3: Introduction to multiparameter models

Taylor

University of Virginia

Introduction

We discuss a few examples of models with more than one parameter.

A noninformative prior with a normal likelihood

Consider a normal likelihood

$$
\begin{aligned}
p(y \mid \mu, \sigma^2) &\propto (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\sum_i (y_i - \mu)^2\right] \\
&= (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\sum_i ([y_i - \bar{y}] + [\bar{y} - \mu])^2\right] \\
&= (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_i (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2 + 0\right\}\right] \\
&= (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y} - \mu)^2\right\}\right]
\end{aligned}
$$

and the noninformative, improper prior $p(\mu, \sigma^2) \propto \sigma^{-2}$. Clearly

$$
p(\mu, \sigma^2 \mid y) \propto (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y} - \mu)^2\right\}\right]
$$

A noninformative prior with a normal likelihood

Suppose instead that σ^2 is a nuisance parameter, and we're only interested in μ. Then, we want the marginal posterior.

Let $z = \frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\} = \frac{A}{2\sigma^2}$. Then

$$
\begin{aligned}
p(\mu \mid y) &\propto \int (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right] \mathrm{d}\sigma^2 \\
&= \int_{\infty}^{0} (A/2)^{-(n+2)/2} z^{(n+2)/2} \exp\left[-z\right] (-A/2) z^{-2} \mathrm{d}z \\
&= (A/2)^{-n/2} \underbrace{\int_0^{\infty} z^{n/2-1} \exp\left[-z\right] \mathrm{d}z}_{\Gamma(n/2)}
\end{aligned}
$$

A noninformative prior with a normal likelihood

So

$$
\begin{aligned}
p(\mu|y) &\propto (A/2)^{-n/2} \\
&\propto A^{-n/2} \\
&\propto A^{-n/2}[(n-1)s^2]^{n/2} \\
&\propto \left(1 + \frac{(\bar{y}-\mu)^2}{(n-1)s^2/n}\right)^{-n/2}
\end{aligned}
$$

$\mu \mid y \sim t_{n-1}(\bar{y}, s^2/n)$

A noninformative prior with a normal likelihood

Suppose that μ is a nuisance parameter, and we're only interested in σ^2. Then, we want he marginal posterior:

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right] \mathrm{d}\mu \\
&= (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{(n-1)}{2\sigma^2}s^2\right] \int \exp\left[-\frac{1}{2\sigma^2} n(\mu-\bar{y})^2\right] \mathrm{d}\mu \\
&\propto (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{(n-1)}{2\sigma^2}s^2\right] (\sigma^2)^{1/2} \\
&= (\sigma^2)^{-[(n-1)/2+1]} \exp\left[-\frac{(n-1)s^2}{2\sigma^2}\right]
\end{aligned}
$$

$\sigma^2 \mid y \sim \text{Inv-Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

A noninformative prior with a normal likelihood

Recall the joint posterior:

$$p(\mu, \sigma^2 \mid y) \propto (\sigma^2)^{-(n+2)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right]$$

Clearly:

$$p(\mu \mid \sigma^2, y) \propto \exp\left[-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right]$$

A noninformative prior with a normal likelihood

Recall the joint posterior:

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We also have $p(\sigma^2 \mid y)$ from the last slide. This means that we can figure out the normalizing constants for the joint posterior if we multiply these two known densities together:

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y).$$

Sometimes this is called a **normal-inverse-gamma** distribution.

A noninformative prior with a normal likelihood

After we have figured out the joint posterior, we may be interested in predicting new observations with the **posterior predictive distribution**:

$$p(\tilde{y} \mid y) = \iint p(\tilde{y} \mid \mu, \sigma^2) p(\mu, \sigma^2 \mid y) \mathrm{d}\mu \mathrm{d}\sigma^2.$$

A noninformative prior with a normal likelihood

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We can simulate $\tilde{y}_i$ as follows:

1. draw $\sigma_i^2 \mid y \sim p(\sigma^2 \mid y)$
2. draw $\mu_i \mid \sigma_i^2, y \sim p(\mu \mid \sigma_i^2, y)$
3. draw $\tilde{y}_i \mid \mu_i, \sigma_i^2 \sim p(\tilde{y} \mid \mu_i, \sigma_i^2)$

A noninformative prior with a normal likelihood

After we have figured out the joint posterior, we may be interested in predicting new observations with the **posterior predictive distribution**:

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A noninformative prior with a normal likelihood

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It's a homework question to show that

$$\tilde{y} \mid y \sim t_{n-1}\left(\bar{y}, s^2\left(1 + \frac{1}{n}\right)\right)$$

A noninformative prior with a normal likelihood

Let's get some practice simulating predictions, which will come in handy when we are dealing with more complicated scenarios where a closed-form posterior predictive distribution isn't available. We can simulate each $\tilde{y}_i$ as follows:

Sampling Strategy

For $i = 1, 2, \ldots$

1. draw $\sigma_i^2 \mid y \sim p(\sigma^2 \mid y)$
2. draw $\mu_i \mid \sigma_i^2, y \sim p(\mu \mid \sigma_i^2, y)$
3. draw $\tilde{y}_i \mid \mu_i, \sigma_i^2 \sim p(\tilde{y} \mid \mu_i, \sigma_i^2)$

Each triple
$(\tilde{y}_i, \mu_i, \sigma_i^2) \sim p(\tilde{y}, \mu, \sigma^2 \mid y) = p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2 \mid y) p(\sigma^2 \mid y)$.

So $\tilde{y}_i \sim p(\tilde{y} \mid y) = \iint p(\tilde{y} \mid \mu, \sigma^2) p(\mu, \sigma^2 \mid y) \mathrm{d}\mu \mathrm{d}\sigma^2$

Tip 1: If the joint is easier to sample from

If you simulate $(\tilde{y}^i, \theta_1^i, \theta_2^i)_{i=1}^n \sim p(\tilde{y}, \theta_1, \theta_2 \mid y)$, then ignoring pieces of each sample is analogous to sampling from the marginal:

$$n^{-1} \sum_{i=1}^{n} h(\tilde{y}^i) \to E_{\tilde{y},\theta_1,\theta_2}[h(\tilde{y}^i)] = E_{\tilde{y}}[h(\tilde{y}^i)]$$

Tip 2: if the "top" factor of a joint is tractable

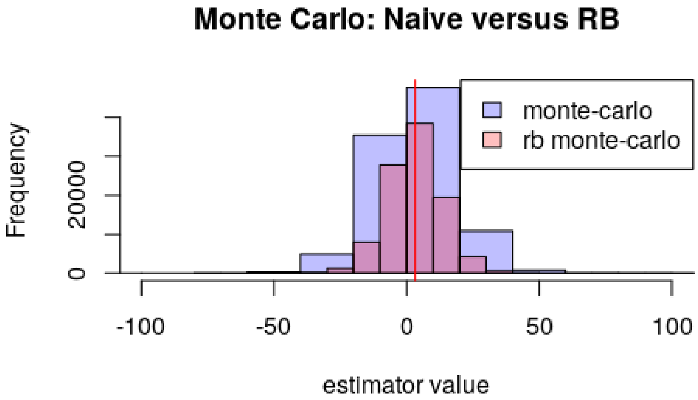
If $p(\tilde{y}, \theta_1, \theta_2 \mid y) = p(\tilde{y} \mid \theta_1, \theta_2, y) p(\theta_1, \theta_2 \mid y)$, then

$$
\begin{aligned}
n^{-1} \sum_{i=1}^{n} E[h(\tilde{y}, \theta_1^i, \theta_2^i) \mid \theta_1^i, \theta_2^i, y] &\to E\left(E[h(\tilde{y}, \theta_1, \theta_2) \mid \theta_1, \theta_2, y]\right) \\
&= E[h(\tilde{y}, \theta_1, \theta_2) \mid y]
\end{aligned}
$$

If you can derive expectations of $p(\tilde{y} \mid \theta_1, \theta_2, y)$, and you can sample from the other piece, then this **Rao-Blackwellization** or **marginalization** strategy can be a useful variance reduction technique.

A comparison in R

See `3.r` for details:



Another multiparameter example of conjugacy: Dirichlet-multinomial

Let $y = (y_1, y_2, \ldots, y_k)$ be a vector of counts. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ be the probabilities of any trial resulting in each of the k outcomes. We assume that there is a known total count (which means $\sum_i y_i = n$) and that the only possible outcomes are these k outcomes $\sum_i \theta_i = 1$.

The likelihood is a multinomial distribution

$$p(y \mid \theta) \propto \prod_{i=1}^{k} \theta_i^{y_i},$$

and the prior is a Dirichlet distribution

$$p(\theta \mid \alpha) \propto \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}.$$

The hyper-parameters have a very nice interpretation of counts!

Multivariate Normal Observations

Let each observation y follow a multivariate normal distribution. The likelihood $p(y_1, \ldots, y_n \mid \mu, \Sigma)$ is usefully written with a few properties of the trace operator:

$$\propto \det(\Sigma)^{-n/2} \exp\left(-\frac{1}{2}\sum_i (y_i - \mu)'\Sigma^{-1}(y_i - \mu)\right)$$

$$= \det(\Sigma)^{-n/2} \exp\left[-\frac{1}{2}\sum_i \mathrm{tr}\{(y_i - \mu)'\Sigma^{-1}(y_i - \mu)\}\right]$$

$$= \det(\Sigma)^{-n/2} \exp\left[-\frac{1}{2}\sum_i \mathrm{tr}\{\Sigma^{-1}(y_i - \mu)(y_i - \mu)'\}\right]$$

$$= \det(\Sigma)^{-n/2} \exp\left[-\frac{1}{2}\mathrm{tr}\left\{\Sigma^{-1}\overbrace{\sum_i (y_i - \mu)(y_i - \mu)'}^{S_0}\right\}\right]$$

Multivariate Normal Observations with known covariance matrix

A conjugate prior for $p(y \mid \mu) = \det(\Sigma)^{-n/2} \exp\left[-\frac{1}{2}\mathrm{tr}\left\{\overbrace{\Sigma^{-1}}^{\text{known}} S_0\right\}\right]$ is

$$p(\mu \mid \mu_0, \Lambda_0) = \det(\Lambda_0)^{-1/2} \exp\left[(\mu - \mu_0)'\Lambda^{-1}(\mu - \mu_0)\right]$$

This makes the posterior distribution (homework question exercise 3.13) normal with mean and precision

$$\mu_n = (\Lambda_0 + n\Sigma^{-1})^{-1}(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y})$$

$$\Lambda_n^{-1} = \Lambda_0^{-1} + n\Sigma^{-1}.$$