M367K Topology Notebook

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1 Introduction

This is my digital notebook for Topology Through Inquiry. Throughout this notebook there will be listed theorems that we will be assigned from class. These theorems do not come with a proof, but instead we are to fill them out ourselves. Let the games begin!

2 Cardinality

2.2 Cardinality and Countable Sets

Theorem 2.9. *Every infinite set has a countably infinite subset.*

Proof. Assume S is an infinite set. We want to show that S has an countably infinite subset; that is, there exist a subset of S that has the same cardinality as the natural numbers. We can show this by constructing a one-to-one correspondence of the elements of S.

Let the elements of S be denoted in any order as S. Take an arbitrarily chosen element $s_1 \in S$, then we can correspond it to $1 \in \mathbb{N}$. We then correspond $2 \in \mathbb{N}$ to any arbitrarily chosen element of $s_2 \in S \setminus s_1$, the previous set without the element s_1. We can repeat this cycle for all n. Now take the set of elements we used and we have the subset of S, $S' = \{s_1, s_2, \dots\}$, the chosen elements that correspond to the natural numbers $\mathbb{N}$. Therefore S' is a countably infinite subset. $\square$

Theorem 2.10. *A set is infinite if and only if there is an injection from the set into a proper subset of itself.*

Proof. We will use Theorem 2.9; that is, there exist a countably infinite subset.

$\Rightarrow$ Assume that a set S is infinite. We want to show that there is an injection from the set into a proper subset of itself. By the previous theorem 2.9 we can say that there exists a countably infinite subset of S, call it F. Let $F = \{f_1, f_2, \dots\} \subset S$, then we can take the map $S \to F$ such that $f_n \to f_{n+1}$, and for all the elements in S and not in F the mapping goes to it self, $s_n \to s_n$. Clearly this mapping is injective into itself.

$\Leftarrow$ Assume that there is an injection from the set, S, into a proper subset, F, of itself. We want to show that S is infinite. Suppose that S is finite. We have that F is a proper subset of S, so $|F| < |S|$, then the injection $S \to F$ is not possible since the domain is larger than the codomain, our contradiction. Therefore S is infinite. $\square$

Theorem 2.11. *The union of two countable sets is countable*

Proof. Let A and B be two countable sets. We want to show that $S = A \cup B$ is countable. We can do this by showing a one-to-one correspondence with the natural numbers. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$, then our correspondence can be, for all $k \in \mathbb{N}$:

$$1 - a_1$$

$$2 - b_1$$

$$3 - a_2$$

$$4 - b_2$$

$$\dots$$

$$(2k - 1) - a_k$$

$$2k - b_k$$

Thus we have shown that S is countable. □

Theorem 2.12. *The union of countably many countable sets is countable.*

Proof. Let us list the countable sets:

$$S_1 = \{s_{1,1}, s_{2,1}, s_{3,1}, \dots\}$$

$$S_2 = \{s_{1,2}, s_{2,2}, s_{3,2}, \dots\}$$

$$\dots$$

$$S_n = \{s_{1,n}, s_{2,n}, s_{3,n}, \dots\}$$

$$\dots$$

We want to find a correspondence with the set of these sets and the natural numbers. We can construct one by starting at the top left element and walking down the set in a zig-zag pattern counting the diagonals. For example the elements of the naturals corresponds as follows:

$$1 - s_{1,1}$$

$$2 - s_{2,1}$$

$$3 - s_{1,2}$$

$$4 - s_{1,3}$$

$$5 - s_{2,2}$$

$$6 - s_{3,1}$$

$$\dots$$

and so forth. Thus the set of sets is countable. □

Theorem 2.13. *The set $\mathbb{Q}$ is countable.*

Proof. The proof follows directly from the previous theorems. Let us list the set $\mathbb{Z}_n = \{\dots, -1/n, 0, 1/n, 2/n, 3/n, \dots\}$ for all $n \in \mathbb{N}$. Then we have

$$\mathbb{Z}_1 = \{\dots, -1, 0, 1, \dots\}$$

$$\mathbb{Z}_2 = \{\dots, -1/2, 0, 1/2, \dots\}$$
$$\mathbb{Z}_3 = \{\dots, -1/3, 0, 1/3, \dots\}$$
$$\dots$$

We have shown in Theorem 2.7 that $\mathbb{Z}$ is countable, so we write the rationals as a union of these countably many of countable sets. By Theorem 2.12 we know that a union of countably many countable sets is countable, and therefore $\mathbb{Z}$ is countable. What a tongue twister. □

Theorem 2.14. *The set of all finite subsets of a countable set is countable.*

Proof. Before we prove this theorem let us make a claim.

Claim: Any set of finite subsets of order n is countable.

We will prove this by induction. Let $n = 1$ then we have the set of singletons, which is in bijection to the natural numbers. Clearly this set is countable. Suppose S_k is countable. We want to show that S_{k+1} is countable. We list the set $S_{k,n}$ as the set S_k with the element n added to each subset of the set. Now we can list countably the number of countable S_k sets as $R = \{S_{1,k}, S_{1,k}, \dots, \}$. By Theorem 2.12 we can say that a union of countably many countable set is countable. Thus $S_{k+1} = R \cup S_k$ is countable.

We can prove this theorem by using Theorem 2.12. Let us list the set of subsets of size n, S_n. Then we can list out the set of set of subsets countably as $S = \{S_1, S_2, S_3, \dots\}$. By Theorem 2.12 we know that a union of countably many countable sets is countable. Therefore $\cup S_j$ for all $j \in \mathbb{N}$ is countable.

□

Exercise 2.15. Suppose a submarine is moving in the plane along a straight line at a constant speed such that at each hour, the submarine is at a lattice point, that is, a point whose two coordinates are both integers. Suppose at each hour you can explode one depth charge at a lattice point that will hit the submarine if it is there. You do not know the submarine's direction, speed, or its current position. Prove that you can explode one depth charge each hour in such a way that you will be guaranteed to eventually hit the submarine.

We have that a submarine is moving in a straight path at a constant speed, so for any given path at time t_i we can determine its position (this can easily be done by using parameters). However, we do not know which path the submarine is on, and let us list the options as the set $P = \{p_1, p_2, \dots\}$ for any path p_j. The number of paths are created by two points on our lattice which is countable. Remember for each p_j we can determine its position at t_i. As previously stated in Theorem 2.12 a union of countable sets is countable. Clearly each straight line is countable, so we have a union of countably many countable sets. In order to strike the submarine we must countably go down the set P dropping a depth charge where the submarine would be in each path if it took that path, for example we would start t_1 at the position the submarine would be on p_1 at t_1, and next move on to t_2 and dropping the depth charge where the submarine would be in position of t_2 on p_2, and so forth for the rest of our natural numbers (in this case we are using our natural numbers as time). And in countably infinite time we should surely strike the submarine with a depth charge.

Theorem 2.16. *The cardinality of the set of natural numbers is not the same as the cardinality of the set of real numbers. That is, the set of real numbers is uncountable.*

Proof. We can prove this by contradiction. Suppose the the set of real numbers is countable. Then we can correspond the real numbers, r_i, to the natural numbers, n_i. Let us list this correspondence:

$$1 - r_1$$

$$2 - r_2$$

$$\dots$$

$$n - r_n$$

Now our goal is to construct a number that is not on this list; this will show our contradiction. Let us write the number $r_i = d_{i,1}d_{i,2}d_{i,3} \dots d_{i,n}$ as the decimal places of r_i in scientific notation. For example if $r_i = 1.23$ then $d_{i,1} = 1$, $d_{i,2} = 2$, $d_{i,3} = 3$. The decimal places are in bijection with the natural numbers, so are countable. We know by Theorem 2.12 that a union of countably many countable sets is countable. Now we use this to construct a number not on the list, r_x. We go down the list of r_i and make the ith decimal place of the ith number different from out constructions; that is, $d_{x,j} \neq d_{i,j}$ and not equal to the previous decimals place, $d_{x,j} \neq d_{x,j-1}$ for all $j \in \mathbb{N}$. Since the decimal place is different from each number we get a new real number that is not on the list, a contradiction.

Note: I included that $d_{x,j} \neq d_{x,j-1}$ in order to get rid of the case with repeating decimals $.\overline{9} = 1$. This would be a counterexample of the proof without it. □

2.3 The Axiom of Choice

Exercise 2.17. Show that $\mathbb{R}$ with the usual ordering is totally ordered but not well-ordered.

It is clear that $\mathbb{R}$ is partially ordered; that is, for all x, y, z in $\mathbb{R}$ it follows these three properties:

1. $x \leq x$,

2. if $x \leq y$ and $y \leq z$, then $x \leq z$

3. if $x \leq y$ and $y \leq x$, then $x = y$

This is true for all the elements in $\mathbb{R}$, thus is totally ordered. We can even construct a number line showing the elements ordered. We can't list them all, but we can represent where they would be on the number line. However $\mathbb{R}$ is not well-ordered since every subset does not have a smallest element. An example is the interval from 0 to 1, let us denote $(0, 1)$. We can see that this set has no smallest element, and if anyone tells us they have found the smallest element we will be able to find a smaller element. For example, let us call this smallest element $\epsilon \in (0,1)$, then we take $\epsilon_o = \epsilon/10$ and we have a new element that is smaller than our original and in our interval $(0,1)$.

Theorem 2.18. *Zorn's Lemma, the Axiom of Choice, and the Well-Ordering Principle are equivalent.*

Proof. The proof will not be listed here. However assume it true for the following problems. $\square$

2.4 Power Sets

Exercise 2.19. Suppose $A = \{a, b, c\}$. Explicitly write out 2^A, the power set of A.

By writing out all the subsets, and union of the subsets we get the power set

$$Pow_A = \{\{\emptyset\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

Note: We can determine that the order of this set is $2^3 = 8$, by the next Theorem.

Theorem 2.20. *If a set A is finite, then the power set of A has cardinality $2^{|A|}$, that is, $|2^A| = 2^{|A|}$*

Proof. We can prove this by induction.

Let $A = \{\emptyset\}$, then $|A| = 0$ and $|2^A| = 2^0 = 1$. The power set is $Pow_A = A$, and by counting the elements we can see that $|Pow_A| = 1$. This is our "base case".

Now suppose that for any set $|S| = k$ for some $k \in \mathbb{N}$. Assume $|2^S| = 2^k$. We want to show the cardinality argument is true for any set $|R| = k+1$. Let us list the elements of $S = \{s_1, s_2, \ldots, s_k\}$. We define $R := S \cup \{r\}$, so $|R| = k+1$. Then the power set of R is $Pow_R = \{Pow_S, Pow_S \cup \{r\}\}$; that is, the set of subsets of S with the element r and the set of subsets of S without r. We see the cardinality is $|Pow_R| = 2^k + 2^k = 2(2^k) = 2^{k+1}$. Therefore is true for R. This is our proved inductive hypothesis.

By the two previous arguments we can see, by induction, that $|2^A| = 2^{|A|}$ is true for any $|A| = k$ where $k \in \mathbb{N}$ or $|A| = 0$. $\square$

Theorem 2.21. *For any set A, there is an injection from A into 2^A.*

Proof. Let A be any set. Let us list the elements of $A = \{a_1, a_2, \ldots\}$. We define the function

$$\phi : A \to 2^A$$

such that $\phi(a_i) = \{a_i\}$; that is $\phi(a_i)$ is the singleton set of the element $\{a_i\}$ for any $i \in \mathbb{N}$. We can see that this function is injective since $a_i \neq a_j$ for $j \neq i$ for any $j \in \mathbb{N}$. $\square$

Theorem 2.22. *For a set A, let P be the set of all functions from A to the two point set $\{0, 1\}$. Then $|P| = |2^A|$*

Proof. We can prove this by induction.

We will write the set of functions from A to $\{0,1\}$ as $P_A = \{\phi_1, \phi_2, \dots, \phi_n\}$ for $|P_A| = n$. Let A be a set such that $|A| = 1$, we list it's element $A = \{a_1\}$. There are only two functions such that $\phi_i : A \to \{0,1\}$; that is $\phi_1(a_1) = 0$ and $\phi_2(a_2) = 1$. Thus we can see the for $|A| = 1$, $|P_A| = 2^{|A|}$.

Suppose for every set $|A| = k$, $|P_A| = 2^k$. We want to show that for a set $|B| = k+1$, $P_B = 2^{k+1}$. We define the set $B = A \cup \{b\}$. We can see that $P_B = \{\{P_A \cup \{b\} : \phi(b) = 0\}, \{P_A \cup \{b\} : \phi(b) = 1\}\}$, and by counting up the elements we can see that $|P_B| = 2^k + 2^k = 2^{k+1} = 2^{|B|}$. Therefore this is true for all $|A| = k$ for $k \in \mathbb{N}$. $\square$

Theorem 2.23. *There is a one-to-one correspondence between $2^{\mathbb{N}}$ and the set of all infinite sequences of 0's and 1's.*

Proof. This can be seen as representing the elements (subsets of $\mathbb{N}$) of $2^{\mathbb{N}}$ with an element of the infinite sequence, $\mathbb{B}$, of 1's and 0's. For example, let us take the element $S \in 2^{\mathbb{N}}$, and represent it as $S = \{s_1, s_2, \dots\}$ where $s_i \in \mathbb{N}$. Let us take the element $B \in \mathbb{B}$, and represent it as $B = \{b_1, b_2, \dots\}$ where $b_i \in \{0,1\}$. Let us define the map

$$\phi : \mathbb{B} \to 2^{\mathbb{N}}$$

such that

$$\phi(B) = \{i : b_i = 1\} \text{ for any } i \in \mathbb{N} \text{ and } b_i \in B.$$

We can see this map is injective since $\phi(B_i) \neq \phi(B_j)$ for $i \neq j$ and $B_i, B_j \in \mathbb{B}$. Since there is no greatest element in $2^{\mathbb{N}}$ we can also say that this mapping is surjective. We can see this by taking any element of $S \in 2^{\mathbb{N}}$ and we have an element in $\mathbb{B}$ that corresponds to it. It is the element with 1's for each b_i such that $i \in S$ and 0's for each b_j such that j is not in S. $\square$

Theorem 2.24. *(Cantors Power set Theorem). There is no surjection from a set A onto 2^A. That is, for any set A the cardinality of A is not the same as the cardinality of its power set. In other words, $|A| \neq |2^A|$.*

Proof. Suppose there exist a surjection from $f : A \to 2^A$. We then define the set $B = \{x \in A : x \notin f(x)\}$; that is, the set of element of A that do not map to a set with x in that set. Remember the map f is sending elements of A to a subset of A. Since f is a surjection there must be some x_o in A such that $f(x_o) = B$. Let us identify this x_o.

If x_o is in $f(x_o) = B$ then by how we define B, $x_o \notin B = f(x_o)$ a contradiction. So x_o cannot be in $B = f(x_o)$, but that means $x_o \notin f(x_o) = B$ therefore making $x_o \in B = f(x_o)$ another contradiction. Therefore we can see that $x_o \in B \Leftrightarrow x_o \notin B$.

We conclude that there exist no surjection f. $\square$

Exercise 2.25. Consider $A = [0,1]$ and $B = [0,1)$ and injections $f(x) = x/3$ from A to B and $g(x) = x$ from B to A. Construct a bijection h from A to B such that on some points x of A, $h(x) = f(x)$, and for the other points x in A, $h(x) = g^{-1}(x)$.

We define the function $h : A \to B$, such that $h(x) := x/3$ for $x \in \{\frac{1}{3}^{n-1} : n \in \mathbb{N}\} = C \subset [0,1]$ and $h(x) := x$ for $x \notin C$. This is a bijective function.

Theorem 2.26. *(Schroeder-Bernstein). If A and B are sets such that there exist injections f from A into B and g from B into A, then $|A| = |B|$.*

Proof. Let us construct a mapping that shows a bijection between A and B as in the previous problem. We define this function to be $h : A \to B$ such that $h(x) := f(x)$ for $x \in [g \circ f(x)]^n(A - g(B))$, otherwise $h(x) := g^{-1}(x)$. $\square$

Theorem 2.27. $|\mathbb{R}| = |(0, 1)| = |[0, 1]|$.

Proof. We will directly prove this by the Schroeder-Bernstein Theorem (2.26). We define $A = (0, 1)$, $B = [0, 1]$. We also define the mapping $f : A \to B \subset \mathbb{R}$, such that $f(a) = a$. Clearly this function is injective. We define the function $g : \mathbb{R} \to A \subset B$

$$g(x) = \begin{cases} \frac{1}{x} & \text{if } 4 \le x \\ \frac{x}{16} + 1/4 & \text{if } 0 \le x < 4 \\ -\frac{x}{16} + 1/2 & \text{if } -4 \le x < 0 \\ 1 + \frac{1}{x} & \text{if } x \le -4 \end{cases}$$

$\square$

This function is injective. Lastly we define the function $h : B \to A$ such that $h(b) = b/3$ for $b \neq 0$ and $h(0) = .9$ another injective function. By Schroeder-Bernstein we conclude that these sets have the same cardinality.

Theorem 2.28. *Let $(0, 1) \times (0, 1)$ denote the Cartesian product of two open unit intervals. Then $|(0, 1) \times (0, 1)| = |(0, 1)|$.*

Proof. We can prove this by using Theorem 2.27 by showing there exist an injections $f : (0, 1) \to (0, 1) \times (0, 1)$ and $g : (0, 1) \times (0, 1) \to (0, 1)$. We have that f is trivial as we can define it as $f(a) = (a, 0)$. The challenge to this problem is defining g. However we can do this by alternating the digits of each number. Let $(a, b) \in (0, 1) \times (0, 1)$. We write the decimals of a and b as $a = a_1.a_2a_3...$ and $b = b_1.b_2b_3....$ Next we define the injection $g(a, b) = a_1.b_1a_2b_2....$ We conclude that $|(0, 1)| = |(0, 1) \times (0, 1)|$. $\square$

Theorem 2.29. $|\mathbb{R}| = |2^{\mathbb{N}}|$.

Proof. We can do this by showing that there exist an injection from $f : \mathbb{R} \to 2^{\mathbb{N}}$ and $g : 2^{\mathbb{N}} \to \mathbb{R}$. However this will be easier if we use the infinite sequences of 1's and 0's. In Theorem 2.23 we have already shown that $|2^{\mathbb{N}}| = |\{0, 1\}^{\mathbb{N}}|$. To make things easier we will define this bijection as $h : 2^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$, where we denote as $h(S)$ as the i^{th} coordiantes being equal to 1 for $i \in S$ and the other coordiantes not in S equal to 0.

Note: the way to represent the elements of infinite sequence of 1's and 0's more directly would be writing out $S = \{0, 1, 0, 1, 1, 1, 1, 1, ...\}$ for $S \in \{0, 1\}^{\mathbb{N}}$. To be even more clear we would have $h(\{1, 2, 3\}) = \{1, 1, 1, 0, 0, 0, ..., 0, ...\}$. We can see that the first three elements are 1 and the rest are 0's.

We can simply turn these infinite 1's and 0's into decimals and we have found our injection g. For example, $g = i(h(\{1, 2, 5\}))$ where $i(\{1, 0, 1, 0, 0, 1, 0, 0, ..., 0, ...\}) = .10100100...$ We simply represented it as a decimal.

Next we must find an f. This can be done by construction, however for this proof we will use the binary expansion. We define f as writing the real number r in binary expansion and converting it into an sequence of 0's and 1's with the function i denoted above. Note: we will have to be more particular with the decimal positioning.

We conclude that $|2^{\mathbb{N}}| = |\{0, 1\}^{\mathbb{N}}| = |\mathbb{R}|$. □

Theorem 2.30. *There are infinitely many different infinite cardinalities.*

Proof. We can prove this by contradiction. Suppose there is a set with the largest cardinality call it A; that is, no set has a larger cardinality than A. We take the power set of A call it 2^A. By Theorem 2.24 we know that $|A| \neq |2^A|$, and by Theorem 2.21 there is an injection from A to 2^A, thus $|A| < |2^A|$. We conclude that $|A| < |2^A|$, our contraction. □

3 Topological Spaces

3.2 Open Sets and Topologies

Theorem 3.1. *Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in a topological space $(X, \mathfrak{T})$. Then $\cap_{i=1}^n U_i$ is open.*

Proof. We can prove this by using Axiom 3) and induction. That is to say we have that for the open sets U_1 and U_2 we have that $U_1 \cap U_2$ is open. We then define $U_j = U_i \cap U_{i+1}$ is open and show that implies that $U_{j+1} = U_j \cap U_{i+2}$ is open for any U_i, U_j in $\mathfrak{T}$. We have that the collection of open sets is in the topology are open and it follows that the intersection is open. Thus we have for all $i \in \mathbb{N}$ that U_i is open. □

Theorem 3.2. *A set U is open in a topological space $(X, \mathfrak{T})$ if and only if for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$.*

Proof. ($\Rightarrow$ Assume that U is open in $\mathfrak{T}$. We want to show for all $x \in U$ there exist an open set U_x such that $x \in U_x \subset U$. However we have that for all $x \in U$ that $x \in U \subset U$.

$\Leftarrow$) Assume instead that for all x in a set U we have the property that $x \in U_x \subset U$. We then apply Axiom 4) of a Topology to show that $\cup_{x \in U} U_x = U$ and it follows that $U \in \mathfrak{T}$.

Note: I didn't prove that $\cup_{x \in U} U_x = U$, however it can be proved by showing each is a subset of the other. However both directions are trivial, so we can see it will be easier to see it by inspection. □

Theorem 3.3. *Verify that $\mathfrak{T}_{std}$ is a topology on $\mathbb{R}^n$; in other words, it satisfies the four conditions of the definition of a topology.*

Proof. The proof will be left as an exercise to the reader. In order to prove this it will be needed to show that the 4 axioms of a topology hold. The first two are trivial. □

Exercise 3.4. Verify that the finite complement topology is indeed a topology on any set X.

In order to show that a finite complement topology, $\mathfrak{T}$ is a topology on X since the four axioms hold; that is:

$$\begin{aligned}
&1)\ \emptyset \in \mathfrak{T} \text{ (Given)}\\
&2)\ X \in \mathfrak{T} \text{ (since } X - X = \emptyset \text{ is finite.)}\\
&3)\ V, U \in \mathfrak{T} \Rightarrow V \cap U \in \mathfrak{T}\\
&4)\ \{U_\alpha\}_{\alpha\in\lambda} \in \mathfrak{T} \Rightarrow \bigcup_{\alpha\in\lambda} U_\alpha \in \mathfrak{T}
\end{aligned}$$

We have that 1) is trivial. We have 2) since $X - X = \emptyset$ and the empty set is finite.

3) Let $V, U \in \mathfrak{T}$ then we can say that $X - V = V^c = F_1$ and $X - U = U^c = F_2$ for some finite sets F_1 and F_2. We want to show that $V \cap U$ is in $\mathfrak{T}$. We have $X - V \cap U = (V \cap U)^c = V^c \cup U^c = F_1 \cup F_2$ which is finite. If we extend it to an arbitrarily number of sets the statement still holds.

4) Let $V, U \in \mathfrak{T}$ then we can say that $X - V = V^c = F_1$ and $X - U = U^c = F_2$ for some finite sets F_1 and F_2. We want to show that $V \cup U$ is in $\mathfrak{T}$. We have $X - V \cup U = (V \cup U)^c = V^c \cap U^c = F_1 \cap F_2$ which is at most finite. If we extend it to an arbitrarily number of sets we can see the statement still holds, and will result in $X - \cup_{\alpha\in\lambda} U_\alpha = (\cup_{\alpha\in\lambda} U_\alpha)^c = \cap_{\alpha\in\lambda} U_\alpha^c = F_1 \cap F_2 \cap \cdots \cap F_\alpha \cap \cdots$. which is at most finite since for all F_α are finite. Note: I wrote out the finite sets F_i for $i \in \mathbb{N}$, however it doesn't mean the collection of these sets is countable. I just wanted to give an easy example to the reader.

Exercise 3.5. Describe some of the open sets you get if $\mathbb{R}$ is endowed with the topologies described above (standard, discrete, indiscrete, co-finite, and countable complement). Specifically, identify sets that demonstrate the differences among these topologies, that is, find sets that are open in some topologies but not in others. For each of the topologies, determine if the interval $(0, 1)$ is an open set in that topology.

First let us start off with the open sets of a standard topology for $\mathbb{R}$, denoted $\mathfrak{T}_{std}$. Recall that a subset U of $\mathbb{R}$ is open if and only if for each point p of U there is some ϵ_p such that there is an open ball $B(p, \epsilon_p)$ that contains x. Therefore we have intervals of the form (a, b) for any $a, b \in \mathbb{R}$, and we can see that $(0, 1)$ is of this form. Therefore $(0, 1)$ is in the standard topology.

Let us take the discrete topology space, denoted $(X, 2^X)$. Any subset of X is an open set of X. For the topology endowed on $\mathbb{R}$ we have $(\mathbb{R}, 2^{\mathbb{R}})$. Examples of open sets include $\{r_1, r_2, r_3, ...\}$ for $r_i \in \mathbb{R}$. However we also have intervals of the form $[a, b]$ or (a, b) for real numbers a and b. Also we include the arbitrary union of either of these sets.

The indiscrete topology is trivial, for the topology is $(\mathbb{R}, \{\emptyset, \mathbb{R}\})$.

For the co-finite topology for X, for any finite set F the open sets are of the form F^c. We have for the co-finite topology endowed on $\mathbb{R}$, we have open sets similar to that of $\{0,1\}^c = (-\infty, 0] \cup (0,1) \cup [1, \infty)$.

Exercise 3.6. Give an example of a topological space and a collection of open sets in that topological space to show that the infinite intersection of open sets need not be open.

The easiest one we learn is in real-analysis. It is the standard topology $(\mathbb{R}, \mathcal{T}_{std})$ for the infinite intersection of the collection of sets $U_n = (0, 1 + \frac{1}{n})$. It can be proved that $\cap_{\mathbb{N}} U_n = (0, 1]$ which is not a closed set in this topology.

3.3 Limit Points and Closed Sets

Exercise 3.7. Let $X = \mathbb{R}$ and $A = (1, 2)$. Verify that 0 is a limit point of A in the indiscrete topology and finite complement topology, but not in the standard topology or discrete topology on $\mathbb{R}$.

For the indiscrete topology we have that the only open set that contains 0 is $\mathbb{R}$. And certainly $(\mathbb{R} - \{0\}) \cap A = A \neq \emptyset$. However this is not the case on the other topologies.

Theorem 3.8. *Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is not a limit point of A if and only if there exists an open set U with $p \in U$ and $U \cap A = \emptyset$*

Proof. This proof is directly from the definition of limit point. Therefore it is trivial and will left as an exercise to the reader. □

Exercise 3.9. If p is an isolated point of a set A in a topological space X, then there is an open set U such that $U \cap A = \{p\}$.

Exercise 3.10. Give examples of sets A in various topological spaces $(X, \mathcal{T})$ with

1. a limit point of A that is an element of A;
2. a limit point of A that is not an element of A;
3. an isolated point of A;
4. a point not in A that is not a limit point of A.

Exercise 3.11.

1. Which sets are closed in a set X with the discrete topology?
2. Which sets are closed in a set X with the indiscrete topology?
3. Which sets are closed in a set X with the finite complement topology?
4. Which sets are closed in a set X with the countable complement topology?

Theorem 3.12. *For any topological space $(X, \mathfrak{T})$ and $A \subset X$, $\bar{A}$ is closed.*

Proof. We will prove this by contradiction. In order to show that $\bar{A}$ is close we must show that it contains its limit points. We will denote A' as the set of limit points of A. Let $p \notin \bar{A}$ be a limit point of $\bar{A}$ not in $\bar{A}$ and not a limit point of A; that is, $U \cap A = \emptyset$. Then we can say by Theorem 3.8 there exist an open set U such that $U \cap A = \emptyset$. Since p is a limit point of $\bar{A}$ we have for all open sets U containing p that $U \cap \bar{A} \neq \emptyset$. However $U \cap \bar{A} = U \cap (A \cup A') = (U \cap A) \cup (U \cap A') = U \cap A' \neq \emptyset$. Let y be in $U \cap A'$, and since $y \in A'$ we have that $(U - \{y\}) \cap A \neq \emptyset$, our contradiction. Thus there are no limit points p not in $\bar{A}$. $\square$

Theorem 3.13. *Let $(X, \mathfrak{T})$ be a topological space. Then the set A is closed if and only if $X - A$ is open.*

Proof. $\Rightarrow$ Let A be closed. We want to show that $X - A = A^c$ is open. Since A is closed it contains all its limit points and every point outside of A there exist an open set that doesn't have any point of A. We take the arbitrary union of these sets and we have an open set that equals A^c. Thus A^c is open.

$\Leftarrow$ Let A^c be open. Then we know that every point a_c of A^c there exist an open set U_c that contains a_c and is contained in A^c; that is, for every point not in A there exist an open set that does not intersect A. This implies that there are no limit points outside of A, and A contains all its limit points and is closed. $\square$

Theorem 3.14. *Let $(X, \mathfrak{T})$ be a topological space, and let U be an open set and A be a closed subset of X. Then the set $U - A$ is open and the set $A - U$ is closed.*

Proof. We have $U - A = U \cap A^c$. We have that A is closed so by Theorem 3.13 A^c is open. Thus $U \cap A^c$ is open since a finite intersection of open sets is open. On the other hand, we have $A - U = A \cap U^c$. Let $A \cap U^c = C$, then $C^c = (A \cap U^c)^c = A^c \cup U$. We know that A is closed, so by Theorem 3.13 A^c is open. Also U is open so C^c is open since any union of open sets is open. By Theorem 3.13 we can conclude that C is closed. $\square$

Exercise 3.17. Give examples of topological spaces and sets in them that:

1. are closed, but not open;

2. are open, but not closed;

3. are both open and closed;

4. are neither open nor closed.

1. Take the topological space $(\mathbb{R}, \mathfrak{T}_{std})$. The sets that are closed but not open are the intervals of the form $[a, b]$ such that $a, b \in \mathbb{R}$

2. Take the topological space $(\mathbb{R}, \mathfrak{T}_{std})$. The open sets that are open and not closed are the complements of the previous example, and of the form (a, b) where $a, b \in \mathbb{R}$

3. Take the set X on the indescrete topology $\mathcal{T}$. Both of the sets contained, $\emptyset$ and X are both closed and open.

4. If we go back to our standard topology on $\mathbb{R}$ we have intervals of the form $[a, b)$ for $a, b \in \mathbb{R}$ and the complement of these intervals.

Theorem 3.21. *Let A, B be subsets of a topological space X. Then*

1. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$

2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof.

1. We have that $A \subset B$. We want to show that $\overline{A} \subset \overline{B}$. We can do this by showing that $\overline{A} = A \cup A'$ is a subset of $\overline{B} = B \cup B'$ (A' and B' are the set limit points of the set A and B respectively). We are given that $A \subset B$, so all we need to show is $A' \subset \overline{B}$. If we take a limit point $p \in A'$, then by definition for all open sets U containing p we have that $(U - \{p\}) \cap A \neq \emptyset$. However $A \subset B$ so we can say that for all open sets U containing p it is also true that $(U - \{p\}) \cap B \neq \emptyset$. Thus p is a limit point of B and $p \in B'$. We conclude that $A' \subset B'$ and $\overline{A} \subset \overline{B}$.

2. We want to show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. First let us look at the limit points of the union of sets, namely $(A \cup B)'$. Let $p \in (A \cup B)'$ then we can say by definition of a limit point that for all open sets U that contain p we have $(U - \{p\}) \cap (A \cup B) \neq \emptyset$. Also $(U - \{p\}) \cap (A \cup B) = [(U - \{p\}) \cap A] \cup [(U - \{p\}) \cap B]$. We can see that this is equivalent to being a limit point of the set A or B; that is, $(A \cup B)' = A' \cup B'$. Going back to the closure of the union two sets we have $\overline{A \cup B} = (A \cup B) \cup (A \cup B)' = A \cup B \cup A' \cup B' = (A \cup A') \cup (B \cup B') = \overline{A} \cup \overline{B}$. We conclude that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

□

Exercise 3.22. Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of subsets of a topological space X. Then the following statement is true?

$$\overline{\bigcup_{\alpha \in \lambda} A_\alpha} = \bigcup_{\alpha \in \lambda} \overline{A}_\alpha.$$

This statement is not true. We will show by a counterexample. By Theorem 3.12 we showed that the closure of a set is closed, so $\overline{\bigcup_{\alpha \in \lambda} A_\alpha}$ is closed. However $\bigcup_{\alpha \in \lambda} \overline{A}_\alpha$ is not always closed, for example take the collection of subsets of $\mathbb{R}$ such that $A_n = [1/n, 1]$. We can see that 0 is a limit point of $\bigcup_{\alpha \in \lambda} \overline{A}_\alpha$ and it is not contained in $\bigcup_{\alpha \in \lambda} \overline{A}_\alpha$, so this set is not closed. We conclude that these two sets are not equal.

Exercise 3.23. In $\mathbb{R}^2$ with the standard topology, describe the limit points and closure of each of the following two sets:

1. The topologist's sine curve:

$$S = \left\{ \left(x, sin\left(\frac{1}{x}\right) \right) : x \in (0,1) \right\}$$

2. The topologist's comb:

$$C = \{(x,0) : x \in [0,1]\} \cup \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, y \right) : y \in [0,1] \right\}$$

1. We have the set S. It's limit point are the point from the set S, the points $C = \{(0,y) : y \in [-1,1]\}$, and the point $x = (1, sin(1))$. The closure would be the set $S \cup C \cup \{x\}$.

2. The set C is a union of two sets. First we have the limit points of the first set which are $D = \{(x,0) : x \in [0,1]\}$ and this set has no limit points as is closed. The second set we have the limit points $E = \{(0,y) : y \in [0,1]\}$. The closure of C is the set $C \cup E$.

Exercise 3.24. In the standard topology on $\mathbb{R}$, there exists a non-empty subset C of the closed unit interval $[0,1]$ that is closed, contains no non-empty open interval, and where no point of C is an isolated point.

Theorem 3.27. *Let A be a subset of a topological space X. Then $Int(A)$, $Bd(A)$, and $Int(A^c)$ are disjoint sets whose union is X.*

Proof. We want to show that A is partitioned by the $Int(A)$, $Bd(A)$, and $Int(A^c)$; that is, intersection of these sets is empty and their union is the set A. We have that $Int(A) \cap Int(A^c) = Bd(A) \cap Int(A) = Bd(A) \cap Int(A^c) = \emptyset$. Also if we take the union of these sets we have $Int(A) \cup Int(A^c) \cup Bd(A)$. We can see that $Int(A) \cup Bd(A) = \bar{A}$ and the complement of $\bar{A}^c = Int(A^c)$. Thus $\bar{A} \cup \bar{A}^c = A$ and we have that $Int(A)$, $Bd(A)$, and $Int(A^c)$ partitions the set A.

□

Theorem 3.30. *Let A be a subset of the topological space X. Suppose $x \notin A$, but there is a sequence $\{x_i\}_{i \in \mathbb{N}} \subset A$ such that $x_i \to x$. Then x is in the closure of A.*

Proof. We have that there is a sequence $\{x_i\}_{i \in \mathbb{N}}$ in A, such that $x_i \to x$ for some $i > N \in \mathbb{N}$. We want to show that x is in the closure of A; that is, $x \in \bar{A} = A \cup A'$. We know that $x \notin A$, so x can only be in A'. We want to show that x is a limit point of A, however we have the sequence $\{x_i\} \in A$ and $x_i \to x$. By definition of convergence we can say that for all open sets that contain x there is another point in A. Thus x is a limit point and contained in the closure of A. □

Exercise 3.32. 1. Consider sequences in $\mathbb{R}$ with the finite complement topology. Which sequences converge? To what value(s) do they converge?

The sequences in $\mathbb{R}$ with the finite complement topology that converge are continuous function with points of discontinuity. For example take the sequence $a_r = 1/r$ where $r \in \mathbb{R}$. The points of discontinuity are the points of convergence since they are limit points of the image.

4 Creating New Spaces

4.1 Bases

Theorem 4.1. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if:*

1) $\mathcal{B} \subset \mathcal{T}$, and

2) for each set U in $\mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subset U$.

Proof.

$\Rightarrow$ Assume that $\mathcal{B}$ is a basis, then every open set of $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$. We want to show 1) and 2) are true. By definition of a basis we can say that 1) is true; that is, every element of $\mathcal{B}$ is an open set. Let U be a set in $\mathcal{T}$ and p a point in U, then we can write U as a union of elements from $\mathcal{B}$ since $\mathcal{B}$ is a basis for $\mathcal{T}$. We write $\cup_i B_i = U$ for $B_i \in \mathcal{T}$ and $i \in \mathbb{N}$. And so $p \in \cup_i B_I = U \subset U$. Thus 2) is true.

$\Leftarrow$ Instead, assume that 1) and 2), from above, are true. We want to show that $\mathcal{B}$ is a basis. Let U be an non-empty open set in $\mathcal{T}$. We will construct a cover for U. Take a point p_1 in U, then there exist a V_1 such that $p_1 \in V_1 \subset U$. Next we take a point p_2 in $U - V_1$, then either there exist a V_2 such that $p_2 \in V_2 \subset U - V_1$ or the set is empty. As we do this many times, until U is empty, we can see that $\cup_i V_i = U$. Since U was an arbitrarily chosen non-empty open set, we conclude that $\mathcal{B}$ is a basis for $\mathcal{T}$. $\square$

Theorem 4.3. *Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if:*

1) each point of X is in some element of $\mathcal{B}$, and

2) if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.

Proof.

$\Rightarrow$ Assume that $\mathcal{B}$ is a basis for some topology on X. We want to show 1) and 2) above are true. We have that every open set in X can be written as a union of elements from $\mathcal{B}$, since $\mathcal{B}$ is a basis. Thus X can be written as a union of elements of $\mathcal{B}$, and this implies that 1) is true. Next we have that topologies are closed under finite intersections. Thus for a point p in the intersection of two sets V and U we have $W = V \cap U$ can be written as a union of elements of $\mathcal{B}$. Also, it is easy to see that $p \in W = V \cap U \subset V \cap U$.

$\Leftarrow$ Suppose that for a collection $\mathcal{B}$ of subsets of X satisfies 1) and 2). We want to show that $\mathcal{B}$ is a basis for some topology. We generate the set $\mathcal{T} = \{\cup b_{\alpha \in \lambda} : b_\alpha \in \mathcal{B} \text{ or } b_\alpha = \emptyset\}$. We want to show that $\mathcal{T}$ is a topology. We already have that $\emptyset \in \mathcal{T}$. By 1) we can take a union of the open sets that contains each element of X; that is, $X = \cup_{x \in X} b_x$ where b_x is the open set that contains x, so $X \in \mathcal{T}$. By construction we have that arbitrary unions of $\mathcal{B}$ is in $\mathcal{T}$. To show $\mathcal{T}$ is closed for finite intersection sets U and V we take the arbitrary union of the open sets in the intersection of U_i and V_j; that is, $U_i \cap V_j = \cup_{x \in U_i \cap V_j} W_x$ where W_x is the open set in that contains x and satisfies 2). By induction we show this is true for any finite number of sets. Therefore, we conclude that $\mathcal{T}$ is a topology. □

Exercise 4.4. Show that the basis proposed above for the lower limit topology is in face a basis. The basis above proposed were the open sets of the form $[a, b)$ where $a, b \in \mathbb{R}$.

We want to show that the basis proposed above for the lower limit topology is in fact a basis; that is, any element of the topology $(\mathbb{R}, \mathbb{R}_{LL})$ can be written as a union of elements of the collection of open subsets $\mathcal{S} = \{[a, b) : a, b \in \mathbb{R}\}$. To show that $\mathcal{S}$ is a basis we have to show that every open set of $\mathbb{R}_{LL}$ can be written as a union of elements of $\mathcal{S}$, however $\mathcal{S}$ is the set of open sets of the lower limit topology. Therefore the only thing left to show is that we have $\mathbb{R}$. If we take the union of $S_n = \{[n-1, n+1)\}$, we have $\cup_{n \in \mathbb{N}} S_n = X$.

Theorem 4.5. *Every open set in $\mathbb{R}_{std}$ is an open set in $\mathbb{R}_{LL}$, but not vice versa.*

Proof. We want to show that every open set in $\mathbb{R}_{std}$ is an open set in $\mathbb{R}_{LL}$. We can do this by taking the countable union of open sets of $\mathbb{R}_{LL}$. Take an arbitrarily chosen open set from $\mathbb{R}_{std}$ and call it $U_{a,b}$, then it is of the form $U_{a,b} = \{(a, b) : a, b \in \mathbb{R}\}$. On the other hand we have the open sets from $\mathbb{R}_{LL}$ that we will denote $V_{a,b}$, and is of the form $V_{a,b} = \{[a, b) : a, b \in \mathbb{R}\}$. By using the arbitrary union property of a topology we have $\cup_{n \in \mathbb{N}} V_{a+1/n,b} = \cup_{n \in \mathbb{N}} [a + 1/n, b) = (a, b) = U_{a,b}$. Therefore any open set of $\mathbb{R}_{std}$ is open in $\mathbb{R}_{LL}$. However it is not true that every open set of $\mathbb{R}_{LL}$ is open in $\mathbb{R}_{std}$. This is because a topology is only closed under finite intersections, and in fact the open sets of the lower limit topology is neither closed, nor open in the standard topology. □

Exercise 4.9.

1) In $\mathbb{H}_{bub}$, what is the closure of the set of rational points on the x-axis?

2) In $\mathbb{H}_{bub}$, which subsets of the x-axis are closed sets?

3) In $\mathbb{H}_{bub}$, let A be a countable set on the x-axis and let z be a point on the x-axis not in A. Then there exist disjoint open sets U and V such that $A \subset U$ and $z \in V$.

4) In $\mathbb{H}_{bub}$, let A be a countable set on the x-axis and let B be a disjoint countable set of points on the x-axis. Then there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

5) In $\mathbb{H}_{bub}$, let A be the rational numbers and let B be the irrational numbers. Do there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$?

1) In $\mathbb{H}_{bub}$ the closure of the set of rational points on the x-axis is the set of rational points on the x-axis. This is because any point on the x-axis of $\mathbb{H}_{bub}$ is an isolated point by how the open sets of this topology are defined.

2) Any subset is closed on the x-axis is closed, since all the points are isolated points, so contain no limit points.

3) Let A be the countable set, so $A = \{a_1, a_2, \dots\}$. We start off by construction an open set containing v. Then go down the list of elements of A constructing an open set containing each element and being disjoint of the open set containing v.

4) We do a similar construction as part 3). By part 3) we can construct a disjoint set that contains a point and a countable set. We now apply this to each element and alternate the elements of the countable set. For example, let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be our countable sets. Then we start by constructing an open set containing a_1. Next we construct an open set containing b_1 disjoint from the open set containing a_1. Then continue this process for all a_n and b_n where $n \in \mathbb{N}$.

5) We can prove this by contradiction. Assume that it is possible to construct U and V that are open disjoint sets and contain A and B respectively. We take two sequence $\{a_n\}$ and $\{b_n\}$ such that each $a_i \in A$ and $b_i \in B$ for any $n, i \in \mathbb{N}$. We construct each sequence such that $a_{n+1} \in \{(x, 0) : x \in (a_n, b_n)\}$ (WLOG let $a_n < b_n$). We construct another sequence $\{\epsilon_k\}$ dependent on the previous two sequences as the radius of each open ball tangent to the axis that contains each a_i and b_i. By further inspection we see that $\{\epsilon_k\} \to 0$ and so the alternating sequence $\{a_1, b_1, a_2, b_2, ...\} \to l$. We can see that l is a limit point of both U and V, thus there exist no open set disjoint from U and V that contains l. However we have that since l is on the x-axis it must be in A or B. Thus we reach a contradiction and conclude that there are no two disjoint sets U and V that contain A and B respectively.

Exercise 4.10. Check that the set of arithmetic progressions $\{az + b : a \neq 0, z \in \mathbb{Z}\}$ forms a basis for a topology, $\mathbb{Z}_{arith}$.

We have all the open sets. We need to show $\mathbb{Z}$ is in the arithmetic progressions. However take the set $B = \{z + 1 : z \in \mathbb{Z}\}$. We can see that $B = \mathbb{Z}$. Thus the set of arithmetic progressions forms a basis for a topology, $\mathbb{Z}_{arith}$.

Theorem 4.11. *There are infinitely many primes in $\mathbb{Z}$.*

Proof. We will prove this by contradiction. We have the sets $p\mathbb{Z}$ where p is a prime in the integers. This set is closed since it contains all of its limit points (in order to prove this we can show that each point not in $p\mathbb{Z}$ is an interior of $(p\mathbb{Z})^c$). This set is not open since we cannot write $p\mathbb{Z}$ as a union of basis elements (it will result in $gcd(p, 0) = p \neq 1$) . Suppose that there are a finite number of primes, then the union of primes p is $\bigcup_{p\in\mathbb{P}} p\mathbb{Z} \cup \{-1, 1\} = \mathbb{Z}$ where $\mathbb{P}$ is the set of primes. However this is a contradiction since a finite union of closed sets that are not open cannot be an open set, and $\mathbb{Z}$ is an open set. □

4.2 Subbases

4.3 Order Topology

Exercise 4.20. In the lexicographically ordered square find the closures of the following subsets:

$$A = \left\{ \left(\frac{1}{n}, 0 \right) | n \in \mathbb{N} \right\}.$$
$$B = \left\{ \left(1 - \frac{1}{n}, \frac{1}{2} \right) | n \in \mathbb{N} \right\}.$$
$$C = \{(x, 0) \mid 0 < x < 1\}.$$
$$D = \left\{ \left(x, \frac{1}{2} \right) \mid 0 < x < 1 \right\}.$$
$$E = \left\{ \left(\frac{1}{2}, y \right) \mid 0 < y < 1 \right\}.$$

1) $\overline{A} = A \cup (0, 1)$

2) $\overline{B} = B \cup (1, 0)$

3) $\overline{C} = C \cup \{(x, 1) | x \in (0, 1)\}$

4) $\overline{D} = D \cup (0, 1) \cup (1, 1)$

5) $\overline{E} = E \cup (1/2, 0) \cup (1/2, 1)$

Exercise 4.21. Assume that $\mathbb{N}$ has the usual order. Let $\mathbb{N}^\omega$ denote the Cartesian product of a countable number spaces $\mathbb{N}$. It can be endowed with the dictionary order in a natural way. Show that $\mathbb{N}^\omega$ is uncountable, is not well-ordered, and any set that does not have a least element does not have a limit point.

We want to show that $\mathbb{N}^\omega$ is uncountable. We can do this by showing it has the same carnality as $\mathbb{R}$ by showing there exist an injection to each other. We take the map

$$f : \mathbb{N}^\omega \to \mathbb{R}$$

where for the element $n \in \mathbb{N}^\omega$ and $n = (n_1, n_2, \dots)$ for $i \in \mathbb{N}$. We define $f(n) = \sum_{i=1}^{\infty} (n_i) 10^{-i+1}$. This map is clearly injective. Next we know that the real number are in bijection to the set of infinite sequence of 1's and 0's to the infinite sequence of 1's and 2's. We define the injective map

$$g : \mathbb{R} \to \{1, 2\}^{\mathbb{N}}$$

This map is injective since the map to infinite sequence of 0's and 1's is injective.Therefore, $|\mathbb{R}| = |\mathbb{N}^\omega|$ and the set $\mathbb{N}^\omega$ is uncountable.

Next we want to show that $\mathbb{N}^\omega$ is not well-ordered. We can do that by showing that all subsets do not have a least element. An example is the set $\mathbb{N}^\omega \backslash s$, where s is the minimum element of $\mathbb{N}^\omega$ with all 1's in each projection map $\pi_{\alpha\in\omega}(s) = 1$ for all $\alpha \in \omega$. And each set that does not have a smallest element has a limit point. We can see this by constructing a sequence whose limit approaches the infimum of the set (it would look something like $(1, 1, 1, 1, 1, 1, ..., 2, ...)$ where 2 is the last element however we can always find another spot further in the sequence to put the 2 therefore there is no least element). This shows that every set does not have a least element, and therefore is not well-ordered. Two birds, one stone (in this case both of the birds were the same).

4.4 Subspaces

Theorem 4.27. *Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if there is a set $D \subset X$, closed in $(X, \mathcal{T})$, such that $C = D \cap Y$.*

Proof. $\Rightarrow$ Assume C is closed in $(Y, \mathcal{T}_Y)$. We want to show that there exist a closed set $D \subset X$ in $(X, \mathcal{T}_X)$ such that $C = D \cap Y$. Let $D = Cl_x(C)$ and we have that C is closed in D, and the intersection with Y is C which is closed.

$\Leftarrow$ Suppose there exists a closed D in X. We want to show that $C = D \cap Y$ is closed in Y. We have that D^c is open in X and so $C^c = D^c \cap Y$ is open in Y. Therefore the complement of C^c in Y is closed.

□

Exercise 4.28. Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if $Cl_X(C) \cap Y = C$.

It is enough to take the observation that any limit point of C in Y is also in X. Therefore the intersection with this closed set and Y is a closed set, since it contains all of its limit points in Y and X.

Theorem 4.29. *Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.*

Proof. This follows directly from the definition of a subspace. All the elements of a subspace are the open sets of X intersected with Y. Take an arbitrarily chosen open set V in Y we know there exist an open set U in X, then we can write it as a union of open sets $\cup_i B_i$ in the set $\mathcal{B}$ and $U \cap Y$ is open in Y. However we can see that $\cup_i B_i \cap Y = \cup_i (B_i \cap Y)$. We conclude that $\mathcal{B}_Y$ is a basis for Y. □

Exercise 4.30. Consider the following subspaces of the lexicographically ordered square:

1. $D = \{(x, \frac{1}{2}) \mid 0 < x < 1\}$.

2. $E = \{(\frac{1}{2}, y) \mid 0 < y < 1\}$.

3. $F = \{(x, 1) \mid 0 < x < 1\}$.

As sets they are all lines. Describe their relative topologies.

For this exercise I will state the subset in which generates the topology.

1. $Y_D = \{((0,1), 1/2)\}$ intersected with the open sets from the lexicographically ordered square.

2. $Y_E = \{(1/2, (0,1))\}$ intersected with the open sets from the lexicographically ordered square.

3. $Y_F = \{((0,1), 1)\}$ intersected with the open sets from the lexicographically ordered square.

4.5 Product Spaces

Exercise 4.33. Show that the product topology on $X \times Y$ is the same as the topology whose subbasis consists of all sets that are either of the form $\pi_X^{-1}(U)$ where U is an open set in X or of the form $\pi_Y^{-1}(V)$ where V is an open set in Y.

Take an arbitrarily chosen element from $\mathcal{T}_X \times \mathcal{T}_Y$ call it $V_{x,y} = (U_x, U_Y)$ for open sets U_x in X and U_y in Y. We can see that $\pi_x(V_{x,y}) = U_x$ and $\pi_y(V_{x,y}) = U_y$. Thus $V_{x,y} = \pi_x^{-1}(U_x) \cap \pi_y^{-1}(U_Y)$, and the subbasis consisting of all sets listed as above are the same as the product topology.

Exercise 4.36. Let $\mathcal{T}$ be the topology on 2^X with sub-basis $\mathcal{S}$.

1. Every basic open set in 2^X is both open and closed.

2. Show that if a collection of subbasic open sets of 2^X has the property that every point of 2^X lies in at least one of those subbasic open sets, then there are two subbasic open sets in that collection such that every point of 2^X lies in one of those two subbasic sets.

3. Show that if a collection of basic open sets of 2^X has the property that every point of 2^X lies in at least one of those basic open sets, then there are a finite number of basic open sets in that collection of such that every point of 2^X lies in one of those basic sets

Exercise 4.37. In the product space $2^{\mathbb{R}}$, what is the closure of the set A consisting of all elements of $2^{\mathbb{R}}$ that are 0 on every rational coordinate, but may be 0 or 1 on any irrational coordinate?

Exercise 4.38. Let $\mathbb{R}^\omega = \prod_{i=1}^{\infty} \mathbb{R}$, the countable product of copies of $\mathbb{R}$, so that every point in $\mathbb{R}^\omega$ is a sequence $(x_1, x_2, x_3, \dots)$. Let $A \subset \mathbb{R}^\omega$ be the set consisting of all points with only positive coordinates. Then in the product topology $O = (0,0,0,\dots)$ is a limit point of the set A, and there is a sequence of points in A converging to O. However, in the box topology $O = (0,0,0,\dots)$ is a limit point of the set A, but there is no sequence of points in A converging to O.

5 Separation Properties

Theorem 5.1. *A space* $(X, \mathcal{T})$ *is* T_1 *if and only if every point in* X *is a closed set.*

Proof. $\Rightarrow$ Assume that $(X, \mathcal{T})$ is T_1, then for every point x in X and y in X we can find an open set U that contains x and an open set V that contains y such that U does not contain y and V does not contain x. This implies that every point not equal to x is an interior point of the complement and not a limit point of x. Therefore x contains its limit points and is closed.

$\Leftarrow$ Assume instead that every point x of X is closed. That means that every point y not equal to x and in X there exist an open set that contains y, but does not contain x. This implies that the space is T_1 since we can do the same for y; that is, we can find an open set that contains x and not y. We denote these open sets V and U, respectively. $\square$

Exercise 5.2. Let X be a space with the finite complement topology. Show that X is T_1.

We have the cofinite topology space $(X, \mathcal{T})$. We want to show that this is T_1; that is, for two points x, y in X we can find open sets U, V in $\mathcal{T}$ such that U contains x and not y and V contains y and not x. However we can do this by letting $U = X\{y\}$ and $V = X\{x\}$. Therefore, $(X, \mathcal{T})$ is T_1.

Exercise 5.3. Show that $\mathbb{R}_{std}$ is Hausdorff.

We want to show $(\mathbb{R}, \mathcal{T}_{std})$ is Hausdorff. Take two arbitrarily chosen points x and y in X, then we want to show that there exist disjoint open sets U and V such that U contains x and not y and V contains y and not x. Let $U = N_r(x)$ and $V = N_r(y)$ where $N_r(x_o) = (x_o - r, x_o + r)$. Let $r = (x + y)/2$ and we can see that these two open sets are disjoint and contain x and y, respectively.

Exercise 5.4. Show that $\mathbb{H}_{bub}$ is regular.

Exercise 5.5. Show that $\mathbb{R}_{LL}$ is normal.

The closed sets in $\mathbb{R}_{LL}$ are of the form $[a, b]$, $(-\infty, a]$, or $[a, \infty)$ where a and b are in $\mathbb{R}$. Suppose we have two disjoint closed sets $A = [a_1, a_2]$ and $B = [b_1, b_2]$ in $\mathbb{R}_{LL}$, then in order to show this space is normal we need to show that there exist two disjoint open sets U and V that contain A and B respectively. Since A and B are disjoint we can use the fact that $a_2 < b_1$. We take $U = [a_1, b_1)$ and $V = [b_1, \infty)$, and we can see that $U \cap V = \emptyset$. Also $A \subset U$ and $B \subset V$. Therefore $\mathbb{R}_{LL}$ is normal.

Exercise 5.6. 1. Consider $\mathbb{R}^2$ with the standard topolgy. Let $p \in \mathbb{R}^2$ be a point not in a closed set A. Show that $\inf\{d(a, p) : a \in A\} > 0$.

2. Show that $\mathbb{R}^2$ with the standard topology is regular.

3. Find two disjoint closed subsets A and B of $\mathbb{R}^2$ with the standard topology such that $\epsilon = \inf\{d(a, b) : a \in A \text{ and } b \in B\} = 0$.

4. Show that $\mathbb{R}^2$ with the standard topology is normal

1. Take $\epsilon = \inf\{d(a, p) : a \in A\}$, then we know that $\epsilon \geq 0$ by the properties of metrics. We want to show that $d(a, p) \neq 0$ for all $a \in A$. We can do this by contradiction. Suppose $d(a, p) = 0$, then $a = p$ for some $a \in A$ and this implies that $p \in A$. However by assumption we have that $p \notin A$ our contradiction.

2. We have already shown that $\epsilon > 0$. Now take the same p and A from part 1., and we want to show that there exist two disjoint open sets U and V such that the contain p and A, respectively. However if we take $U = N_{\epsilon/2}(p)$ and $V = \overline{V}^c$ then we have two open sets such that $U \cap V = \emptyset$ and $p \in U$ and $A \subset V$. Note: $N_r(p)$ is the neighborhood of points less that the distance of r, or open ball with radius r, around the point p.

3.

4.

Theorem 5.7. *1. A T_2-space (Hausdorff) is a T_1-space.*

2. A T_3-space (regular and T_1) is a Hausdorff space, that is, a T_2-space.

3. A T_4-space (normal and T_1) is regular and T_1, that is, a T_3-space.

Proof. 1. We have a Hausdorff space. We want to show that this is a T_1 space. Let $(X, \mathcal{T})$ be our toopology that is a Hausdorff space. We want to show that for all $x, y \in X$ there exist two open sets U and V that contain x and y respectively. However we have that $(X, \mathcal{T})$ is Hausdorff, so we know there exist two disjoint open sets U_x and V_x such that contain x and y respectively. Let $U = U_x$ and $V = V_x$ and we have that $y \notin U$ and $x \notin V$ since U and V are disjoint.

2. Instead let $(X, \mathcal{T})$ be a T_3-space. We want to show that $(X, \mathcal{T})$ is a Hausdorff space. Take two points x and y in X such that $x \neq y$, then we want to find two disjoint open sets U and V that contain x and y respectively. By theorem 5.1 we showed that points are closed sets. Therefore we let $A = \{x\}$ and we apply the properties of T_3-space; that is, there exist U and V that contain A and y respectively. However $A = \{x\}$ and we conclude that $(X, \mathcal{T})$ is a Hausdorff space.

3. Instead let $(X, \mathcal{T})$ be a T_4-space. We want to show that it is a T_3-space. We take a closed set A in X and a point p in X that is disjoint from A. By theorem 5.1 we can say that isolated points are closed sets and we let $B = \{p\}$. We have that $(X, \mathcal{T})$ is a T_4-space so we can find disjoint open sets U and V that contain A and B respectively. However $B = \{p\}$, so we conclude that $(X, \mathcal{T})$ is a T_3-space.

□

Theorem 5.8. *A topological space X is regular if and only if for each point p in X and open set U containing p there is an open set V such that $p \in V$ and $\overline{V} \subset U$.*

Proof. $\Rightarrow$ Assume that $(X, \mathfrak{T})$ is regular, then we want to show that for all $x \in X$ and $p \in U \in \mathfrak{T}$ that there exist a $V \in \mathfrak{T}$ such that $p \in V \subset \overline{V} \subset U$. Let x be in X and a open set U such that $x \in U$. Since $x \in U$ we know that $x \notin U^c$, and U^c is closed since U is open. We apply our properties of a space being regular and we know we can find disjoint open sets V_{U^c} and V_x such that contain U^c and x respectively. We let $V = V_x$ and we can see that $V \cap U^c = \emptyset$ since $U^c \subset V_{U^c}$. And actually for the same reason there are no limit points of $V = V_x$ in U. We can also see that $V \cap U = V$ since $V \cap X = (V \cap U^c) \cup (V \cap U) = \emptyset \cup V$. We conclude that $p \in V \subset \overline{V} \subset U$.

$\Leftarrow$ Assume that for a topological space $(X, \mathfrak{T})$ that for each point p in X and open set U containing p there is an open set V such that $p \in V$ and $\overline{V} \subset U$. We want to show that this space is regular. However take a p in X and a closed set A in X disjoint from p. Then by the properties of our topology we take an open set V that contains p and $\overline{V}^c$ that contains A. We do this in a way where $V \cap A = \emptyset$. We can do this by taking open sets with in the open sets until the set is small enough not to intersect A. Once we have found our V and $\overline{V}^c$ we conclude that these sets open sets are disjoint and our space is regular. $\square$

Theorem 5.9. *A topological space X is normal if and only if for each closed set A in X and open set U containing A there is an open set V such that $A \subset V$, and $\overline{V} \subset U$.*

Proof. $\Rightarrow$ Assume that $(X, \mathfrak{T})$ is normal. Let A be a closed set in X and U be an open set that contains A. We want to show there exist some V such that $A \subset V \subset \overline{V} \subset U$. However take the closed sets A and U^c, then since our space is normal we know there exist disjoint open sets U_A and U_c such that contain the closed sets respectively. We take $V = U_A$ and we can see that $A \subset V \subset \overline{V} \subset U$. *Note: there are disjoint open sets, so the limit points of V cannot be in U.

$\Leftarrow$ Assume instead that in our topology $(X, \mathfrak{T})$ for each closed set A in X and open set U containing A there is an open set V such that $A \subset V$, and $\overline{V} \subset U$. Let A be a closed set in X such that $A \subset U$ where U is an open set. Let B be a disjoint closed set from A. By the properties of our topological space we know there exist some V_i in U. We also know there is some V_{i+1} in V_i. We repeat this process until we find a V that is disjoint from B. Then we take the open sets V and $\overline{V}^c$ and clearly these sets are disjoint, and contain A and B respectively. We conclude that our topology is normal. $\square$

Theorem 5.10. *A topological space X is normal if and only if for each pair of disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.*

Proof. Have you ever seen a loaded chicken sandwich with a side of barbeque sauce? Well this problem is nothing like that.

$\Rightarrow$ We will start off by assuming that X is normal. Let there be two disjoint closed sets A and B in X. Since X is normal we know there exist some open disjoint sets U and V in X. We apply Theorem 5.9 to show there exists an open set U_0 and V_0 that contain A and B respectively, and $\overline{U_0} \subset U$ and $\overline{V_0} \subset V$. We conclude that since $U \cap V = \emptyset$ we have that $\overline{U_0}$ and $\overline{V_0}$ are disjoint.

$\Leftarrow$ Assume the ladder for some topological space X. Then we want to show that X is normal. However this direction is trivial, since it follows from the definition of our properties of X. $\square$

Theorem 5.11. *A topological space X is normal if and only if for each pair of open sets U, V such that $U \cup V = X$, there exist open sets U', V' such that $\overline{U'} \subset U$ and $\overline{V'} \subset V$, and $\overline{U'} \cup \overline{V'} = X$.*

Proof. $\Rightarrow$ Assume that X is normal. We have that $U \cup V = X$, so by demorgan's laws we have that $U^c \cap V^c = \emptyset$. By the properties of our topology we can find two disjoint open sets such that U_o and V_o that contain U^c and V^c respectively. We have that $U^c \subset U_o$ and $V^c \subset V_o$, and once we take the complement we have that $U_o^c \subset U$ and $V_o^c \subset V$. The rest of the proof follows directly from Theorem 5.9.

$\Leftarrow$ Instead, let us assume the ladder property for a topological space X. We want to show that X is normal. Let A and B be two closed disjoint sets, then A^c and B^c are open, and $A^c \cup B^c = X$ since the original sets were disjoint. By the properties of the topology we know there exist U_A and U_B such that $U_A \subset \overline{U_A} \subset A^c$ and $U_B \subset \overline{U_B} \subset B^c$. And so by Demorgans laws we have that $A \subset \overline{U}_A^c \subset U_A^c$ and $B \subset \overline{U}_B^c \subset U_B^c$. $\square$

Exercise 5.12. 1. Describe an example of a topological space that is T_1 but not T_2.

2. Describe an example of a topological space that is T_2 but not T_3.

3. Describe an example of a topological space that is T_3 but not T_4.

1. We want to show an example of a topological space that is T_1 but not T_2. However if we take the set $\mathbb{Z}$ with the co-finite topology, then we can see that we will never find disjoint open sets U and V for any two points a and b. This is because U and V have to have finite complements and so there will always be some point in $\mathbb{Z}$ that is contained in both sets U and V.

2. We want to describe an example of a topological space that is T_2 but not T_3. ?

3. We want to describe an example of a topological space that is T_3 but not T_4. Take the $\mathbb{H}_{bub}$ topology. As we will show later in the homework this is not normal.

Exercise 5.13. Construct a chart listing our examples of topological spaces along the top and listing the separation properties down the side. In each box answer the question of whether the example of the column has the property of the row. Here are the examples to use as column heads:

1. $\mathbb{R}_{std}$
2. $\mathbb{R}^n_{std}$
3. indiscrete topology

4. discrete topology

5. finite complement topology

6. countable complement topology

7. lower limit topology, $\mathbb{R}_{LL}$

8. double headed snake, $\mathbb{R}_{+00}$

9. $\mathbb{R}_{har}$

10. Sticky Bubble Topology, $\mathbb{H}_{bub}$

11. arithmetic progression topology, $\mathbb{Z}_{arith}$

12. lexicographically ordered square

13. 2^X

Here are the properties to use as labels of the rows:

1. T_1

2. Hausdorff

3. regular

4. normal

1. $\mathbb{R}_{std}$ is normal therefore 1, 2, and 3.

2. $\mathbb{R}^n_{std}$ is normal. It can be seen by using projection maps. Each projection map will result in the standard topology, and once we piece up the projections we will find that this topology is normal and therefore 1, 2, and 3.

3. indiscrete topology is the trivial topology. There are no such thing as disjoint points or sets, so it doesn't make sense to consider whether it is 1, 2, 3, or 4.

4. discrete topology is normal since every point is an isolated point, and therefore the closed sets are open sets. So if we take two disjoint closed sets those sets are the open sets that are disjoint. Therefore it is also 1, 2, and 3.

5. Finite complement topology is as shown above an example of a space that is T_1, but not Hausdorff. Therefore this space is only T_1.

6. Countable complement topology is similar to the discrete topology. The closed sets are also open, so for any two disjoint closed sets they are contained in themselves, the open sets which by assumption are disjoint. Therefore this space is normal and 1, 2, and 3.

7. lower limit topology, $\mathbb{R}_{LL}$ is not normal, however it is regular. Therefore we have that it is 1 and 2.

8. double headed snake, $\mathbb{R}_{+00}$ is actually only T_1. This comes from the fact that you can't find disjoint open sets when the points are the distinct $0'$ and $0''$. Therefore this space is only T_1.

9.

Theorem 5.16. *Every Hausdorff space is hereditarily Hausdorff.*

Proof. Assume a topological space X is a Hausdorff space. We want to show that a sub space Y of X is also Hausdorff. Take two points a and b in Y, then these points are in X since Y is a subspace of X. Since X is Hausdorff we know there exist two disjoint open sets U and V such that $U \cap V = \emptyset$. However this implies that $U_y = U \cap Y$ and $V_y = V \cap Y$ are disjoint open sets in Y that contain a and b respectively, Also $U_y \cap V_y = Y \cap (U \cap V) = \emptyset$. We conclude that Y is Hausdorff. □

Theorem 5.17. *Every regular space is hereditarily regular.*

Proof. Let X be a regular space and Y be a subspace of X. Let p be a point of Y and A be a closed set of Y. We want to show there exist disjoint open sets U and V that contain p and A respectively. However if we take the closure of A in X, then we can use that there exist disjoint open sets V_x that contains A and U_x that contain p. Also we let $U = U_x \cap Y$ and $V = V_x \cap Y$ and these sets are clearly open. These sets are also disjoint since $U \cap V = U_x \cap V_x \cap Y = \emptyset$. Therefore we conclude that Y is regular. □

Theorem 5.20. *The space X is a completely normal space if and only if X is hereditarily normal.*

Proof. $\Rightarrow$ Suppose that X is completely normal, then we want to show that X is hereditarily normal. Let Y be a relative topology to X. Let A and B be closed sets in Y. We want to find open disjoint sets U and V that cover A and B respectively. Let us take the sets $A \subset Y$ and $B \subset Y$ in X. If we can show these sets are separated, then by the properties of completely normal topology we have that there exist some U and V. However $Cl_x(A \cap Y) \cap B = \emptyset$ since $B \subset Y$ and $A \subset Y$, so any limit point of A is not in Y and therefore does not intersect B. For the same argument we have that $Cl_x(B \cap Y) \cap A = \emptyset$. So we have that A and B are seperated in X, and thus there exist disjoint open sets U_A and U_B that contain A and B respectively in X. Also they are disjoint in Y and we let $U = U_A \cap Y$ and $V = U_B \cap Y$.

$\Leftarrow$ Suppose that X is herditarily normal. We want to show that X is completely normal. Let A and B be two separated sets in X. We want to find disjoint open sets U and V that contain A and B respectively. Let us take the relative topology $Y = Cl_X(A \cap B)^c$ of X. We can see that Y is an open set. We have that Y is normal since X is heredtiraily normal, and so we know there exist some disjoint open sets U_A and U_B such that contain $Cl_X(A) \cap Y$ and $Cl_X(B) \cap Y$. We let $U = U_A$ and $V = U_B$, and we can see that $A \subset U$ and $B \subset V$. Also we have that U and V are open in X since Y is open. We conclude that X is completely normal.

□

Theorem 5.22. *Order topologies are hereditarily normal.*

Proof. It follows from proving that ordered topologies are completely normal. From there we use Theorem 5.20 to show that the space is hereditarily normal. □

Theorem 5.26. *Let A and B be subsets of a topological space X and let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be two collections of open sets such that*

1. $A \subset \bigcup_{i\in\mathbb{N}} U_i$
2. $B \subset \bigcup_{i\in\mathbb{N}} V_i$
3. for each i in $\mathbb{N}$, $\overline{U}_i \cap B = \emptyset$, and $\overline{V}_i \cap A = \emptyset$.
Then there are open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Proof. Let $\cup_{i\in\mathbb{N}} U_i = U_u$ and $\cup_{i\in\mathbb{N}} V_i = V_v$ (two open sets). We take $\overline{U}^c = U_c$ and $\overline{V}^c = V_c$. I do not know how to show that $\overline{V}_v^c$ and $\overline{U}_u^c$. However if this is true we can intersect them with the original open covers and we should have our U and V. □

6 2nd Countable Spaces

Exercise 6.1. Show that the definition that A is dense in X is equivalent to saying that every non-empty open set of X contains a point of A

We can see that every point of X is a limit point of A or a point in A. Another way to see this is to notice that there are not points in the interior of the complement, since for all open sets there contains some element of A.

Exercise 6.2. Show that $\mathbb{R}_{std}$ is separable. With which of the topologies on $\mathbb{R}$ that you have studied is $\mathbb{R}$ not seperable?

We have that the co-finite topology on $\mathbb{R}$ is not separable. This is because the open sets are $\mathbb{R} - [\{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_k\}]$ for some $k \in \mathbb{N}$. However there are more than an uncountable combinations of open sets; that is, for any countable set C, there will exist an open set that does not contain a point from C.

Exercise 6.3. Add 'separable' as a new property in your chart, and complete your chart by deciding which of the spaces we've studied are saparable.

This section will be left out.

Exercise 6.4. Find a separable space that contains a subspace that is not separable in the subspace topology.

We take the set $Q^2 = \{(q_1, q_2) : q_1, q_2 \in \mathbb{Q} \text{ and } q_1 \geq 0, q_2 \geq 0\}$. This set is separable in the topology $\mathbb{H}_{bub}$, however not in the discrete sub-topology on the x-axis.

Theorem 6.5. *If X and Y are separable spaces, then $X \times Y$ is separable.*

Proof. Let the basis for the topology X be the open sets U_α where $\alpha \in \Gamma$. Also let the basis for the topology Y be the open sets V_α where $\alpha \in \Gamma$. Then we can say that the basis for the topology $X \times Y$ are the open sets of the form $\{(U_\alpha, V_\beta) : \text{ for some } \alpha, \beta \in \Gamma\}$. We know that X and Y are separable, therefore it follows that each open set of the basis contains the element (x_o, y_o) where $x_o \in U_\alpha$ and $y_o \in V_\beta$. □

Theorem 6.6. *The space $2^{\mathbb{R}}$ is separable.*

Proof. □

Theorem 6.7. *Let $\{X_\beta\}_{\beta \in \mu}$ be a collection of separable spaces where $|\mu| \leq 2^{\omega_0}$, then $\prod_{\beta \in \mu} X_\beta$ is separable.*

Proof. □

Theorem 6.8. *If X is separable, Hausdorff space, then $|X| \leq |2^{2^{\mathbb{N}}}|$.*

Proof. We will directly prove this by showing there exist an injection from X to $2^{\mathbb{R}}$. First we have shown in previous theorems that $|2^{\mathbb{N}}| = |\mathbb{R}|$. We have also shown that if we can find an injection from one set A to a set B then $|A| \leq |B|$. Let Q be our countably dense subset of X. Then let us define a map from X to $2^{\mathbb{R}}$ as :

$$\phi(x) : X \to 2^{\mathbb{R}}$$

$$\phi(x) = \{U \cap Q : U \text{ is open and contains x}\}$$

We must show that this map is injective. However we can do this by using the fact that our space is Hausdorff. Take two different elements of X call them x_0 and x_1, then we know there exist two disjoint open sets U_0 and U_1 that contain x_0 and x_1 respectively. We also know that X is separable, so we know there exist elements from Q in the set U_0 and U_1. Therefore there contains an open set U_0 in $\phi(x_0)$ that is not in $\phi(x_1)$, and vice versa. We conclude that this map is an injection and the carnality argument holds.

□

Theorem 6.9. *Let X be a 2nd countable space, then X is separable.*

Proof. We can prove this by constructing a dense set. We do this by taking a point from each of the basis elements (basic open sets) which there are countable number of them, and we end up with a countable set that is dense. □

Exercise 6.10. 1. The space $\mathbb{R}_{std}$ is the 2nd countable (and hence separable).

2. The space $\mathbb{R}_{LL}$ is separable but not 2nd countable.

3. The space $\mathbb{H}_{bub}$ is separable but not 2nd countable.

1. We have that $\mathbb{R}_{std}$ has the open sets of the form (a, b) where $a, b \in \mathbb{R}$. We want to show that $\mathbb{R}_{std}$ is 2nd countable; that is, it has a countable basis. We can do this by using the collection of open sets (p, q) where $p, q \in \mathbb{Q}$ is a countable basis. Using the previous theorems on a previous homework we know that the countable union of countable sets is countable, thus the collection $\mathcal{B} = \{(p, q) : p, q \in \mathbb{Q}\}$ is countable. Now we need to show that in fact it is a basis. Take any arbitrarily chosen open set (a, b) in $\mathbb{R}_{std}$. Then we can write it as a union of sets of $\mathcal{B}$ by $(a, b) = \bigcup_{a \leq p < q \leq b}(p, q)$. We conclude that $\mathbb{R}_{std}$ is 2nd countable.

2. The space $\mathbb{R}_{LL}$ however is not 2nd countable. We can see this by the open sets of the form $[a, b)$ where a is an irrational number. This means we can no longer use the rationals as a basis, and we much include all of the irrational number. The irrational numbers however are uncountable, so this will include an uncountable number of disjoint open sets. However if we take the rationals in $\mathbb{R}_{LL}$ then we still have a countable dense subset.

3. First let us look in to the fact that $\mathbb{H}_{bub}$ is not 2nd countable. This is because every point on the x-axis is an isolated point with some excess bubble (circle) above it. To include each point on the x-axis with some arbitrary bubble above it, the number of open sets would be uncountable because of the points on the x-axis are uncountable. Therefore $\mathbb{H}_{bub}$ is not 2nd countable. However we have the $\mathbb{H}_{bub}$ is separable. Refer to Exercise 6.4.

Theorem 6.11. *Every uncountable set in a 2nd countable space has a limit point.*

Proof. We will prove this by contradiction. Suppose that a uncountable set A in a 2nd countable space X had no limit points, then there would exist disjoint open sets containing each point. However this leads to a contradiction since the space X is 2nd countable; that is, it has a countable basis. □

Theorem 6.12. *A 2nd countable space is hereditary 2nd countable.*

Proof. We will prove this directly. Take some 2nd countable space X and some subspace Y. We want to show that Y is 2nd countable. Take $\mathcal{B}$ the countable basis for the space X. Then we generate the basis for Y as $\mathcal{C} = \{B \cap Y : B \in \mathcal{B}\}$. Since $\mathcal{B}$ is countable $\mathcal{C}$ is countable and in fact $|\mathcal{C}| \leq |\mathcal{B}|$. It should be straight forward that $\mathcal{C}$ is a basis for Y. □

Theorem 6.13. *If X and Y are 2nd countable spaces, then $X \times Y$ is 2nd countable.*

Proof. This follows directly from the theorem that states that the countable union of countable sets is countable. We let $\mathcal{B}_X$ be the countable basis for X and let $\mathcal{B}_Y$ be the countable basis for Y. We construct the new countable basis $\mathcal{C} = \{(B_x, B_y) : B_x \in \mathcal{B}_X, B_y \in \mathcal{B}_Y\}$. This is set is countable since it is a countable union of countable sets, and is a basis for $X \times Y$ since any open set of $X \times Y$ is the product of two open sets which can be covered by our basis. We conclude that $X \times Y$ is 2nd countable. □

Theorem 6.14. *Let X be a 2nd countable space. Then X is 1st countable.*

Proof. We will directly prove this. Assume that a space X is 2nd countable; that is, it has a countable basis. Let us call the basis of open sets $\mathcal{B}$. We want to show that X is 1st countable; that is, every point of X has a countable neighborhood basis. Let x_o be an element of X, then there exist some smallest open set B of $\mathcal{B}$ that contains x_o. Also any open set that contains x_o must contain B, since B is a basic open set. We conclude that B is a countable neighborhood basis for x_o, and therefore all points of X have a countable neighborhood basis and we conclude that X is 1st countable. $\square$

Theorem 6.15. *If X is a topological space, $p \in X$, and p has a countable neighborhood basis, then p has a nested countable neighborhood basis.*

Proof. We can show this by construction. We take the topological space X that has a countable neighborhood basis $\{U_i\}_{i\in\mathbb{N}}$ around the point p. We want to show that is has a nested countable neighborhood basis. If $\{U_i\}$ are already nested then we are done. Otherwise we define the nested countable neighborhood basis as $V_n = \bigcup_{k\geq n} U_k$ for $k \in \mathbb{N}$. Each $p \in V_i \subset V_{i+1}$. This set is also countable since a countable union of countable sets is countable. Therefore we conclude that $\{V_i\}_{i\in\mathbb{N}}$ is a nested countable neighborhood basis for p. $\square$

Exercise 6.16. 1. The space $\mathbb{R}_{LL}$ is 1st countable.

2. The space $\mathbb{H}_{bub}$ is 1st countable.

3. The space $2^{\mathbb{R}}$ is not 1st countable.

1. We want to show that the space $\mathbb{R}_{LL}$ is 1st countable. Take an arbitrarily chosen point x in $\mathbb{R}_{LL}$, and we want to show that it has a neighborhood basis. Take the collection of open sets $\{U_i\}$ such that $U_i = [x, x + 1/i)$ for some $i \in \mathbb{N}$. We have that $1/i$ converges to 0, so for any open set that contains x contains some U_i. Also each U_i contains x by definition. Therefore we conclude that for all points of $\mathbb{R}_{LL}$ we have that there exist a countable neighborhood basis, and $\mathbb{R}_{LL}$ is 1st countable.

2. We want to show that the space $\mathbb{H}_{bub}$ is 1st countable. Take an arbitrarily chosen point (x, y) in $\mathbb{R}_{LL}$, and we want to show that is has a neighborhood basis. Suppose first that $y \neq 0$, it is off the x-axis, then we take the collection of open sets $\{U_i\}$ such that $U_i = B((x, y), 1/i)$ for some $i \in \mathbb{N}$. Every open set contains at least some U_i since $1/i$ converges to 0. Also, every open ball that contains (x, y) contains some U_i. Now instead let us suppose that $y = 0$, then we let our collection of open sets $\{U'_i\}$ where $U'_i = B((x, 1/i), 1/i)$. All of these open sets, U_i for all $i \in \mathbb{N}$ will include $(x, 0)$ by how we define open sets, also every open set that contains $(x, 0)$ contains some U'_i. We conclude that $\mathbb{H}_{bub}$ is 1st countable.

3. We want to show the space $2^{\mathbb{R}}$ is not 1st countable. We can do this by showing there exist some point x in $2^{\mathbb{R}}$ that does not contain some countable neighborhood basis.

Exercise 6.17. You may as well extend your table of spaces and properties by adding new columns for the properties 1st countable and 2nd countable and determining those properties for each of your spaces.

This exercise will not be done here. Refer to notes for more information.

Theorem 6.18. *Suppose x is a limit point of the set A in a 1st countable space X. Then there is a sequence of points $\{a_i\}_{i\in\mathbb{N}}$ in A that converges to x.*

Proof. We want to show that for a limit point x in the set A of a 1st countable space X there is a sequence $\{a_i\}_{i\in\mathbb{N}}$ in A that conberges to x. Since X is 1st countable we take the countable neighborhood of open sets to be $\{U_i\}_{i\in\mathbb{N}}$. By Theorem 6.15 we let this be a nested countable neighborhood basis. We construct the elements a_i to be any element of $U_i \cap A$ but not in $U_{i+1} \cap A$ for all $i \in \mathbb{N}$. This sequence converges exist since x is a limit point of A, and converges to x. Therefore we conclude that there exist a sequence that converges to x. $\square$

Theorem 6.19. *A 1st countable space is hereditarily 1st countable*

Proof. We will prove this directly. Let X be our 1st countable space, and Y or subspace. For any point $y \in Y$ we want to show that there exist some countable neighborhood basis. However $Y \subset X$, so we take the neighborhood basis of y in X and we let the countable neighborhood basis be $\{U_i\}_{i\in\mathbb{N}}$. We define the countable neighborhood basis in Y to be $V_i = U_i \cap Y$. The collection of $\{V_i\}$ satisfy the properties of a countable neighborhood basis, and we conclude that Y is 1st countable. $\square$

Theorem 6.20. *If X and Y are 1st countable spaces, then $X \times Y$ is 1st countable.*

Proof. We want to show that the product of two 1st countable space $X \times Y$ is 1st countable. Take an arbitrarily chosen point (x, y) of $X \times Y$. Let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be the nested countable neighborhood basis of x and y in X and Y respectively. The we let $W_i = (U_i, V_i)$ for all $i \in \mathbb{N}$. Our collection of $\{W_i\}$ satisfy the properties of a countable neighborhood basis, and therefore $X \times Y$ is 1st countable. $\square$

7 Compactness

Theorem 7.1. *Let X be a finite topological space. Then X is compact.*

Proof. We want to show that a topological space X is compact. We can do this by showing that for any open cover C of X there exist a finite subcover C'. Suppose that there exist an open cover C of X. We define X as the finite points $X = \{x_1, x_2, \ldots, x_k\}$ for some $k \in \mathbb{N}$. We construct the new open cover C' by taking the open sets of C that contains x_i for each $i \in \mathbb{N}$ such that $i \leq k$. Now we have our finite subcover C'. $\square$

Theorem 7.2. *If X is compact and E is a subset of X with no limit points, then E must be finite.*

Proof. We want to show that for a compact space X and E a subset of X with no limit points that E must be finite. We can show this directly. Since E has no limit points it must have only isolated points; that is, each point has an open set around it. We also know that we can cover E with a finite subcover since it is a subset of X. Therefore E has at most finite open sets that cover all of the isolated points of E, and thus E has only finitely many isolated points. We conclude that E is finite. □

Corollary 7.3. *If X is compact, then every infinite subset of X has a limit point.*

Proof. By Theorem 7.2 we can see that a subset of a compact set with no limit points must be finite. Therefore we will have a contradiction if E is infinite, and therefore there must exist a limit point. □

Theorem 7.4. *A space X is compact if and only if every collection of closed sets with the finite intersection property as a nonempty intersection.*

Proof. ($\Rightarrow$

Assume that a space X is compact. We want to show that for all collection of closed sets with the finite intersection property has a nonempty intersection. Let $\mathcal{C}$ be a collection of closed sets in X with the finite intersection property. We want to show that $\mathcal{C}$ has a nonempty intersection. We can prove this by contradiction. Suppose $\bigcap_{C\in\mathcal{C}} C = \emptyset$, then $(\cap C)^c = \cup C^c = X$. Since X is compact we can find a finite subcover of C, call it K, and $\cup K^c = X$ which implies once we take the complement that there exist a finite intersection of closed sets that have no intersection $(\cup K^c)^c = \cap K = \emptyset$. This contradicts our assumption with C having the finite intersection property.

$\Leftarrow$)

Instead assume that X is a space with the property that for all collection of closed sets C in X with the finite intersection property has a non-empty intersection. We want to show that X is compact. We will show this by contradiction. Suppose X isn't compact. Let C be an arbitrary cover for X, and there doesn't exist a finite subcover of C for X. Then C has the finite intersection property and $\cap_C C^c = \emptyset$. This contradicts our hypothesis. We conclude that X is compact.

□

Exercise 7.6. If A and B are compact subsets of X, then $A \cup B$ is compact. Suggest and prove a generalization.

We can find a finite subcover for A and B for any cover of A and B since X is compact. Also when we union A and B we also have a sum of two finite subcovers. In fact we can prove by induction that we can do this for a finitely many union of compact subsets. Also if X is compact we can do it for uncountably many unions.

Theorem 7.7. *Let A be a closed subspace of a compact space. Then A is compact.*

Proof. Let A be a closed subspace of a compact space X. Let C be a cover for A. We have that A^c is an open set in X, so $C \cup A^c$ is a cover for X. We have that X is compact, so we know there exist a finite subcover for X, namely $K \cup A^c$ for some collection of open sets K. We then conclude that K is a finite subcover for A, and A is compact. □

Theorem 7.8. *Let A be a compact subspace of a Hausdorff space X. Then A is closed.*

Proof. Let A be a compact subspace of a Hausdorff space X, then we want to show that A is closed. We can show this by showing that every point not in A is contained in a disjoint open set from A. Take some point x in X but not in A; that is, $x \in A^c$. For every point in a in A we can find disjoint open sets around x and a call them U_x and U_a. We take the cover $\cup_{a \in A} U_a$ and since A is compact we can find a finite union of U_a to cover A. This corresponds to a finite number of U_x disjoint open sets covering the point x. We take $\cap U_x$ corresponding to the finite subcover. We conclude that this intersection is open and every point outside of A is an interior point of A^c. Therefore A is closed. □

Exercise 7.9. Construct an example of a compact subset of a topological space that is not closed

Exercise 7.10. Must the intersection of two compact sets be compact?

Theorem 7.11. *The subspace $[0, 1]$ of $\mathbb{R}_{std}$ is compact.*

Proof. Let C be an arbitrary cover for the subspace $[0, 1]$. We define $A = \{[0, x] : x \in [0, 1]$ where $[0, x]$ is compact for the cover $C\}$. Note: this set is non-empty since any cover around the point $[0, 0]$ has a finite subcover. We then define a to be the least upper bound of A. Next we will prove by contradiction that a has to be greater than one. In fact we will show that a has to be greater than any integer (This means that this will work for any closed set of the form $[c, d]$ where $c, d \in \mathbb{R}$.) Nevertheless let us get back to the proof. Suppose that $a < 1$, then we have that $A = [0, a]$ is the only interval of the form that can be covered by a finite subcover. However we know that C is a cover for A, so we take the open set that contains a, call it U_a. We join it to our finite collection and BOOM! Contradiction. This is because our new collection is finite, and contradicts the fact that a is the least upper bound. In fact A should be the set $A = [0, a] \cup U_a$, and U_a contains an element greater than a. We can see that this works for any element a up to 1. □

Theorem 7.12. *(Heine-Borel Theorem) Let A be a subset of $\mathbb{R}$ with the standard topology. Then A is compact if and only if A is closed and bounded.*

Proof. ($\Rightarrow$ Assume that A is compact and a subset of $\mathbb{R}$. We want to show that A is closed and bounded. By Theorem 7.8 compact subset of Hausdorff spaces are closed. We have that $\mathbb{R}$ is Hausdorff therefore A is closed. Next we will show that A must be bounded by contradiction. Suppose A is not bounded. Take any arbitrary cover C for A, then we take any finite collection of open sets from C call it K. Then for $\cup K$ we can find some element of A that K does not cover. Therefore there exists no finite subcover for A, and A must be bounded.

$\Leftarrow$) Assume instead that A is a closed and bounded subset of $\mathbb{R}$. We want to show that A is compact. However the proof follows directly from Theorem 7.11 except instead we bring in a greatest upper bound, and apply it to the set $A = \{[a, b] : a, b \in \mathbb{R} where [a, b] can be cover by a finite cover\}$. We then find that any closed and bounded interval is compact and we apply Exercise 7.6 to show that a finite union of closed intervals is as well compact (and also closed and bounded). We conclude that A is closed and bounded. □

Exercise 7.13. Consider the rationals $\mathbb{Q}$ with the subspace topology inherited from $\mathbb{R}$. Find a set A in $\mathbb{Q}$ which is closed and bounded but not compact.

Exercise 7.14. Every compact subset C of $\mathbb{R}$ contains maximum that is in the set C, i.e., there is an $m \in C$ such that for any $x \in C$, $x \leq m$.

Theorem 7.15. *Every countably compact and Linedelof space is compact.*

Proof. Let X be a compact and Lindelof space. Let C be an arbitrarily chosen cover for X. We want to show that there exist a finite subcover of C, call it K. We have that X is Lindelof, so C has a countable subcover Q. We also have that X is countably compact, therefore Q has a finite subcover K. Therefore we have K a finite subcover of C of X, and we conclude that X is compact. $\square$

Theorem 7.16. *Every 2nd countable space is Lindelof.*

Proof. Let X be a 2nd countable space. We want to show that X is Lindelof; that is, for any cover there is a countable subcover. We know that X has a countable basis since X is 2nd countable call it B. We have that B is a cover for X, and is countable. We take C to be a cover for X. We construct the countable set Q by taking all the elements of B that are in C. Therefore $Q \subset B$ and thus we have that Q is a countable subcover for X. $\square$

Theorem 7.17. *Let B be a basis for a space X. Then X is compact (respectively, Lindelof) if and only if every cover of X by basic open sets in B has a finite (respectively, countable) subcover.*

8 Continuous Functions

Theorem 8.1. *Let $f : X \to Y$ be a function. Then the following are equivalent:*

1. *f is continuous,*
2. *for every closed set K in Y, $f^{-1}(K)$ is closed in X,*
3. *if p is a limit point of A in X, then $f(p)$ belongs to $\overline{f(A)}$.*
4. *for each $x \in X$ and each open set V containing $f(x)$, there is an open set U containing x such that $f(U) \subset V$.*

Proof. 1 implies 2: Suppose that f is continuous. Then K^c is open and $f^{-1}(K^c) = f^{-1}(K)^c$ is open and $f^{-1}(K)$ is closed.

Theorem 8.5. *A function $f : \mathbb{R}_{std} \to \mathbb{R}_{std}$ is continuous if and only if for each point x in $\mathbb{R}$ and $\epsilon > 0$, there is a $\delta > 0$ such that for each $y \in \mathbb{R}$ with $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$.*

Proof. ($\Rightarrow$ By definition of continuous, for each open set in $Y \in \mathbb{R}_{std}$ $f^{-1}(Y)$ is open in $\mathbb{R}_{std}$. In other words the open sets of $\mathbb{R}$ are points $d(f(x), f(y)) < \epsilon$, and once we apply our definition of continuous we know that there exist $d(x, y) < \delta$, an open set.

$\Leftarrow$) Assume instead that for each point x in $\mathbb{R}$ and $\epsilon > 0$, there is a $\delta > 0$ such that for each $y \in \mathbb{R}$ with $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$. We want to show that this function is continuous. However the function is continuous since we have that for any two points $d(f(x), f(y)) < \epsilon$ there is an open set $d(x, y) < \epsilon$, furthermore the inverse image of a open set is an open set. $\square$

Theorem 8.6. *Let X be a $1st$ countable space and Y be a topological space. Then a function $f : X \to Y$ and $g : X \to Y$ is continuous if and only if for each convergent sequence $x_n \to x$ in X, $f(x_n)$ converges to $f(x)$ in Y.*

$\square$