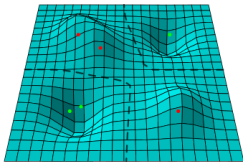


# Introduction to Machine Learning

## The Gaussian RBF Kernel



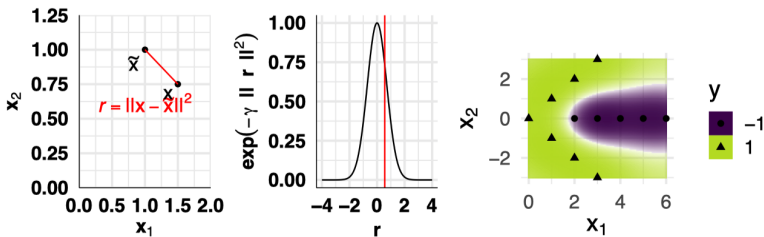
### Learning goals

- Know the Gaussian (RBF) kernel
- Understand that all data sets are separable with this kernel
- Understand the effect of the kernel hyperparameter  $\sigma$

# RBF KERNEL

The “radial” **Gaussian kernel** is defined as

$$k(\mathbf{x}, \tilde{\mathbf{x}}) = \exp\left(-\frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|^2}{2\sigma^2}\right) \quad \text{or} \quad k(\mathbf{x}, \tilde{\mathbf{x}}) = \exp(-\gamma\|\mathbf{x} - \tilde{\mathbf{x}}\|^2)$$



A straightforward extension is

$$k(\mathbf{x}, \tilde{\mathbf{x}}) = \exp\left(-(\mathbf{x} - \tilde{\mathbf{x}})^T C (\mathbf{x} - \tilde{\mathbf{x}})\right)$$

for a symmetric, positive definite matrix  $C$ .

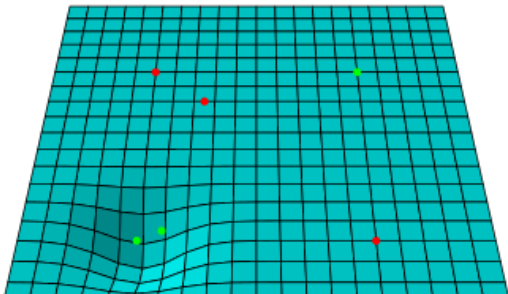
# RBF KERNEL

- With a Gaussian kernel, all RKHS basis functions  $\phi(\mathbf{x}) = k(\mathbf{x}, \cdot)$  are linearly independent - which we will not prove here.
- This means that all (finite) data sets are linearly separable!
- Do we then need soft-margin machines? The answer is “yes”. The roles of the nonlinear feature map and the soft-margin constraints are very different:
  - The purpose of the kernel (and its feature map) is to make learning “easy”.
  - Even in an infinite-dimensional feature space we may want some margin violators because we should not trust noisy data. A hard-margin SVM with Gaussian kernels may be able to separate any dataset but will usually overfit.

# WEIGHTED MIXTURE OF GAUSSIANS

Via the RKHS / basis function intuition we can understand the effect of the RBF kernel much better as a local model.

$$f(\mathbf{x}) = \sum_{i=1}^n \alpha_i y^{(i)} k(\mathbf{x}^{(i)}, \mathbf{x}) + \theta_0$$

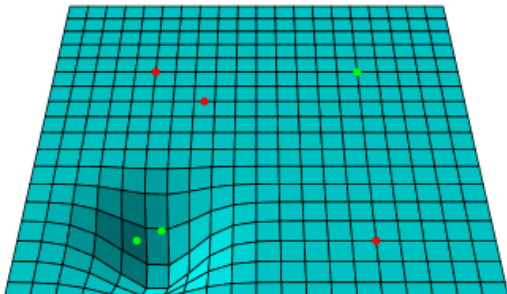


All support vectors are assigned RBF "bumps", these are weighted with the dual variables / Lagrange multipliers  $\alpha_i$  and labels  $y^{(i)}$ . We then "mix" these bumps together to form the decision score function. Which becomes a bumpy surface.

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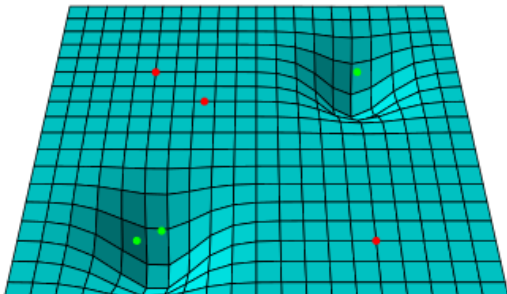


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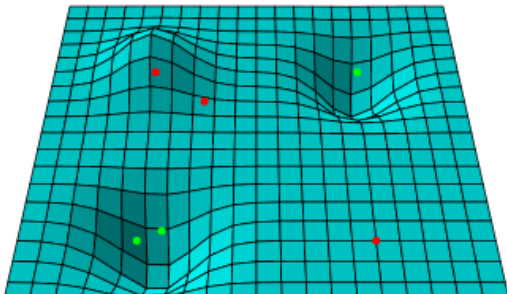


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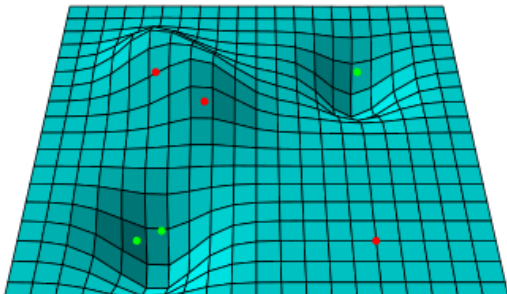


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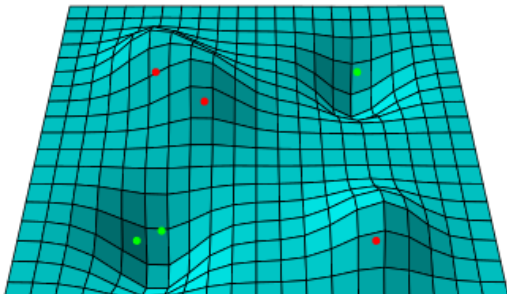
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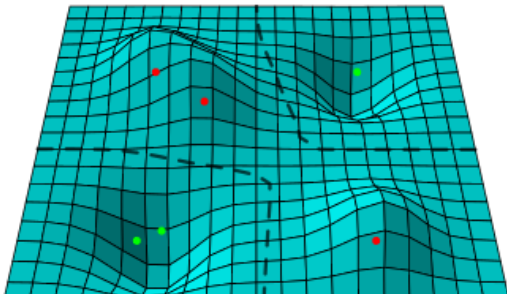


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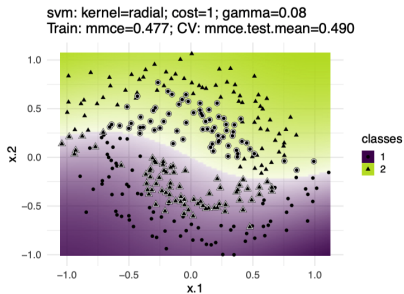
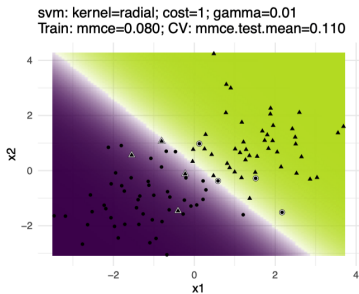
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# RBF KERNEL WIDTH

A large  $\sigma$  (or a small  $\gamma$ ) will make the decision boundary very smooth and in the limit almost linear.



# RBF KERNEL WIDTH

A small  $\sigma$  parameter makes the function more “wiggly”, in the limit we totally over fit the data by basically modelling each training data point - and maximal uncertainty at all other test points.

