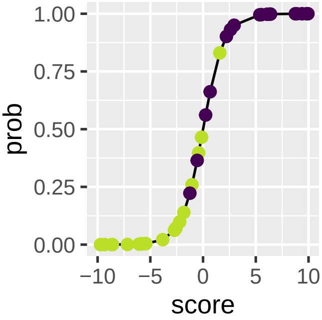
Introduction to Machine Learning

Classification: Logistic Regression



Learning goals

- Understand the definition of the logit model
- Understand how a reasonable loss function for binary classification can be derived
- Know the hypothesis space that belongs to the logit model

MOTIVATION

A **discriminant** approach for directly modeling the posterior probabilities $\pi(\mathbf{x} \mid \boldsymbol{\theta})$ of the labels is **logistic regression**.
For now, let's focus on the binary case $y \in \{0, 1\}$ and use empirical risk minimization.

$$\arg\min_{\boldsymbol{\theta}\in\Theta} \mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) = \arg\min_{\boldsymbol{\theta}\in\Theta} \sum_{i=1}^{n} L\left(y^{(i)}, \pi\left(\mathbf{x}^{(i)} \mid \boldsymbol{\theta}\right)\right).$$

A naive approach would be to model

$$\pi(\mathbf{x} \mid \boldsymbol{\theta}) = \boldsymbol{\theta}^T\mathbf{x}.$$

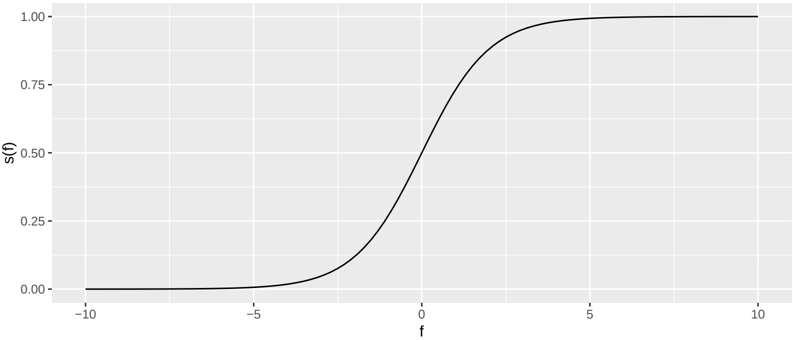
NB: We will often suppress the intercept in notation.

Obviously this could result in predicted probabilities $\pi(\mathbf{x} \mid \boldsymbol{\theta}) \notin [0, 1]$.

LOGISTIC FUNCTION

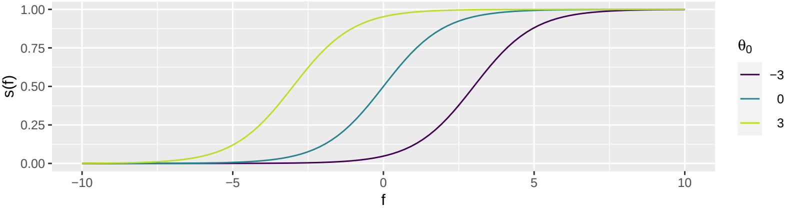
To avoid this, logistic regression “squashes” the estimated linear scores $\boldsymbol{\theta}^T\mathbf{x}$ to $[0, 1]$ through the **logistic function** s:

$$\pi(\mathbf{x} \mid \boldsymbol{\theta}) = \frac{\exp\left(\boldsymbol{\theta}^T\mathbf{x}\right)}{1+\exp\left(\boldsymbol{\theta}^T\mathbf{x}\right)} = \frac{1}{1+\exp\left(-\boldsymbol{\theta}^T\mathbf{x}\right)} = s\left(\boldsymbol{\theta}^T\mathbf{x}\right)$$

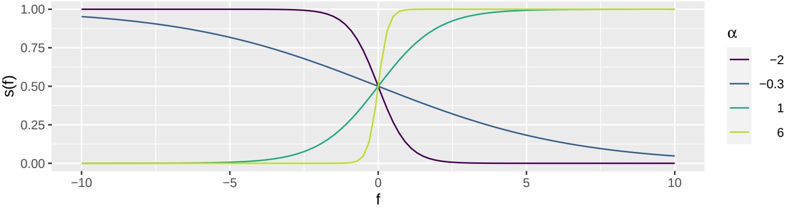


LOGISTIC FUNCTION

The intercept shifts $s(f)$ horizontally $s(\theta_0 + f) = \frac{\exp(\theta_0 + f)}{1 + \exp(\theta_0 + f)}$



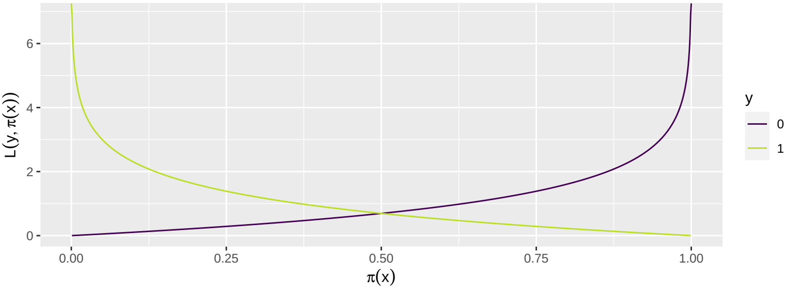
Scaling f like $s(\alpha f) = \frac{\exp(\alpha f)}{1 + \exp(\alpha f)}$ controls the slope and direction.



BERNOULLI / LOG LOSS

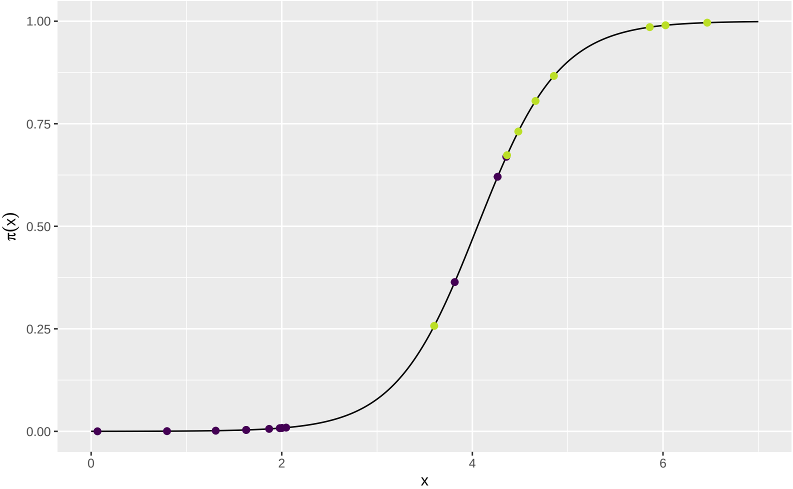
We need to define a loss function for the ERM approach:

- $L(y, \pi(\mathbf{x})) = -y \ln(\pi(\mathbf{x})) - (1 - y) \ln(1 - \pi(\mathbf{x}))$
- Penalizes confidently wrong predictions heavily
- Called Bernoulli, log or cross-entropy loss
- We can derive it from the negative log-likelihood of Bernoulli / logistic regression model in statistics
- Used for many other classifiers, e.g., in NNs or boosting



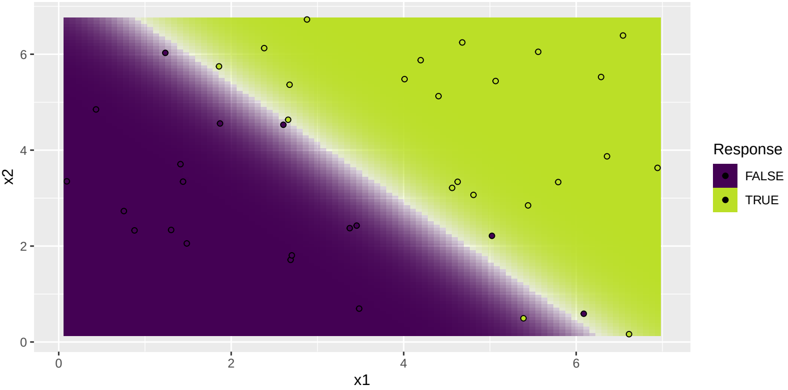
LOGISTIC REGRESSION IN 1D

With one feature $\mathbf{x} \in \mathbb{R}$. The figure shows data and $\mathbf{x} \mapsto \pi(\mathbf{x})$.

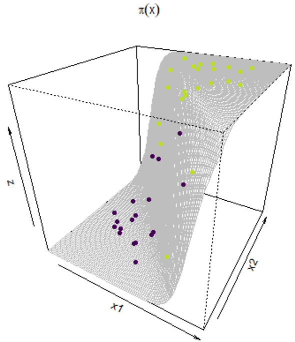


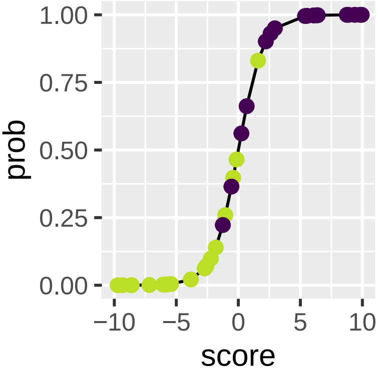
LOGISTIC REGRESSION IN 2D

Obviously, logistic regression is a linear classifier, as $\pi(\mathbf{x} \mid \boldsymbol{\theta}) = s\left(\boldsymbol{\theta}^T\mathbf{x}\right)$ and s is isotonic.



LOGISTIC REGRESSION IN 2D





SUMMARY

Hypothesis Space:

$$\mathcal{H} = \left\{\pi : \mathcal{X} \to [0,1] \mid \pi(\mathbf{x}) = s(\boldsymbol{\theta}^T\mathbf{x})\right\}$$

Risk: Logistic/Bernoulli loss function.

$$L\left(y, \pi(\mathbf{x})\right) = -y \ln(\pi(\mathbf{x})) - (1-y)\ln(1-\pi(\mathbf{x}))$$

Optimization: Numerical optimization, typically gradient-based methods.