Zenith OAR: Week Five

The Tech Club - Robotics Team

Introduction

Controllability and observability represent two major concepts of modern control system theory

Consider a car to be a control system. The state of the system would be defined by the velocity, acceleration, fuel left and location of the car etc. The input for the car would be the acceleration pedal, brake pedal, clutch and gear. The output would be the odometer, fuel meter and temperature meter. Now imagine the car not having any one input. We wouldn't be able to properly control the car in its entire state space. Imagine we didn't have the odometer. We would not be able to know the car's acceleration right?

Controllability in its essence is how much control we have over the system. A completely controllable system is such that we have appropriate actuators and control inputs to control the system over its entire state space.

Observability on the other hand gives a sense of how much information we have about the system. A completely observable system is a system where we have sensors such that we can observe the entire state of the system.

State controllability and observability are key properties in linear input–output systems in state-space form. In the state-space approach, the relation between inputs and outputs is represented using the state variables of the system. A natural question is then to what extent it is possible to manipulate the values of the state vector by means of an appropriate choice of the input function. The concepts of controllability, reachability, and null controllability address this issue. Another important question is whether it is possible to uniquely determine the values of the state vector from knowledge of the input and output signals over a given time interval. This question is dealt with using the concept of observability.

Observability

Consider a system described by a set of differential equations

$$\frac{dx}{dt} = Ax$$
$$y = Cx$$

where $x \subset R^n$ is the state, $u \subset R^p$ the input and $y \subset R^q$ the measured output.

We wish to estimate the state of the system from its inputs and outputs. In some situations we will assume that there is only one measured signal, i.e., that the signal y is a scalar and that C is a (row) vector.

Definition: A linear system is observable if for any $T > 0$, it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ in the interval $[0, T]$.

The problem of observability is one that has many important applications, even outside feedback systems. If a system is observable, then there are no “hidden” dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs.

The problem of observability is of significant practical interest because it will determine if a set of sensors is sufficient for controlling a system. Sensors combined with a mathematical model can also be viewed as a “virtual sensor” that gives information about variables that are not measured directly. The process of reconciling signals from many sensors with mathematical models is also called sensor fusion.

Testing for Observability

For convenience, we initially neglect the input and focus on the autonomous system,

$$\begin{aligned} \frac{dx}{dt} &= Ax \\ y &= Cx \end{aligned}$$

We wish to understand when it is possible to determine the state from observations of the output. The output itself gives the projection of the state on vectors that are rows of the matrix C. The observability problem can immediately be solved if the matrix C is invertible. If the matrix is not invertible, we can take derivatives of the output to obtain the following equation.

$$\begin{aligned} \frac{dy}{dt} &= C\frac{dx}{dt} \\ \frac{dy}{dt} &= CAx \end{aligned}$$

From the derivative of the output we thus get the projection of the state on vectors that are rows of the matrix CA. Proceeding in this way, we get the following.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x\_0 \qquad [0.1]$$

If we know the output of the system, we can obtain the initial state of the system x_0 by using the output equation of the state space representation as follows

$$\begin{aligned} y_0 &= Cx_0 \\ y_1 &= Cx_1 = CAx_0 \\ &\vdots \\ y_{n-1} &= CA^{n-1}x_0 \end{aligned}$$

We thus find that the state can be determined if the observability matrix

$$\Omega = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{0.2}$$

has n independent rows, i.e the rank of ω is n

Controllability

Consider a system described by a set of differential equations

$$\frac{dx}{dt} = Ax + Bu$$

where $x \in R_n$ is the state, $u \in R$ is the input vector, A is an n x n matrix and B a column vector.

A fundamental question is whether it is possible to find control signals so that any point in the state space can be reached through some choice of input.

Definition: A linear system is reachable if for any x_0, $x_f \subset \mathrm{R}^n$ there exists a $T > 0$ and $u : [0, T] \longrightarrow R$ such that the corresponding solution satisfies, $x(0) = x_0 andx(T) = x_f$

The definition of reachability addresses whether it is possible to reach all points in the state space in a transient fashion. In many applications, the set of points that we are most interested in reaching is the set of equilibrium points of the system (since we can remain at those points once we get there).

Consider a system with initial state, X_0 and you want to reach $X^\star$. This can be achieved by a series of control signals $U_1, U_2, \ldots\ldots U_n$.

Now for unit time, lets consider the steps

$$\begin{gathered} x_1 = Ax_0 + Bu_0 = Bu_0 \\ x_2 = Ax_1 + Bu_1 = ABu\_0 + Bu_1 \\ x_3 = Ax_2 + Bu_2 = A^2Bu_0 + ABu_1 + Bu_2 \\ \vdots \\ x_n = A^{n-1}Bu_0 + \ldots + Bu_{n-1} \end{gathered}$$

To get the control signals, we solve for the U's

$$x^\star = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u_{n-1} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}$$

Testing for Controllability

A system is completely controllable iff it is possible to calculate the U matrix in the equation above. This is only possible if the rank of the controllability matrix Γ has a rank of n.

$$\Gamma = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$