3: Introduction to multiparameter models

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Introduction

We discuss a few examples of models with more than one parameter.

A noninformative prior with a normal likelihood

Consider a normal likelihood

$$
\begin{aligned}
p(y \mid \mu, \sigma^2) &\propto (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\sum_i (y_i - \mu)^2\right] \\
&= (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\sum_i ([y_i - \bar{y}] + [\bar{y} - \mu])^2\right] \\
&= (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_i (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2 + 0\right\}\right] \\
&= (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y} - \mu)^2\right\}\right]
\end{aligned}
$$

and the noninformative, improper prior $p(\mu, \sigma^2) \propto \sigma^{-2}$. Clearly

$$
p(\mu, \sigma^2 \mid y) \propto (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y} - \mu)^2\right\}\right]
$$

A noninformative prior with a normal likelihood

Suppose instead that σ^2 is a nuisance parameter, and we're only interested in μ. Then, we want the marginal posterior.

Let $z = \frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\} = \frac{A}{2\sigma^2}$. Then

$$\begin{aligned}
p(\mu \mid y) &\propto \int (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right] d\sigma^2 \\
&= \int_\infty^0 (A/2)^{-(n+2)/2} z^{(n+2)/2} \exp\left[-z\right] (-A/2) z^{-2} dz \\
&= (A/2)^{-n/2} \underbrace{\int_0^\infty z^{n/2-1} \exp\left[-z\right] dz}_{\Gamma(n/2)}
\end{aligned}$$

A noninformative prior with a normal likelihood

So

$$\begin{aligned} p(\mu|y) &\propto (A/2)^{-n/2} \\ &\propto A^{-n/2} \\ &\propto A^{-n/2}[(n-1)s^2]^{n/2} \\ &\propto \left(1 + \frac{(\bar{y}-\mu)^2}{(n-1)s^2/n}\right)^{-n/2} \end{aligned}$$

$\mu \mid y \sim t_{n-1}(\bar{y}, s^2/n)$

A noninformative prior with a normal likelihood

Suppose that μ is a nuisance parameter, and we're only interested in σ^2. Then, we want he marginal posterior:

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right] \mathrm{d}\mu \\
&= (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{(n-1)}{2\sigma^2}s^2\right] \int \exp\left[-\frac{1}{2\sigma^2} n(\mu-\bar{y})^2\right] \mathrm{d}\mu \\
&\propto (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{(n-1)}{2\sigma^2}s^2\right] (\sigma^2)^{1/2} \\
&= (\sigma^2)^{-[(n-1)/2+1]} \exp\left[-\frac{(n-1)s^2}{2\sigma^2}\right]
\end{aligned}
$$

$\sigma^2 \mid y \sim \text{Inv-Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

A noninformative prior with a normal likelihood

Recall the joint posterior:

$$p(\mu, \sigma^2 \mid y) \propto (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right]$$

Clearly:

$$p(\mu \mid \sigma^2, y) \propto \exp\left[-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right]$$

A noninformative prior with a normal likelihood

Recall the joint posterior:

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Clearly:

$$p(\mu \mid \sigma^2, y) \propto \exp\left[-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right]$$

We also have $p(\sigma^2 \mid y)$ from the last slide. This means that we can figure out the normalizing constants for the joint posterior if we multiply these two known densities together:

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y)p(\sigma^2 \mid y).$$

Sometimes this is called a **normal-inverse-gamma** distribution.

A noninformative prior with a normal likelihood

After we have figured out the joint posterior, we may be interested in predicting new observations with the **posterior predictive distribution**:

$$p(\tilde{y} \mid y) = \iint p(\tilde{y} \mid \mu, \sigma^2) p(\mu, \sigma^2 \mid y) \mathrm{d}\mu \mathrm{d}\sigma^2.$$

A noninformative prior with a normal likelihood

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We can simulate $\tilde{y}_i$ as follows:

1. draw $\sigma_i^2 \mid y \sim p(\sigma^2 \mid y)$
2. draw $\mu_i \mid \sigma_i^2, y \sim p(\mu \mid \sigma_i^2, y)$
3. draw $\tilde{y}_i \mid \mu_i, \sigma_i^2 \sim p(\tilde{y} \mid \mu_i, \sigma_i^2)$

A noninformative prior with a normal likelihood

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A noninformative prior with a normal likelihood

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It's a homework question to show that

$$\tilde{y} \mid y \sim t_{n-1}\left(\bar{y}, s^2\left(1 + \frac{1}{n}\right)\right)$$

A noninformative prior with a normal likelihood

Let's get some practice simulating predictions, which will come in handy when we are dealing with more complicated scenarios where a closed-form posterior predictive distribution isn't available. We can simulate each $\tilde{y}_i$ as follows:

Sampling Strategy

For $i = 1, 2, \dots$

1. draw $\sigma_i^2 \mid y \sim p(\sigma^2 \mid y)$
2. draw $\mu_i \mid \sigma_i^2, y \sim p(\mu \mid \sigma_i^2, y)$
3. draw $\tilde{y}_i \mid \mu_i, \sigma_i^2 \sim p(\tilde{y} \mid \mu_i, \sigma_i^2)$

Each triple
$(\tilde{y}_i, \mu_i, \sigma_i^2) \sim p(\tilde{y}, \mu, \sigma^2 \mid y) = p(\tilde{y} \mid \mu, \sigma^2)p(\mu \mid \sigma^2 \mid y)p(\sigma^2 \mid y)$.

So $\tilde{y}_i \sim p(\tilde{y} \mid y) = \iint p(\tilde{y} \mid \mu, \sigma^2)p(\mu, \sigma^2 \mid y)\mathrm{d}\mu\mathrm{d}\sigma^2$

Tip 1: If the joint is easier to sample from

If you simulate $(\tilde{y}^i, \theta_1^i, \theta_2^i)_{i=1}^n \sim p(\tilde{y}, \theta_1, \theta_2 \mid y)$, then ignoring pieces of each sample is analogous to sampling from the marginal:

$$n^{-1} \sum_{i=1}^{n} h(\tilde{y}^i) \to E_{\tilde{y}, \theta_1, \theta_2}[h(\tilde{y}^i)] = E_{\tilde{y}}[h(\tilde{y}^i)]$$

Tip 2: if the "top" factor of a joint is tractable

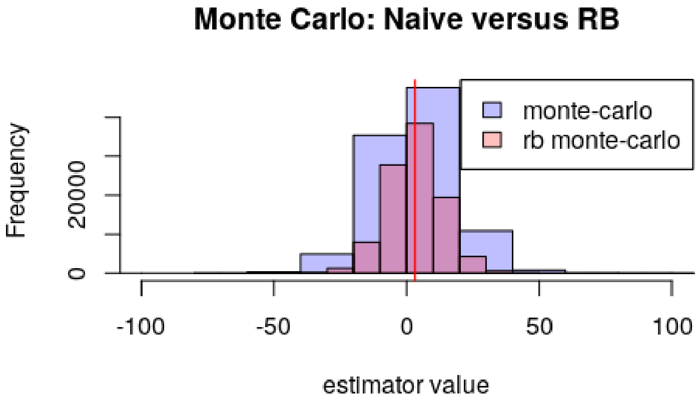
If $p(\tilde{y}, \theta_1, \theta_2 \mid y) = p(\tilde{y} \mid \theta_1, \theta_2, y)p(\theta_1, \theta_2 \mid y)$, then

$$
\begin{aligned}
n^{-1} \sum_{i=1}^{n} E[h(\tilde{y}, \theta_1^i, \theta_2^i) \mid \theta_1^i, \theta_2^i, y] &\to E\left(E[h(\tilde{y}, \theta_1, \theta_2) \mid \theta_1, \theta_2, y]\right) \\
&= E[h(\tilde{y}, \theta_1, \theta_2) \mid y]
\end{aligned}
$$

If you can derive expectations of $p(\tilde{y} \mid \theta_1, \theta_2, y)$, and you can sample from the other piece, then this **Rao-Blackwellization** or **marginalization** strategy can be a useful variance reduction technique.

A comparison in R

See `3.r` for details:



Another multiparameter example of conjugacy: Dirichlet-multinomial

Let $y = (y_1, y_2, \ldots, y_k)$ be a vector of counts. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ be the probabilities of any trial resulting in each of the k outcomes. We assume that there is a known total count (which means $\sum_i y_i = n$) and that the only possible outcomes are these k outcomes $\sum_i \theta_i = 1$.

The likelihood is a multinomial distribution

$$p(y \mid \theta) \propto \prod_{i=1}^{k} \theta_i^{y_i},$$

and the prior is a Dirichlet distribution

$$p(\theta \mid \alpha) \propto \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}.$$

The hyper-parameters have a very nice interpretation of counts!

Multivariate Normal Observations

Let each observation y follow a multivariate normal distribution. The likelihood $p(y_1, \ldots, y_n \mid \mu, \Sigma)$ is usefully written with a few properties of the trace operator:

$$
\begin{aligned}
&\propto \det(\Sigma)^{-n/2} \exp\left(-\frac{1}{2}\sum_i (y_i - \mu)'\Sigma^{-1}(y_i - \mu)\right) \\
&= \det(\Sigma)^{-n/2} \exp\left[-\frac{1}{2}\sum_i \mathrm{tr}\{(y_i - \mu)'\Sigma^{-1}(y_i - \mu)\}\right] \\
&= \det(\Sigma)^{-n/2} \exp\left[-\frac{1}{2}\sum_i \mathrm{tr}\{\Sigma^{-1}(y_i - \mu)(y_i - \mu)'\}\right] \\
&= \det(\Sigma)^{-n/2} \exp\left[-\frac{1}{2}\mathrm{tr}\left\{\Sigma^{-1}\overbrace{\sum_i (y_i - \mu)(y_i - \mu)'}^{S_0}\right\}\right]
\end{aligned}
$$

Multivariate Normal Observations with known covariance matrix

A conjugate prior for $p(y \mid \mu) \propto \det(\Sigma)^{-n/2} \exp\left[-\frac{1}{2}\text{tr}\left\{\overbrace{\Sigma^{-1}}^{\text{known}} S_0\right\}\right]$ is

$$p(\mu \mid \mu_0, \Lambda_0) = \det(\Lambda_0)^{-1/2} \exp\left[(\mu - \mu_0)'\Lambda_0^{-1}(\mu - \mu_0)\right]$$

This makes the posterior distribution (homework question exercise 3.13) normal with mean and precision

$$\mu_n = (\Lambda_0 + n\Sigma^{-1})^{-1}(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y})$$

$$\Lambda_n^{-1} = \Lambda_0^{-1} + n\Sigma^{-1}.$$

Multivariate Normal Observations with unknown covariance matrix

When all of the elements of Σ are unknown, we need a prior for that as well. This prior must put zero mass on matrices that aren't positive definite or aren't symmetric.

A popular option is the **inverse Wishart** distribution, which is analagous to the inverse-Gamma distribution. It has a degrees of freedom parameter: ν_0. And it has a scale matrix parameter Λ_0.

Multivariate Normal Observations with unknown covariance matrix

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A popular option is the **inverse Wishart** distribution, which is analagous to the inverse-Gamma distribution. It has a degrees of freedom parameter: ν_0. And it has a scale matrix parameter Λ_0.

If $\Sigma \in \mathbb{R}^{d \times d}$, we will write

$$\Sigma \sim \text{Inv-Wishart}_{\nu_0}(\Lambda_0^{-1})$$

and we can write (something proportional to) the density as

$$p(\Sigma) \propto \det(\Sigma)^{-(\nu_0+d+1)/2} \exp\left(-\frac{1}{2}\text{tr}\left[\Lambda_0 \Sigma^{-1}\right]\right)$$

Multivariate Normal Observations with unknown covariance matrix

The following is a conjugate prior

$$
\begin{aligned}
p(\mu \mid \Sigma)p(\Sigma) &= \mathrm{N}(\mu_0, \Sigma/\kappa_0)\text{Inv-Wishart}_{\nu_0}(\Lambda_0^{-1}) \\
&\propto \left[\det(\Sigma)^{-1/2} \exp\left(-\frac{\kappa_0}{2}(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)\right)\right] \times \\
&\quad \left[\det(\Sigma)^{-(\nu_0+d+1)/2} \exp\left(-\frac{1}{2}\mathrm{tr}\left[\Lambda_0\Sigma^{-1}\right]\right)\right]
\end{aligned}
$$

Multivariate Normal Observations with unknown covariance matrix

Here's the posterior:

$$
\begin{aligned}
p(\mu, \Sigma \mid y) &\propto p(y \mid \mu, \Sigma) p(\mu \mid \Sigma) p(\Sigma) \\
&\propto \det(\Sigma)^{-n/2} \exp\left[-\frac{1}{2} \text{tr} \left\{ \Sigma^{-1} \sum_i (\mu - y_i)(\mu - y_i)' \right\} \right] \times \\
&\quad \det(\Sigma)^{-1/2} \exp\left(-\frac{\kappa_0}{2} (\mu - \mu_0)' \Sigma^{-1} (\mu - \mu_0) \right) \times \\
&\quad \det(\Sigma)^{-(\nu_0 + d + 1)/2} \exp\left(-\frac{1}{2} \text{tr}\left[\Lambda_0 \Sigma^{-1} \right] \right) \\
&= \cdots
\end{aligned}
$$

Multivariate Normal Observations with unknown covariance matrix

It helps to recognize $p(\mu \mid \Sigma, y)$ first, and then $p(\Sigma \mid y)$. Here is negative twice the log of the exponent:

$$
\begin{aligned}
&\operatorname{tr}\left\{\Sigma^{-1}\sum_i(\mu - y_i)(\mu - y_i)' + \kappa_0(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)\right\} + c_1 \\
&= \sum_i(\mu - y_i)'\Sigma^{-1}(\mu - y_i) + \kappa_0(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0) + c_1 \\
&= n\mu'\Sigma^{-1}\mu - 2n\mu'\Sigma^{-1}\bar{y} + \kappa_0\mu'\Sigma^{-1}\mu - 2\kappa_0\mu'\Sigma^{-1}\mu_0 + c_2 \\
&= \mu'\left[(\Sigma/n)^{-1} + (\Sigma/\kappa_0)^{-1}\right]\mu - 2\mu'\left[(\Sigma/n)^{-1}\bar{y} + (\Sigma/\kappa_0)^{-1}\mu_0\right] + c_2 \\
&= (\mu-\mu_n)'B(\mu-\mu_n) + c_3
\end{aligned}
$$

where $B = (\Sigma/n)^{-1} + (\Sigma/\kappa_0)^{-1}$ and $\mu_n = B^{-1}\left[(\Sigma/n)^{-1}\bar{y} + (\Sigma/\kappa_0)^{-1}\mu_0\right]$

Multivariate Normal Observations with unknown covariance matrix

Clearly

$$B = (\Sigma/n)^{-1} + (\Sigma/\kappa_0)^{-1} = \Sigma^{-1}(n + \kappa_0)$$

and

$$\mu_n = B^{-1}\left[(\Sigma/n)^{-1}\bar{y} + (\Sigma/\kappa_0)^{-1}\mu_0\right] = \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y}$$

Multivariate Normal Observations with unknown covariance matrix

Back to neg. twice the log-exponent of the *entire* posterior (can't ignore Σ anymore so keep track of c_1, c_2, c_3)

$$(\mu - \mu_n)' B (\mu - \mu_n) - \mu_n' B \mu_n + \operatorname{tr}\left[\Lambda_0 \Sigma^{-1} + \sum_i y_i' \Sigma^{-1} y_i + \kappa_0 \mu_0 \mu_0' \Sigma^{-1}\right]$$

$$= (\mu - \mu_n)' B (\mu - \mu_n) - \mu_n' B \mu_n + \operatorname{tr}\left[\left(\Lambda_0 + \sum_i y_i y_i' + \kappa_0 \mu_0 \mu_0'\right) \Sigma^{-1}\right]$$

$$= (\mu - \mu_n)' B (\mu - \mu_n) +$$

$$\operatorname{tr}\left[\underbrace{\left(\Lambda_0 + \sum_i y_i y_i' + \kappa_0 \mu_0 \mu_0' - (n + \kappa_0)\mu_n \mu_n'\right)}_{\text{hw is to show that this equals } \Lambda_n} \Sigma^{-1}\right]$$

A few notes on example 3.7

1. The setup of example 3.7 will be re-used again and again in later chapters
2. It's a logistic regression model with two parameters: slope and intercept
3. Groups: $i = 1, 2, 3, 4$
4. For each group, sample size n_i is known
5. For each group, y_i is a count (tumors, deaths, etc.)
6. For each group, x_i is a dose (continuous amount of treatment for each group)

A few notes on example 3.7

For each group:

$$y_i \mid \alpha, \beta \sim \text{Binomial}(n_i, \text{invlogit}(\alpha + \beta x_i))$$

We can write $\theta_i = \text{invlogit}(\alpha + \beta x_i)$ to make it cleaner, but note that this isn't introducing more parameters. The likelihood is

$$p(y \mid \alpha, \beta) = \prod_{i=1}^{4} \theta_i^{y_i} (1 - \theta_i)^{n_i - y_i}$$

A few notes on example 3.7

For each group:

$$y_i \mid \alpha, \beta \sim \text{Binomial}(n_i, \text{invlogit}(\alpha + \beta x_i)) \tag{1}$$

1. The **dose-response** is the relationship between x_i and θ_i (which is assumed the same for each group i).
2. **LD-50** is the unknown quantity $-\alpha/\beta$. It only makes sense when $\beta > 0$, and it is the value of x_i that yields $\theta_i = .5$ (plug it into eqn (1) above). Sometimes scientists are more interested in estimating this than they are in estimating individual parameters.
3. One of the authors has provided `R` code: `https://github.com/avehtari/BDA_R_demos/tree/master/demos_ch3`