Global Sensitivity (GS)
Laplace Mechanism Applied to Proportions vs. Frequencies

September 4, 2018

1 Laplace Mechanism Applied to Frequencies

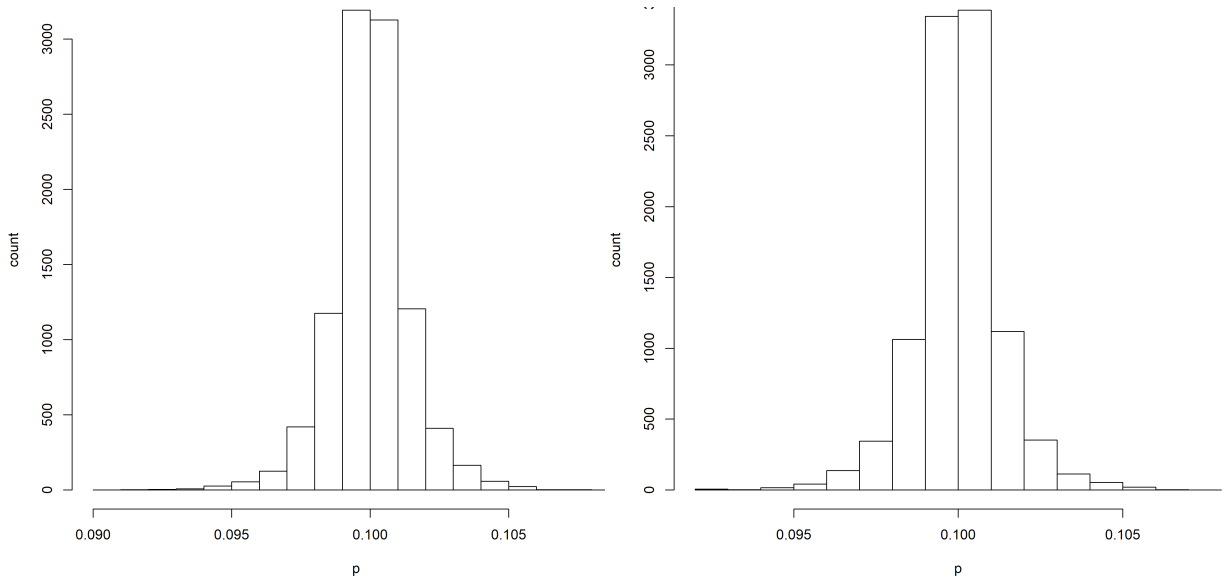
- Let k_i be the frequency of label i of a given attribute, n be the total frequency of the attribute
- GS: $\Delta f()$=1
- Laplace mechanism: $\lambda = \frac{\epsilon}{\Delta f()} \Rightarrow \text{Lap}(t, \lambda) = \frac{\epsilon}{2}\text{e}^{\epsilon}$
- 0.99 UB at $1 - \text{e}^{-\epsilon t} = 0.495 \Rightarrow t = -\frac{\ln\ 0.505}{\epsilon} \approx \frac{0.6832}{\epsilon}$
- Proportions are on interval $\quad \frac{k_i}{n} - \frac{0.6832}{n\epsilon} \leq p_i \leq \frac{k_i}{n} + \frac{0.6832}{n\epsilon}$

2 Laplace Mechanism Applied to Proportions

- Let k_i be the frequency of label i of a given attribute, n be the total frequency of the attribute
- GS: $\Delta f() = \max_i \left(\frac{k_i}{n} - \frac{k_i - 1}{n-1}\right) = \max_i \left(\frac{n-k_i}{n(n-1)}\right) = \max_i \left(\frac{1}{n-1} - \frac{k_i}{n(n-1)}\right)$
- $0 < k_i < n \Rightarrow 0 < \Delta f() < \frac{1}{n}$
- $\Delta f()$ maximized at $\min_i (k_i)$ and increases with decrease in $\min_i (k_i)$
- Laplace parameter: $\lambda = \frac{\epsilon}{\Delta f()}$, minimized at $\max \Delta f()$
- Laplace 0.99 UB at $1 - \text{e}^{-\lambda t} = 0.495 \Rightarrow t = -\frac{\ln\ 0.505}{\lambda} = \frac{0.6832\Delta f()}{\epsilon}$
- Proportions are on interval $\quad \frac{k_i}{n} - \frac{0.6832\Delta f()}{\epsilon} \leq p_i \leq \frac{k_i}{n} + \frac{0.6832\Delta f()}{\epsilon}$
  $\Rightarrow \frac{k_i}{n} - \frac{0.6832}{n\epsilon}\frac{n-k_m}{n-1} \leq p_i \leq \frac{k_i}{n} + \frac{0.6832}{n\epsilon}\frac{n-k_m}{n-1}$, where $k_m = \min(k_i)$
- Since $\frac{n-k_m}{n-1} \leq 1$ the interval for proportions with Laplace mechanism applied is within, and possibly much smaller than, the corresponding interval of proportions computed by applying the Laplace mechanism to frequencies

3 Graphical Comparison of the Laplace Mechanism Applied to Frequencies vs. Proportions

Following are histograms and Q-Q plots that compare the distribution of proportions with the Laplace mechanism applied to frequencies vs. applying the Laplace mechanism directly to computed proportions. As stated in the previous sections, the interval is at least as wide for proportions computed from Laplace adjusted frequencies than that resulting from applying the Laplace mechanism directly to proportions.



(a) Laplace mechanism applied to frequencies

(b) Laplace mechanism applied to proportions

Figure 1: Comparison of Laplace mechanism distribution, applied to frequencies vs. proportions, n=1000, min(k)=100

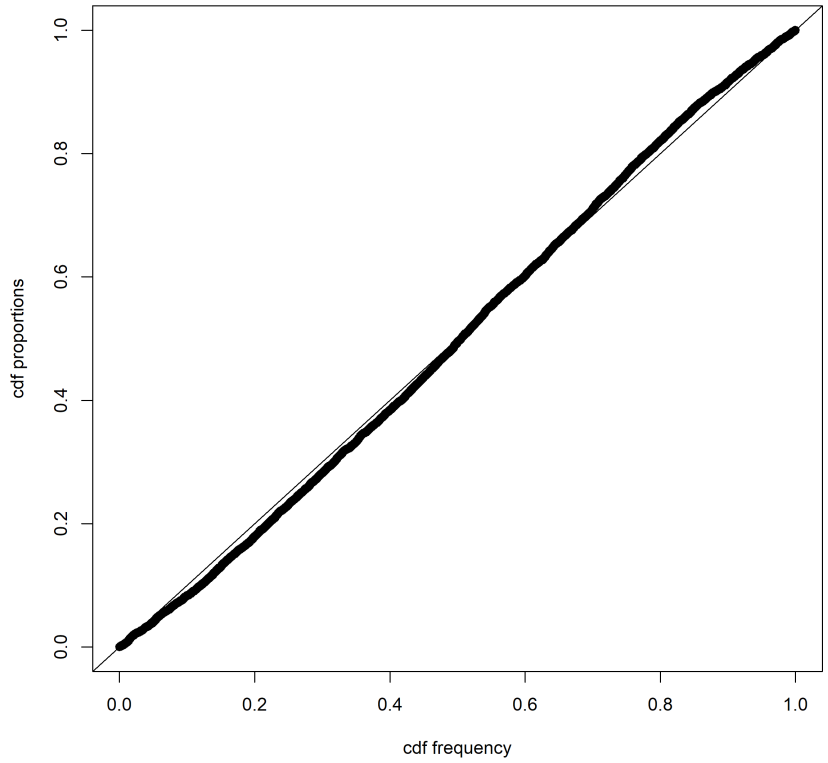
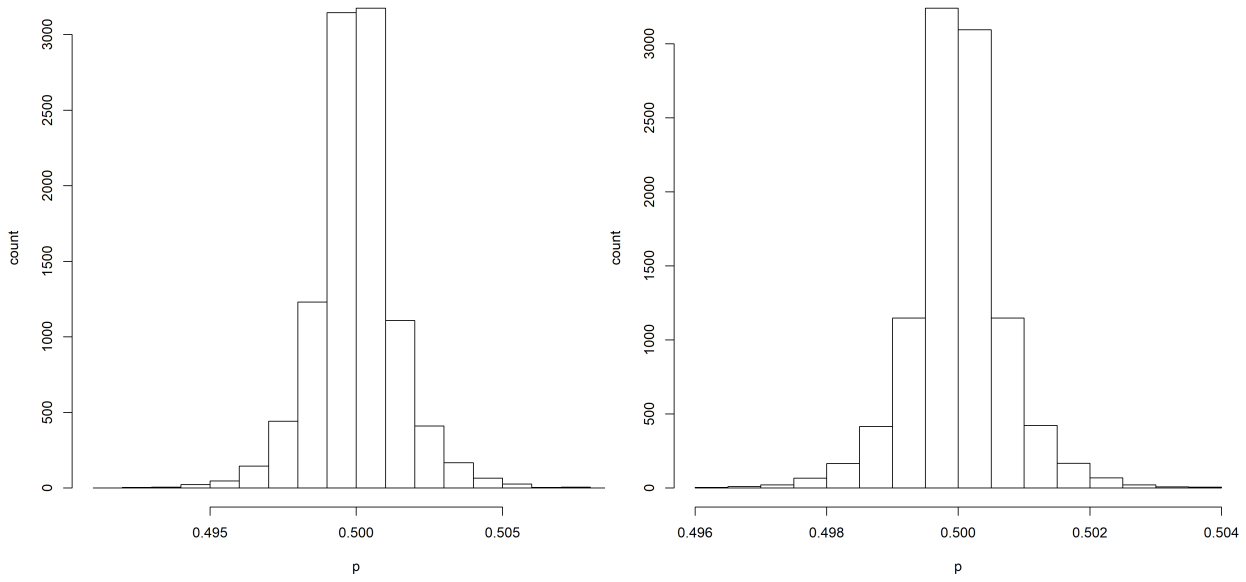


Figure 2: Q-Q plot, Laplace mechanism applied to proportions vs. frequencies, n=1000, min(k)=100



(a) Laplace mechanism applied to frequencies

(b) Laplace mechanism applied to proportions

Figure 3: Comparison of Laplace mechanism distribution, applied to frequencies vs. proportions, n=1000, min(k)=500

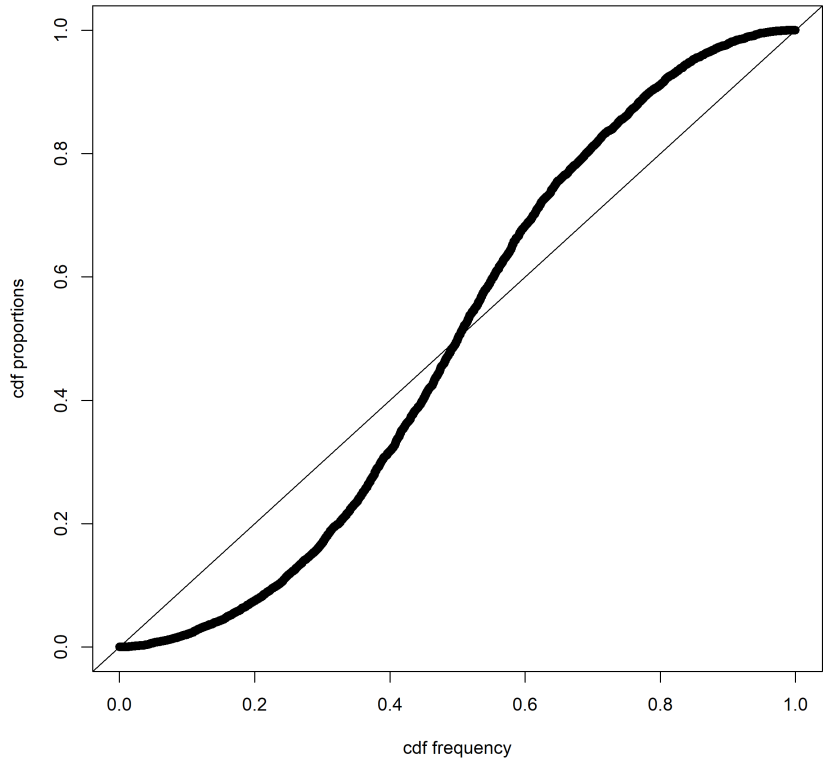


Figure 4: Q-Q plot, Laplace mechanism applied to proportions vs. frequencies, n=1000, min(k)=500