7: Evaluating, comparing and expanding models

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Introduction

This chapter focuses mostly on quantifying a model's predictive capabilities for the purposes of model selection and expansion.

New Notation!

1. f is the true model
2. y is the data we use to estimate our model
3. $\tilde{y}$ is the future (time series) or alternative (not time series) data that we test our predictions on
4. $p_{\text{post}}(\tilde{y}) = p(\tilde{y} \mid y)$
5. $p_{\text{post}}(\theta) = p(\theta \mid y)$
6. $E_{\text{post}}[\cdot]$ is taken with respect to $p(\theta \mid y)$

Definitions

A **scoring rule/function** $S(p, \tilde{y})$ is a function that takes

1. the distribution you're using to forecast p (ppd, or likelihood with estimated parameters), and
2. a realized value $\tilde{y}$

and then gives you a real-valued number/score/utility. Higher is better, although this convention isn't always followed in the literature.

Keep in mind that the realized value cannot be used to fit the data.

Examples

Example: $S(p, \tilde{y}) = -(\tilde{y} - E_p[\tilde{y}])^2$

Example: $S(p, \tilde{y}) = \log p(\tilde{y})$

Definitions

Future/unseen data is unknown, so we must take the expected score under the true distribution f:

$$E_f[S(p, \tilde{y})].$$

A scoring rule is **proper** if the above expectation is minimized when $f = p$.

A scoring rule is **local** if $S(p, \tilde{y})$ only depends on $p(\tilde{y})$ (don't care about events that didn't happen).

Note, when we are dealing with a logarithmic scoring rule, $E[-2 \log p(\tilde{y})]$ is often called an **information criterion.** The book switches back and forth between dealing with expected score, and information criteria.

Examples

Example: $S(p, \tilde{y}) = -(\tilde{y} - E_p[\tilde{y}])^2$
Most common, perhaps not local or proper for non-Gaussian data.

Example: $S(p, \tilde{y}) = \log p(\tilde{y})$
Obviously local. Proper, too (homework question).

Problem

We are generally not able to evaluate the expectation because we don't know f. However, we may be able to wait for new out-of-sample data and use a Monte-Carlo approach:

$$n^{-1} \sum_{i=1}^{n} S(p, \tilde{y}^i) \to E_f[S(p, \tilde{y})]$$

as $n \to \infty$

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as $n \to \infty$

If we can afford to wait for an infinite amount of data, though, what is the point of trying to predict it?

Problem

NB: the textbook focuses on $S(p, \tilde{y}) = \log p(\tilde{y})$, and the data are iid (after conditioning on the parameter). They call the following quantity the "elppd:"

expected log pointwise predictive density

$$E_f[\log p(\tilde{y})] = E_f\left[\log \prod_i p(\tilde{y}_i)\right]$$
$$= \sum_{i=1}^{n} E_f\left[\log p(\tilde{y}_i)\right]$$

Problem

For the moment let's use $p(\tilde{y}) = p_{\text{post}}(\tilde{y})$

The "elppd" is not obtainable because

1. you don't know f (can't directly integrate)
2. you don't have $\tilde{y}$ (no Monte-Carlo)

Problem

For the moment let's use $p(\tilde{y}) = p_{\text{post}}(\tilde{y})$

The "elppd" is not obtainable because

1. you don't know f (can't directly integrate)
2. you don't have $\tilde{y}$ (no Monte-Carlo)

Using y for $\tilde{y}$, we can come up with a rough elppd estimate called the "lppd"

log pointwise predictive density

$$\text{lppd} = \log p_{\text{post}}(y) = \sum_{i=1}^{n} \log p_{\text{post}}(y_i)$$

Problem

There'a also the problem that arises where we cannot evaluate

$$p_{\text{post}}(y) = \int p(y \mid \theta) p(\theta \mid y) \mathrm{d}\theta = E_{\text{post}}[p(y \mid \theta)]$$

The "computed lppd" again uses y for $\tilde{y}$, but it also uses Monte-Carlo to sample from the posterior

log pointwise predictive density

$$\text{computed lppd} = \log \hat{p}_{\text{post}}(y) = \sum_{i=1}^{n} \log \left(\frac{1}{S} \sum_{j=1}^{n} p(y_i \mid \theta^j) \right)$$

-Biased and probably high variance, though.

Three problems

Don't know f, don't want to wait for $\tilde{y}$...

and unfortunately, plugging the same data that we used for estimation into the predictive distribution might lead us to overfit because this strategy overestimates the average predictive score. What do we do?

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However, we can get around this in two ways generally:

1. plug in the already-used y data, but then add an extra penalty term (e.g. AIC, DIC, WAIC, etc.)
2. Cross-Validation: split the data y, many different ways, into a train and test set; estimate and evaluate on each split.

Information Criteria

AIC stands for "an information criterion" or "Akaike's Information Criterion." Let k be the number of parameters:

$$\widehat{\text{elpd}}_{\text{AIC}} = \log p(y \mid \hat{\theta}_{\text{MLE}}) - \overbrace{k}^{\text{penalty}}$$

or

$$\text{AIC} = \underbrace{-2 \log p(y \mid \hat{\theta}_{\text{MLE}})}_{\text{a deviance}} + 2k$$

We estimate $\hat{\theta}_{\text{MLE}}$ using y, and we plug y into the log likelihood.

Information Criteria

DIC replaces the point estimate with $\hat{\theta}_{\text{Bayes}} = E[\theta \mid y]$, and replaces the penalty term with p_{DIC}

$$\widehat{\text{elpd}}_{\text{DIC}} = \log p(y \mid \hat{\theta}_{\text{Bayes}}) - p_{\text{DIC}}$$

or

$$\text{DIC} = -2 \log p(y \mid \hat{\theta}_{\text{Bayes}}) + 2p_{\text{DIC}}$$

Information Criteria

The book gives two ways to estimate p_{DIC}:

1. $p_{\text{DIC}} = 2\left(\log p(y \mid \hat{\theta}_{\text{Bayes}}) - E_{\text{post}}\left[\log p(y \mid \theta)\right]\right)$
2. $p_{\text{DIC alt}} = 2\,\text{Var}_{\text{post}}\left[\log p(y \mid \theta)\right]$

Both of these can be approximated using samples from the posterior.

Information Criteria

Motivation for p_{DIC}

$$
\begin{aligned}
& E_{\tilde{y}}\left[-2\log p(\tilde{y} \mid \hat{\theta}_{\text{Bayes}})\right] \\
&= -2\log p(y \mid \hat{\theta}_{\text{Bayes}}) + E_{\tilde{y}}\left[-2\log p(\tilde{y} \mid \hat{\theta}_{\text{Bayes}})\right] + 2\log p(y \mid \hat{\theta}_{\text{Bayes}}) \\
&\approx -2\log p(y \mid \hat{\theta}_{\text{Bayes}}) + E_{\theta|y}\left[-2\log p(y \mid \theta)\right] + 2\log p(y \mid \hat{\theta}_{\text{Bayes}}) \\
&= -2\log p(y \mid \hat{\theta}_{\text{Bayes}}) + p_{\text{DIC}}
\end{aligned}
$$

Information Criteria

p_{WAIC} either stands for "widely applicable information criterion" or "Watanabe-Akaike information criterion."

The book refers to it as the most "fully Bayesian" of the three, probably because it doesn't plug in point estimates into the likelihood instead of integrating.

$$\widehat{\text{elppd}}_{\text{WAIC}} = \text{lppd} - p_{\text{WAIC}}$$

or

$$\text{WAIC} = -2\text{lppd} + 2p_{\text{WAIC}}$$

where $\text{lppd} = \sum_{i=1}^{n} \log \left(\frac{1}{S} \sum_{s=1}^{S} p(y_i \mid \theta^s) \right)$

Information Criteria

Two ways to estimate

1. $p_{\text{WAIC 1}} = 2\left(\log p(y \mid y) - E_{\theta|y}\left\{\log p(y \mid \theta)\right\}\right)$
2. $p_{\text{WAIC 2}} = \sum_{i=1}^{n} var_{\text{post}}(\log p(y_i \mid \theta))$

Both of these can be approximated using samples from the posterior.

Information Criteria

Motivation for p_{WAIC}:

$$
\begin{aligned}
& E_{\tilde{y}}\left[-2\log p(\tilde{y} \mid y)\right] \\
& E_{\tilde{y}}\left[-2\log E_{\theta|y}(p(\tilde{y} \mid \theta))\right] \\
&= -2\log p(y \mid y) + 2\left(\log p(y \mid y) - E_{\tilde{y}}\left[\log E_{\theta|y}(p(\tilde{y} \mid \theta))\right]\right) \\
&\approx -2\log p(y \mid y) + 2\left(\log p(y \mid y) - E_{\tilde{y}}\left[E_{\theta|y}(\log p(\tilde{y} \mid \theta))\right]\right) \\
&= -2\log p(y \mid y) + 2\left(\log p(y \mid y) - E_{\theta|y}\left[E_{\tilde{y}}(\log p(\tilde{y} \mid \theta))\right]\right) \\
&\approx -2\log p(y \mid y) + 2\left(\log p(y \mid y) - E_{\theta|y}\left[\log p(y \mid \theta)\right]\right) \\
&= -2\log p(y \mid y) + 2\left(\log \prod_i p(y_i \mid y) - E_{\theta|y}\left[\log \prod_i p(y_i \mid \theta)\right]\right) \\
&= -2\log p(y \mid y) + 2\sum_i \left(\log p(y_i \mid y) - E_{\theta|y}\left[\log p(y_i \mid \theta)\right]\right) \\
&= -2\log p(y \mid y) + p_{\text{WAIC1}}
\end{aligned}
$$

Cross-Validation

To assess prediction performance, one may also use **cross-validation**. Here the data is repeatedly partitioned into different training-set-test-set pairs (aka **folds**).

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1. The partitions are nonrandom, test sets are disjoint
2. for each split/estimation/prediction, we never use a data point twice
3. for each split/estimation/prediction, we lose parameter estimation accuracy because each training set is smaller than the full set
4. however, we get to average over many prediction scores, which reduces variance
5. there is still a bias that we have to estimate (but it's usually smaller than AIC/DIC/WAIC/etc.)
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The logo of this QA website illustrates the idea nicely!

Cross-Validation

leave-one-out cross-validation (loo-cv) is a special case where each test set is of size 1.

This necessarily implies that each training set is of size $n - 1$, and there are n possible splits.

If this ends up being too computationally expensive, it is also possible to do k-**fold cross-validation**, which selects k splits/folds. This means the size of each test set is n/k, and the size of each training set is $n - n/k$

Cross-Validation Notation

We only discuss loo-cv...

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If this ppd isn't tractable, we can use draws from the posterior as follows:

$$p_{\text{post}(-i)}(y_i) = \frac{1}{S}\sum_{s=1}^{S} p(y_i \mid \theta^s)$$

where θ^s are draws from $p_{\text{post}(-i)}(\theta)$

Cross-Validation

The Bayesian loo-cv estimate for out-of-sample predictive fit is

$$\text{lppd}_{\text{loo-cv}} = \sum_{i=1}^{n} \log p_{\text{post}(-i)}(y_i)$$

There are also bias-corrected versions as well.

Bayes Factors

Bayes factors are another way to compare models, two at a time. You compare each model's prior predictive distribution/marginal likelihood/integrated likelihood/evidence:

Bayes Factors

$$B_{2,1} = \frac{p(y \mid H_2)}{p(y \mid H_1)} = \frac{\int p(y \mid \theta_2, H_2) p(\theta_2 \mid H_2) \mathrm{d}\theta_2}{\int p(y \mid \theta_1, H_1) p(\theta_1 \mid H_1) \mathrm{d}\theta_1}$$

assuming $0 < p(y \mid H_i) < \infty$

NB1: models do not have to be nested, and the parameters can be of varying dimension.
NB2: Unlike frequentist hypothesis testing, it measures the *strength* of one hypothesis over another.

Bayes Factors

$$B_{2,1} = \frac{p(y|H_2)}{p(y|H_1)}$$

$\log_{10}(B_{10})$	B_{10}	Evidence against H_0
0 to 1/2	1 to 3.2	Not worth more than a bare mention
1/2 to 1	3.2 to 10	Substantial
1 to 2	10 to 100	Strong
>2	>100	Decisive

From `http://www.andrew.cmu.edu/user/kk3n/simplicity/KassRaftery1995.pdf`

Bayes Factors

The reason they call it a Bayes factor is because

$$\text{posterior odds} = \text{Bayes factor} \times \text{prior odds}$$

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$$\text{posterior odds} = \text{Bayes factor} \times \text{prior odds}$$

$$\begin{aligned}
\text{posterior odds} &= \frac{p(H_2 \mid y)}{p(H_1 \mid y)} \\
&= \frac{p(y \mid H_2)p(H_2)/p(y)}{p(y \mid H_1)p(H_1)/p(y)} && \text{(Bayes rule)} \\
&= \frac{p(y \mid H_2)}{p(y \mid H_1)} \frac{p(H_2)}{p(H_1)} \\
&= \text{Bayes factor} \times \text{prior odds}
\end{aligned}$$

Bayes Factors

You should not use improper priors when you calculate Bayes factors because

$$p(y \mid H_1) = \int p(y \mid \theta_1, H_1) p(\theta_1 \mid H_1) \mathrm{d}\theta_1$$

is not a density (homework question), and the normalizing constant will be ambiguous.

Bayes Factors

Even noninformative proper priors can be "biased" towards one of the hypotheses.

Consider the following example of the **Jeffreys-Lindley's paradox**:

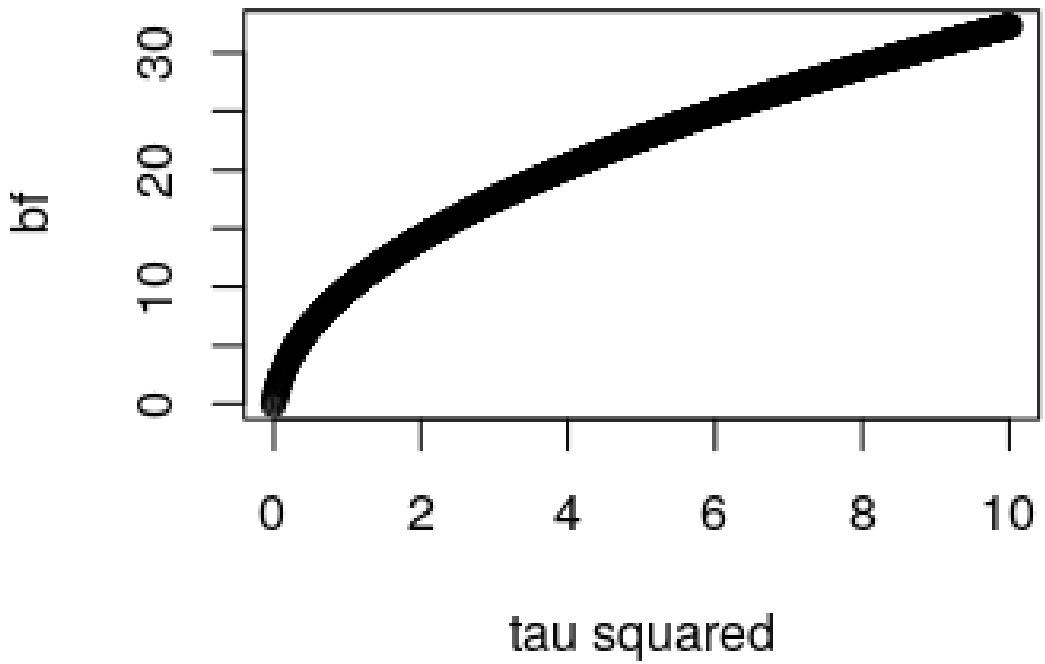
1. under H_1: $\theta = 0$ with prior probability 1
2. $p(\bar{y} \mid H_1) = (2\pi)^{-n/2} n^{n/2} \exp\left[-\frac{n}{2}\bar{y}^2\right]$
3. under H_2: $p(\theta) = N(0, \tau^2)$
4. $p(\bar{y} \mid H_2) = \int p(\bar{y} \mid \theta, H_2) p(\theta \mid H_2) \mathrm{d}\theta =$
   $[2\pi(\tau^2 + n^{-1})]^{-n/2} \exp\left[-\frac{1}{2(\tau^2+n^{-1})}\bar{y}^2\right]$

so

$$B_{1,2} = (n\tau^2 + 1)^{1/2} \exp\left[-\frac{\bar{y}^2}{2}\left(n - \frac{1}{(\tau^2 + n^{-1})}\right)\right]$$

Bayes Factors: The Jeffreys-Lindley's paradox

Say $\bar{y} = 1.5$ and $n = 10$. Then our p-value for the null is $2.101436e-06$, but



Different decisions based on whether we are frequentist or Bayesian?!

Bayes Factors

If you can't derive $p(y \mid H_i)$, then it must be approximated. Noticing that the joint $p(y \mid \theta_i, H_i)p(\theta_i \mid H_i)$ is an unnormalized target, here is the justification behind importance sampling:

$$\begin{aligned} p(y \mid H_i) &= \int p(y \mid \theta_i, H_i)p(\theta_i \mid H_i)\mathrm{d}\theta_i \\ &= \int \frac{p(y \mid \theta_i, H_i)p(\theta_i \mid H_i)}{q(\theta_i)} q(\theta_i)\mathrm{d}\theta_i \\ &\leftarrow \sum_{s=1}^{n} \frac{p(y \mid \theta_i^s, H_i)p(\theta_i^s \mid H_i)}{q(\theta_i^s)} \end{aligned}$$

where $\theta_i^s \sim q(\theta_i)$.

Importance sampling will be discussed further in chapter 10.

Bayes Factors and the "Worst Monte Carlo Method Ever"

One might tempted to use the posterior samples, too:

$$
\begin{aligned}
p(y \mid H_i) &= \left[\frac{1}{p(y \mid H_i)} \int p(\theta_i \mid H_i) \mathrm{d}\theta_i \right]^{-1} \\
&= \left[\int \frac{p(y \mid \theta_i, H_i) p(\theta_i \mid H_i)}{p(y \mid H_i) p(y \mid \theta_i, H_i)} \mathrm{d}\theta_i \right]^{-1} \\
&= \left[\int \frac{p(\theta_i \mid y, H_i)}{p(y \mid \theta_i, H_i)} \mathrm{d}\theta_i \right]^{-1} \\
&\leftarrow \left[\frac{1}{S} \sum_{s=1}^{S} \frac{1}{p(y \mid \theta_i^s, H_i)} \right]^{-1}
\end{aligned}
$$

where $\theta_i^s \sim p(\theta_i \mid y, H_i)$ are samples from the posterior.

However, this estimator often has infinite variance. There are a number of adjustments to this approach.

Bayes Factors

Under certain conditions, the **Bayesian Information Criterion** or **Schwarz Information Criterion** approximates the log of integrated likelihood.

$$BIC(H_i) = \log p(y \mid \hat{\theta}, H_i) - k \log(n)$$

where n is the number of data points, and k is the dimension of θ.

NB: you don't even need to specify a prior.