Rodrigues' rotation formula

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This article is about the Rodrigues' rotation formula, which is distinct from Euler–Rodrigues parameters and The Euler-Rodrigues formula for 3D rotation.

In the theory of three-dimensional rotation, the **Rodrigues rotation formula** (named after Olinde Rodrigues) is a formula for rotating a vector in space, given an axis and angle of rotation. If **v** is a vector in R^3 and **z** is a unit vector describing an axis of rotation about which we want to rotate **v** by an angle θ (in a right-handed sense), the Rodrigues formula is:

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{v}\cos\theta + (\mathbf{z}\times\mathbf{v})\sin\theta + \mathbf{z}(\mathbf{z}\cdot\mathbf{v})(1-\cos\theta).$$

By extension, this can be used to transform all three basis vectors to compute a rotation matrix from an axis-angle representation. In other words, the Rodrigues formula is an algorithm to compute the exponential map from so(3) to SO(3).

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Derivation

Given a unit vector, **z** and a vector **v** that we wish to rotate about **z**, the vector

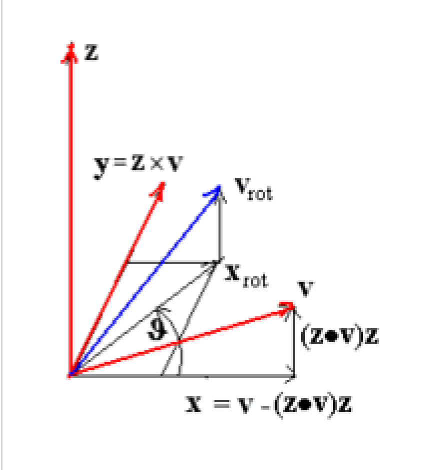
$$\mathbf{x} = \mathbf{v} - \mathbf{z}(\mathbf{z}\cdot\mathbf{v})$$

is the projection of **v** onto the plane orthogonal to **z**. Next let

$$\mathbf{y} = \mathbf{z}\times\mathbf{v}.$$

Using trigonometry, we can rotate **x** by θ around **z** to obtain the projection of the rotated vector $\mathbf{v}_{\mathrm{rot}}$:

$$\begin{aligned}\mathbf{x}_{\mathrm{rot}} &= \mathbf{x}\cos\theta + \mathbf{y}\sin\theta\\ &= (\mathbf{v} - (\mathbf{z}\cdot\mathbf{v})\mathbf{z})\cos\theta + (\mathbf{z}\times\mathbf{v})\sin\theta.\end{aligned}$$



The Rodrigues rotation formula rotates **v** around **z** by decomposing it into its component parallel to **z** and perpendicular (**x**) to **z**, rotating the perpendicular part in the plane orthogonal to **z** and adding back the parallel part to get $\mathbf{v}_{\mathrm{rot}}$.

Notice that both the vectors **x** and **y** have the same length. This follows from the following formulae for the scalar product and cross product:

$$\begin{aligned}&\mathbf{z}\cdot\mathbf{v} = |\mathbf{z}|\,|\mathbf{v}|\cos\phi\\ &|\mathbf{z}\times\mathbf{v}| = |\mathbf{z}|\,|\mathbf{v}|\sin\phi\end{aligned}$$

where Φ denotes the angle between **z** and **v**. Since **z** has unit length it immediately follows from the second formula that

$$|\mathbf{y}| = |\mathbf{z}\times\mathbf{v}| = |\mathbf{v}|\sin\phi.$$

The first formula gives the length of **z**(**z•v**) and because this forms a right triangle with **v** and **x** it becomes clear that this is also the length of **x**.

To get the rotated vector **v**, we have to add back the component of **v** parallel to **z**:

$$(\mathbf{z}\cdot\mathbf{v})\mathbf{z}.$$

So the result is

$$\begin{aligned}\mathbf{v}_{\mathrm{rot}} &= (\mathbf{v} - (\mathbf{z}\cdot\mathbf{v})\mathbf{z})\cos\theta + (\mathbf{z}\times\mathbf{v})\sin\theta + (\mathbf{z}\cdot\mathbf{v})\mathbf{z}\\ &= \mathbf{v}\cos\theta + (\mathbf{z}\times\mathbf{v})\sin\theta + \mathbf{z}(\mathbf{z}\cdot\mathbf{v})(1-\cos\theta),\end{aligned}$$

as required. Or in matrix notation:

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{v}\cos\theta + \mathbf{z}\times\mathbf{v}\sin\theta + \mathbf{z}\mathbf{z}^{\top}\mathbf{v}(1-\cos\theta).$$

Computing a rotation matrix

Denoting by $[\mathbf{z}]_{\times}$ the "cross-product matrix" for $\mathbf{z}$, i.e.,

$$[\mathbf{z}]_{\times}\mathbf{v} = \mathbf{z}\times\mathbf{v} = \begin{bmatrix} 0 & -z_3 & z_2\\ z_3 & 0 & -z_1\\ -z_2 & z_1 & 0\end{bmatrix}\mathbf{v},$$

the equation can be written as

$$\begin{aligned}\mathbf{v}_{\mathrm{rot}} &= (I\cos\theta)\mathbf{v} + ([\mathbf{z}]_\times \sin\theta)\mathbf{v} + (1-\cos\theta)\mathbf{z}\mathbf{z}^\top\mathbf{v} \\ &= \left(I\cos\theta + [\mathbf{z}]_\times \sin\theta + (1-\cos\theta)\mathbf{z}\mathbf{z}^T\right)\mathbf{v} \\ &= R\mathbf{v}\end{aligned}$$

where *I* is the 3×3 identity matrix. Thus we have a formula for the rotation matrix, *R*, corresponding to an axis angle vector, **z**:

$$R = I\cos\theta + [\mathbf{z}]_\times \sin\theta + (1-\cos\theta)\mathbf{z}\mathbf{z}^\top.$$

Noting that, using the outer product $\mathbf{z}\mathbf{z}^\top = [\mathbf{z}]_\times^2 + I$, we have

$$R = I + [\mathbf{z}]_\times \sin\theta + (1-\cos\theta)[\mathbf{z}]_\times^2$$

or, equivalently,

$$R = I + \sin\theta[\mathbf{z}]_\times + (1-\cos\theta)(\mathbf{z}\mathbf{z}^\top - I).$$

For the inverse mapping, see Log map from SO(3) to so(3).

See also

- Axis angle
- Rotation (mathematics)
- SO(3) and SO(4)
- History of quaternions

References

- Don Koks, (2006) *Explorations in Mathematical Physics*, Springer Science+Business Media,LLC. ISBN 0-387-30943-8. Ch.4, pps 147 et seq. *A Roundabout Route to Geometric Algebra'*

External links

- Weisstein, Eric W., "Rodrigues' Rotation Formula (http://mathworld.wolfram.com/RodriguesRotationFormula.html) " from MathWorld.
- Johan E. Mebius, Derivation of the Euler-Rodrigues formula for three-dimensional rotations from the general formula for four-dimensional rotations. (http://arxiv.org/abs/math/0701759) , *arXiv General Mathematics* 2007.
- For a proof that begins with the xyz frame see http://myyn.org/m/article/proof-of-rodrigues-rotation-formula/
- For another descriptive example see http://chrishecker.com/Rigid_Body_Dynamics#Physics_Articles, Chris Hecker, physics section, part 4. "The Third Dimension" -- on page 3, section ``Axis and Angle*, http://chrishecker.com/images/b/bb/Gdmphys4.pdf*