# **HTW Model**

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2024-02-15

In project 1, we applied model-based techniques to quantify and control for the similarity between training and testing experience, which in turn enabled us to account for the difference between varied and constant training via an extended version of a similarity based generalization model. In project 2, we will go a step further, implementing a full process model capable of both 1) producing novel responses and 2) modeling behavior in both the learning and testing stages of the experiment. Project 2 also places a greater emphasis on extrapolation performance following training - as varied training has often been purported to be particularly beneficial in such situations.

## Introduction

In project 1, I applied model-based techniques to quantify and control for the similarity between training and testing experience, which in turn enabled us to account for the difference between varied and constant training via an extended version of a similarity based generalization model. In project 2, we will go a step further, implementing a full process model capable of both 1) producing novel responses and 2) modeling behavior in both the learning and testing stages of the experiment. Project 2 also places a greater emphasis on extrapolation performance following training. Although varied training has

often been purported to be particularly beneficial for generalization or transfer, few experiments have compared varied and constant training in contexts with unambiguous extrapolation testing.

## **Function Learning and Extrapolation**

The study of human function learning investigates how people learn relationships between continuous input and output values. Function learning is studied both in tasks where individuals are exposed to a sequence of input/output pairs (?; ?), or situations where observers are presented with a an incomplete scatterplot or line graph and make predictions about regions of the plot that don't contain data (?; ?; ?).

Carroll (?) conducted the earliest work on function learning. Input stimuli and output responses were both lines of varying length. The correct output response was related to the length of the input line by a linear, quadratic, or random function. Participants in the linear and quadratic performed above chance levels during extrapolation testing, with those in the linear condition performing the best overall. Carroll argued that these results were best explained by a ruled based model wherein learners form an abstract representation of the underlying function. Subsequent work by Brehmer (?), testing a wider array of functional forms, provided further evidence for superior extrapolation in tasks with linear functions. Brehmer argued that individuals start out with an assumption of a linear function, but given sufficient error will progressively test alternative hypothesis with polynomials of greater degree. Koh & Meyer (?) employed a visuomotor function learning task, wherein participants were trained on examples from an unknown function relating the length of an input line to the duration of a response (time between keystrokes). In this domain, participants performed best when the relation between line length and response duration was determined by a power, as opposed to linear function. Koh & Meyer developed the log-polynomial adaptive-regression model to account for their results.

The first significant challenge to the rule-based accounts of function learning was put forth by DeLosh et al. (?) . In

their task, participants learned to associate stimulus magnitudes with response magnitudes that were related via either linear, exponential, or quadratic function. Participants approached ceiling performance by the end of training in each function condition, and were able to correctly respond in interpolation testing trials. All three conditions demonstrated some capacity for extrapolation, however participants in the linear condition tended to underestimate the true function, while exponential and quadratic participants reliably overestimated the true function on extrapolation trials. Extrapolation and interpolation performance are depicted in Figure ??.

The authors evaluated both of the rule-based models introduced in earlier research (with some modifications enabling trial-by-trial learning). The polynomial hypothesis testing model (?; ?) tended to mimic the true function closely in extrapolation, and thus offered a poor account of the human data. The log-polynomial adaptive regression model (?) was able to mimic some of the systematic deviations produced by human subjects, but also predicted overestimation in cases where underestimation occurred.

The authors also introduced two new function-learning models. The Associative Learning Model (ALM) and the extrapolationassociation model (EXAM). ALM is a two layer connectionist model adapted from the ALCOVE model in the category learning literature (?). ALM belongs to the general class of radial-basis function neural networks, and can be considered a similarity-based model in the sense that the nodes in the input layer of the network are activated as a function of distance. The EXAM model retains the same similarity based activation and associative learning mechanisms as ALM, while being augmented with a linear rule response mechanism. When presented with novel stimuli, EXAM will retrieve the most similar inputoutput examples encountered during training, and from those examples compute a local slope. ALM was able to provide a good account of participant training and interpolation data in all three function conditions, however it was unable to extrapolate. EXAM, on the other hand, was able to reproduce both the extrapolation underestimation, as well as the quadratic and exponential overestimation patterns exhibited by the human participants. Subsequent research identified some limitations

in EXAM's ability to account for cases where human participants learn and extrapolate sinusoidal function Bott & Heit (?) or to scenarios where different functions apply to different regions of the input space Kalish et al. (?), though EXAM has been shown to provide a good account of human learning and extrapolation in tasks with bi-linear, V shaped input spaces Mcdaniel et al. (?).

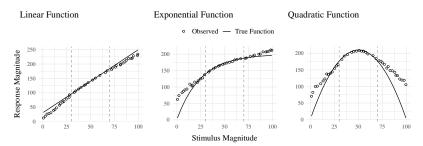


Figure 1: Generalization reproduced patterns from DeLosh et al. (1997) Figure 3. Stimulii that fall within the dashed lines are interpolations of the training examples.

# Methods

#### **Participants**

Data was collected from 647 participants (after exclusions). The results shown below consider data from subjects in our initial experiment, which consisted of 196 participants (106 constant, 90 varied). The follow-up experiments entailed minor manipulations: 1) reversing the velocity bands that were trained on vs. novel during testing; 2) providing ordinal rather than numerical feedback during training (e.g. correct, too low, too high). The data from these subsequent experiments are largely consistently with our initial results shown below.

### **Task**

We developed a novel visuomotor extrapolation task, termed the Hit The Wall task, wherein participants learned to launch a projectile such that it hit a rectangle at the far end of the screen with an appropriate amount of force. Although the projectile had both x and y velocity components, only the x-dimension was relevant for the task. Link to task demo

### **Procedure**

The HTW task involved launching projectiles to hit a target displayed on the computer screen. Participants completed a total of 90 trials during the training stage. In the varied training condition, participants encountered three velocity bands (800-1000, 1000-1200, and 1200-1400). In contrast, participants in the constant training condition encountered only one velocity band (800-1000).

During the training stage, participants in both conditions also completed "no feedback" trials, where they received no information about their performance. These trials were randomly interleaved with the regular training trials.

Following the training stage, participants proceeded to the testing stage, which consisted of three phases. In the first phase, participants completed "no-feedback" testing from three novel extrapolation bands (100-300, 350-550, and 600-800), with each band consisting of 15 trials.

In the second phase of testing, participants completed "no-feedback" testing from the three velocity bands used during the training stage (800-1000, 1000-1200, and 1200-1400). In the constant training condition, two of these bands were novel, while in the varied training condition, all three bands were encountered during training.

The third and final phase of testing involved "feedback" testing for each of the three extrapolation bands (100-300, 350-550, and 600-800), with each band consisting of 10 trials. Participants received feedback on their performance during this phase.

Throughout the experiment, participants' performance was measured by calculating the distance between the produced x-velocity of the projectiles and the closest edge of the current velocity band. Lower distances indicated better performance.

After completing the experiment, participants were debriefed and provided with an opportunity to ask questions about the study.

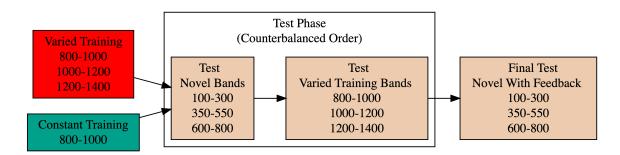


Figure 2: Experiment 1 Design. Constant and Varied participants complete different training conditions.

## **Analyses Strategy**

All data processing and statistical analyses were performed in R version 4.31 Team (?). To assess differences between groups, we used Bayesian Mixed Effects Regression. Model fitting was performed with the brms package in R Bürkner (?), and descriptive stats and tables were extracted with the BayestestR package Makowski et al. (?). Mixed effects regression enables us to take advantage of partial pooling, simultaneously estimating parameters at the individual and group level. Our use of Bayesian, rather than frequentist methods allows us to directly quantify the uncertainty in our parameter estimates, as well as circumventing convergence issues common to the frequentist analogues of our mixed models. For each model, we report the median values of the posterior distribution, and 95% credible intervals.

Each model was set to run with 4 chains, 5000 iterations per chain, with the first 2500 of which were discarded as warmup chains. Rhat values were generally within an acceptable range, with values  $\leq 1.02$  (see appendix for diagnostic plots). We

Table 1: Testing Deviation - Empirical Summary

(a) Full datasets

Band	Band Type	Mean	Median	Sd
100-300	Extrapolation	254	148	298
350 - 550	Extrapolation	191	110	229
600-800	Extrapolation	150	84	184
800-1000	Trained	184	106	242
1000-1200	Extrapolation	233	157	282
1200 - 1400	Extrapolation	287	214	290

(b) Intersection of samples with all labels available

Band	Band Type	Mean	Median	$\operatorname{Sd}$
100-300	Extrapolation	386	233	426
350 - 550	Extrapolation	285	149	340
600-800	Extrapolation	234	144	270
800-1000	Trained	221	149	248
1000-1200	Trained	208	142	226
1200-1400	Trained	242	182	235

used uninformative priors for the fixed effects of the model (condition and velocity band), and weakly informative Student T distributions for for the random effects.

We compared varied and constant performance across two measures, deviation and discrimination. Deviation was quantified as the absolute deviation from the nearest boundary of the velocity band, or set to 0 if the throw velocity fell anywhere inside the target band. Thus, when the target band was 600-800, throws of 400, 650, and 1100 would result in deviation values of 200, 0, and 300, respectively. Discrimination was measured by fitting a linear model to the testing throws of each subjects, with the lower end of the target velocity band as the predicted variable, and the x velocity produced by the participants as the predictor variable. Participants who reliably discriminated between velocity bands tended to have positive slopes with values ~1, while participants who made throws irrespective of the current target band would have slopes ~0.

## Results

### Testing Phase - No feedback.

In the first part of the testing phase, participants are tested from each of the velocity bands, and receive no feedback after each throw.

## **Deviation From Target Band**

Descriptive summaries testing deviation data are provided in Table ?? and Figure ??. To model differences in accuracy between groups, we used Bayesian mixed effects regression models to the trial level data from the testing phase. The primary model predicted the absolute deviation from the target velocity band (dist) as a function of training condition (condit), target velocity band (band), and their interaction, with random intercepts and slopes for each participant (id).

$$dist_{ij} = \beta_0 + \beta_1 \cdot condit_{ij} + \beta_2 \cdot band_{ij} + \beta_3 \cdot condit_{ij} \cdot band_{ij} + b_{0i} + b_{1i} \cdot band_{ij} + \epsilon_{ij} \tag{1}$$

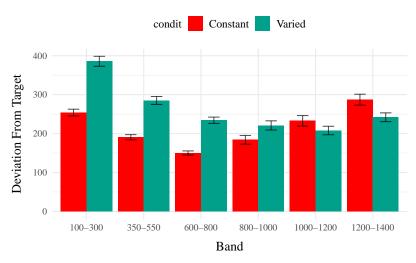


Figure 3: E1. Deviations from target band during testing without feedback stage.

Table 2: Experiment 1. Bayesian Mixed Model predicting absolute deviation as a function of condition (Constant vs. Varied) and Velocity Band

#### (a) Constant Testing1 - Deviation

Term	Estimate	95% CrI Lower	95% CrI Upper	pd
Intercept	205.09	136.86	274.06	1.00
${\bf condit} {\bf Varied}$	157.44	60.53	254.90	1.00
Band	0.01	-0.07	0.08	0.57
condit*Band	-0.16	-0.26	-0.06	1.00

(b) Varied Testing - Deviation

contrast	Band	value	lower	upper	pd
Constant - Varied	100	-141.49	-229.2	-53.83	1.00
Constant - Varied	350	-101.79	-165.6	-36.32	1.00
Constant - Varied	600	-62.02	-106.2	-14.77	1.00
Constant - Varied	800	-30.11	-65.1	6.98	0.94
Constant - Varied	1000	2.05	-33.5	38.41	0.54
Constant - Varied	1200	33.96	-11.9	81.01	0.92

The model predicting absolute deviation (dist) showed clear effects of both training condition and target velocity band (Table X). Overall, the varied training group showed a larger deviation relative to the constant training group ( = 157.44, 95% CI [60.53, 254.9]). Deviation also depended on target velocity band, with lower bands showing less deviation. See Table ?? for full model output.

#### Discrimination between bands

In addition to accuracy/deviation, we also assessed the ability of participants to reliably discriminate between the velocity bands (i.e. responding differently when prompted for band 600-800 than when prompted for band 150-350). Table ?? shows descriptive statistics of this measure, and Figure 1 visualizes the full distributions of throws for each combination of condition and velocity band. To quantify discrimination, we again fit Bayesian Mixed Models as above, but this time the dependent variable was the raw x velocity generated by participants

on each testing trial.

$$vx_{ij} = \beta_0 + \beta_1 \cdot condit_{ij} + \beta_2 \cdot bandInt_{ij} + \beta_3 \cdot condit_{ij} \cdot bandInt_{ij} + b_{0i} + b_{1i} \cdot bandInt_{ij} + \epsilon_{ij}$$
 (2)

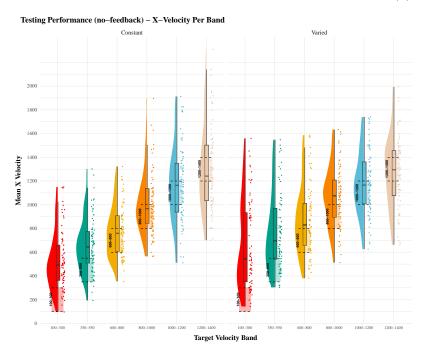


Figure 4: E1 testing x velocities. Translucent bands with dash lines indicate the correct range for each velocity band.

See Table ?? for the full model results. The estimated coefficient for training condition ( $B=164.05,\,95\%$  CrI [45.5, 278.85]) suggests that the varied group tends to produce harder throws than the constant group, but is not in and of itself useful for assessing discrimination. Most relevant to the issue of discrimination is the slope on Velocity Band ( $B=0.71,\,95\%$  CrI [0.62, 0.8]). Although the median slope does fall underneath the ideal of value of 1, the fact that the 95% credible interval does not contain 0 provides strong evidence that participants exhibited some discrimination between bands. The estimate for the interaction between slope and condition ( $B=-0.14,\,95\%$  CrI [-0.26, -0.01]), suggests that the discrimination was somewhat modulated by training condition, with the varied participants

Table 3: Testing vx - Empirical Summary

(a) Constant

Band	Band Type	Mean	Median	Sd
100-300	Extrapolation	524	448	327
350 - 550	Extrapolation	659	624	303
600-800	Extrapolation	770	724	300
800-1000	Trained	1001	940	357
1000-1200	Extrapolation	1167	1104	430
1200-1400	Extrapolation	1283	1225	483

(b) Varied

Band	Band Type	Mean	Median	Sd
100-300	Extrapolation	664	533	448
350 - 550	Extrapolation	768	677	402
600-800	Extrapolation	876	813	390
800-1000	Trained	1064	1029	370
1000-1200	Trained	1180	1179	372
1200-1400	Trained	1265	1249	412

Table 4: Experiment 1. Bayesian Mixed Model Predicting Vx as a function of condition (Constant vs. Varied) and Velocity Band

## (a) Model fit to all 6 bands

Term	Estimate	95% CrI Lower	95% CrI Upper	pd
Intercept	408.55	327.00	490.61	1.00
${\rm condit} {\rm Varied}$	164.05	45.50	278.85	1.00
Band	0.71	0.62	0.80	1.00
condit*Band	-0.14	-0.26	-0.01	0.98

# (b) Model fit to 3 extrapolation bands

Term	Estimate	95% CrI Lower	95% CrI Upper	pd
Intercept	497.49	431.26	566.17	1.00
${\bf condit} {\bf Varied}$	124.79	26.61	224.75	0.99
Band	0.49	0.42	0.56	1.00
condit*Band	-0.06	-0.16	0.04	0.88

showing less senitivity between vands than the constant condition. This difference is depicted visually in Figure ??.@tbl-e1-slope-quartile shows the average slope coefficients for varied and constant participants separately for each quartile. The constant participant participants appear to have larger slopes across quartiles, but the difference between conditions may be less pronounced for the top quartiles of subjects who show the strongest discrimination. Figure Figure ?? shows the distributions of slope values for each participant, and the compares the probability density of slope coefficients between training conditions. Figure ??

The second model, which focused solely on extrapolation bands, revealed similar patterns. The Velocity Band term (B=0.49, 95% CrI [0.42, 0.56]) still demonstrates a high degree of discrimination ability. However, the posterior distribution for interaction term (B=-0.06, 95% CrI [-0.16, 0.04]) does across over 0, suggesting that the evidence for decreased discrimination ability for the varied participants is not as strong when considering only the three extrapolation bands.

Table 5: Slope coefficients by quartile, per condition

Condition	$Q\_0\%\_mean$	$Q\_25\%\_mean$	$Q\_50\%\_mean$	$Q\_75\%\_mean$	Q_100%_mean
Constant	-0.109	0.479	0.691	0.930	1.39
Varied	-0.202	0.268	0.588	0.902	1.30

test

# **Experiment 2**

Figure ?? illustrates the design of Experiment 2. The stages of the experiment (i.e. training, testing no-feedback, test with feedback), are identical to that of Experiment 1. The only change is that Experiment 2 participants train, and then test, on bands in the reverse order of Experiment 1 (i.e. training on the softer bands; and testing on the harder bands).

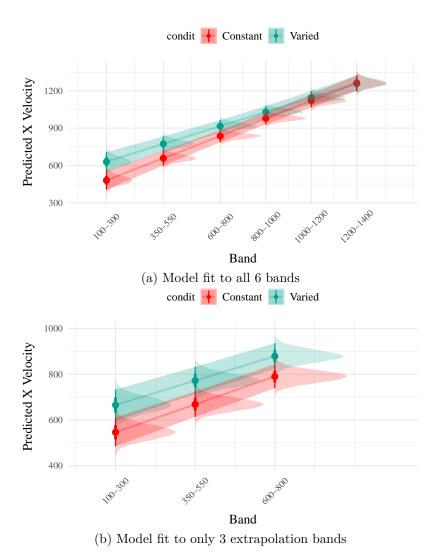
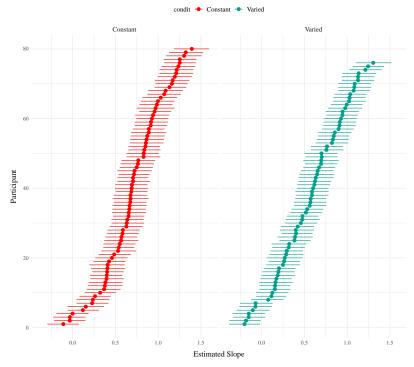
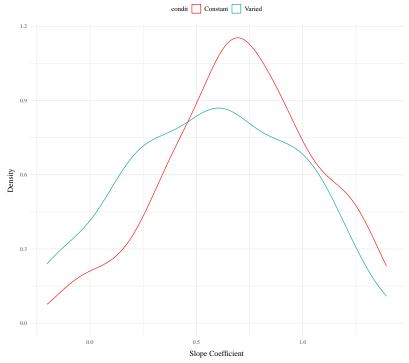


Figure 5: Conditional effect of training condition and Band. Ribbons indicate 95% HDI. The steepness of the lines serves as an indicator of how well participants discriminated between velocity bands.



(a) Slope estimates by participant - ordered from lowest to highest within each condition.



(b) Destiny of slope coefficients by training group

Figure 6: Slope distributions between condition

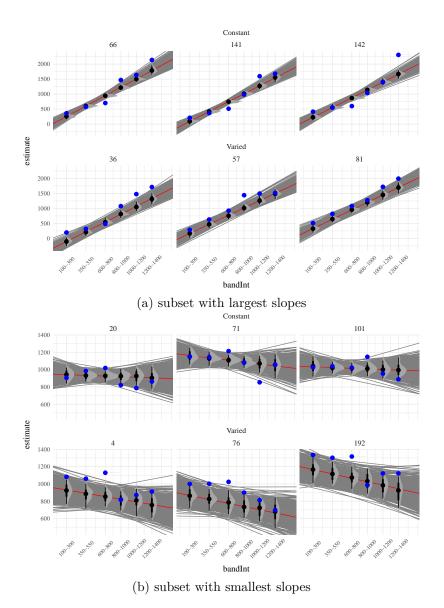


Figure 7: Subset of Varied and Constant Participants with the smallest and largest estimated slope values. Red lines represent the best fitting line for each participant, gray lines are 200 random samples from the posterior distribution. Colored points and intervals at each band represent the empirical median and 95% HDI.

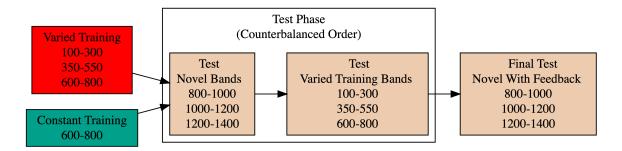


Figure 8: Experiment 2 Design. Constant and Varied participants complete different training conditions. The training and testing bands are the reverse of Experiment 1.

#### **E2** Results

#### Testing Phase - No feedback.

In the first part of the testing phase, participants are tested from each of the velocity bands, and receive no feedback after each throw.

## **Deviation From Target Band**

Descriptive summaries testing deviation data are provided in Table ?? and Figure ??. To model differences in accuracy between groups, we used Bayesian mixed effects regression models to the trial level data from the testing phase. The primary model predicted the absolute deviation from the target velocity band (dist) as a function of training condition (condit), target velocity band (band), and their interaction, with random intercepts and slopes for each participant (id).

$$dist_{ij} = \beta_0 + \beta_1 \cdot condit_{ij} + \beta_2 \cdot band_{ij} + \beta_3 \cdot condit_{ij} \cdot band_{ij} + b_{0i} + b_{1i} \cdot band_{ij} + \epsilon_{ij}$$

$$(3)$$

The model predicting absolute deviation showed a modest tendency for the varied training group to have lower deviation compared to the constant training group ( = -70.33, 95% CI [-156.87, 16.66]),with 94% of the posterior distribution being less than 0. This suggests a potential benefit of training with variation, though the evidence is not definitive.

Table 6: Testing Deviation - Empirical Summary

## (a) Constant Testing - Deviation

Band	Band Type	Mean	Median	Sd
100-300	Extrapolation	206	48	317
350 - 550	Extrapolation	194	86	268
600-800	Trained	182	112	240
800-1000	Extrapolation	200	129	233
1000-1200	Extrapolation	238	190	234
1200 - 1400	Extrapolation	311	254	288

## (b) Varied Testing - Deviation

Band	Band Type	Mean	Median	$\operatorname{Sd}$
100-300	Trained	153	25	266
350 - 550	Trained	138	53	233
600-800	Trained	160	120	183
800-1000	Extrapolation	261	207	257
1000 - 1200	Extrapolation	305	258	273
1200-1400	Extrapolation	363	314	297

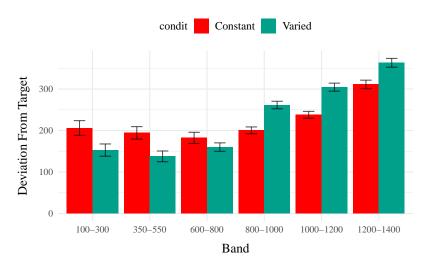


Figure 9: E2. Deviations from target band during testing without feedback stage.

Table 7: Experiment 2. Bayesian Mixed Model predicting absolute deviation as a function of condition (Constant vs. Varied) and Velocity Band

(a) Model fits

Term	Estimate	95% CrI Lower	95% CrI Upper	pd
Intercept	151.71	90.51	215.86	1.00
${\rm condit} {\rm Varied}$	-70.33	-156.87	16.66	0.94
Band	0.10	0.02	0.18	1.00
condit*Band	0.12	0.02	0.23	0.99

(b) Contrasts

contrast	Band	value	lower	upper	$\operatorname{pd}$
Constant - Varied	100	57.6	-20.5	135.32	0.93
Constant - Varied	350	26.6	-30.9	83.84	0.83
Constant - Varied	600	-4.3	-46.7	38.52	0.58
Constant - Varied	800	-29.3	-69.4	11.29	0.92
Constant - Varied	1000	-54.6	-101.1	-5.32	0.98
Constant - Varied	1200	-79.6	-139.5	-15.45	0.99

### Discrimination between Velocity Bands

In addition to accuracy/deviation. We also assessed the ability of participants to reliably discriminate between the velocity bands (i.e. responding differently when prompted for band 600-800 than when prompted for band 150-350). Table ?? shows descriptive statistics of this measure, and Figure 1 visualizes the full distributions of throws for each combination of condition and velocity band. To quantify discrimination, we again fit Bayesian Mixed Models as above, but this time the dependent variable was the raw x velocity generated by participants.

$$vx_{ij} = \beta_0 + \beta_1 \cdot condit_{ij} + \beta_2 \cdot bandInt_{ij} + \beta_3 \cdot condit_{ij} \cdot bandInt_{ij} + b_{0i} + b_{1i} \cdot bandInt_{ij} + \epsilon_{ij}$$
 
$$(4)$$

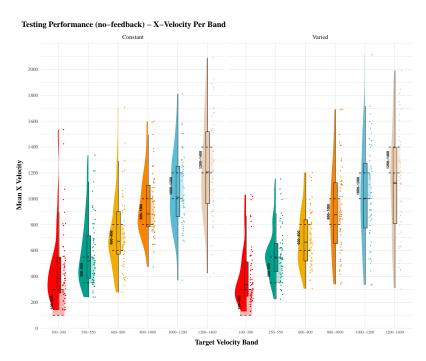


Figure 10: E2 testing x velocities. Translucent bands with dash lines indicate the correct range for each velocity band.

Table 8: Testing vx - Empirical Summary

(a) Constant Testing - vx

Band Type	Mean	Median	$\operatorname{Sd}$
Extrapolation	457	346	354
Extrapolation	597	485	368
Trained	728	673	367
Extrapolation	953	913	375
Extrapolation	1064	1012	408
Extrapalation	1213	1139	493
Extrapolation	1210	1100	100
(b) Varied Tes			100
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(b) Varied Tes	sting - vx Mean	Median	Sd
(b) Varied Tes Band Type Trained	sting - vx Mean 410	Median 323	Sd 297
(b) Varied Test Band Type Trained Trained	Mean 410 582	Median 323 530	Sd 297 303
	Extrapolation Extrapolation Trained Extrapolation Extrapolation	Extrapolation 457 Extrapolation 597 Trained 728 Extrapolation 953 Extrapolation 1064	Extrapolation       457       346         Extrapolation       597       485         Trained       728       673         Extrapolation       953       913         Extrapolation       1064       1012

Table 9: Experiment 2. Bayesian Mixed Model Predicting Vx as a function of condition (Constant vs. Varied) and Velocity Band

1095

1051

510

Extrapolation

1200-1400

Term	Estimate	95% C	CrI Lower	95%	CrI Upper	$\overline{\mathrm{pd}}$	
Intercept	362	2.64	27	4.85	45	0.02	1
conditVar	ied -8	3.56	-13	3.97	11	3.98	(
Band	(	0.71		0.58		0.84	1
condit*Ba	and -(	0.06	_	0.24		0.13	(

See Table ?? for the full model results.

When examining discrimination ability using the model predicting raw x-velocity, the results were less clear than those of the absolute deviation analysis. The slope on Velocity Band ( = 0.71, 95% CrI [0.58, 0.84]) indicates that participants showed good discrimination between bands overall. However, the interaction term suggested this effect was not modulated by training

condition ( = -0.06, 95% CrI [-0.24, 0.13]) Thus, while varied training may provide some advantage for accuracy, both training conditions seem to have similar abilities to discriminate between velocity bands.

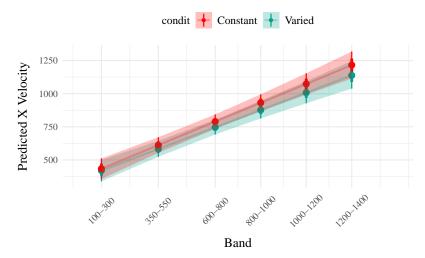


Figure 11: Conditional effect of training condition and Band. Ribbons indicate 95% HDI.

# **Experiment 3**

The major manipulation adjustment of experiment 3 is for participants to receive ordinal feedback during training, in contrast to the continuous feedback of the earlier experiments. Ordinal feedback informs participants whether a throw was too soft, too hard, or fell within the target velocity range. Experiment 3 participants were randomly assigned to both a training condition (Constant vs. Varied) and a Band Order condition (original order used in Experiment 1, or the Reverse order of Experiment 2).