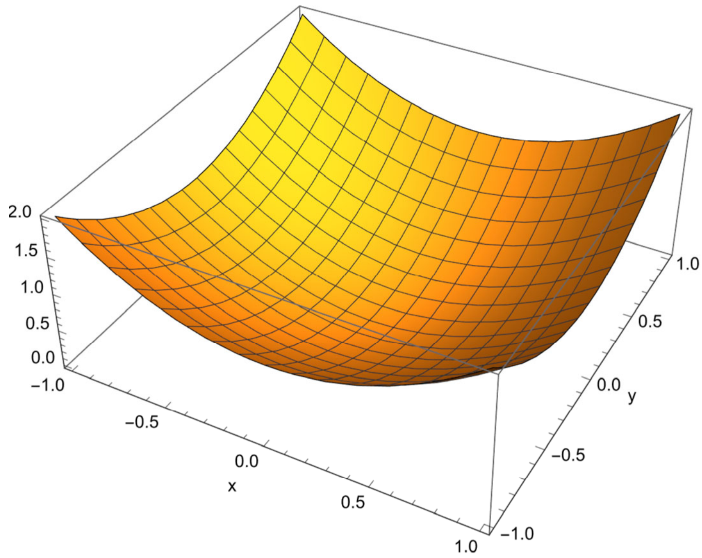
In[ ]:=
```
Plot[Sin[102 x], {x, 0, 2 Pi}, PlotPoints → 100]
```

Out[ ]=



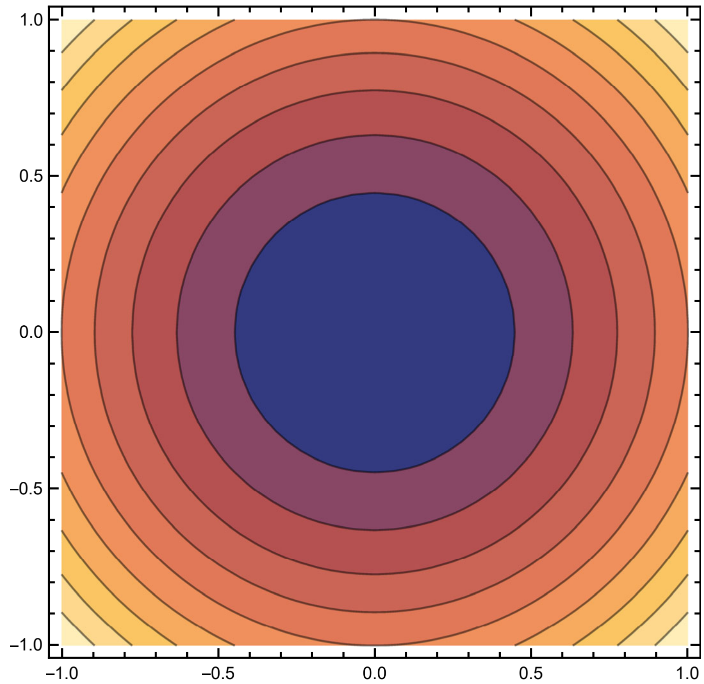
In[ ]:=
```
Plot3D[x^2 + y^2, {x, -1, 1}, {y, -1, 1}, AxesLabel → {"x", "y"}]
```

Out[ ]=



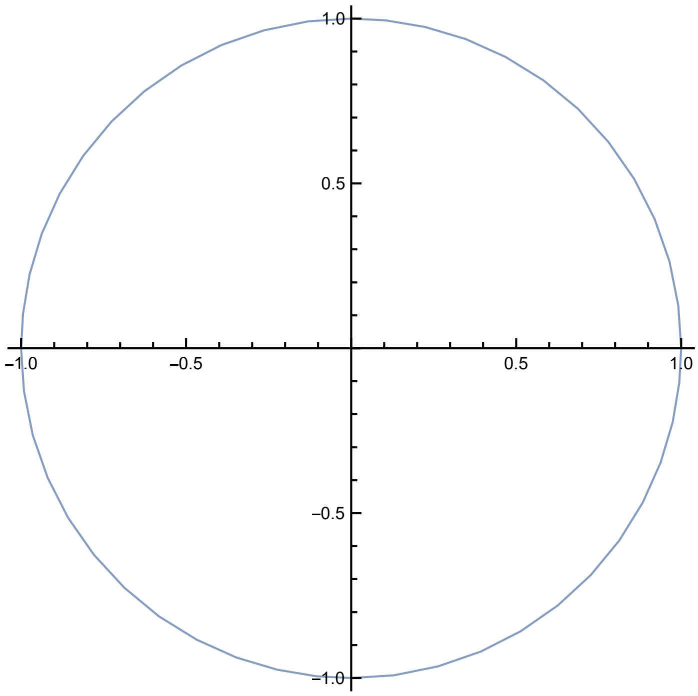
In[ ]:=
```
ContourPlot[x^2 + y^2, {x, -1, 1}, {y, -1, 1}]
```

Out[ ]=



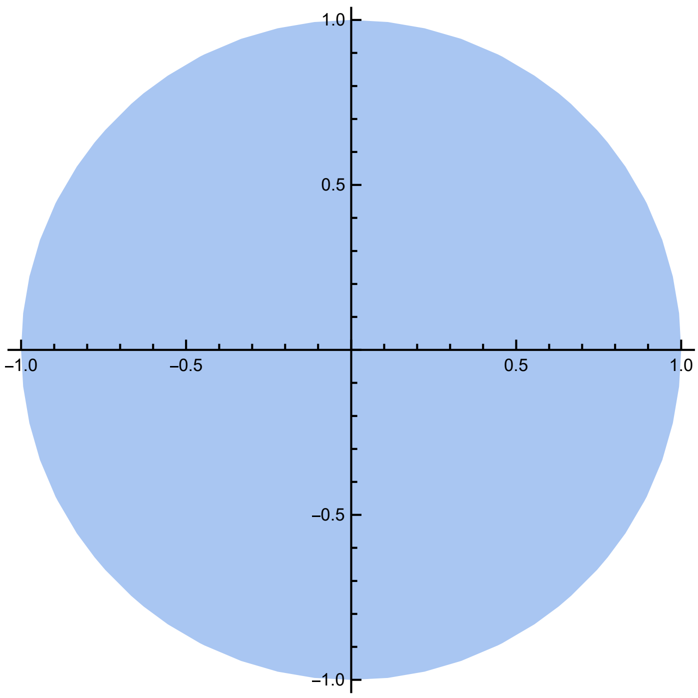
In[◦]:=
```
Clear[x, y];
reg1 = ImplicitRegion[x^2 + y^2 == 1, {x, y}];
Region[reg1, Axes → True]
```

Out[◦]=



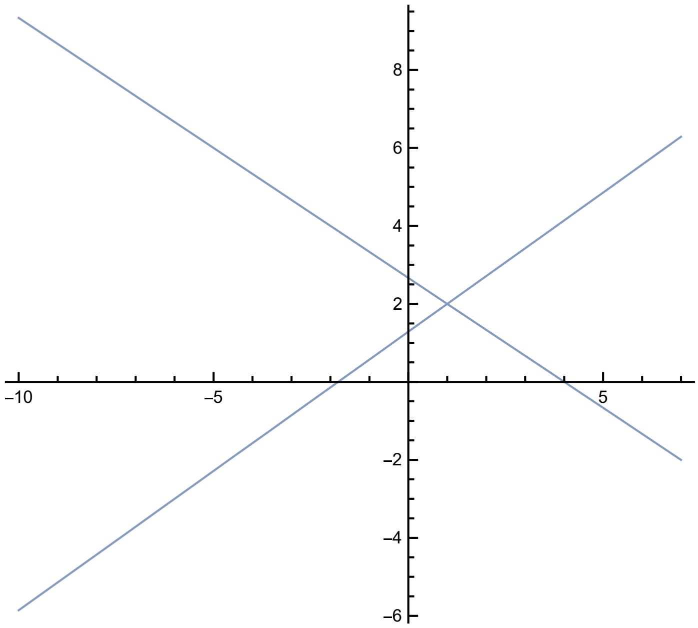
In[◦]:=
```
reg2 = ImplicitRegion[x^2 + y^2 ≤ 1, {x, y}];
Region[reg2, Axes → True]
```

Out[◦]=



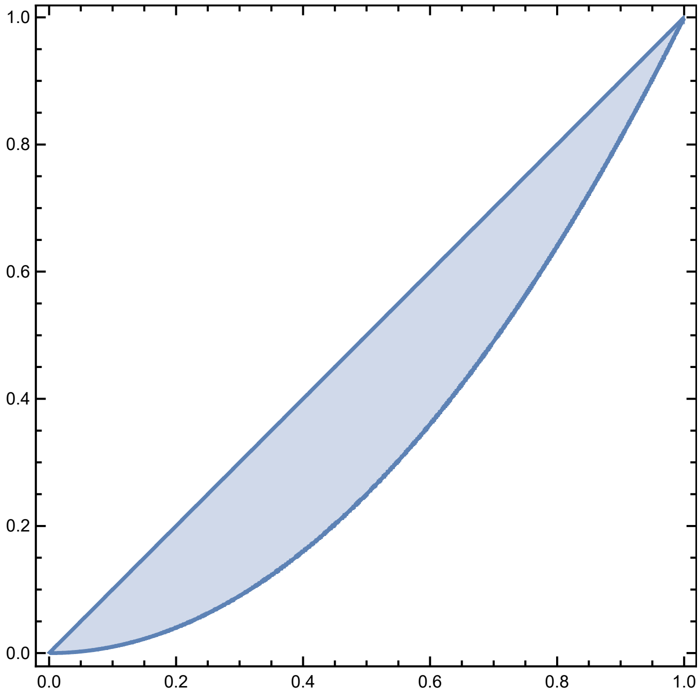
In[ ]:=
```
reg = ImplicitRegion[Or[2 x + 3 y == 8, 5 x - 7 y == -9], {x, y}];
Region[reg, Axes → True]
```

Out[ ]=

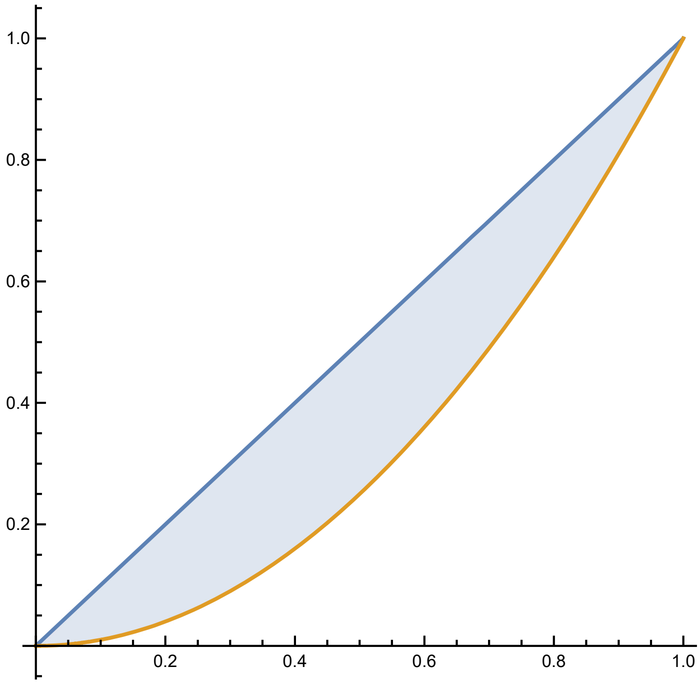


In[ ]:=
```
reg = ImplicitRegion[And[y ≤ x, y ≥ x^2], {x, y}];
RegionPlot[reg]                          (* ① *)
Plot[{x, x^2}, {x, 0, 1}, Filling → {1 → {2}}, AspectRatio → 1] (* ② *)
```

Out[ ]=



Out[ ]=



In[ ]:=
```
{RegionMeasure[reg1], RegionMeasure[reg2]}
```

Out[ ]=

$\{2\pi, \pi\}$

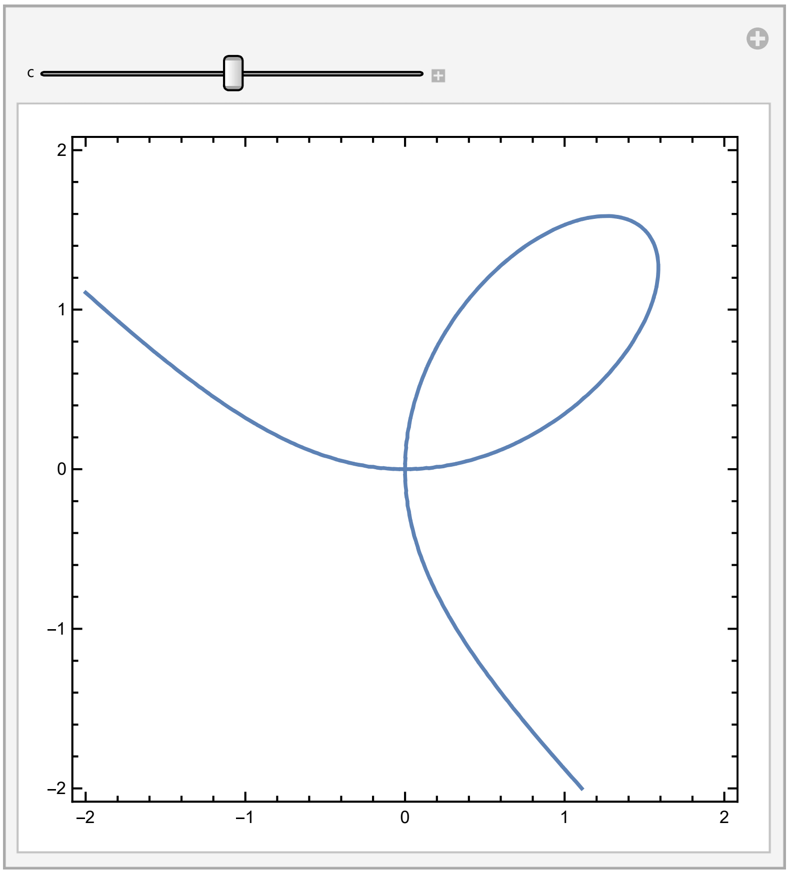
In[∘]:=
```
RegionMeasure[reg]
```
Out[∘]=

$$\frac{1}{6}$$

In[∘]:=
```
Clear[x];
{a, b} = Sort[SolveValues[{x == x^2}, x]];
Integrate[x - x^2, {x, a, b}]
```
Out[∘]=

$$\frac{1}{6}$$

In[∘]:=
```
Manipulate[
 ContourPlot[x^3 + y^3 - 3 x y == c, {x, -2, 2}, {y, -2, 2}],
 {{c, 0}, -1, 1}] (* cは-1以上1以下で, 初期値は0 *)
```
Out[∘]=



In[∘]:=
```
Clear["Global`*"];
```
In[∘]:=
```
SolveValues[x^2 + 2 x - 4 == 0, x]
```
Out[∘]=

$$\left\{-1-\sqrt{5},\ -1+\sqrt{5}\right\}$$

In[•]:=
```
{a, b} = SolveValues[x^2 + 2 x - 4 == 0, x]
a + b

tmp = Solve[x^2 + 2 x - 4 == 0, x]
{a, b} = x /. tmp;
a + b

tmp = Reduce[x^2 + 2 x - 4 == 0, x]
{a, b} = x /. {ToRules[tmp]};
a + b
```

Out[•]=

$\{-1-\sqrt{5}, -1+\sqrt{5}\}$

Out[•]=

-2

Out[•]=

$\{\{x \to -1-\sqrt{5}\}, \{x \to -1+\sqrt{5}\}\}$

Out[•]=

-2

Out[•]=

$x == -1-\sqrt{5} \;||\; x == -1+\sqrt{5}$

Out[•]=

-2

In[•]:=
```
n = 3; Simplify[Total[SolveValues[x^n + 2 x - 4 == 0, x]]]
```

Out[•]=

0

In[•]:=
```
Clear["Global`*"];
```

In[•]:=
```
sol = SolveValues[{2 x + 3 y == 8, 5 x - 7 y == -9}, {x, y}]
```

Out[•]=

{{1, 2}}

In[•]:=
```
{{x1, y1}} = sol; x1 + y1
```

Out[•]=

3

In[•]:=
```
Clear["Global`*"];
```

In[•]:=
```
f[x_] := 2^x + Sin[x]
FindRoot[f[x] == 0, {x, 0}]
```

Out[•]=

$\{x \to -0.676182\}$

In[•]:=
```
Clear["Global`*"];
```

In[•]:=
```
Reduce[x^2 + 2 x - 4 < 0, x]
```

Out[•]=

$-1-\sqrt{5} < x < -1+\sqrt{5}$

5 論理式

```
In[•]:= Clear["Global`*"];

In[•]:= expr = Exists[x, Element[x, Reals], x^2 == 2];
        Reduce[expr]

Out[•]= True

In[•]:= Reduce[Implies[x > 10, x > 11]]

Out[•]= x ≤ 10 || x > 11

In[•]:= Reduce[ForAll[x, Element[x, Reals], Implies[x > 10, x > 11]]]

Out[•]= False

In[•]:= BooleanConvert[Implies[A, B], "OR"] (* 含意 *)

Out[•]= !A || B

In[•]:= BooleanConvert[And[A, B], "OR"] (* 論理積 *)

Out[•]= ! (!A || !B)

In[•]:= {BooleanConvert[Not[A]] == BooleanConvert[Nand[A, A]],
         BooleanConvert[Or[A, B]] == BooleanConvert[Nand[Not[A], Not[B]]]}

Out[•]= {True, True}

In[•]:= Clear["Global`*"];

In[•]:= Reduce[Exists[x, Element[x, Reals], x^2 == 2]]

Out[•]= True

In[•]:= Reduce[Exists[x, x^2 == 2], Reals]

Out[•]= True

In[•]:= Reduce[Exists[x, Element[x, Rationals], x^2 == 2]] (* False *)
        Reduce[Exists[x, x^2 == 2], Rationals]      (* False *)

Out[•]= False

Out[•]= False

In[•]:= Clear["Global`*"];
```

In[ ]:=
```
expr = ForAll[b, Element[b, Reals], Exists[n, Element[n, Integers], n > b]];
Reduce[expr]
```

Out[ ]=
```
True
```

In[ ]:=
```
expr1 = ForAll[b,
  Element[b, Reals], Exists[n, And[Element[n, Integers], n > b]]];
Reduce[expr1] (* True *)

expr2 = ForAll[b,
  Implies[Element[b, Reals], Exists[n, Element[n, Integers], n > b]]];
Reduce[expr2] (* 失敗 *)
```

Out[ ]=
```
True
```

Reduce: -- Message text not found --

Out[ ]=

Reduce$[\forall_b \ (b \in \mathbb{R} \Rightarrow \exists_{n, n\in\mathbb{Z}} \ n > b)]$

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
Reduce[Exists[x, a x + b == 0]]
```

Out[ ]=

$(b == 0 \,\&\&\, a == 0) \,||\, a \neq 0$

In[ ]:=
```
Reduce[Exists[x, Element[x, Reals], x^2 + a^2 == 0]]
```

Out[ ]=

$\mathrm{Re}[a] == 0$

In[ ]:=
```
Reduce[Exists[x, Element[x, Reals], x^2 + a^2 < 0]] (* False *)
```

Out[ ]=
```
False
```

In[ ]:=
```
Reduce[Exists[x, Element[x, Reals], x^2 + a^2 < 0], Complexes] // Simplify
```

Out[ ]=

$\mathrm{Re}[a] == 0 \,\&\&\, \mathrm{Im}[a] \neq 0$

In[ ]:=
```
Reduce[Not[Exists[{n, a, b, c}, And[n ≥ 3, a^n + b^n == c^n]]],
 PositiveIntegers]
```

Out[ ]=
```
True
```

In[ ]:=
```
Reduce[Not[Exists[{a, b, c}, a^4 + b^4 == c^4]], PositiveIntegers]
```

Out[ ]=
```
True
```

In[ ]:=
```
Reduce[Not[Exists[{a, b, c}, a^4 + b^4 == c^2]], PositiveIntegers] (* 失敗 *)
```

Reduce: -- Message text not found --

Out[ ]=

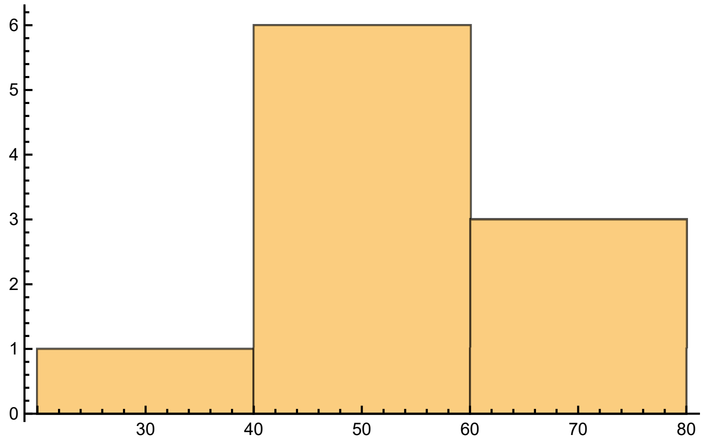
Reduce$[\forall_{\{a,b,c\}} \ a^4 + b^4 \neq c^2, \mathbb{Z}_{>0}]$

6 1次元のデータ

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
a = {36, 43, 53, 55, 56, 56, 57, 60, 61, 73};
b = {34, 39, 39, 49, 50, 52, 52, 55, 83, 97};
Histogram[a]
```

Out[∘]=



In[∘]:=
```
HistogramList[a, {20, 80, 20}]
```

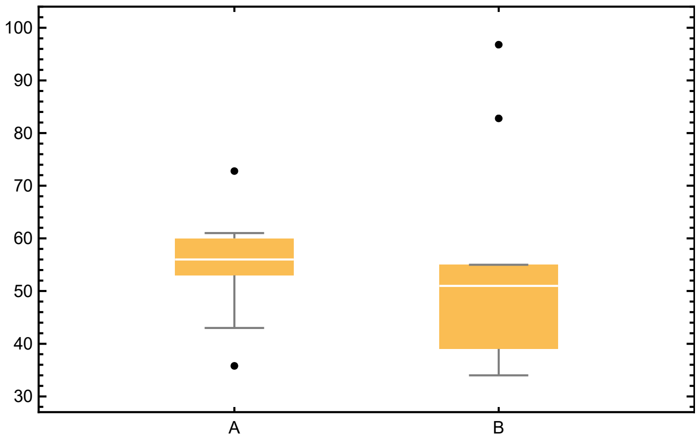
Out[∘]=
```
{{20, 40, 60, 80}, {1, 6, 3}}
```

In[∘]:=
```
x = {7, 3, 1, 3, 4, 7, 7, 7, 10, 3};
f = Counts[x]
```

Out[∘]=
```
<|7 → 4, 3 → 3, 1 → 1, 4 → 1, 10 → 1|>
```

In[∘]:=
```
BoxWhiskerChart[{a, b}, "Outliers", ChartLabels → {"A", "B"}]
```

Out[∘]=



In[∘]:=
```
Clear["Global`*"];
```

In[ ]:=
```
a = {36, 43, 53, 55, 56, 56, 57, 60, 61, 73};
Mean[a]
```
Out[ ]=
```
55
```
In[ ]:=
```
b = {34, 39, 39, 49, 50, 52, 52, 55, 83, 97};
Total[b] / Length[b]
```
Out[ ]=
```
55
```
In[ ]:=
```
Mean[a - Mean[a]]
```
Out[ ]=
```
0
```
In[ ]:=
```
Variance[a]
```
Out[ ]=
```
100
```
In[ ]:=
```
Total[(b - Mean[b])^2] / (Length[b] - 1) // N
```
Out[ ]=
```
397.778
```
In[ ]:=
```
z = Standardize[a]
```
Out[ ]=

$$\left\{-\frac{19}{10}, -\frac{6}{5}, -\frac{1}{5}, 0, \frac{1}{10}, \frac{1}{10}, \frac{1}{5}, \frac{1}{2}, \frac{3}{5}, \frac{9}{5}\right\}$$

In[ ]:=
```
{Mean[z], StandardDeviation[z]}
```
Out[ ]=
```
{0, 1}
```
In[ ]:=
```
(a - Mean[a]) / StandardDeviation[a]
```
Out[ ]=

$$\left\{-\frac{19}{10}, -\frac{6}{5}, -\frac{1}{5}, 0, \frac{1}{10}, \frac{1}{10}, \frac{1}{5}, \frac{1}{2}, \frac{3}{5}, \frac{9}{5}\right\}$$

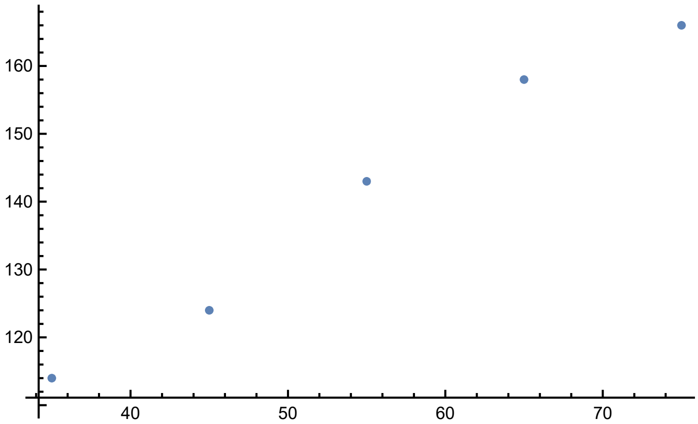
In[ ]:=
```
StandardDeviation[a] z + Mean[a]
```
Out[ ]=
```
{36, 43, 53, 55, 56, 56, 57, 60, 61, 73}
```
In[ ]:=
```
10 * z + 50
```
Out[ ]=
```
{31, 38, 48, 50, 51, 51, 52, 55, 56, 68}
```

7 2次元のデータ

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
x = {35, 45, 55, 65, 75}; y = {114, 124, 143, 158, 166};
ListPlot[Transpose[{x, y}]]
```

Out[ ]=



In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
x = {35, 45, 55, 65, 75}; y = {114, 124, 143, 158, 166};
Covariance[x, y]
```

Out[ ]=
```
345
```

In[ ]:=
```
Covariance[Transpose[{x, y}]]
```

Out[ ]=
```
{{250, 345}, {345, 484}}
```

In[ ]:=
```
(x - Mean[x]) . (y - Mean[y]) / (Length[x] - 1)
```

Out[ ]=
```
345
```

In[ ]:=
```
Correlation[x, y] // N
```

Out[ ]=
```
0.991805
```

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
x = {35, 45, 55, 65, 75}; y = {114, 124, 143, 158, 166};
data = Thread[{x, y}]; (* x, yを列とする行列 *)
model = LinearModelFit[data, X, X]
```

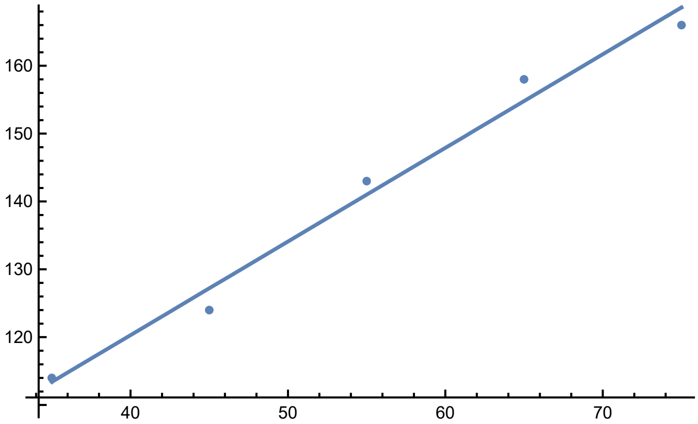
Out[ ]=
```
FittedModel[ 65.1 + 1.38 X ]
```

In[ ]:=
```
model[40]
```

Out[ ]=
```
120.3
```

In[ ]:=
```
Show[ListPlot[data], Plot[model[x], {x, 35, 75}]]
```

Out[ ]=



In[ ]:=
```
L = Total[(y - (a x + b))^2]
```

Out[ ]=

$$(166 - 75\,a - b)^2 + (158 - 65\,a - b)^2 + (143 - 55\,a - b)^2 + (124 - 45\,a - b)^2 + (114 - 35\,a - b)^2$$

In[ ]:=
```
sol = SolveAlways[L == p (a - q)^2 + r (b - (s a + t))^2 + u, {a, b}]
{q, s q + t} /. sol[[1]]
```

Out[ ]=

$$\left\{\left\{u \to \frac{158}{5},\ p \to 1000,\ q \to \frac{69}{50},\ s \to -55,\ t \to 141,\ r \to 5\right\}\right\}$$

Out[ ]=

$$\left\{\frac{69}{50}, \frac{651}{10}\right\}$$

In[ ]:=
```
a = Covariance[x, y]/Variance[x]; b = Mean[y] - a Mean[x];
{a, b} // N
```

Out[ ]=

{1.38, 65.1}

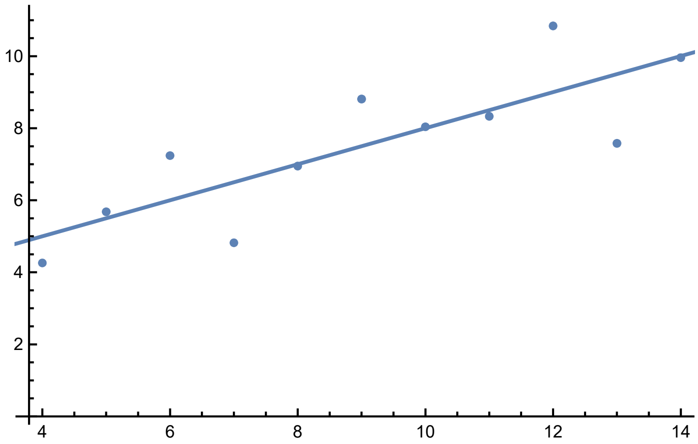
In[ ]:=
```
Clear["Global`*"];
```

In[∘]:=
```
anscombe = ExampleData[{"Statistics", "AnscombeRegressionLines"}];
x1 = anscombe⟦All, 1⟧; y1 = anscombe⟦All, 5⟧; data = Thread[{x1, y1}];
Correlation[x1, y1]
model = LinearModelFit[data, X, X]
Show[ListPlot[data], Plot[model[x], {x, 0, 21}]]
```

Out[∘]=
```
0.816421
```

Out[∘]=
```
FittedModel[ 3. + 0.5 X ]
```

Out[∘]=



8 確率変数と確率分布

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
dist = DiscreteUniformDistribution[{1, 6}];
PDF[dist][2]
```

Out[∘]=

$$\frac{1}{6}$$

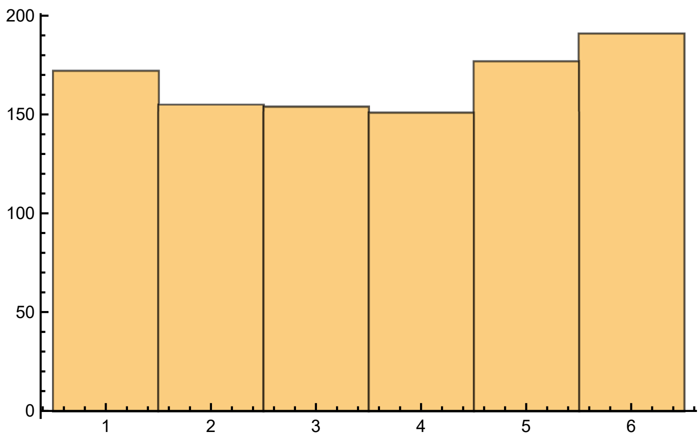
In[∘]:=
```
Probability[X == 2, Distributed[X, dist]]
```

Out[∘]=

$$\frac{1}{6}$$

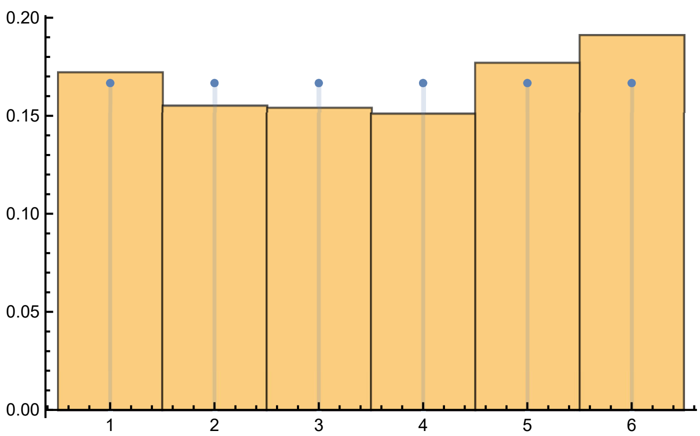
In[∘]:=
```
data = RandomVariate[dist, 1000];
Histogram[data] (* 結果は割愛 *)
```

Out[∘]=



In[∘]:=
```
Show[Histogram[data, {0.5, 6.5, 1}, "PDF"],
 DiscretePlot[PDF[dist][x], {x, 1, 6}]]
```

Out[∘]=



```
In[ ]:= dist = BernoulliDistribution[3/10];
        data = RandomVariate[dist, 1000];
        Counts[data]
Out[ ]=
        <|0 → 711, 1 → 289|>
```

```
In[ ]:= dist = BinomialDistribution[10, 3/10];
        PDF[dist][3]
Out[ ]=
```

$$\frac{66\,706\,983}{250\,000\,000}$$

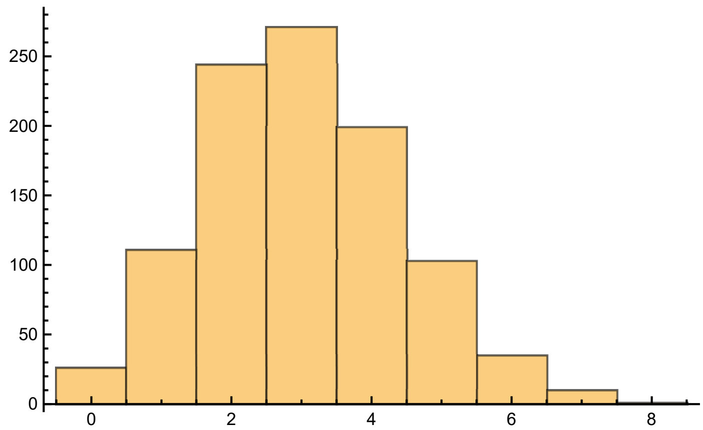
```
In[ ]:= Probability[X == 3, Distributed[X, dist]]
Out[ ]=
```

$$\frac{66\,706\,983}{250\,000\,000}$$

```
In[ ]:= dist = BinomialDistribution[n, p];
        PDF[dist]
Out[ ]=
```

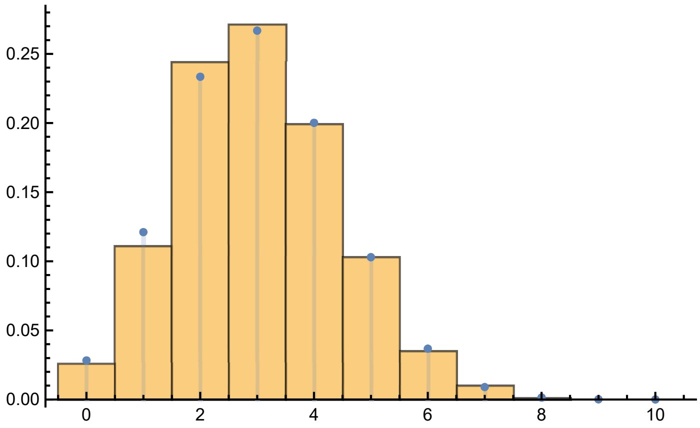
$$\text{Function}\left[x, \begin{cases} (1-p)^{-x+n}\, p^{x}\, \text{Binomial}[n, x] & 0 \le x \le n \\ 0 & \text{True} \end{cases}, \text{Listable}\right]$$

```
In[ ]:= n = 10; p = 3/10; dist = BinomialDistribution[n, p];
        data = RandomVariate[dist, 1000];
        Histogram[data] (* 結果は割愛 *)
Out[ ]=
```



In[ ]:=
```
Show[Histogram[data, {-0.5, n + 0.5, 1}, "PDF"],
 DiscretePlot[PDF[dist][x], {x, 0, n}]]
```

Out[ ]=



In[ ]:=
```
dist = BinomialDistribution[10, 3/10];
CDF[dist][3]
```

Out[ ]=

$$\frac{406\,006\,699}{625\,000\,000}$$

In[ ]:=
```
Probability[X ≤ 3, Distributed[X, dist]]
```

Out[ ]=

$$\frac{406\,006\,699}{625\,000\,000}$$

In[ ]:=
```
Sum[PDF[dist][k], {k, 0, 3}]
```

Out[ ]=

$$\frac{406\,006\,699}{625\,000\,000}$$

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
dist = UniformDistribution[{0, 360}];
{CDF[dist][200], CDF[dist][150], CDF[dist][200] - CDF[dist][150]}
```

Out[ ]=

$$\left\{\frac{5}{9}, \frac{5}{12}, \frac{5}{36}\right\}$$

In[ ]:=
```
Probability[150 ≤ X ≤ 200, Distributed[X, dist]]
```

Out[ ]=

$$\frac{5}{36}$$

In[ ]:=
```
Integrate[PDF[dist][x], {x, 150, 200}]
```

Out[ ]=

$$\frac{5}{36}$$

In[∘]:=
```
Integrate[PDF[dist][t], {t, 0, x},
 Assumptions → Element[x, Reals]] (* xは実数と仮定する. *)
```

Out[∘]=

$$\begin{cases} 1 & x > 360 \\ \frac{x}{360} & 0 < x \leq 360 \\ 0 & \text{True} \end{cases}$$

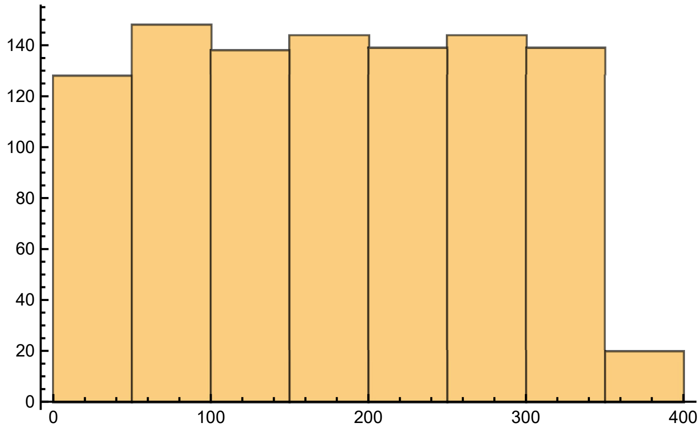
In[∘]:=
```
D[x / 360, x]
```

Out[∘]=

$$\frac{1}{360}$$

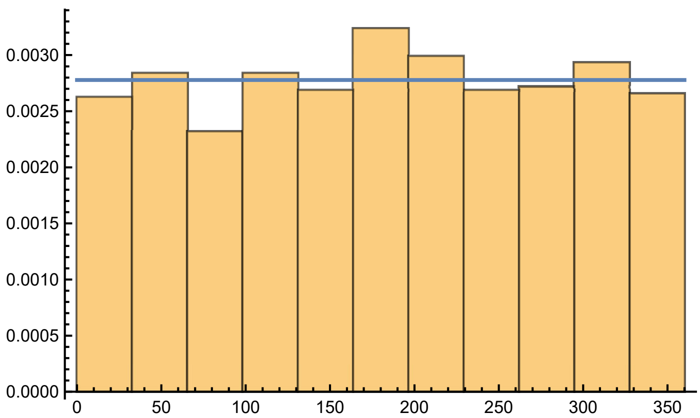
In[∘]:=
```
data = RandomVariate[dist, 1000];
Histogram[data] (* 結果は割愛 *)
```

Out[∘]=



In[∘]:=
```
data = RandomVariate[dist, 1000];
Show[Histogram[data, {"Raw", "Sturges"}, "PDF"],
 Plot[PDF[dist][x], {x, 0, 360}]]
```

Out[∘]=



In[∘]:=
```
dist = NormalDistribution[6, 2];
CDF[dist][6 + 3×2] - CDF[dist][6 - 3×2] // N
```

Out[∘]=

0.9973

In[ ]:=
```
Probability[6 - 3×2 ≤ X ≤ 6 + 3×2, Distributed[X, dist]] // N
```
Out[ ]=
```
0.9973
```

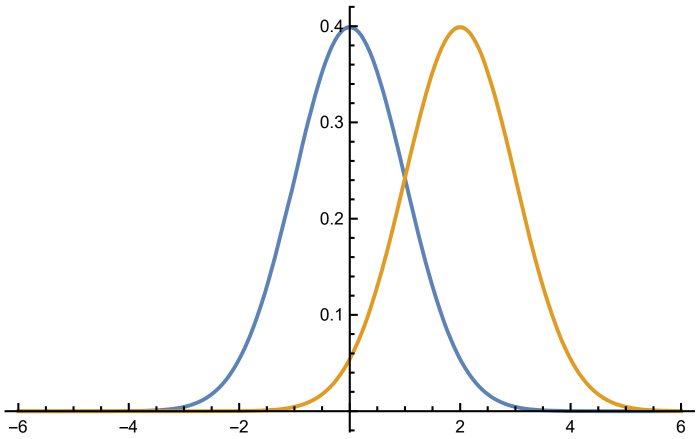
In[ ]:=
```
Integrate[PDF[dist][x], {x, 6 - 3×2, 6 + 3×2}] // N
```
Out[ ]=
```
0.9973
```

In[ ]:=
```
Clear[mu, sigma, x];
dist = NormalDistribution[mu, sigma];
{a, b} = {mu - 3 sigma, mu + 3 sigma};
CDF[dist][b] - CDF[dist][a] // N          (* 方法1 *)
Probability[a ≤ X ≤ b, Distributed[X, dist]] // N (* 方法2 *)
Integrate[PDF[dist][x], {x, a, b}] // N      (* 方法3 *)
```
Out[ ]=
```
0.9973
```
Out[ ]=
```
0.9973
```
Out[ ]=
```
0.9973
```

In[ ]:=
```
dist = NormalDistribution[mu, sigma];
PDF[dist][x]
```
Out[ ]=

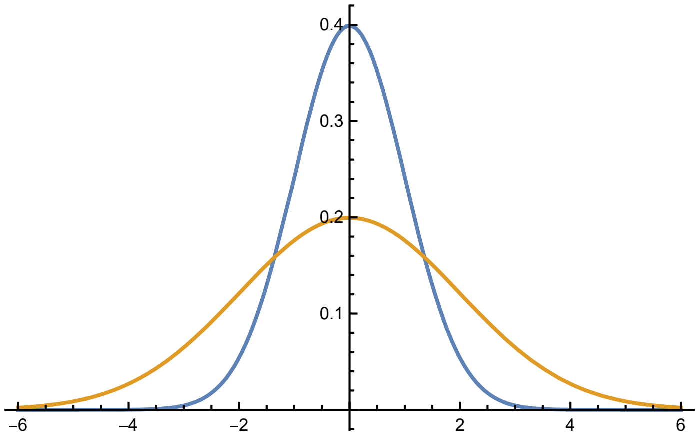
$$\frac{e^{-\frac{(-mu+x)^2}{2\,sigma^2}}}{\sqrt{2\pi}\,sigma}$$

In[ ]:=
```
Plot[{PDF[NormalDistribution[0, 1]][x],
  PDF[NormalDistribution[2, 1]][x]}, {x, -6, 6}]
```
Out[ ]=



In[ ]:=
```
Plot[{PDF[NormalDistribution[0, 1]][x],
  PDF[NormalDistribution[0, 2]][x]}, {x, -6, 6}]
```

Out[ ]=



In[ ]:=
```
Clear["Global`*"];
```

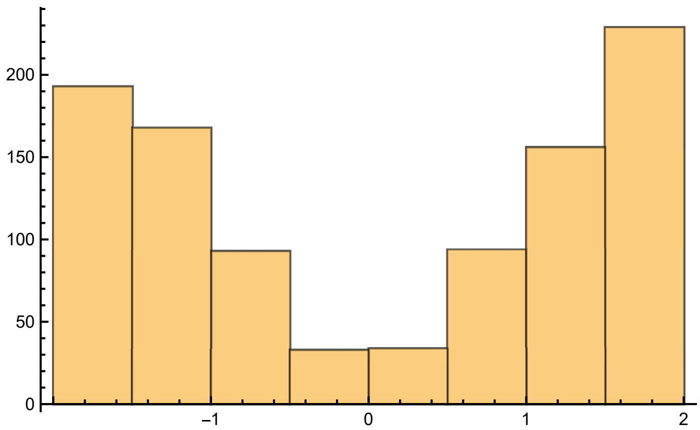
In[ ]:=
```
Xs = {0, 100, 1000, 10000}; Ps = {0.9, 0.08, 0.015, 0.005};
tmp = Piecewise[Thread[{Ps, Thread[x == Xs]}]];
dist = ProbabilityDistribution[tmp, {x, 0, 10000, 1}, (* 確率分布の定義 *)
  Method → "Normalize"]; (* 念のため合計を1にする. *)
data = RandomVariate[dist, 1000];
Counts[data]
```

Out[ ]=

```
<|0 → 909, 100 → 71, 1000 → 14, 10000 → 6|>
```

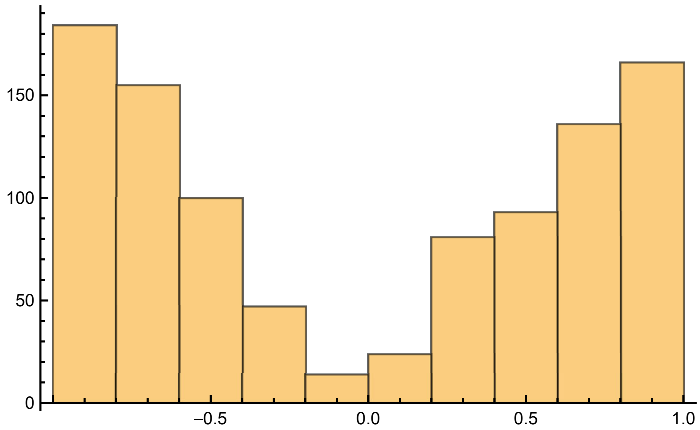
In[ ]:=
```
dist = ProbabilityDistribution[Abs[x], {x, -2, 2}, (* 確率分布の定義 *)
  Method → "Normalize"]; (*“全確率”が1にならない場合の備え *)
data = RandomVariate[dist, 1000];
Histogram[data]
```

Out[ ]=



In[ ]:=
```
Clear[f, F, t, x];
f[x_] := RealAbs[x]                    (* 手順1 *)
F[x_] := Evaluate[Integrate[f[t], {t, -1, x}]] (* 手順2 *)
Finv = InverseFunction[F];             (* 手順3 *)
data = Table[Finv[RandomReal[]], {1000}];   (* 手順4 *)
Histogram[data]                    (* 手順5 *)
```

Out[ ]=



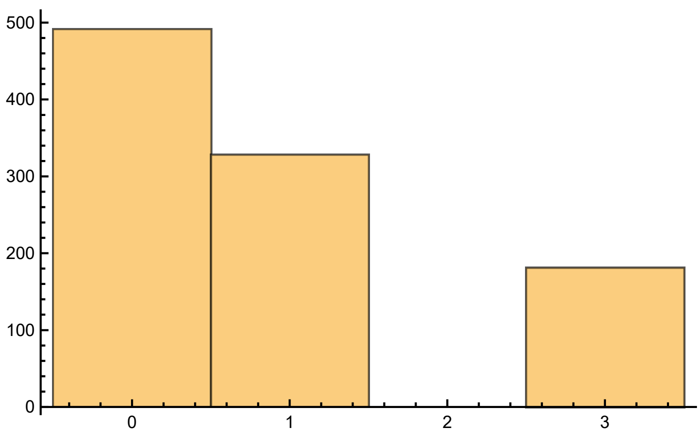
In[ ]:=
```
distY = UniformDistribution[{0, 1}];
distX = TransformedDistribution[
  Piecewise[{{-Sqrt[1 - 2 Y], Y ≤ 1/2}}, Sqrt[-1 + 2 Y]],
  Distributed[Y, distY]];
PDF[distX]
```

Out[ ]=

$$\text{Function}\left[x,\ \begin{cases} -x & -1 < x \leq 0 \\ x & 0 < x \leq 1 \\ 0 & \text{True} \end{cases},\ \text{Listable}\right]$$

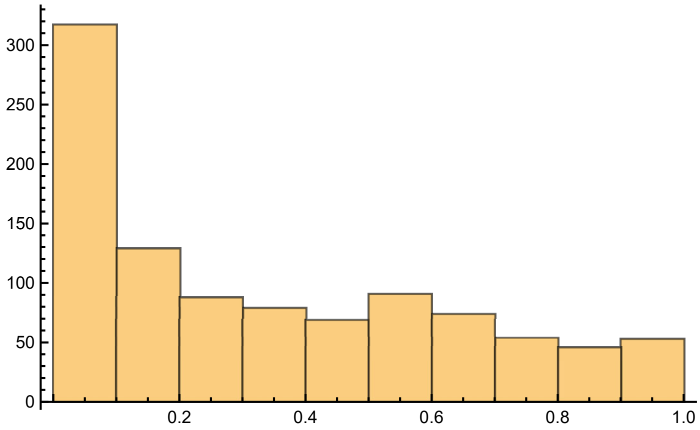
In[ ]:=
```
distX = DiscreteUniformDistribution[{1, 6}];
distY = TransformedDistribution[Mod[X^3, 4], Distributed[X, distX]];
data = RandomVariate[distY, 1000];
Histogram[data]
```

Out[ ]=



In[∘]:=
```
distX = UniformDistribution[{0, 1}];
distY = TransformedDistribution[X^2, Distributed[X, distX]];
data = RandomVariate[distY, 1000];
Histogram[data]
```

Out[∘]=



In[∘]:=
```
PDF[distY]
```

Out[∘]=

Function$\left[x, \begin{cases} \frac{1}{2\sqrt{x}} & 0 < x < 1 \\ 0 & \text{True} \end{cases}, \text{Listable}\right]$

In[∘]:=
```
distX = ProbabilityDistribution[Abs[x], {x, -1, 1}];
distY = TransformedDistribution[X^2, Distributed[X, distX]];
PDF[distY]
```

Out[∘]=

Function$\left[x, \begin{cases} 1 & 0 < x < 1 \\ 0 & x > 1 \,||\, x < 0 \\ \text{Indeterminate} & \text{True} \end{cases}, \text{Listable}\right]$

In[∘]:=
```
dist = NormalDistribution[mu, sigma]; Clear[a, b];
TransformedDistribution[a X + b, Distributed[X, dist]]
```

Out[∘]=

```
NormalDistribution[b + a mu, sigma Abs[a]]
```

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
Xs = {0, 100, 1000, 10000}; Ps = {0.9, 0.08, 0.015, 0.005};
tmp = Piecewise[Thread[{Ps, Thread[x == Xs]}]];
dist = ProbabilityDistribution[tmp, {x, 0, 10000, 1}];
Expectation[X, Distributed[X, dist]]
```

Out[∘]=

73.

In[∘]:=
```
Mean[dist]
```

Out[∘]=

73.

In[∘]:=
```
Sum[x PDF[dist][x], {x, Xs}]
```
Out[∘]=
```
73.
```

In[∘]:=
```
Xs . Ps
```
Out[∘]=
```
73.
```

In[∘]:=
```
Mean[RandomVariate[dist, 500000]] // N
```
Out[∘]=
```
72.1082
```

In[∘]:=
```
Clear[n, p];
dist = BinomialDistribution[n, p];
Expectation[X, Distributed[X, dist]]   (* 方法1 *)
Mean[dist]                 (* 方法2 *)
Sum[x PDF[dist][x], {x, 0, n}] // Simplify (* 方法3 *)
```
Out[∘]=
```
n p
```
Out[∘]=
```
n p
```
Out[∘]=

$$\begin{cases} n\,p & n \geq 1 \\ 0 & \text{True} \end{cases}$$

In[∘]:=
```
dist = ProbabilityDistribution[Abs[x], {x, -1, 1}];
Integrate[x PDF[dist][x], {x, -1, 1}]
```
Out[∘]=
```
0
```

In[∘]:=
```
Xs = {0, 100, 1000, 10000}; Ps = {0.9, 0.08, 0.015, 0.005};
tmp = Piecewise[Thread[{Ps, Thread[x == Xs]}]];
dist = ProbabilityDistribution[tmp, {x, 0, 10000, 1}];
Variance[dist]
```
Out[∘]=
```
510471.
```

In[∘]:=
```
Expectation[(X - Mean[dist])^2, Distributed[X, dist]]
```
Out[∘]=
```
510471.
```

In[∘]:=
```
Sum[(x - Mean[dist])^2 PDF[dist][x], {x, Xs}]
```
Out[∘]=
```
510471.
```

In[∘]:=
```
((Xs - Xs . Ps)^2) . Ps
```
Out[∘]=
```
510471.
```

In[ ]:=
```
Clear[n, p];
dist = BinomialDistribution[n, p];
Variance[dist]                          (* 方法1 *)
Expectation[(X - Mean[dist])^2, Distributed[X, dist]]    (* 方法2 *)
Sum[(x - Mean[dist])^2 PDF[dist][x], {x, 0, n}] // Simplify (* 方法3 *)
```

Out[ ]=

$n (1 - p) p$

Out[ ]=

$n p - n p^2$

Out[ ]=

$-n (-1 + p) p$

In[ ]:=
```
dist = ProbabilityDistribution[Abs[x], {x, -1, 1}];
Integrate[(x - Mean[dist])^2 PDF[dist][x], {x, -1, 1}]
```

Out[ ]=

$\frac{1}{2}$

9 多次元の確率分布

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
distX = DiscreteUniformDistribution[{1, 6}];
dist = TransformedDistribution[{Max[X1, X2], Min[X1, X2]},
  {Distributed[X1, distX], Distributed[X2, distX]}];
probs = Table[{
  Probability[{X, Y} == {x, y}, Distributed[{X, Y}, dist]], (* 確率 *)
  {X, Y} == {x, y}},                    (* 条件 *)
 {x, 1, 6}, {y, 1, 6}]

dist = ProbabilityDistribution[Piecewise[Flatten[probs, 1]], (* 作り直し *)
  {X, 1, 6, 1}, {Y, 1, 6, 1}];
```

Out[∘]=

$\{\{\{\frac{1}{36}, \{X, Y\} == \{1, 1\}\}, \{0, \{X, Y\} == \{1, 2\}\}, \{0, \{X, Y\} == \{1, 3\}\},$
$\{0, \{X, Y\} == \{1, 4\}\}, \{0, \{X, Y\} == \{1, 5\}\}, \{0, \{X, Y\} == \{1, 6\}\}\},$
$\{\{\frac{1}{18}, \{X, Y\} == \{2, 1\}\}, \{\frac{1}{36}, \{X, Y\} == \{2, 2\}\}, \{0, \{X, Y\} == \{2, 3\}\},$
$\{0, \{X, Y\} == \{2, 4\}\}, \{0, \{X, Y\} == \{2, 5\}\}, \{0, \{X, Y\} == \{2, 6\}\}\},$
$\{\{\frac{1}{18}, \{X, Y\} == \{3, 1\}\}, \{\frac{1}{18}, \{X, Y\} == \{3, 2\}\}, \{\frac{1}{36}, \{X, Y\} == \{3, 3\}\},$
$\{0, \{X, Y\} == \{3, 4\}\}, \{0, \{X, Y\} == \{3, 5\}\}, \{0, \{X, Y\} == \{3, 6\}\}\},$
$\{\{\frac{1}{18}, \{X, Y\} == \{4, 1\}\}, \{\frac{1}{18}, \{X, Y\} == \{4, 2\}\}, \{\frac{1}{18}, \{X, Y\} == \{4, 3\}\},$
$\{\frac{1}{36}, \{X, Y\} == \{4, 4\}\}, \{0, \{X, Y\} == \{4, 5\}\}, \{0, \{X, Y\} == \{4, 6\}\}\},$
$\{\{\frac{1}{18}, \{X, Y\} == \{5, 1\}\}, \{\frac{1}{18}, \{X, Y\} == \{5, 2\}\}, \{\frac{1}{18}, \{X, Y\} == \{5, 3\}\},$
$\{\frac{1}{18}, \{X, Y\} == \{5, 4\}\}, \{\frac{1}{36}, \{X, Y\} == \{5, 5\}\}, \{0, \{X, Y\} == \{5, 6\}\}\},$
$\{\{\frac{1}{18}, \{X, Y\} == \{6, 1\}\}, \{\frac{1}{18}, \{X, Y\} == \{6, 2\}\}, \{\frac{1}{18}, \{X, Y\} == \{6, 3\}\},$
$\{\frac{1}{18}, \{X, Y\} == \{6, 4\}\}, \{\frac{1}{18}, \{X, Y\} == \{6, 5\}\}, \{\frac{1}{36}, \{X, Y\} == \{6, 6\}\}\}\}$

In[ ]:=
```
PDF[MarginalDistribution[dist, 1]][x] // Simplify
PDF[MarginalDistribution[dist, 2]][y] // Simplify
```

Out[ ]=

$$\begin{cases} \frac{1}{36} & x == 1 \\ \frac{1}{12} & x == 2 \\ \frac{5}{36} & x == 3 \\ \frac{7}{36} & x == 4 \\ \frac{1}{4} & x == 5 \\ \frac{11}{36} & x == 6 \\ 0 & \text{True} \end{cases}$$

Out[ ]=

$$\begin{cases} \frac{11}{36} & y == 1 \\ \frac{1}{4} & y == 2 \\ \frac{7}{36} & y == 3 \\ \frac{5}{36} & y == 4 \\ \frac{1}{12} & y == 5 \\ \frac{1}{36} & y == 6 \\ 0 & \text{True} \end{cases}$$

In[ ]:=
```
Table[CDF[dist][{x, y}], {y, 1, 6}, {x, 1, 6}] // TableForm
```

Out[ ]//TableForm=

$\frac{1}{36}$	$\frac{1}{12}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{1}{4}$	$\frac{11}{36}$
$\frac{1}{36}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$
$\frac{1}{36}$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{3}{4}$
$\frac{1}{36}$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{4}{9}$	$\frac{2}{3}$	$\frac{8}{9}$
$\frac{1}{36}$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{4}{9}$	$\frac{25}{36}$	$\frac{35}{36}$
$\frac{1}{36}$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{4}{9}$	$\frac{25}{36}$	1

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
c = Counts[Flatten[Table[{Max[x, y], Min[x, y]}, {x, 1, 6}, {y, 1, 6}], 1]]/36;
dist = ProbabilityDistribution[Piecewise[KeyValueMap[{#2, {X, Y} == #1} &, c]],
  {X, 1, 6, 1}, {Y, 1, 6, 1}];

Mean[dist]        (* 平均 *)
Variance[dist]      (* 分散 *)
StandardDeviation[dist]  (* 標準偏差 *)
Covariance[dist][[1, 2]]  (* 共分散 *)
Correlation[dist][[1, 2]] (* 相関係数 *)
```

Out[ ]=
$$\left\{\frac{161}{36}, \frac{91}{36}\right\}$$

Out[ ]=
$$\left\{\frac{2555}{1296}, \frac{2555}{1296}\right\}$$

Out[ ]=
$$\left\{\frac{\sqrt{2555}}{36}, \frac{\sqrt{2555}}{36}\right\}$$

Out[ ]=
$$\frac{1225}{1296}$$

Out[ ]=
$$\frac{35}{73}$$

In[ ]:=
```
{uX, uY} = Mean[dist]; {sX, sY} = StandardDeviation[dist];
Expectation[{X, Y,      (* 平均 *)
  (X - uX)^2, (Y - uY)^2,  (* 分散 *)
  (X - uX) (Y - uY),     (* 共分散 *)
  (X - uX) (Y - uY) / sX / sY}, (* 相関係数 *)
 Distributed[{X, Y}, dist]]
```

Out[ ]=
$$\left\{\frac{161}{36}, \frac{91}{36}, \frac{2555}{1296}, \frac{2555}{1296}, \frac{1225}{1296}, \frac{35}{73}\right\}$$

In[ ]:=
```
Sum[x Probability[X == x, Distributed[{X, Y}, dist]], {x, 1, 6}] (* 平均 *)
Sum[(x - uX) (y - uY) PDF[dist][{x, y}], {x, 1, 6}, {y, 1, 6}] (* 共分散 *)
```

Out[ ]=
$$\frac{161}{36}$$

Out[ ]=
$$\frac{1225}{1296}$$

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
dist = DiscreteUniformDistribution[{1, 6}];
Probability[Conditioned[X == 2, X ≤ 3], Distributed[X, dist]]
```
Out[ ]=

$$\frac{1}{3}$$

In[ ]:=
```
c = Counts[Flatten[Table[{Max[x, y], Min[x, y]}, {x, 1, 6}, {y, 1, 6}], 1]]/36;
dist = ProbabilityDistribution[Piecewise[KeyValueMap[{#2, {X, Y} == #1} &, c]],
  {X, 1, 6, 1}, {Y, 1, 6, 1}];
rule = Distributed[{X, Y}, dist];

Table[Probability[Conditioned[X == x, Y == 3], rule], {x, 1, 6}]
```
Out[ ]=

$$\left\{0, 0, \frac{1}{7}, \frac{2}{7}, \frac{2}{7}, \frac{2}{7}\right\}$$

In[ ]:=
```
Table[
 Probability[And[X == x, Y == 3], rule]/Probability[Y == 3, rule], {x, 1, 6}]
```
Out[ ]=

$$\left\{0, 0, \frac{1}{7}, \frac{2}{7}, \frac{2}{7}, \frac{2}{7}\right\}$$

In[ ]:=
```
Expectation[Conditioned[X, Y == 3], rule]
```
Out[ ]=

$$\frac{33}{7}$$

In[ ]:=
```
Sum[x Probability[Conditioned[X == x, Y == 3], rule], {x, 1, 6}]
```
Out[ ]=

$$\frac{33}{7}$$

In[ ]:=
```
Table[Probability[And[X ≤ x, Y ≤ y], rule], {x, 1, 6}, {y, 1, 6}] ==
 Table[Probability[X ≤ x, rule] × Probability[Y ≤ y, rule],
 {x, 1, 6}, {y, 1, 6}]
```
Out[ ]=

False

In[ ]:=
```
distU = DiscreteUniformDistribution[{1, 6}];
distXY = TransformedDistribution[{Mod[U, 2], Mod[U, 3]},
  Distributed[U, distU]];
rule = Distributed[{X, Y}, distXY];
Table[Probability[And[X ≤ x, Y ≤ y], rule], {x, 0, 1}, {y, 0, 2}] ==
 Table[Probability[X ≤ x, rule] × Probability[Y ≤ y, rule],
 {x, 0, 1}, {y, 0, 2}]
```
Out[ ]=

True

In[ ]:=
```
distX = BinomialDistribution[3, 1/2];
distXY = TransformedDistribution[
  {X, Piecewise[{{1, Or[X == 0, X == 3]}}, 2]}, Distributed[X, distX]];
Covariance[distXY][[1, 2]]
```
Out[ ]=
```
0
```

In[ ]:=
```
rule = Distributed[{X, Y}, distXY];
Table[Probability[And[X ≤ x, Y ≤ y], rule], {x, 0, 3}, {y, 1, 2}] ==
 Table[Probability[X ≤ x, rule] × Probability[Y ≤ y, rule],
  {x, 0, 3}, {y, 1, 2}]
```
Out[ ]=
```
False
```

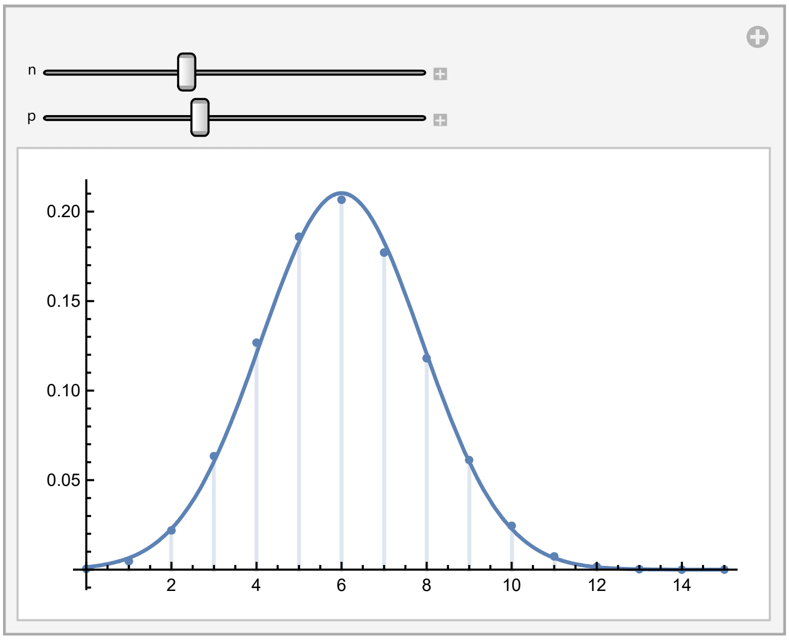
In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
c = Counts[Flatten[Table[{Max[x, y], Min[x, y]}, {x, 1, 6}, {y, 1, 6}], 1]]/36;
dist = ProbabilityDistribution[Piecewise[KeyValueMap[{#2, {X, Y} == #1} &, c]],
  {X, 1, 6, 1}, {Y, 1, 6, 1}];
rule = Distributed[{X, Y}, dist];

{Expectation[X + Y, rule],
 Expectation[X, rule] + Expectation[Y, rule]} (* 平均1 *)

{distX, distY} = Table[MarginalDistribution[dist, i], {i, 2}];
distXplusY = TransformedDistribution[X + Y, rule];

{Mean[distXplusY], Mean[distX] + Mean[distY]} (* 平均2 *)
{Variance[distXplusY],
 Variance[distX] + Variance[distY] + 2 Covariance[dist][[1, 2]]} (* 分散 *)
```
Out[ ]=
```
{7, 7}
```
Out[ ]=
```
{7, 7}
```
Out[ ]=

$\left\{\frac{35}{6}, \frac{35}{6}\right\}$

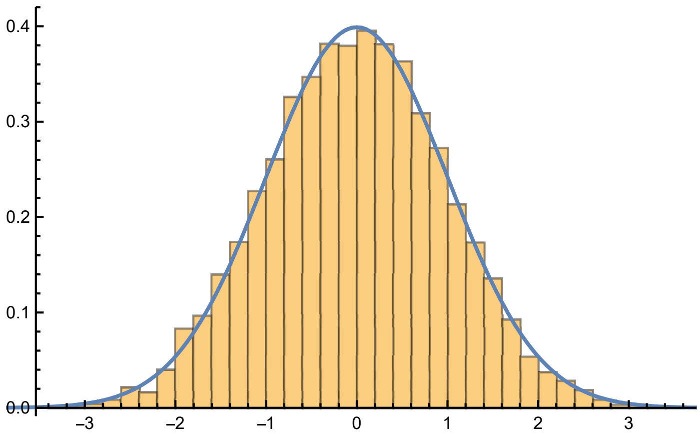
In[ ]:=
```
Manipulate[
 distY = BinomialDistribution[n, p];
 mu = Mean[distY]; sigma = StandardDeviation[distY];
 distZ = NormalDistribution[mu, sigma];
 Show[DiscretePlot[PDF[distY][x], {x, 0, n}], Plot[PDF[distZ][x], {x, 0, n}]],
 {{n, 15}, 1, 40, 1}, {{p, 4/10}, 0, 1}]
```

Out[ ]=



In[ ]:=
```
distX = UniformDistribution[]; distZ = NormalDistribution[];
data = Table[Total[RandomVariate[distX, 12]] - 6, {10000}];
Show[Histogram[data, Automatic, "PDF"], Plot[PDF[distZ][x], {x, -4, 4}]]
```

Out[ ]=



In[ ]:=
```
Clear["Global`*"];
```

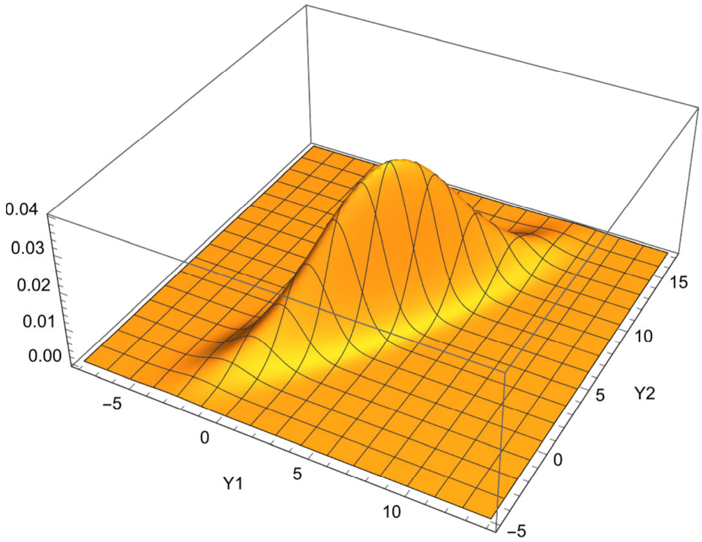
In[ ]:=
```
dist1 = NormalDistribution[0, 2]; dist2 = NormalDistribution[1, 1];
TransformedDistribution[{X1 + X2 + 2, X1 + 3 X2 + 3},
 {Distributed[X1, dist1], Distributed[X2, dist2]}]
```

Out[ ]=

```
MultinormalDistribution[{3, 6}, {{5, 7}, {7, 13}}]
```

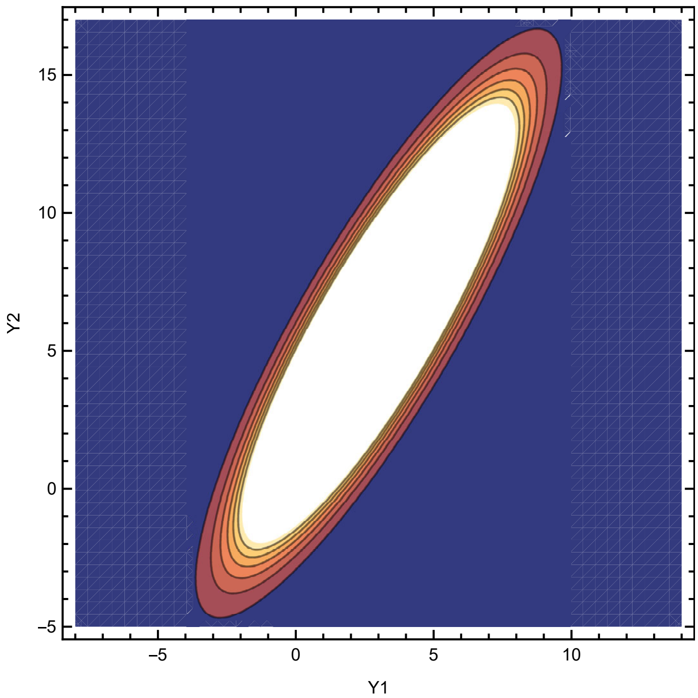
In[ ]:=
```
mu = {3, 6}; Sigma = {{5, 7}, {7, 13}};
dist = MultinormalDistribution[mu, Sigma];
Plot3D[PDF[dist][{Y1, Y2}], {Y1, -8, 14}, {Y2, -5, 17},
 PlotPoints → 100, PlotRange → All, AxesLabel → Automatic]
```

Out[ ]=



In[ ]:=
```
ContourPlot[PDF[dist][{Y1, Y2}], {Y1, -8, 14}, {Y2, -5, 17},
 PlotPoints → 50, FrameLabel → Automatic]
```

Out[ ]=



In[ ]:=
```
sol = Solve[{Y1 == X1 + X2 + 2, Y2 == X1 + 3 X2 + 3}, {X1, X2}][[1]]
```

Out[ ]=

$$\left\{X1 \to \frac{1}{2}\ (-3 + 3\ Y1 - Y2),\ X2 \to \frac{1}{2}\ (-1 - Y1 + Y2)\right\}$$

In[ ]:=
```
J = D[{X1, X2} /. sol, {{Y1, Y2}}];
absj = Abs[Det[J]]
```

Out[ ]=

$\frac{1}{2}$

In[ ]:=
```
f1 = PDF[dist1]; f2 = PDF[dist2];
PDF[dist][{Y1, Y2}] == f1[X1] × f2[X2] absj /. sol // Simplify
```

Out[ ]=

True

In[ ]:=
```
{MarginalDistribution[dist, 1], MarginalDistribution[dist, 2]}
```

Out[ ]=

$\left\{\text{NormalDistribution}\left[3, \sqrt{5}\right], \text{NormalDistribution}\left[6, \sqrt{13}\right]\right\}$

In[ ]:=
```
dist = MultinormalDistribution[{u1, u2}, {{v1, 0}, {0, v2}}];
d1 = MarginalDistribution[dist, 1]; d2 = MarginalDistribution[dist, 2];
Simplify[CDF[dist][{x1, x2}] == CDF[d1][x1] × CDF[d2][x2],
 And[v1 ≥ 0, v2 ≥ 0]]
```

Out[ ]=

True

10 推測統計

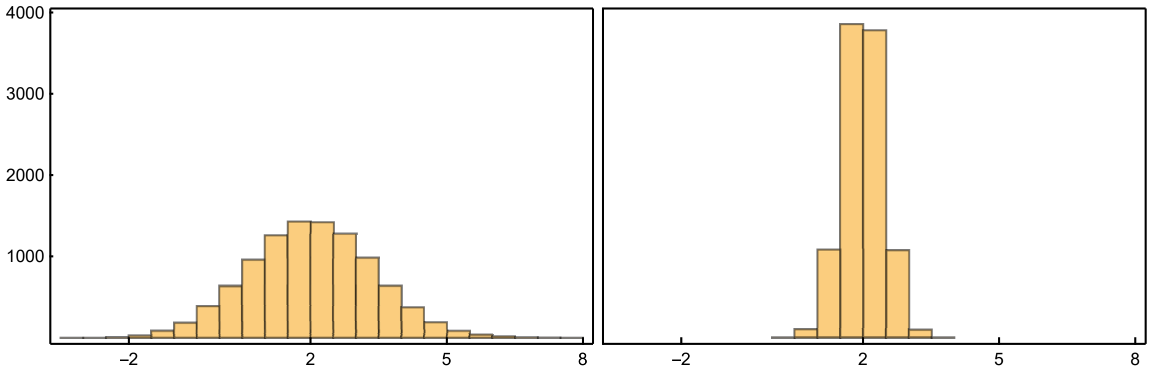
In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
dist = NormalDistribution[2, 3];
data1 = Table[Mean[RandomVariate[dist, 5]], 10000];
data2 = Table[Mean[RandomVariate[dist, 50]], 10000];
{{Mean[data1], Variance[data1]}, {Mean[data2], Variance[data2]}}
```

Out[∘]=
```
{{2.01413, 1.83253}, {1.99802, 0.182192}}
```

In[∘]:=
```
Histogram[{data1, data2}, ChartLayout → "Row"]
```

Out[∘]=

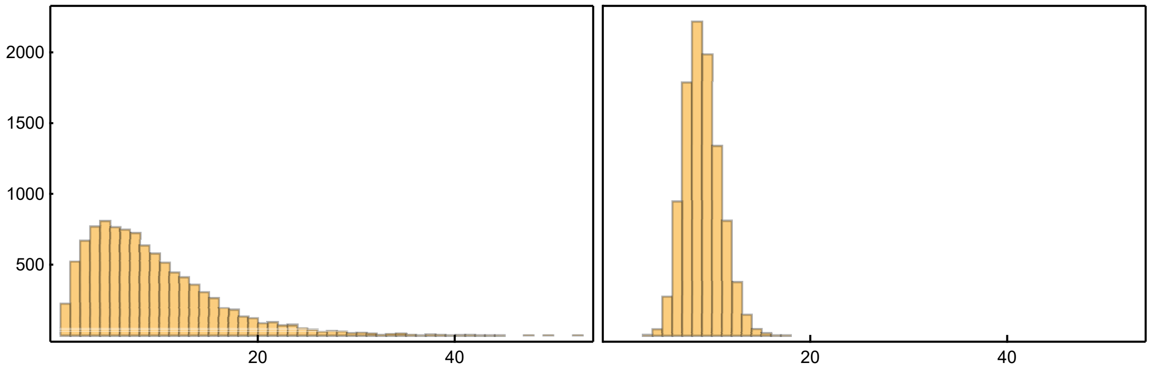


In[∘]:=
```
dist = NormalDistribution[2, 3];
data1 = Table[Variance[RandomVariate[dist, 5]], 10000];
data2 = Table[Variance[RandomVariate[dist, 50]], 10000];
{{Mean[data1], Variance[data1]}, {Mean[data2], Variance[data2]}}
```

Out[∘]=
```
{{9.05559, 40.3025}, {9.00464, 3.26093}}
```

In[∘]:=
```
Histogram[{data1, data2}, ChartLayout → "Row"]
```

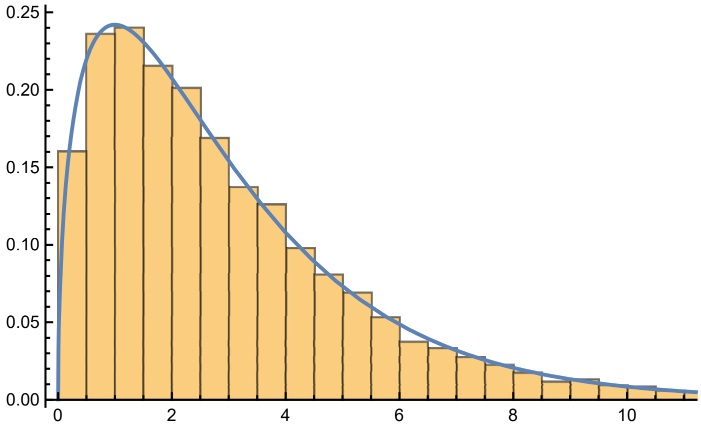
Out[∘]=



In[∘]:=
```
Clear["Global`*"];
```

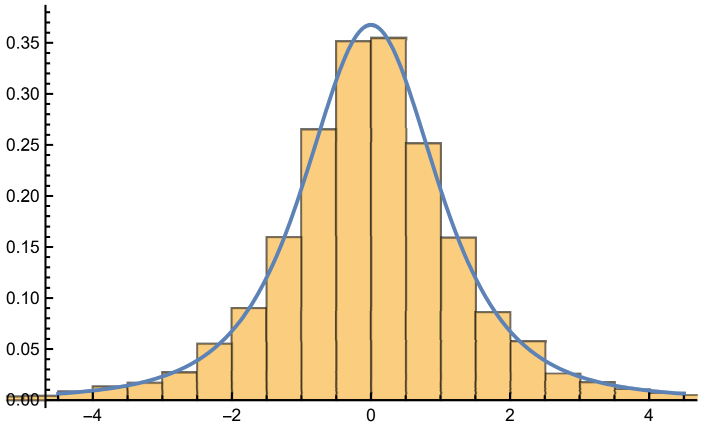
In[∘]:=
```
n = 4; mu = 5; sigma = 7; dist := NormalDistribution[mu, sigma];
f[x_] := (n - 1) Variance[x] / sigma^2
data = Table[f[RandomVariate[dist, n]], 10000];
Show[Histogram[data, Automatic, "PDF"],
 Plot[PDF[ChiSquareDistribution[n - 1]][x], {x, 0, Max[data]}]]
```

Out[∘]=



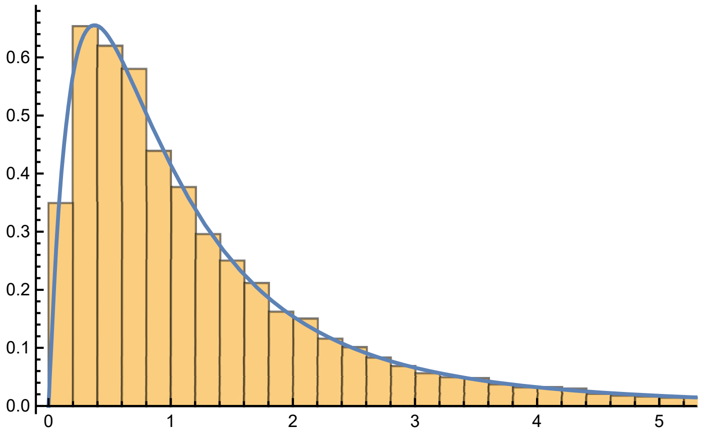
In[∘]:=
```
n = 4; mu = 5; sigma = 7; ndist = NormalDistribution[mu, sigma];
t = Function[{x}, (Mean[x] - mu) / Sqrt[Variance[x] / n]];
data = Table[t[RandomVariate[ndist, n]], 10000];
Show[Histogram[data, Automatic, "PDF"],
 Plot[PDF[StudentTDistribution[n - 1]][x], {x, -4.5, 4.5}]]
```

Out[∘]=



In[∘]:=
```
m = 5; muX = 2; sigmaX = 3; distX = NormalDistribution[muX, sigmaX];
n = 7; muY = 3; sigmaY = 2; distY = NormalDistribution[muY, sigmaY];
f[x_, y_] := (Variance[x] / sigmaX^2) / (Variance[y] / sigmaY^2)
data = Table[f[RandomVariate[distX, m], RandomVariate[distY, n]], {10000}];
Show[Histogram[data, Automatic, "PDF"],
 Plot[PDF[FRatioDistribution[m - 1, n - 1]][x], {x, 0, 7}]]
```

Out[∘]=



In[∘]:=
```
Clear[k, T];
TransformedDistribution[T^2, Distributed[T, StudentTDistribution[k]]]
```

Out[∘]=

```
FRatioDistribution[1, k]
```

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
n = 15; p0 = 4 / 10; dist = BinomialDistribution[n, p0];
tmp = Table[PDF[dist][x], {x, 0, n}];
Total[Cases[tmp, p_ /; p ≤ PDF[dist][2]]] // N
```

Out[∘]=

```
0.0364617
```

In[∘]:=
```
CDF[dist][2] // N
```

Out[∘]=

```
0.027114
```

In[∘]:=
```
n = 15; p0 = 4 / 10; dist = NormalDistribution[n p, Sqrt[n p (1 - p)]];
2 CDF[dist /. p → p0][2] // N
```

Out[∘]=

```
0.035015
```

In[∘]:=
```
alpha = 5 / 100; InverseCDF[dist /. {p → p0}, {alpha / 2, 1 - alpha / 2}] // N
```

Out[∘]=

```
{2.28123, 9.71877}
```

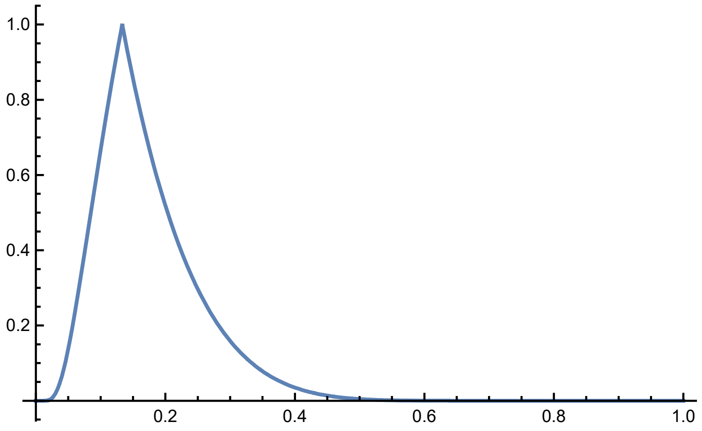
In[∘]:=
```
N[Reduce[InverseCDF[dist, alpha / 2] ≤ 2 ≤ InverseCDF[dist, 1 - alpha / 2], p]]
```

Out[∘]=

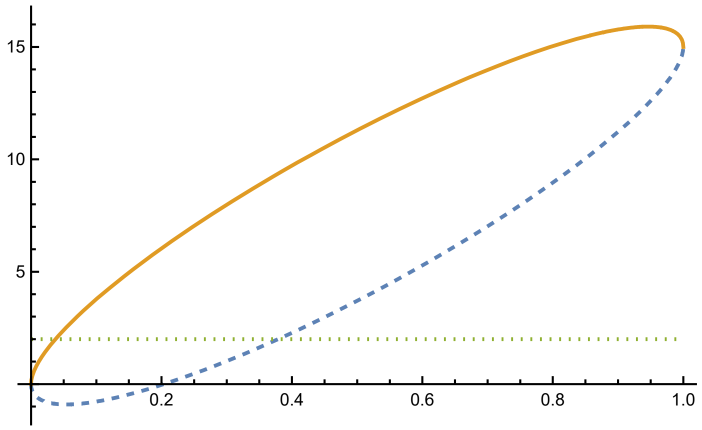
```
0.0373613 ≤ p ≤ 0.37882
```

In[∘]:=
```
pvalue[p0_] := With[{c = CDF[dist][2] /. p → p0}, 2 Min[c, 1 - c]]
Plot[pvalue[p0], {p0, 0, 1}]

Plot[{InverseCDF[dist, alpha / 2], InverseCDF[dist, 1 - alpha / 2], 2},
 {p, 0, 1}, PlotStyle → {Dashed, Thick, Dotted}]
```

Out[∘]=



Out[∘]=



In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
x = {24.2, 25.3, 26.2, 25.7, 24.4, 25.1, 25.6}; mu0 = 25;
TTest[x, mu0]
```

Out[∘]=

```
0.458101
```

In[∘]:=
```
m = Mean[x]; s2 = Variance[x]; n = Length[x];
t := (m - mu0) / Sqrt[s2 / n];
dist = StudentTDistribution[n - 1]; c = CDF[dist][t];
2 Min[c, 1 - c]
```

Out[∘]=

```
0.458101
```

In[∘]:=
```
alpha = 5 / 100;
{a, b} = InverseCDF[dist, {alpha / 2, 1 - alpha / 2}] // N
```

Out[∘]=

```
{-2.44691, 2.44691}
```

In[ ]:=
```
Needs["HypothesisTesting`"] (* 「`」はシングルクォートではなくバッククォート *)
MeanCI[x]
```

Out[ ]=
```
{24.5529, 25.8757}
```

In[ ]:=
```
Clear[mu0]; Reduce[a ≤ t ≤ b, mu0]
```

```
Reduce: -- Message text not found --
```

Out[ ]=
```
24.5529 ≤ mu0 ≤ 25.8757
```

In[ ]:=
```
dist = StudentTDistribution[n - 1];
Reduce[InverseCDF[dist, alpha / 2] ≤ t ≤ InverseCDF[dist, 1 - alpha / 2]]
```

```
Reduce: -- Message text not found --
```

Out[ ]=
```
24.5529 ≤ mu0 ≤ 25.8757
```

In[ ]:=
```
x = {25, 24, 25, 26}; y = {23, 18, 22, 28, 17, 25, 19, 16};
TTest[{x, y}, 0, AlternativeHypothesis → "Greater",
 VerifyTestAssumptions → "EqualVariance" → False]
```

Out[ ]=
```
0.0160194
```

In[ ]:=
```
alpha = 5 / 100;
m = Length[x]; n = Length[y]; sx2 = Variance[x]; sy2 = Variance[y];
s2 = ((m - 1) sx2 + (n - 1) sy2) / (m + n - 2);
T = (Mean[x] - Mean[y] - d) / Sqrt[s2 (1 / m + 1 / n)]; (* t統計量 *)
t := T /. d → 0                  (* t値 *)
df = m + n - 2;                  (* 自由度 *)
dist := StudentTDistribution[df];        (* t分布 *)
P := 1 - CDF[dist][t];            (* P値 *)
a := InverseCDF[dist, 1 - alpha];        (* 採択域の上限 *)
interval := Reduce[T ≤ a, d]          (* 信頼区間 *)
{t, P, a, interval} // N
```

Out[ ]=
```
{1.84017, 0.0477856, 1.81246, d ≥ 0.0602415}
```

In[ ]:=
```
T = (Mean[x] - Mean[y] - d) / Sqrt[sx2 / m + sy2 / n];
df = (sx2 / m + sy2 / n) ^2 / ((sx2 / m) ^2 / (m - 1) + (sy2 / n) ^2 / (n - 1)) // N;
{t, P, a, interval} // N
```

```
Reduce: -- Message text not found --
```

Out[ ]=
```
{2.5923, 0.0160194, 1.85992, d ≥ 1.13009}
```

In[ ]:=
```
x = {25, 24, 25, 26}; y = {23, 18, 22, 28, 17, 25, 19, 16};
VarianceTest[{x, y}, 1, "HypothesisTestData"]["TestDataTable"]
```

Out[ ]=

	Statistic	P-Value
Fisher Ratio	0.0376344	0.021215

In[ ]:=
```
m = Length[x]; n = Length[y]; dist = FRatioDistribution[m - 1, n - 1];
F = Variance[x] / Variance[y] / r; f = F /. r → 1;
c = CDF[dist][f];
{f, 2 Min[c, 1 - c]} // N
```

Out[ ]=
```
{0.0376344, 0.021215}
```

In[ ]:=
```
alpha = 5 / 100;
{a, b} = InverseCDF[dist, {alpha / 2, 1 - alpha / 2}] // N
```

Out[ ]=
```
{0.0683789, 5.88982}
```

In[ ]:=
```
Needs["HypothesisTesting`"] (* 「`」はシングルクォートではなくバッククォート *)
VarianceRatioCI[x, y]
```

Out[ ]=
```
{0.00638974, 0.55038}
```

In[ ]:=
```
Reduce[a ≤ F ≤ b, r]
```

Reduce : -- Message text not found --

Out[ ]=
```
0.00638974 ≤ r ≤ 0.55038
```

11 線形回帰分析

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
data = {{1, 2, 3}, {1, 3, 6}, {2, 5, 3}, {3, 7, 6}};
model = LinearModelFit[data, {X1, X2}, {X1, X2}]
model["BestFitParameters"]
```

Out[∘]=
```
FittedModel[ 3. - 4. ≪2≫ + 2. X2 ]
```

Out[∘]=
```
{3., -4., 2.}
```

In[∘]:=
```
model[1.5, 4]
```

Out[∘]=
```
5.
```

In[∘]:=
```
x1 = {1, 3, 6, 10}; y = {7, 1, 6, 14};
e = y - (b0 + b1 x1);
L = e.e; (* 内積 *)
FindMinimum[L, {{b0, 0}, {b1, 0}}]
```

Out[∘]=
```
{40., {b0 → 2., b1 → 1.}}
```

In[∘]:=
```
Minimize[L, {b0, b1}] (* 解析的な結果 *)
```

Out[∘]=
```
{40, {b0 → 2, b1 → 1}}
```

In[∘]:=
```
L = Total[Abs[e]]; (* 差の絶対値の和 *)
Minimize[L, {b0, b1}] // N
```

Out[∘]=
```
{10.2857, {b0 → -4.57143, b1 → 1.85714}}
```

In[∘]:=
```
e = x1 - (y - b0) / b1;
L = e.e;
Minimize[L, {b0, b1}] // N
```

Out[∘]=
```
{21.3953, {b0 → -2.34783, b1 → 1.86957}}
```

In[∘]:=
```
line = Module[{x1, y}, ImplicitRegion[y == b0 + b1 x1, {x1, y}]];
L = Sum[RegionDistance[line, p]^2, {p, Thread[{x1, y}]}];
Minimize[L, {b0, b1}] // Simplify // N
```

Out[∘]=
```
{15.8403, {b0 → -0.626059, b1 → 1.52521}}
```

```
In[ ]:= data = {{1, 2, 3}, {1, 3, 6}, {2, 5, 3}, {3, 7, 6}};
        X = DesignMatrix[data, {X1, X2}, {X1, X2}];
        y = data[[All, -1]];
        Inverse[Transpose[X] . X] . Transpose[X] . y
Out[ ]=
        {3, -4, 2}

In[ ]:= PseudoInverse[X] . y
Out[ ]=
        {3, -4, 2}

In[ ]:= b = {b0, b1, b2};
        L = (y - X . b) . (y - X . b);
        Reduce[{D[L, {b}] == 0 b}]
Out[ ]=
        b2 == 2 && b1 == -4 && b0 == 3

In[ ]:= D[L, {b}] == -2 Transpose[X] . y + 2 Transpose[X] . X . b // Simplify
Out[ ]=
        True

In[ ]:= Clear["Global`*"];

In[ ]:= data = {{1, 2, 3}, {1, 3, 6}, {2, 5, 3}, {3, 7, 6}};
        model = LinearModelFit[data, {X1, X2}, {X1, X2}];
        model["RSquared"]
Out[ ]=
        0.333333

In[ ]:= model["AdjustedRSquared"]
Out[ ]=
        -1.

In[ ]:= x1 = {1, 3, 6, 10}; y = {7, 1, 6, 14}; data = Thread[{x1, y}];
        X = DesignMatrix[data, X1, X1];
        yh = X . PseudoInverse[X] . y;
        eh = y - yh; fh = yh - Mean[y]; g = y - Mean[y];
        R2 = 1 - eh . eh / g . g; N[R2]
Out[ ]=
        0.534884

In[ ]:= {Mean[eh] == 0,              (* 特徴1 *)
         Mean[yh] == Mean[y],         (* 特徴2 *)
         g . g == fh . fh + eh . eh,  (* 特徴3 *)
         R2 == fh . fh / g . g,       (* 特徴4 *)
         R2 == Correlation[y, yh]^2,  (* 特徴5 *)
         0 ≤ R2 ≤ 1,                  (* 特徴6 *)
         Correlation[y, yh] == Correlation[y, x1]} (* 特徴7 *)
Out[ ]=
        {True, True, True, True, True, True, True}
```

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
data = {{1, 2, 3}, {1, 3, 6}, {2, 5, 3}, {3, 7, 6}};
n := Length[data]                    (* サンプルサイズ *)
p := Length[data[[1]]]                (* 変数の個数 *)
vars := Table[Subscript[x, i], {i, p - 1}]    (* 入力変数（記号）*)
X := DesignMatrix[data, vars, vars]         (* 計画行列 *)
y := data[[All, -1]]                  (* 出力変数の実現値 *)
beta := Table[Subscript[β, i - 1], {i, p}]  (* 回帰係数 *)
epsilon := Table[Subscript[ϵ, i], {i, n}] (* 誤差項 *)
Y := X . beta + epsilon              (* 出力変数（確率変数）*)
betah := PseudoInverse[X] . Y           (* 回帰係数の推定量 *)
betah // Simplify
```

Out[ ]=

$$\left\{\beta_0 + \epsilon_1 + \frac{\epsilon_2}{2} - \frac{\epsilon_4}{2},\ \beta_1 + \frac{1}{6}\left(12\,\epsilon_1 - 13\,\epsilon_2 - 4\,\epsilon_3 + 5\,\epsilon_4\right),\ \frac{1}{6}\left(6\,\beta_2 - 6\,\epsilon_1 + 5\,\epsilon_2 + 2\,\epsilon_3 - \epsilon_4\right)\right\}$$

In[ ]:=
```
Clear[sigma];
udist = UniformDistribution[{-Sqrt[3] sigma, Sqrt[3] sigma}];
udists = Table[Distributed[v, udist], {v, epsilon}];
Expectation[betah, udists]
```

Out[ ]=

$$\{\beta_0,\ \beta_1,\ \beta_2\}$$

In[ ]:=
```
ndist = NormalDistribution[0, sigma];
ndists = Table[Distributed[v, ndist], {v, epsilon}];
TransformedDistribution[betah, ndists] ==
 MultinormalDistribution[beta, sigma^2 Inverse[Transpose[X] . X]]
```

Out[ ]=

True

In[ ]:=
```
model := LinearModelFit[data, vars, vars];
model["EstimatedVariance"]
```

Out[ ]=

6.

In[ ]:=
```
e := Y - X . betah; RSS := e . e; s2 := RSS / (n - p)
s2 // Simplify
```

Out[ ]=

$$\frac{1}{6}\left(\epsilon_2 - 2\,\epsilon_3 + \epsilon_4\right)^2$$

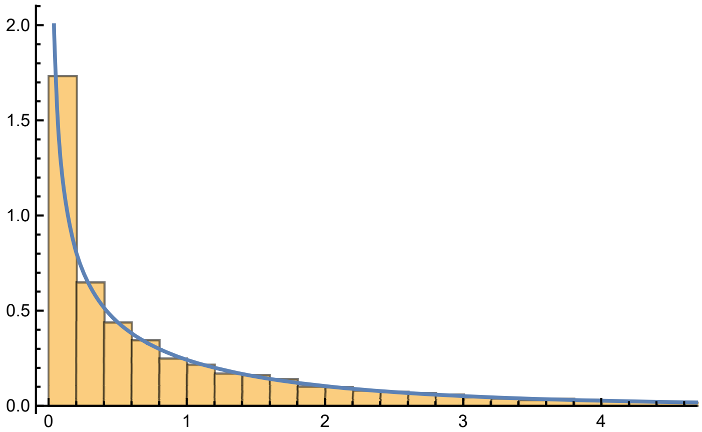
In[ ]:=
```
Expectation[s2, udists]
```

Out[ ]=

$$\text{sigma}^2$$

In[ ]:=
```
tmp = Block[{sigma = 2},
  dist = TransformedDistribution[Simplify[(n - p) s2/sigma^2], ndists];
  RandomVariate[dist, 10000]];
cdist = ChiSquareDistribution[n - p];
Show[Histogram[tmp, Automatic, "PDF"],
 Plot[PDF[cdist][x], {x, 0, 5}, PlotRange → {0, 2}]]
```

Out[ ]=



In[ ]:=
```
uh := Transpose[A] . betah
M := Transpose[A] . Inverse[Transpose[X] . X] . A
r := MatrixRank[A]
F := (uh - u) . Inverse[M] . (uh - u) / r / s2
fdist := FRatioDistribution[r, n - p]
pvalue := 1 - CDF[fdist, F]

Y := y (* この先，実現値のみを扱う． *)
A = Transpose[{{0, 1, 0}, {0, 0, 1}}]; u = {0, 0};
{F, pvalue} // N
```

Out[ ]=

{0.25, 0.816497}

In[ ]:=
```
model["ParameterTable"]
```

Out[ ]=

	Estimate	Standard Error	t-Statistic	P-Value
1	3.	3.	1.	0.5
x_1	-4.	7.68115	-0.520756	0.69435
x_2	2.	3.31662	0.603023	0.654545

In[ ]:=
```
u = {0};
A = Transpose[{{1, 0, 0}}]; pvalue // N (* k = 0 *)
A = Transpose[{{0, 1, 0}}]; pvalue // N (* k = 1 *)
A = Transpose[{{0, 0, 1}}]; pvalue // N (* k = 2 *)
```

Out[ ]=

0.5

Out[ ]=

0.69435

Out[ ]=

0.654545

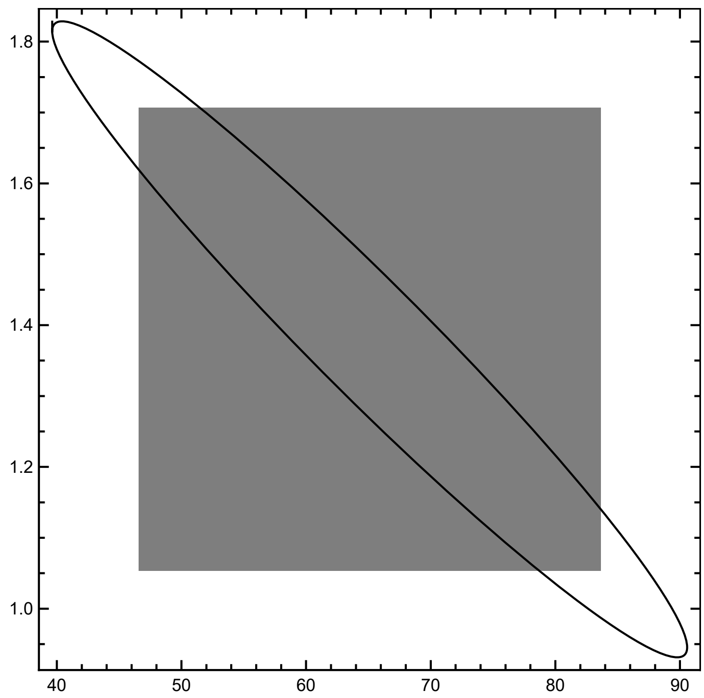
```
In[ ]:= s := Sqrt[s2 Diagonal[Inverse[Transpose[X] . X]]]
        s // N
Out[ ]=
        {3., 7.68115, 3.31662}

In[ ]:= t := betah / s
        t // N
Out[ ]=
        {1., -0.520756, 0.603023}

In[ ]:= tdist := StudentTDistribution[n - p]
        Table[2 Min[CDF[tdist][v], 1 - CDF[tdist][v]], {v, t}] // N
Out[ ]=
        {0.5, 0.69435, 0.654545}

In[ ]:= data = Transpose[{{35, 45, 55, 65, 75}, {114, 124, 143, 158, 166}}];
        alpha = 5 / 100; level := ConfidenceLevel → 1 - alpha
        model["ParameterConfidenceIntervalTable", level]
Out[ ]=
```

	Estimate	Standard Error	Confidence Interval
1	65.1	5.82838	{46.5515, 83.6485}
x_1	1.38	0.102632	{1.05338, 1.70662}

```
In[ ]:= tmp = InverseCDF[tdist, 1 - alpha / 2];
        {betah - s tmp, betah + s tmp} // Transpose // N
Out[ ]=
        {{46.5515, 83.6485}, {1.05338, 1.70662}}

In[ ]:= cond := F ≤ InverseCDF[fdist, 1 - alpha]
        confint := Reduce[cond]
        A = Transpose[{{1, 0}}]; u = {beta0}; confint // N (* k = 0 *)
        A = Transpose[{{0, 1}}]; u = {beta1}; confint // N (* k = 1 *)
Out[ ]=
        46.5515 ≤ beta0 ≤ 83.6485
Out[ ]=
        1.05338 ≤ beta1 ≤ 1.70662
```

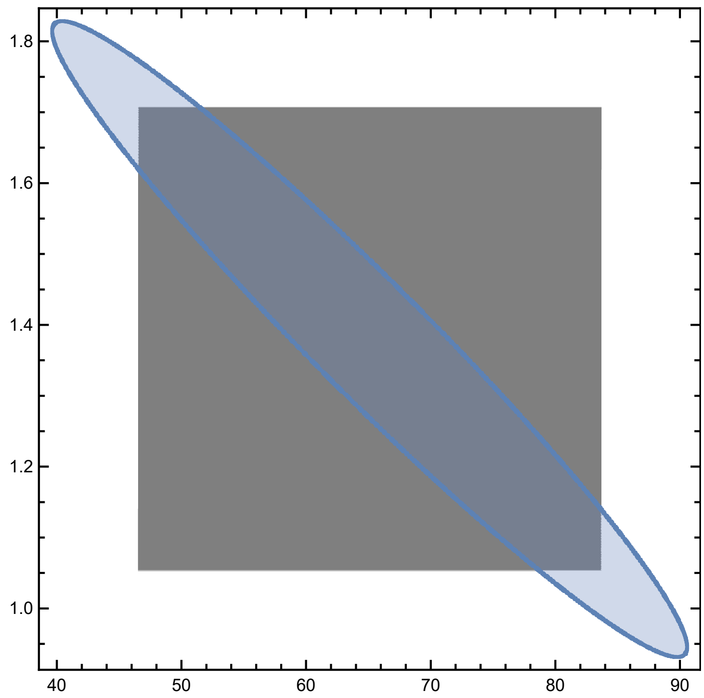
In[ ]:=
```
tmp = model["ParameterConfidenceIntervals", level];
g1 = Graphics[{Gray, Apply[Rectangle, Transpose[tmp]]}];
g2 = Graphics[model["ParameterConfidenceRegion", level]];
Show[g1, g2, AspectRatio → 1, Frame → True]
```

Out[ ]=



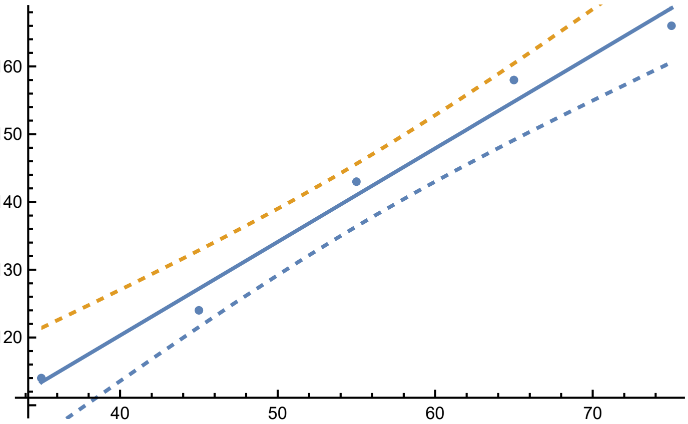
In[ ]:=
```
A = {{1, 0}, {0, 1}}; u = {beta0, beta1};
g3 = RegionPlot[ImplicitRegion[N[cond], Evaluate[u]]];
Show[g1, g3, AspectRatio → 1, Frame → True]
```

Out[ ]=



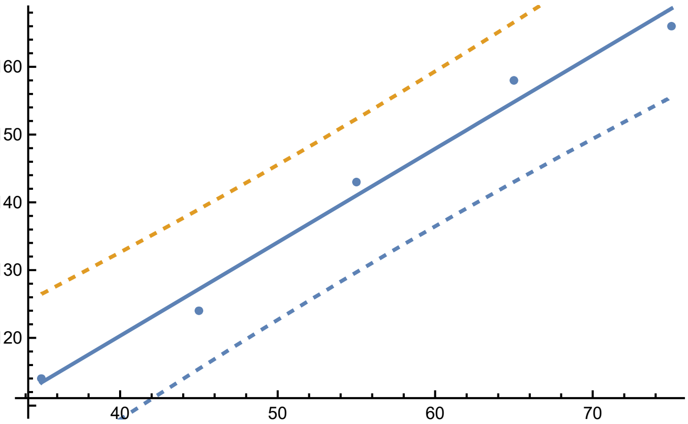
In[ ]:=
```
data = Transpose[{{35, 45, 55, 65, 75}, {114, 124, 143, 158, 166}}];
g = Show[ListPlot[data], Plot[model[x1], {x1, 35, 75}],
 Plot[Evaluate[model["MeanPredictionBands", level]],
  Evaluate[{vars⟦1⟧, 35, 75}], PlotStyle → Dashed]]
```

Out[ ]=



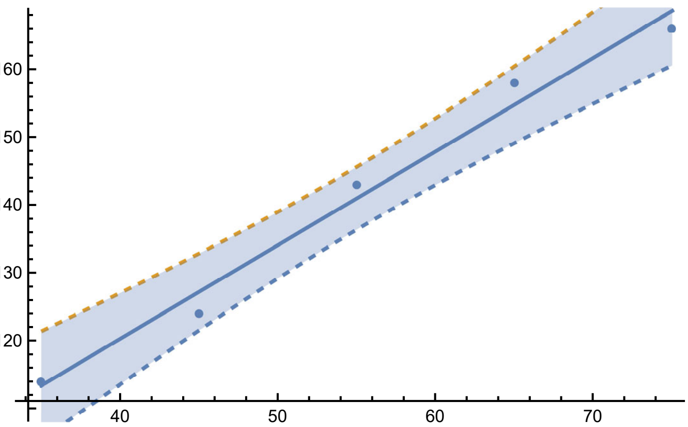
In[ ]:=
```
Show[ListPlot[data], Plot[model[x1], {x1, 35, 75}],
  Plot[Evaluate[model["SinglePredictionBands", level]],
   Evaluate[{vars⟦1⟧, 35, 75}], PlotStyle → Dashed]]
```

Out[ ]=



In[ ]:=
```
A = {{1}, {vars⟦1⟧}}; u = {Yp};
Show[g, RegionPlot[Evaluate[cond],
 Evaluate[{vars⟦1⟧, 35, 75}], {Yp, 0, 200}, BoundaryStyle → None]]
```

Out[ ]=



12 関数の極限と連続性

In[•]:=
```
Clear["Global`*"];
```

In[•]:=
```
f[x_] := 2 x - 3
Limit[f[x], x → 1]
```

Out[•]=
```
-1
```

In[•]:=
```
Limit[2 x - 3, x → 1]
```

Out[•]=
```
-1
```

In[•]:=
```
f[x_] := Piecewise[{{x^2, x ≠ 2}, {3, x == 2}}]
Limit[f[x], x → 2]
```

Out[•]=
```
4
```

In[•]:=
```
g[x_] := (x^2 - 2) / (x - Sqrt[2])
Limit[g[x], x → Sqrt[2]]
```

Out[•]=

$2\sqrt{2}$

In[•]:=
```
A := ForAll[epsilon, epsilon > 0, Exists[delta, delta > 0, B]];
B := ForAll[x, Element[x, Reals],
 Implies[0 < Norm[x - a] < delta, Norm[f[x] - alpha] < epsilon]]

f[x_] := 2 x - 3; a = 1; alpha = -1;
Reduce[A, Reals]
```

Out[•]=
```
True
```

In[•]:=
```
Simplify[Reduce[B, Reals], epsilon > 0]
```

Out[•]=
```
2 delta ≤ epsilon
```

In[•]:=
```
Clear[alpha];
Reduce[A, Reals]
```

Out[•]=
```
alpha == -1
```

In[•]:=
```
Limit[(1 + 1 / x)^x, x → Infinity]
```

Out[•]=
```
ⅇ
```

In[•]:=
```
Limit[1 / x^2, x → 0]
```

Out[•]=
```
∞
```

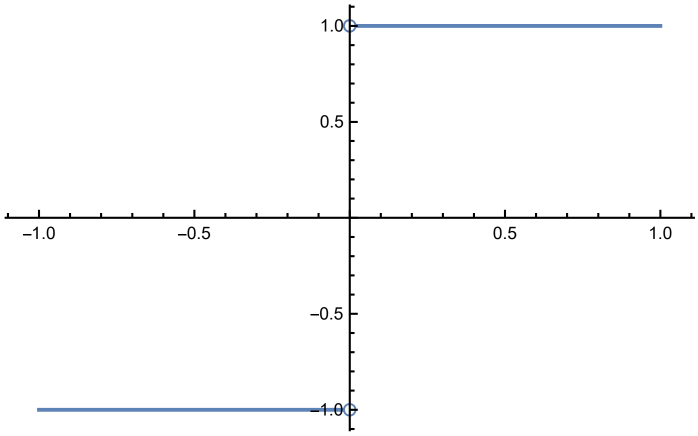
In[ ]:=
```
{Limit[RealAbs[x] / x, x → 0, Direction → "FromAbove"],
 Limit[RealAbs[x] / x, x → 0, Direction → "FromBelow"]}
```

Out[ ]=
```
{1, -1}
```

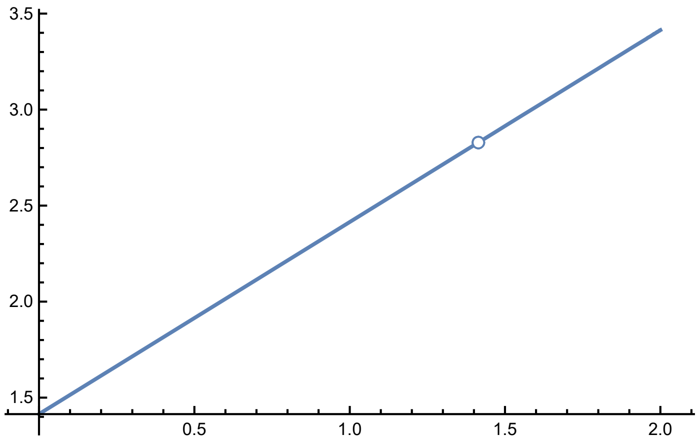
In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
Clear[f, g, x];
f[x_] := Piecewise[{{RealAbs[x] / x, x ≠ 0}}, Undefined]
g[x_] := Piecewise[{{(x^2 - 2) / (x - Sqrt[2]), x ≠ Sqrt[2]}}, Undefined]

ResourceFunction["EnhancedPlot"][f[x], {x, -1, 1}, "FindExceptions" → True]
ResourceFunction["EnhancedPlot"][g[x], {x, 0, 2}, "FindExceptions" → True]
```

Out[ ]=



Out[ ]=



In[ ]:=
```
FunctionContinuous[{f[x], x ≠ 0}, x]
FunctionContinuous[{g[x], x ≠ Sqrt[2]}, x]
```

Out[ ]=
```
True
```

Out[ ]=
```
True
```

13 微分

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
f[x_] := x^3
f'[1]
```

Out[ ]=

3

In[ ]:=
```
a = 1;
Limit[(f[a + h] - f[a])/h, h → 0]
```

Out[ ]=

3

In[ ]:=
```
f[x_] := x^3
f'[x]
```

Out[ ]=

$3 x^2$

In[ ]:=
```
f[x_] := x^3
f1 = f'    (* 方法1 *)
Derivative[1][f] (* 方法2 *)
f2 = f'[x]   (* 方法1 *)
D[f[x], x]   (* 方法2 *)
```

Out[ ]=

$3\ \#1^2\ \&$

Out[ ]=

$3\ \#1^2\ \&$

Out[ ]=

$3 x^2$

Out[ ]=

$3 x^2$

In[ ]:=
```
{f1[1], f2 /. x → 1}
```

Out[ ]=

{3, 3}

In[ ]:=
```
D[x^3, {x, 2}]
```

Out[ ]=

$6 x$

In[ ]:=
```
Clear[a, b, f, g];
f[t_] := t^2
g[x_] := a x + b
Composition[f, g]'[x]          (* ① *)
D[f[g[x]], x]            (* ② *)
(D[f[t], t] /. t → g[x]) D[g[x], x] (* ③ *)
f'[g[x]] g'[x]            (* ④ *)
```

Out[ ]= $2\,a\,(b+a\,x)$

Out[ ]= $2\,a\,(b+a\,x)$

Out[ ]= $2\,a\,(b+a\,x)$

Out[ ]= $2\,a\,(b+a\,x)$

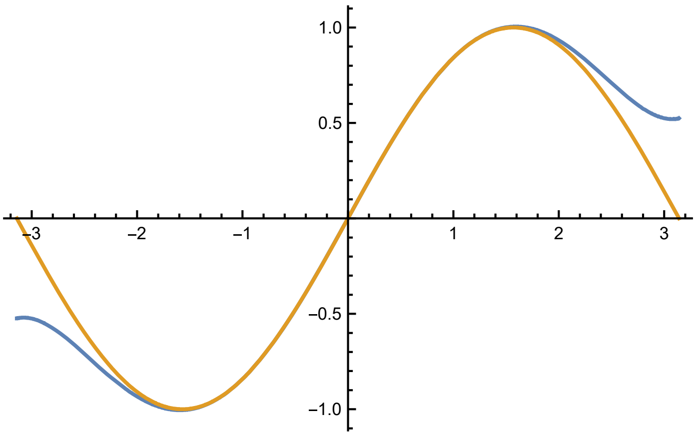
In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
tmp = Series[Sin[x], {x, 0, 5}]
```

Out[ ]= $$x-\frac{x^3}{6}+\frac{x^5}{120}+O[x]^6$$

In[ ]:=
```
Plot[Evaluate[{Normal[tmp], Sin[x]}], {x, -Pi, Pi}]
```

Out[ ]=



In[ ]:=
```
a = 0; Sum[Derivative[k][Sin][a] (x - a)^k/k!, {k, 0, 5}]
```

Out[ ]= $$x-\frac{x^3}{6}+\frac{x^5}{120}$$

In[ ]:=
```
f[x_] := Sqrt[1 + x]
Series[f[x], {x, 0, 7}]
```

Out[ ]= $$1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5\,x^4}{128}+\frac{7\,x^5}{256}-\frac{21\,x^6}{1024}+\frac{33\,x^7}{2048}+O[x]^8$$

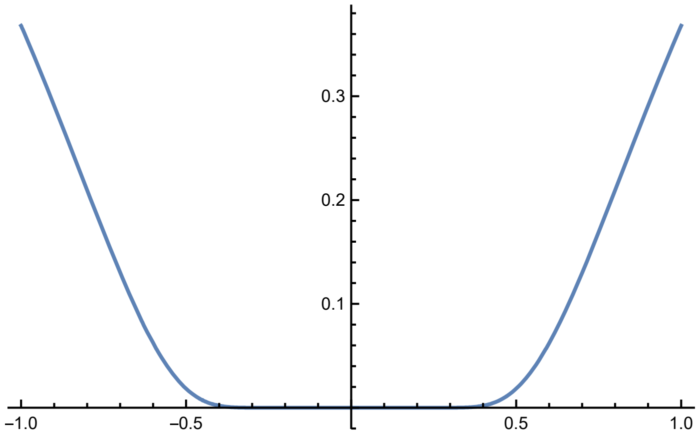
In[∘]:=
```
fn[n_, a_, x_] := Sum[Derivative[k][f][a] (x - a)^k/k!, {k, 0, n},
 GenerateConditions → True]
Reduce[f[x] == fn[Infinity, 0, x], Reals]
```

Out[∘]=

$-1 \leq x \leq 1$

In[∘]:=
```
f[x_] := Piecewise[{{Exp[-1/x^2], x ≠ 0}}, 0]
Plot[f[x], {x, -1, 1}]
Reduce[f[x] == fn[Infinity, 0, x], Reals]
```

Out[∘]=



Out[∘]=

$x == 0$

In[∘]:=
```
Derivative[k][f][0]
```

Out[∘]=

0

In[∘]:=
```
f[x_] := x^3 - 12 x
ResourceFunction["LocalExtrema"][f[x], x]
```

Out[∘]=

<|Minima → {{-16, {x → 2}}}, Maxima → {{16, {x → -2}}}|>

In[∘]:=
```
sol = SolveValues[f'[x] == 0, x]
Series[f[x], {x, sol[[1]], 2}]
```

Out[∘]=

{-2, 2}

Out[∘]=

$16 - 6\,(x + 2)^2 + O[x + 2]^3$

In[∘]:=
```
Clear[a, delta];
f[x_] := Piecewise[{{Exp[-1/x^2], x ≠ 0}}, 0]
Reduce[Exists[delta, delta > 0, ForAll[x, Element[x, Reals],
  Implies[0 < Norm[x - a] < delta, f[a] < f[x]]]], Reals]
```

Out[∘]=

$a == 0$

14 積分

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
Integrate[-x^2 + 4 x + 1, {x, 1, 4}]
```

Out[ ]=

$$12$$

In[ ]:=
```
f[x_] := -x^2 + 4 x + 1
Clear[x]; a = 1; b = 4; h = (b - a) / n;
s = Sum[f[a + k h] h, {k, 1, n}] // Expand
Limit[s, n → Infinity]
```

Out[ ]=

$$12-\frac{9}{2\,n^2}-\frac{9}{2\,n}$$

Out[ ]=

$$12$$

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
Integrate[-t^2 + 4 t + 1, {t, a, x}]
```

Out[ ]=

$$-a+\frac{a^3}{3}+x-\frac{x^3}{3}+4\left(-\frac{a^2}{2}+\frac{x^2}{2}\right)$$

In[ ]:=
```
Integrate[-x^2 + 4 x + 1, x]
```

Out[ ]=

$$x+2\,x^2-\frac{x^3}{3}$$

In[ ]:=
```
Clear[x, y];
DSolveValue[y'[x] == -x^2 + 4 x + 1, y[x], x]
```

Out[ ]=

$$x+2\,x^2-\frac{x^3}{3}+c_1$$

In[ ]:=
```
DSolveValue[{y'[x] == -x^2 + 4 x + 1, y[0] == 1}, y[x], x]
```

Out[ ]=

$$\frac{1}{3}\left(3+3\,x+6\,x^2-x^3\right)$$

In[ ]:=
```
tmp = DSolveValue[y'[x] == -x y[x], y[x], x]
```

Out[ ]=

$$e^{-\frac{x^2}{2}}\,c_1$$

In[ ]:=
```
Reduce[Integrate[tmp, {x, -Infinity, Infinity}] == 1]
```

Out[ ]=

$$c_1 == \frac{1}{\sqrt{2\,\pi}}$$

In[ ]:=
```
Clear[a, f, t, x];
Function[x, Evaluate[Integrate[f[t], {t, a, x}]]]'
```
Out[ ]=
```
Function[x, f[x]]
```

In[ ]:=
```
D[Integrate[f[t], {t, a, x}], x]
```
Out[ ]=
```
f[x]
```

In[ ]:=
```
F = Integrate[-x^2 + 4 x + 1, x];
(F /. x → 4) - (F /. x → 1)
```
Out[ ]=
```
12
```

In[ ]:=
```
Integrate[Log[Sin[x]], {x, 0, Pi/2}]
```
Out[ ]=

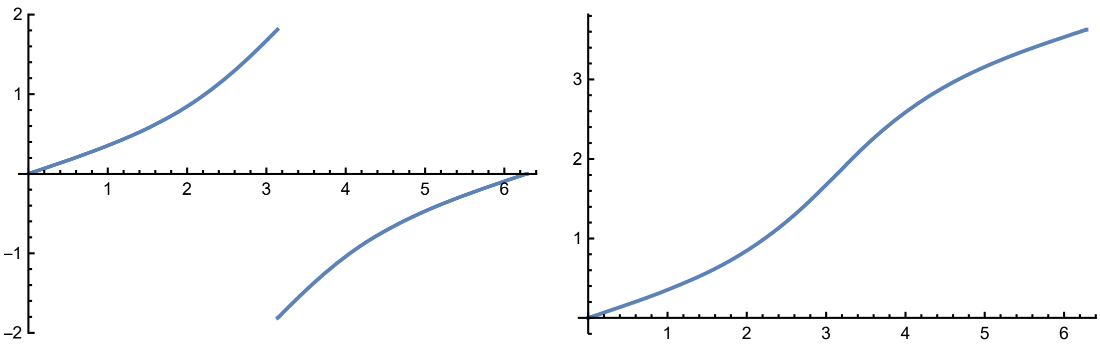
$$-\frac{1}{2}\pi\,\mathrm{Log}[2]$$

In[ ]:=
```
f[x_] := 1/(2 + Cos[x])
F1x = Integrate[f[x], x];
(F1x /. x → 2 Pi) - (F1x /. x → 0) (* 不正解 *)
```
Out[ ]=
```
0
```

In[ ]:=
```
F2x = Integrate[f[t], {t, 0, x}, GenerateConditions → True];
(F2x /. x → 2 Pi) - (F2x /. x → 0) (* 正解 *)
```
Out[ ]=

$$\frac{2\pi}{\sqrt{3}}$$

In[ ]:=
```
GraphicsRow[{Plot[F1x, {x, 0, 2 Pi}], Plot[F2x, {x, 0, 2 Pi}]}]
```
Out[ ]=



In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
Integrate[(p x + q)^100, x]
```
Out[ ]=

$$\frac{(q + p\,x)^{101}}{101\,p}$$

In[ ]:=
```
tmp = IntegrateChangeVariables[
 Inactive[Integrate][(p x + q)^100, x], u, u == p x + q]
Activate[tmp] /. u → p x + q
```

Out[ ]=

$$\int \frac{u^{100}}{p}\, du$$

Out[ ]=

$$\frac{(q + p\,x)^{101}}{101\,p}$$

In[ ]:=
```
Clear[x, y];
IntegrateChangeVariables[
 Inactive[Integrate][1, {x, 0, t}], y, x == Sqrt[y]]
```

Out[ ]=

$$\boxed{\int_0^{t^2} \frac{1}{2\sqrt{y}}\, dy \text{ if } t > 0}$$

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
Integrate[1/x^a, {x, 0, 1}]
```

Out[ ]=

$$\boxed{\frac{1}{1 - a} \text{ if } \mathrm{Re}[a] < 1}$$

In[ ]:=
```
Integrate[1/x^a, {x, 1, Infinity}]
```

Out[ ]=

$$\boxed{\frac{1}{-1 + a} \text{ if } \mathrm{Re}[a] > 1}$$

In[ ]:=
```
Integrate[Exp[-x^2], {x, -Infinity, Infinity}]
```

Out[ ]=

$$\sqrt{\pi}$$

15 多変数関数の微分積分

```
In[ ]:= Clear["Global`*"];

In[ ]:= x = {x1, x2}; f[{x1_, x2_}] := x1 x2^2 / (x1^2 + x2^2)
        Limit[f[x], x → {0, 0}]

Out[ ]= 0

In[ ]:= A := ForAll[epsilon, epsilon > 0, Exists[delta, delta > 0, B]];
        B := ForAll[Evaluate[x], Element[x, Reals],
         Implies[0 < Norm[x - a] < delta, Norm[f[x] - alpha] < epsilon]]

In[ ]:= a = {0, 0}; alpha = 0;
        Reduce[A, Reals]

Out[ ]= True

In[ ]:= Clear[alpha];
        Reduce[A, Reals]

Out[ ]= alpha == 0

In[ ]:= Clear[x, y]; f[x_, y_] := x^2 y / (x^4 + y^2)
        Limit[f[x, y], {x, y} → {0, 0}]

Out[ ]= Indeterminate

In[ ]:= Clear[x, y, r, theta];
        {Limit[Limit[f[x, y], x → 0], y → 0],     (* ① *)
         Limit[Limit[f[x, y], y → 0], x → 0],     (* ② *)
         Limit[f[r Cos[theta], r Sin[theta]], r → 0], (* ③ *)
         Limit[f[x, x^2], x → 0]}          (* ④ *)

Out[ ]= {0, 0, 0, 1/2}

In[ ]:= f[{x1_, x2_}] := Piecewise[{{0, x1 == x2 == 0}}, x1 x2^2 / (x1^2 + x2^2)]
        x = {x1, x2};
        FunctionContinuous[f[x], x]      (* 方法1 *)
        Limit[f[x], x → {0, 0}] == f[{0, 0}] (* 方法2 *)

Out[ ]= True

Out[ ]= True
```

In[ ]:=
```
f[x_, y_] := Piecewise[{{0, x == y == 0}}, x^2 y / (x^4 + y^2)]
Clear[x, y]; FunctionContinuous[f[x, y], {x, y}]
```

Out[ ]=
```
False
```

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
f[x_, y_] := 2 - x^2 - y^2
{D[f[x, y], x], D[f[x, y], y]}
```

Out[ ]=
$\{-2x, -2y\}$

In[ ]:=
```
f[x_, y_] := 2 - x^2 - y^2
{Derivative[1, 0][f], Derivative[0, 1][f]}
```

Out[ ]=
```
{-2 #1 &, -2 #2 &}
```

In[ ]:=
```
g[{x1_, x2_}] := 2 - x1^2 - x2^2
{Derivative[{1, 0}][g], Derivative[{0, 1}][g]}
```

Out[ ]=
```
{-2 #1[[1]] &, -2 #1[[2]] &}
```

In[ ]:=
```
D[f[x, y], {{x, y}}] (* 方法1 *)
Grad[f[x, y], {x, y}] (* 方法2 *)
```

Out[ ]=
$\{-2x, -2y\}$

Out[ ]=
$\{-2x, -2y\}$

In[ ]:=
```
f[x_, y_] := 2 x^3 + 5 x y + 2 y^2
D[f[x, y], {{x, y}, 2}] // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 12x & 5 \\ 5 & 4 \end{pmatrix}$$

In[ ]:=
```
Clear[f, F];
f[{x1_, x2_}] := Sqrt[x1^2 + x2^2]
x = {x1, x2}; a = {1, 1}; h = x - a;
F[t_] := f[a + t h]
expr := Normal[Series[F[t], {t, 0, 2}]] /. t → 1
expr // Simplify
```

Out[ ]=
$$\frac{x1^2 - 2\,x1\,(-2 + x2) + x2\,(4 + x2)}{4\sqrt{2}}$$

In[ ]:=
```
Block[{h = {h1, h2}}, expr /. Thread[h → Map[HoldForm, x - a]]]
```

Out[ ]=
$$\sqrt{2} + \frac{(-1 + x1) + (-1 + x2)}{\sqrt{2}} + \frac{(-1 + x1)^2 - 2\,(-1 + x1)\,(-1 + x2) + (-1 + x2)^2}{4\sqrt{2}}$$

In[ ]:=
```
gradf = D[f[x], {x}] /. Thread[x → a];
H = D[f[x], {x, 2}] /. Thread[x → a];
f[a] + gradf . (x - a) + (x - a) . H . (x - a) / 2 // Simplify
```

Out[ ]=

$$\frac{x1^2 - 2\,x1\,(-2 + x2) + x2\,(4 + x2)}{4\sqrt{2}}$$

In[ ]:=
```
x = {x1, x2}; f[{x1_, x2_}] := 2 x1^3 + x1 x2^2 + 5 x1^2 + x2^2
ResourceFunction["LocalExtrema"][f[x], x]
```

Out[ ]=

$$\langle|\,\text{Minima} \to \{\{0, \{x1 \to 0, x2 \to 0\}\}\},\ \text{Maxima} \to \{\{\frac{125}{27}, \{x1 \to -\frac{5}{3}, x2 \to 0\}\}\}\,|\rangle$$

In[ ]:=
```
points := Solve[D[f[x], {x}] == 0 x, x, Reals]; (* 停留点 *)
H := D[f[x], {x, 2}];              (* ヘッセ行列 *)
Table[With[{h = H /. p},           (* 停留点でのヘッセ行列 *)
 {p, f[x] /. p, Which[
  PositiveDefiniteMatrixQ[h], -1,        (* 極小 *)
  NegativeDefiniteMatrixQ[h], 1,        (* 極大 *)
  IndefiniteMatrixQ[h], 0,          (* 極値ではない *)
  True, Null]}],              (* 不明 *)
 {p, points}]
```

Out[ ]=

$$\{\{\{x1 \to -\frac{5}{3}, x2 \to 0\}, \frac{125}{27}, 1\}, \{\{x1 \to -1, x2 \to -2\}, 3, 0\},$$
$$\{\{x1 \to -1, x2 \to 2\}, 3, 0\}, \{\{x1 \to 0, x2 \to 0\}, 0, -1\}\}$$

In[ ]:=
```
x = {x1, x2}; f[{x1_, x2_}] := x1^2 + x2^4
PositiveDefiniteMatrixQ[H /. Thread[x → {0, 0}]] (* False *)
ResourceFunction["LocalExtrema"][f[x], x]
```

Out[ ]=

False

Out[ ]=

$$\langle|\,\text{Minima} \to \{\{0, \{x1 \to 0, x2 \to 0\}\}\},\ \text{Maxima} \to \{\}\,|\rangle$$

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
d = ImplicitRegion[And[0 ≤ x ≤ 1, 0 ≤ y ≤ x], {x, y}];
f[x_, y_] := x^2 + y^2
Integrate[f[x, y], Element[{x, y}, d]]
```

Out[ ]=

$$\frac{1}{3}$$

In[ ]:=
```
Integrate[Integrate[f[x, y], {y, 0, x}], {x, 0, 1}]
```

Out[ ]=

$$\frac{1}{3}$$

In[ ]:=
```
Integrate[Integrate[f[x, y], {x, y, 1}], {y, 0, 1}]
```

Out[ ]=
$$\frac{1}{3}$$

In[ ]:=
```
Clear[u, v, x, y];
lhs = Inactive[Integrate][f[x, y], Element[{x, y}, d]]
rhs = IntegrateChangeVariables[lhs, {u, v}, {x == 2 u, y == 3 v}]
{Activate[lhs], Activate[rhs]}
```

Out[ ]=
$$\int_{\{x,y\}\in \text{ImplicitRegion}[0\le x\le 1\&\&0\le y\le x,\{x,y\}]} (x^2+y^2)$$

Out[ ]=
$$\int_0^{\frac{1}{2}}\int_0^{\frac{2u}{3}} 6\,(4u^2+9v^2)\,dv\,du$$

Out[ ]=
$$\left\{\frac{1}{3},\frac{1}{3}\right\}$$

In[ ]:=
```
f[x_, y_] := x^2 + y^2
{x, y} = {2 u, 3 v};
J = D[{x, y}, {{u, v}}];
detJ = Det[J]
Integrate[Integrate[f[x, y] Abs[detJ], {v, 0, 2 u/3}], {u, 0, 1/2}]
```

Out[ ]=
6

Out[ ]=
$$\frac{1}{3}$$

In[ ]:=
```
{x, y} = {r Cos[theta], r Sin[theta]};
J = D[{x, y}, {{r, theta}}];
Det[J] // Simplify
```

Out[ ]=
r

In[ ]:=
```
Clear[x, y];
lhs = Inactive[Integrate][Exp[-(x^2 + y^2)],
 {y, -Infinity, Infinity}, {x, -Infinity, Infinity}]
rhs = IntegrateChangeVariables[lhs, {r, theta}, "Cartesian" → "Polar"]
{Activate[lhs], Activate[rhs]}
```

Out[ ]=
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

Out[ ]=
$$\int_0^{\infty}\int_{-\pi}^{\pi} e^{-r^2}\,r\,d\text{theta}\,dr$$

Out[ ]=
$$\{\pi,\pi\}$$

16 ベクトル

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
a = {1/10 + 2/10, 1/10 + 2/10 - 3/10}; b = {3/10, 0};
a == b
```

Out[ ]=
```
True
```

In[ ]:=
```
100 {1, 2} + 10 {3, 1}
```

Out[ ]=
```
{130, 210}
```

In[ ]:=
```
a = {3, 4};
Norm[a]
```

Out[ ]=
```
5
```

In[ ]:=
```
Clear[x, y]; a = {x, y}; Sqrt[a . a]
```

Out[ ]=

$$\sqrt{x^2 + y^2}$$

In[ ]:=
```
Simplify[Norm[{x, y}], Element[x | y, Reals]]
```

Out[ ]=

$$\sqrt{x^2 + y^2}$$

In[ ]:=
```
a = {3, 4};
Normalize[a]
```

Out[ ]=

$$\left\{\frac{3}{5}, \frac{4}{5}\right\}$$

In[ ]:=
```
a = {1, 0}; b = {1, 1};
ArcCos[a . b / (Norm[a] Norm[b])]
```

Out[ ]=

$$\frac{\pi}{4}$$

In[ ]:=
```
VectorAngle[a, b]
```

Out[ ]=

$$\frac{\pi}{4}$$

17 行列

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
MatrixForm[A = {{1, 2, 0}, {0, 3, 4}}]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \end{pmatrix}$$

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
x = {5, 7}; DiagonalMatrix[x] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$$

In[ ]:=
```
SymmetricMatrixQ[{{1, 2}, {2, 3}}]
```

Out[ ]=

```
True
```

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
MatrixForm[A = {{11, 12, 13}, {21, 22, 23}, {31, 32, 33}}]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix}$$

In[ ]:=
```
A[[1 ;; 2, 1 ;; 2]] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 11 & 12 \\ 21 & 22 \end{pmatrix}$$

In[ ]:=
```
A[[All, 3]]
```

Out[ ]=

```
{13, 23, 33}
```

In[ ]:=
```
A[[All, {3}]]
```

Out[ ]=

```
{{13}, {23}, {33}}
```

In[ ]:=
```
A[[2, All]] (* 方法1 *)
A[[2]]   (* 方法2 *)
```

Out[ ]=

```
{21, 22, 23}
```

Out[ ]=

```
{21, 22, 23}
```

```
In[•]:= A⟦{2}, All⟧ (* 方法1 *)
        A⟦{2}⟧   (* 方法2 *)
Out[•]=
        {{21, 22, 23}}
Out[•]=
        {{21, 22, 23}}

In[•]:= Clear["Global`*"];

In[•]:= 10 {{2, 3}, {5, 7}}
Out[•]=
        {{20, 30}, {50, 70}}

In[•]:= {{10, 20}, {30, 40}} + {{2, 3}, {4, 5}}
Out[•]=
        {{12, 23}, {34, 45}}

In[•]:= Clear["Global`*"];

In[•]:= A = {{2, 3}, {5, 7}}; B = {{1, 2}, {3, 4}};
        A . B
Out[•]=
        {{11, 16}, {26, 38}}

In[•]:= A = {{2, 3}, {5, 7}}; B = {{1, 2, 3}, {4, 5, 6}}; S = A . B;
        {p, q} = Dimensions[A]; {r, s} = Dimensions[B];
        S1 = Table[Table[A⟦i, All⟧ . B⟦All, j⟧, {j, 1, s}], {i, 1, p}]; (* ① *)
        S2 = Sum[A⟦All, {j}⟧ . B⟦{j}, All⟧, {j, 1, q}];        (* ② *)
        S3 = Transpose[Table[A . b, {b, Transpose[B]}]];         (* ③ *)
        S4 = Table[a . B, {a, A}];                      (* ④ *)
        {S == S1, S == S2, S == S3, S == S4}
Out[•]=
        {True, True, True, True}

In[•]:= Clear[a1, a2, x1, x2, p, q, r, s];
        x = {x1, x2}; a = {a1, a2};
        G = {{p, q}, {q, s}}; A = {{p, q}, {r, s}};
        D[a . x, {x}] == a
        D[x . G . x, {x}] == 2 G . x // Simplify
        D[(A . x) . (A . x), {x}] == 2 Transpose[A] . A . x // Simplify
Out[•]=
        True
Out[•]=
        True
Out[•]=
        True

In[•]:= Clear["Global`*"];
```

```
In[•]:= Det[{{3, 2}, {1, 2}}]
Out[•]=
        4

In[•]:= RegionMeasure[Parallelepiped[{0, 0}, {{3, 1}, {2, 2}}]]
Out[•]=
        4

In[•]:= RegionMeasure[Parallelepiped[{0, 0, 0}, {{2, 1, 0}, {0, 2, 1}, {1, 1, 1}}]]
Out[•]=
        3

In[•]:= Clear["Global`*"];

In[•]:= Inverse[{{2, 3}, {5, 7}}]
Out[•]=
        {{-7, 3}, {5, -2}}

In[•]:= Clear["Global`*"];

In[•]:= A = {{3, 2}, {1, 2}}; b = {8, 4};
        Inverse[A] . b
Out[•]=
        {2, 1}

In[•]:= RowReduce[{{4, 2, 8}, {2, 1, 4}}]
Out[•]=
```

$$\left\{\left\{1, \frac{1}{2}, 2\right\}, \{0, 0, 0\}\right\}$$

```
In[•]:= A = {{2, 0, 2}, {0, 2, -2}, {2, 2, 0}};
        MatrixRank[A]
Out[•]=
        2
```

18 ベクトル空間

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
a1 = {3, 1}; a2 = {2, 2};
ResourceFunction["LinearlyIndependent"][{a1, a2}]
```

Out[ ]=

True

In[ ]:=
```
Reduce[c1 a1 + c2 a2 == {0, 0}]
```

Out[ ]=

c2 == 0 && c1 == 0

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
A = {{1, 0, 1}, {1, 1, 0}, {0, 1, -1}};
ResourceFunction["ColumnSpace"][A]["Basis"]
```

Out[ ]=

{{1, 1, 0}, {0, 1, 1}}

In[ ]:=
```
A = {{1, 0, 1}, {1, 1, 0}, {0, 1, -1}};
tmp = ResourceFunction["ColumnSpace"][A];
Qt = Orthogonalize[tmp["Basis"]]
```

Out[ ]=

$$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}, \left\{-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right\}\right\}$$

In[ ]:=
```
Q = Transpose[Qt];
Qt . Q
```

Out[ ]=

{{1, 0}, {0, 1}}

In[ ]:=
```
A = {{1, 2}, {1, 2}, {0, 0}}; B = {{1, 0}, {1, 1}, {0, 1}};
{tQa, Ra} = QRDecomposition[A]; Qa = Transpose[tQa]; (* 転置が必要 *)
{tQb, Rb} = QRDecomposition[B]; Qb = Transpose[tQb]; (* 転置が必要 *)
{MatrixForm[Qa], MatrixForm[Ra], A == Qa . Ra,
 MatrixForm[Qb], MatrixForm[Rb], B == Qb . Rb}
```

Out[ ]=

$$\left\{\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix}, \begin{pmatrix}\sqrt{2} & 2\sqrt{2}\end{pmatrix}, \text{True}, \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ 0 & \sqrt{\frac{2}{3}}\end{pmatrix}, \begin{pmatrix}\sqrt{2} & \frac{1}{\sqrt{2}}\\ 0 & \sqrt{\frac{3}{2}}\end{pmatrix}, \text{True}\right\}$$

In[∘]:=
```
qrd[A_] := Module[{m, n, u = Transpose[A], idx = {}, s, Q},
  {m, n} = Dimensions[A];
  Do[Do[u⟦i⟧ = Simplify[u⟦i⟧ - A⟦All, i⟧ . u⟦j⟧ × u⟦j⟧], {j, 1, i - 1}];
   s = Chop[Norm[u⟦i⟧]];
   If[s ≠ 0, u⟦i⟧ /= s; AppendTo[idx, i]], {i, 1, n}];
  Q = If[Length[idx] ≠ 0, Transpose[u⟦idx⟧], IdentityMatrix[m]];
  {Q, Transpose[Q] . A}]

A = {{1, 2}, {1, 2}, {0, 0}}; B = {{1, 0}, {1, 1}, {0, 1}};
Map[MatrixForm, qrd[A]] // Simplify (* 動作確認 *)
Map[MatrixForm, qrd[B]] // Simplify (* 動作確認 *)
```

Out[∘]=

$$\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} & 2\sqrt{2} \end{pmatrix}\right\}$$

Out[∘]=

$$\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \sqrt{\frac{2}{3}} \end{pmatrix}, \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix}\right\}$$

In[∘]:=
```
B = {{1, 0}, {1, 1}, {0, 1}};
{Q, R} = qrd[B];              (* QR分解 *)
tol = 10^-10;
e = IdentityMatrix[Dimensions[Q]⟦2⟧];
{Chop[N[Transpose[Q] . Q] - e, tol] == 0 e,  (* ① *)
 UpperTriangularMatrixQ[R, Tolerance → tol], (* ② *)
 Chop[N[B] - Q . R, tol] == 0 B}       (* ③ *)
(* 誤った転置を検出できないから，①でOrthogonalMatrixQは使えない． *)
```

Out[∘]=

```
{True, True, True}
```

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
A = {{1, 0}, {1, 1}, {0, 1}};
NullSpace[Transpose[A]]
NullSpace[Transpose[N[A]]] (* 正規直交基底 *)
```

Out[∘]=

```
{{1, -1, 1}}
```

Out[∘]=

```
{{0.57735, -0.57735, 0.57735}}
```

In[∘]:=
```
A = {{1, 0}, {1, 1}, {0, 1}};
basis1 = Orthogonalize[Transpose[A]];      (* 列空間 *)
basis2 = Orthogonalize[NullSpace[Transpose[A]]]; (* 直交補空間 *)
MatrixForm[Q = Transpose[Join[basis1, basis2]]]
Transpose[Q] . Q == IdentityMatrix[3]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Out[∘]=

True

In[∘]:=
```
A = {{a, b}, {c, d}};
f[x_] := A . x
R = ParametricRegion[{x, y}, {{x, s, s + u}, {y, t, t + u}}];
Rp = TransformedRegion[R, f];
{RegionMeasure[Rp], Abs[Det[A]] u^2}
```

Out[∘]=

$\{u^2 \, \mathrm{Abs}[-b\,c + a\,d],\ u^2 \, \mathrm{Abs}[-b\,c + a\,d]\}$

19 固有値と固有ベクトル

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
A = {{5, 6, 3}, {0, 9, 2}, {0, 6, 8}}; (* 固有ベクトル（絶対値の降順） *)
{vals, vecs} = Eigensystem[N[A]]   (* 近似値：固有ベクトル（正規） *)
{vals, vecs} = Eigensystem[A]    (* 厳密値：固有ベクトル（非正規）*)
```

Out[∘]=
```
{{12., 5., 5.}, {{0.639602, 0.426401, 0.639602}, {1., 0., 0.}, {0., -0.447214, 0.894427}}}
```

Out[∘]=
```
{{12, 5, 5}, {{3, 2, 3}, {0, -1, 2}, {1, 0, 0}}}
```

In[∘]:=
```
V = Transpose[vecs]; A . V == V . DiagonalMatrix[vals]
```

Out[∘]=
```
True
```

In[∘]:=
```
A = {{5, 6, 3}, {0, 9, 2}, {0, 6, 8}}; n = Length[A];
SolveValues[Det[x IdentityMatrix[n] - A] == 0, x]
```

Out[∘]=
```
{5, 5, 12}
```

In[∘]:=
```
NullSpace[5 IdentityMatrix[n] - A]
```

Out[∘]=
```
{{0, -1, 2}, {1, 0, 0}}
```

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
S = {{2, 2, -2}, {2, 5, -4}, {-2, -4, 5}};
{Q, L, V} = SingularValueDecomposition[S];
{MatrixForm[Q], MatrixForm[L],
 S == Q . L . Transpose[Q] == V . L . Transpose[V]}
```

Out[∘]=

$$\left\{\begin{pmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \end{pmatrix}, \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{True}\right\}$$

In[∘]:=
```
S = {{2, 2, -2}, {2, 5, -4}, {-2, -4, 5}};
{vals, vecs} = Eigensystem[S];    (* ① *)
Q = Transpose[Orthogonalize[vecs]];  (* ②, ③ *)
L = DiagonalMatrix[vals];      (* ④ *)
Chop[N[S] - Q . L . Transpose[Q]] == 0 S (* 近似的な比較 *)
```

Out[∘]=
```
True
```

In[∘]:=
```
Clear["Global`*"];
```

In[∘]:=
```
PositiveSemidefiniteMatrixQ[{{4, 2}, {2, 1}}]
```

Out[∘]=
```
True
```

In[∘]:=
```
A = {{4, 2}, {2, 1}};
AllTrue[Eigenvalues[A], NonNegative]
```

Out[∘]=
```
True
```

In[∘]:=
```
x1 = {1, 3, 6, 10}; y = {7, 1, 6, 14}; X = Transpose[{x1, y}];
n = Length[X]; M = ConstantArray[1, {n, n}] / n;
A = X - M . X;
MatrixForm[S = Transpose[A] . A]
v = Eigenvectors[N[S], 1][[1]] (* 最大固有値に対応する固有ベクトル *)
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 46 & 46 \\ 46 & 86 \end{pmatrix}$$

Out[∘]=
```
{0.548304, 0.836279}
```

In[∘]:=
```
Reduce[Det[{v, {xp - Mean[x1], yp - Mean[y]}}] == 0, yp] // N
```

Out[∘]=
```
yp == -0.626059 + 1.52521 xp
```

In[∘]:=
```
{U, L, V} = SingularValueDecomposition[A]; (* 特異値分解 *)
V[[All, 1]] // N            (* Vの第1列（求めるもの）*)
s2 = Diagonal[L]^2;           (* 特異値の2乗 *)
Accumulate[s2] / Total[s2] // N      (* 累積寄与率（後述）*)
```

Out[∘]=
```
{0.548304, 0.836279}
```

Out[∘]=
```
{0.879998, 1.}
```

```
In[•]:= X = N[Transpose[{{1, 3, 6, 10}, {7, 1, 6, 14}}]];
t = Transpose;
MatrixForm[P = PrincipalComponents[X]]        (* 主成分スコア *)
r = MatrixRank[P]; Pr = P[[All, ;; r]]; tPr = t[Pr];
MatrixForm[tVr1 = Inverse[tPr . Pr] . tPr . X]    (* 主成分（方法1）*)
MatrixForm[tVr2 = (PseudoInverse[P] . X)[[;; r, All]]] (* 主成分（方法2）*)
tVr1[[1]]          (* 第1主成分（求めるもの）（方法1）*)
tVr2[[1]]          (* 第1主成分（求めるもの）（方法2）*)
s2 = Diagonal[Transpose[P] . P]; (* 特異値の2乗 *)
Accumulate[s2]/Total[s2]    (* 累積寄与率（後述）*)
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 2.19321 & 3.34512 \\ 6.11428 & -1.61726 \\ 0.287976 & -1.38458 \\ -8.59547 & -0.343271 \end{pmatrix}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} -0.548304 & -0.836279 \\ -0.836279 & 0.548304 \end{pmatrix}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} -0.548304 & -0.836279 \\ -0.836279 & 0.548304 \end{pmatrix}$$

Out[•]=

{-0.548304, -0.836279}

Out[•]=

{-0.548304, -0.836279}

Out[•]=

{0.879998, 1.}

20 特異値分解と擬似逆行列

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
A = {{1, 0}, {1, 1}, {0, 1}};
{U, S, V} = SingularValueDecomposition[A]; tV = Transpose[V];
{Map[MatrixForm, {U, S, tV}], A == U . S . tV}
```

Out[ ]=

$$\left\{\left\{\begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right\}, \text{True}\right\}$$

In[ ]:=
```
url = "https://github.com/taroyabuki/comath/raw/main/images/boy.jpg";
A = ImageData[ColorConvert[Import[url], "Grayscale"]]; (* 画像の行列への変換 *)
{U, S, V} = SingularValueDecomposition[A];      (* 特異値分解 *)
k = 52;
Ak = U[[All, ;; k]] . S[[;; k, ;; k]] . Transpose[V[[All, ;; k]]]; (* 近似 *)
B = (Ak - Min[Ak]) / (Max[Ak] - Min[Ak]); (* 数値を0～1にする. *)
GraphicsRow[{Image[A], Image[B]}]
```

Out[ ]=


In[ ]:=
```
nonzero[x_, tol_ : 10^-10] := Chop[x, tol] ≠ 0
svd2[A_] := Module[{diag = DiagonalMatrix, eye = IdentityMatrix, t = Transpose,
  gs = Orthogonalize, m, n, G, vals, vecs, s, r, Sr, S, Vr, V, Ur, U},
  {m, n} = Dimensions[A]; G = t[A] . A;          (* ① *)
  {vals, vecs} = Eigensystem[G];                 (* ② *)
  s = Sqrt[Select[vals, nonzero]]; r = Length[s];     (* ③ *)
  If[r ≠ 0,
   Sr = diag[s, 0, {r, r}];                      (* ④ *)
   Vr = t[gs[Take[vecs, r]]];                    (* ⑤ *)
   Ur = A . Vr . diag[1/s, 0, {r, r}];           (* ⑥ *)
   S = diag[s, 0, {m, n}];                       (* ⑦ *)
   V = If[n == r, Vr, Join[Vr, t[gs[NullSpace[t[Vr]]]], 2]]; (* ⑧ *)
   U = If[m == r, Ur, Join[Ur, t[gs[NullSpace[t[Ur]]]], 2]], (* ⑨ *)
   (* else *)
   S = 0 A; V = eye[n]; U = eye[m];
   Sr = {{0}}; Vr = V[[All, {1}]]; Ur = U[[All, {1}]]];
  {Ur, Sr, Vr, U, S, V}]

A = {{1, 0}, {1, 1}, {0, 1}}; Map[MatrixForm, svd2[A]] (* 動作確認 *)
```

Out[ ]=

$$\left\{\begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right\}$$

In[ ]:=
```
tol = 10^-10;
isOrtho[A_] := With[{e = IdentityMatrix[Dimensions[A][[2]]]},
  Chop[Transpose[A] . A - e, tol] == 0 e]
isDiagDesc[A_] := With[{d = Diagonal[A]}, d == Sort[Abs[d], Greater]]
t = Transpose;

A = {{1, 0}, {1, 1}, {0, 1}};
{Ur, Sr, Vr, U, S, V} = svd2[A];        (* 特異値分解 *)
{isOrtho[Ur], isOrtho[Vr], isOrtho[U], isOrtho[V], (* ① *)
 SquareMatrixQ[U], SquareMatrixQ[V],       (* ② *)
 isDiagDesc[Sr], isDiagDesc[S],          (* ③ *)
 Chop[N[A] - Ur . Sr . t[Vr], tol] == 0 A,    (* ④-1 *)
 Chop[N[A] - U . S . t[V], tol] == 0 A}      (* ④-2 *)
```

Out[ ]=

```
{True, True, True, True, True, True, True, True, True, True}
```

In[ ]:=
```
Clear["Global`*"];
```

In[ ]:=
```
A = {{1, 0}, {1, 1}, {0, 1}}; PseudoInverse[A]
```

Out[ ]=

$$\left\{\left\{\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right\}, \left\{-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right\}\right\}$$

In[ ]:=
```
A = {{1, 0}, {1, 1}, {0, 1}}; b = {2, 0, 2};
PseudoInverse[A] . b
```

Out[ ]=

$$\left\{\frac{2}{3}, \frac{2}{3}\right\}$$