

## 7: Evaluating, comparing and expanding models

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This chapter focuses mostly on quantifying a model's predictive capabilities for the purposes of model selection and expansion.

# New Notation!

- 1  $f$  is the true model
- 2  $y$  is the data we use to estimate our model
- 3  $\tilde{y}$  is the future (time series) or alternative (not time series) data that we test our predictions on
- 4  $p_{\text{post}}(\tilde{y}) = p(\tilde{y} \mid y)$
- 5  $p_{\text{post}}(\theta) = p(\theta \mid y)$
- 6  $E_{\text{post}}[\cdot]$  is taken with respect to  $p(\theta \mid y)$

A **scoring rule/function**  $S(p, \tilde{y})$  is a function that takes

- 1 the distribution you're using to forecast  $p$  (ppd, or likelihood with estimated parameters), and
- 2 a realized value  $\tilde{y}$

and then gives you a real-valued number/score/utility. Higher is better, although this convention isn't always followed in the literature.

Keep in mind that the realized value cannot be used to fit the data.

# Examples

Example:  $S(p, \tilde{y}) = -(\tilde{y} - E_p[\tilde{y}])^2$

Example:  $S(p, \tilde{y}) = \log p(\tilde{y})$

Future/unseen data is unknown, so we must take the expected score under the true distribution  $f$ :

$$E_f[S(p, \tilde{y})].$$

A scoring rule is **proper** if the above expectation is minimized when  $f = p$ .

A scoring rule is **local** if  $S(p, \tilde{y})$  only depends on  $p(\tilde{y})$  (don't care about events that didn't happen).

Note, when we are dealing with a logarithmic scoring rule,  $E[-2 \log p(\tilde{y})]$  is often called an **information criterion**. The book switches back and forth between dealing with expected score, and information criteria.

# Examples

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Most common, perhaps not local or proper for non-Gaussian data.

Example:  $S(p, \tilde{y}) = \log p(\tilde{y})$

Obviously local. Proper, too (homework question).

# Problem

We are generally not able to evaluate the expectation because we don't know  $f$ . However, we may be able to wait for new out-of-sample data and use a Monte-Carlo approach:

$$n^{-1} \sum_{i=1}^n S(p, \tilde{y}^i) \rightarrow E_f[S(p, \tilde{y})]$$

as  $n \rightarrow \infty$



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as  $n \rightarrow \infty$

If we can afford to wait for an infinite amount of data, though, what is the point of trying to predict it?

# Problem

NB: the textbook focuses on  $S(p, \tilde{y}) = \log p(\tilde{y})$ , and the data are iid (after conditioning on the parameter). They call the following quantity the “elppd:”

expected log pointwise predictive density

$$\begin{aligned} E_f[\log p(\tilde{y})] &= E_f \left[ \log \prod_i p(\tilde{y}_i) \right] \\ &= \sum_{i=1}^n E_f [\log p(\tilde{y}_i)] \end{aligned}$$

# Problem

For the moment let's use  $p(\tilde{y}) = p_{\text{post}}(\tilde{y})$

The “elppd” is not obtainable because

- ① you don't know  $f$  (can't directly integrate)
- ② you don't have  $\tilde{y}$  (no Monte-Carlo)

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Using  $y$  for  $\tilde{y}$ , we can come up with a rough elppd estimate called the “lppd”

log pointwise predictive density

$$\text{lppd} = \log p_{\text{post}}(y) = \sum_{i=1}^n \log p_{\text{post}}(y_i)$$

# Problem

There's also the problem that arises where we cannot evaluate

$$p_{\text{post}}(y) = \int p(y | \theta) p(\theta | y) d\theta = E_{\text{post}}[p(y | \theta)]$$

The “computed lppd” again uses  $y$  for  $\tilde{y}$ , but it also uses Monte-Carlo to sample from the posterior

log pointwise predictive density

$$\text{computed lppd} = \log \hat{p}_{\text{post}}(y) = \sum_{i=1}^n \log \left( \frac{1}{S} \sum_{j=1}^S p(y_i | \theta^j) \right)$$

-Biased and probably high variance, though.

# Three problems

Don't know  $f$ , don't want to wait for  $\tilde{y}$ ...

and unfortunately, plugging the same data that we used for estimation into the predictive distribution might lead us to overfit because this strategy overestimates the average predictive score. What do we do?

# Three problems

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However, we can get around this in two ways generally:

- 1 plug in the already-used  $y$  data, but then add an extra penalty term (e.g. AIC, DIC, WAIC, etc.)
- 2 Cross-Validation: split the data  $y$ , many different ways, into a train and test set; estimate and evaluate on each split.

**AIC** stands for “an information criterion” or “Akaike’s Information Criterion.” Let  $k$  be the number of parameters:

$$\widehat{\text{elpd}}_{\text{AIC}} = \log p(y \mid \hat{\theta}_{\text{MLE}}) - \underbrace{k}_{\text{penalty}}$$

or

$$\text{AIC} = \underbrace{-2 \log p(y \mid \hat{\theta}_{\text{MLE}})}_{\text{a deviance}} + 2k$$

We estimate  $\hat{\theta}_{\text{MLE}}$  using  $y$ , and we plug  $y$  into the log likelihood.



**DIC** replaces the point estimate with  $\hat{\theta}_{\text{Bayes}} = E[\theta | y]$ , and replaces the penalty term with  $p_{\text{DIC}}$

$$\widehat{\text{elpd}}_{\text{DIC}} = \log p(y | \hat{\theta}_{\text{Bayes}}) - p_{\text{DIC}}$$

or

$$\text{DIC} = -2 \log p(y | \hat{\theta}_{\text{Bayes}}) + 2p_{\text{DIC}}$$

The book gives two ways to estimate  $p_{\text{DIC}}$ :

- ①  $p_{\text{DIC}} = 2 \left( \log p(y \mid \hat{\theta}_{\text{Bayes}}) - E_{\text{post}} [\log p(y \mid \theta)] \right)$
- ②  $p_{\text{DIC alt}} = 2 \text{Var}_{\text{post}} [\log p(y \mid \theta)]$

Both of these can be approximated using samples from the posterior.

## Motivation for $p_{\text{DIC}}$

$$\begin{aligned} & E_{\tilde{y}} \left[ -2 \log p(\tilde{y} \mid \hat{\theta}_{\text{Bayes}}) \right] \\ &= -2 \log p(y \mid \hat{\theta}_{\text{Bayes}}) + E_{\tilde{y}} \left[ -2 \log p(\tilde{y} \mid \hat{\theta}_{\text{Bayes}}) \right] + 2 \log p(y \mid \hat{\theta}_{\text{Bayes}}) \\ &\approx -2 \log p(y \mid \hat{\theta}_{\text{Bayes}}) + E_{\theta|y} \left[ -2 \log p(y \mid \theta) \right] + 2 \log p(y \mid \hat{\theta}_{\text{Bayes}}) \\ &= -2 \log p(y \mid \hat{\theta}_{\text{Bayes}}) + p_{\text{DIC}} \end{aligned}$$

$p_{\text{WAIC}}$  either stands for “widely applicable information criterion” or “Watanabe-Akaike information criterion.”

The book refers to it as the most “fully Bayesian” of the three, probably because it doesn’t plug in point estimates into the likelihood instead of integrating.

$$\widehat{\text{elppd}}_{\text{WAIC}} = \text{lppd} - p_{\text{WAIC}}$$

or

$$\text{WAIC} = -2\text{lppd} + 2p_{\text{WAIC}}$$

where  $\text{lppd} = \sum_{i=1}^n \log \left( \frac{1}{S} \sum_{s=1}^S p(y_i | \theta^s) \right)$

Two ways to estimate

- ①  $p_{\text{WAIC } 1} = 2 (\log p(y | y) - E_{\theta|y} \{\log p(y | \theta)\})$
- ②  $p_{\text{WAIC } 2} = \sum_{i=1}^n \text{var}_{\text{post}}(\log p(y_i | \theta))$

Both of these can be approximated using samples from the posterior.

Motivation for  $p_{\text{WAIC}}$ :

$$\begin{aligned} & E_{\tilde{y}} [-2 \log p(\tilde{y} | y)] \\ & E_{\tilde{y}} [-2 \log E_{\theta|y}(p(\tilde{y} | \theta))] \\ &= -2 \log p(y | y) + 2 (\log p(y | y) - E_{\tilde{y}} [\log E_{\theta|y}(p(\tilde{y} | \theta))]) \\ &\approx -2 \log p(y | y) + 2 (\log p(y | y) - E_{\tilde{y}} [E_{\theta|y}(\log p(\tilde{y} | \theta))]) \\ &= -2 \log p(y | y) + 2 (\log p(y | y) - E_{\theta|y} [E_{\tilde{y}}(\log p(\tilde{y} | \theta))]) \\ &\approx -2 \log p(y | y) + 2 (\log p(y | y) - E_{\theta|y} [\log p(y | \theta)]) \\ &= -2 \log p(y | y) + 2 \left( \log \prod_i p(y_i | y) - E_{\theta|y} \left[ \log \prod_i p(y_i | \theta) \right] \right) \\ &= -2 \log p(y | y) + 2 \sum_i (\log p(y_i | y) - E_{\theta|y} [\log p(y_i | \theta)]) \\ &= -2 \log p(y | y) + p_{\text{WAIC1}} \end{aligned}$$

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- 1 The partitions are nonrandom, test sets are disjoint
- 2 for each split/estimation/prediction, we never use a data point twice
- 3 for each split/estimation/prediction, we lose parameter estimation accuracy because each training set is smaller than the full set
- 4 however, we get to average over many prediction scores, which reduces variance
- 5 there is still a bias that we have to estimate (but it's usually smaller than AIC/DIC/WAIC/etc.)
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The logo of this QA website illustrates the idea nicely!

**leave-one-out cross-validation** (loo-cv) is a special case where each test set is of size 1.

This necessarily implies that each training set is of size  $n - 1$ , and there are  $n$  possible splits.

If this ends up being too computationally expensive, it is also possible to do  **$k$ -fold cross-validation**, which selects  $k$  splits/folds. This means the size of each test set is  $n/k$ , and the size of each training set is  $n - n/k$

# Cross-Validation Notation

We only discuss loo-cv...

$p_{\text{post}(-i)}(y_i)$  is the prediction for the  $i$ th point, using the ppd, which uses the posterior distribution conditioning on all values of the data **except the  $i$ th**

# Cross-Validation Notation

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$p_{\text{post}(-i)}(y_i)$  is the prediction for the  $i$ th point, using the ppd, which uses the posterior distribution conditioning on all values of the data **except the  $i$ th**

If this ppd isn't tractable, we can use draws from the posterior as follows:

$$p_{\text{post}(-i)}(y_i) = \frac{1}{S} \sum_{s=1}^S p(y_i \mid \theta^s)$$

where  $\theta^s$  are draws from  $p_{\text{post}(-i)}(\theta)$

The Bayesian loo-cv estimate for out-of-sample predictive fit is

$$\text{lppd}_{\text{loo-cv}} = \sum_{i=1}^n \log p_{\text{post}(-i)}(y_i)$$

There are also bias-corrected versions as well.

**Bayes factors** are another way to compare models, two at a time. You compare each model's prior predictive distribution

## Bayes Factors

$$\begin{aligned} B_{2,1} &= \frac{p(y \mid H_2)}{p(y \mid H_1)} \\ &= \frac{\int p(y \mid \theta_2, H_2) p(\theta_2 \mid H_2) d\theta_2}{\int p(y \mid \theta_1, H_1) p(\theta_1 \mid H_1) d\theta_1} \end{aligned}$$

assuming  $0 < p(y \mid H_i) < \infty$

NB: models do not have to be nested, and the parameters can be of varying dimension

# Bayes Factors

$$B_{2,1} = \frac{p(y|H_2)}{p(y|H_1)}$$

| $\log_{10}(B_{10})$ | $B_{10}$  | Evidence against $H_0$             |
|---------------------|-----------|------------------------------------|
| 0 to 1/2            | 1 to 3.2  | Not worth more than a bare mention |
| 1/2 to 1            | 3.2 to 10 | Substantial                        |
| 1 to 2              | 10 to 100 | Strong                             |
| >2                  | >100      | Decisive                           |

From <http://www.andrew.cmu.edu/user/kk3n/simplicity/KassRaftery1995.pdf>

# Bayes Factors

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$$\begin{aligned}\text{posterior odds} &= \frac{p(H_2 | y)}{p(H_1 | y)} \\ &= \frac{p(y | H_2)p(H_2)/p(y)}{p(y | H_1)p(H_1)/p(y)} && \text{(Bayes rule)} \\ &= \frac{p(y | H_2)}{p(y | H_1)} \frac{p(H_2)}{p(H_1)} \\ &= \text{Bayes factor} \times \text{prior odds}\end{aligned}$$

You should not use improper priors when you calculate Bayes factors because

$$p(y | H_1) = \int p(y | \theta_1, H_1)p(\theta_1 | H_1)d\theta_1$$

is not a density (homework question), and the normalizing constant will be ambiguous.

Even noninformative proper priors can be “biased” towards one of the hypotheses.

Consider the following example of the **Jeffreys-Lindley's paradox**:

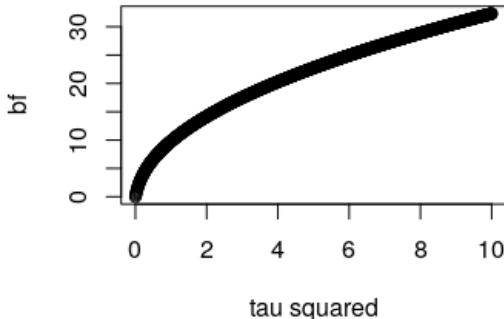
- ①  $p(\bar{y} \mid \theta, H_1) = (2\pi)^{-n/2} n^{n/2} \exp \left[ -\frac{n}{2} \bar{y}^2 \right]$
- ② under  $H_1$ :  $\theta = 0$  with prior probability 1
- ③ under  $H_2$ :  $p(\theta) = N(0, \tau^2)$
- ④  $p(\bar{y} \mid H_2) = [2\pi(\tau^2 + n^{-1})]^{-n/2} \exp \left[ -\frac{1}{2(\tau^2 + n^{-1})} \bar{y}^2 \right]$

so

$$B_{1,2} = (n\tau^2 + 1)^{1/2} \exp \left[ -\frac{\bar{y}^2}{2} \left( n - \frac{1}{(\tau^2 + n^{-1})} \right) \right]$$

# Bayes Factors: The Jeffreys-Lindley's paradox

Say  $\bar{y} = 1.5$  and  $n = 10$ . Then our p-value for the null is  $2.101436e - 06$ , but



Different decisions based on whether we are frequentist or Bayesian?!

# Bayes Factors

If you can't derive  $p(y | H_i)$ , then you can resort to simulation. Notice that the joint  $p(y | \theta_i, H_i)p(\theta_i | H_i)$  is an unnormalized target!

For instance, here is the justification behind importance sampling:

$$\begin{aligned} p(y | H_i) &= \int p(y | \theta_i, H_i)p(\theta_i | H_i)d\theta_i \\ &= \int \frac{p(y | \theta_i, H_i)p(\theta_i | H_i)}{q(\theta_i)}q(\theta_i)d\theta_i \\ &\leftarrow \sum_{s=1}^n \frac{p(y | \theta_i^s, H_i)p(\theta_i^s | H_i)}{q(\theta_i^s)} \end{aligned}$$

where  $\theta_i^s \sim q(\theta_i)$

# Bayes Factors and the “Worst Monte Carlo Method Ever”

One might be tempted to use the posterior samples, too:

$$\begin{aligned} p(y \mid H_i) &= \left[ \frac{1}{p(y \mid H_i)} \int p(\theta_i \mid H_i) d\theta_i \right]^{-1} \\ &= \left[ \int \frac{p(y \mid \theta_i, H_i) p(\theta_i \mid H_i)}{p(y \mid H_i) p(y \mid \theta_i, H_i)} d\theta_i \right]^{-1} \\ &= \left[ \int \frac{p(\theta_i \mid y, H_i)}{p(y \mid \theta_i, H_i)} d\theta_i \right]^{-1} \\ &\leftarrow \left[ \frac{1}{S} \sum_{s=1}^S \frac{1}{p(y \mid \theta_i^s, H_i)} \right]^{-1} \end{aligned}$$

where  $\theta_i^s \sim p(\theta_i \mid y, H_i)$  are samples from the posterior.

However, this estimator often has infinite variance, and so it shouldn't be used.