Calculus
The Mathematics of Modern Science

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Preface

In the Summer of the year 2014 since the Incarnation of the Logos,[1] I began to write the present book. My hope is that some day it will be used in a home school, perhaps even for my younger children, if I should finish the book soon enough.

My mother read Wray G. Brady's and Maynard J. Mansfield's *Calculus* (1960) when she attended the University of St. Thomas Aquinas in Houston, Texas. She did not see the point of studying the calculus. I dislike some of the notation in Brady and Mansfield, but I do like their approach. The authors develop the calculus by theorems from definitions of the set, the function, and the limit. This approach, enabled by mathematical ideas introduced in the 19th Century, frees the calculus from ambiguities and logical inconsistencies that for at least 2,500 years plagued mathematical thought on the concept of the vanishingly small. I shall at least roughly follow Brady and Mansfield, but I shall use different notation; also, I shall add some material from geometry, physics, and philosophy.

Because of my positive high-school experience in simultaneously studying the calculus and calculus-based physics, I have long thought that the calculus ought ideally to be introduced along with some basic physics. So I plan to present both here. The combination might address my mother's dislike for the absence of any recognizable motivation for the calculus. The main results of the calculus, established by Newton and Leibniz more than 200 years before the rigorous approach of the 19th Century was worked out, are well motivated by physical intuition. Newton developed the calculus to serve as the language of physical theory. Because the calculus is arguably most naturally intuited in the context of Newtonian mechanics, I shall develop here along with the calculus some relevant aspects of the mechanics.

Developing the mathematics in the context of the mechanical theory gives me the opportunity to make careful distinctions. In my recent reading of Carl B. Boyer's *The History of the Calculus and Its Conceptual Development*, I was, particularly in his chapter on mathematics in the ancient world, reminded of man's perennial and unfortunate tendency to confuse mathematics with nature. Looking back on the emergence of modern science in the 17th Century, one sees that there are three relevant domains carefully to be distinguished:

1. mathematics,
2. scientific theory, and
3. the natural world.

Surveying the long history of mathematics, Boyer notes "that mathematics is the study of relationships in general and must not be hampered by any preconceived notions, derived from sensory perception, of what these relationships should be."[2] The problem is that logic applied to definitions and postulates can lead to conclusions apparently inconsistent with sense experience. Among the possible responses to the inconsistency are two extremes, each of which seems wrong:

- On the one hand, to throw away a piece of mathematics with intrinsic beauty, internal consistency, and practical utility seems wrong.

[1] Referring to the Logos—the reason behind the existence of all things—as αὐτοῦ (him), the third verse of the Gospel of John in English is, "It was through him that all things came into being, and without him came nothing that has come to be." πάντα δι αὐτοῦ ἐγένετο, καὶ χωρὶς αὐτοῦ ἐγένετο οὐδὲ ἕν ὃ γέγονεν. (Original Greek and English translation from `newadvent.org`). I invoke the Logos here for several reasons.

1. Thinking about the origin of the present book reminds me of origins in general. From the Incarnation of the Logos derives the existence of all space, all past, and all future. Christmas is not the *temporal* origin of things, but Christmas is the *logical* origin of all physical reality, including the present book. Man cannot merely by study of ordinary things in the physical universe be certain of its age (or even, as Aquinas pointed out, whether its age be *finite*). Based on assumptions that can be neither proved nor disproved, the belief in a young age for the universe can be more or less reasonable, but the insistence on everyone's holding such assumptions—lest one risk eternal damnation—is wrong. St. Augustine, the great doctor of the Church, himself argued in the early 400s A.D. both for an old Earth and a for a process of evolution for the human body; see his *De Genesi ad Litteram*. The entire history of the physical world is not so much defined by the conditions at the beginning of time as by the conditions at the Incarnation; unlike the temporal beginning of the physical universe, the logical beginning of all things, including time, is located in a special way in the middle of history.
2. In my experience, the number of the year is too often used without any reference to the origin of the temporal coordinate system. We should at least occasionally mention what the first year is.
3. Because the present book promotes modern mathematics and science, to begin by mentioning the tradition in which they arose is fitting. Doubly fitting is so to begin in the midst of a culture awash in the false perception of a substantial conflict between what is proper to modern science and what is proper to traditional Christianity.
4. The Logos is ultimately what science and mathematics are all about. The idea of comprehensible, logical, consistent reason behind things is the very idea of the Logos. On the one hand, the traditional Christian identifies the Logos as the Second Person of the Trinity. On the other hand, modern science is squarely directed toward the Logos, though in the attempt to connect mathematics to the physical universe, science can never grasp the Logos.

[2] Boyer, *The History of the Calculus and Its Conceptual Development*, p. 13.

- On the other hand, to disregard what is known by sense experience seems wrong.

A moderate approach is to admit that while the seemingly obvious definitions and postulates at the root of a mathematical conclusion do not correspond exactly to the world of sense experience, there is some utility in seeing how far any given correspondendence can be taken. Around the same time as the development of the rigorous approach to the calculus, and after about 200 years of consistency with sense experience, the modern scientific theory of Newtonian mechanics began in the late 19th Century to draw conclusions inconsistent with sense experience. Already with the advent of Maxwell's equations, there was a mathematical inconsistency between the Newtonian mechanics and the new electromagnetic theory; this inconsistency eventually led to Einstein's development of special relativity. The first observational inconsistency, though, appeared in measurements of the radiation emitted by an object of a given temperature. The Newtonian mechanics predicted a spectral energy distribution different from what was observed. The inconsistency arose not because of any error in the mathematics but because technological advancement exposed the error of the scientific postulate—the scientific hypothesis—by which the mathematical is connected to the physical. In this case, the hypothesis is that the same mathematics describing the motion of objects large enough directly to be perceived by the senses also describes the motion of objects too small to be perceived. This hypothesis, however, led to a failure adequately to account for what is in fact observed by the senses. Nature is imperfectly described by scientific theory, and the imperfection in the theory lies in the assertion of a particular connection between mathematics and nature.

Newtonian mechanics and later scientific theories based in one way or another on the calculus are amazingly good at predicting what the observer of a scientific experiment will perceive through the senses. Nevertheless, I intend in the text at least occasionally to point out that modern science can *never* determine what in nature lies beyond sense experience. A modern scientific theory inevitably attempts to connect mathematics to nature at points that are outside the realm of perception. These are precisely the points at which we cannot be certain what—if any—mathematics ought to apply. Regardless of the excellent quality of the predictions made, for example, by quantum field theory, the existence of the electron is not a certainty.[3] At some point in the future, predictions made on the basis of the idea of the electron might fail to match what is perceived by the senses. As Edward Harrison points out in his *Cosmology*, a mask that hides a face should not be mistaken for the face itself. A scientific theory and the sense perceptions that it explains form a mask that covers nature's face. The amazing ability of modern scientific theory, enabled by the calculus, to predict sense experience across a wide range of conditions presents the strong temptation to mistake the mask for the face, but the face will forever remain hidden.

My purpose here is to write a calculus textbook that, through careful development of both the mathematics and some exemplifying aspects of the mechanical theory, makes clear what the calculus is, how it can be applied in the service of scientific theory, and what the limits of scientific theory are.

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[3]The electron might exist only as a mathematical model in the mind and not as a reality in itself. I do not deny the possibility of the electron's existence in itself, but I *do* deny that anyone could ever be certain of such existence. The best scientific theory 1,000 years from now might not refer to the present idea of the electron except as an approximation that is appropriate in certain contexts.

Chapter 1

Preliminaries

Mathematics originated in *reasoning* about number and geometry. Within mathematics, the calculus is a subject involving the idea of an arbitrarily small quantity. We approach the calculus first by reviewing some aspects of reasoning. Then we reason about the *set*, which can provide a basis for mathematical thought. At the end of the chapter, we reason about the *function*, which plays a central role in the calculus.

1.1 Reasoning

In mathematics, every important result can be expressed as an *implication*. The *mind* may *prove* an implication. A deep discussion of the mind is beyond our scope, but we shall discuss three crucial acts of the human mind. These acts culminate in reasoning, which allows the proof of an implication.

The ability of the human mind to reason can be viewed as evidence for the mind's having an immaterial aspect.[1] *Materialism* is the philosophical position according to which everything that exists is matter. Modern science describes the world just as consisting of

- bits of matter,
- the space through which they move, and
- the forces by which they interact.

So *scientism*, according to which everything that exists can be understood in terms of science, is the modern form of materialism. Because materialism is a popular position,[2] the claim of the mind's immateriality is controversial. Below we discuss, as an example of reasoning, Roger Penrose's argument according to which either

1. certain behaviors of the brain cannot be described by quantum mechanics and general relativity, or else
2. the mind is not merely the physical brain.

Although Penrose himself thinks that modern physics fails to describe the brain, one who holds that modern physics *does* well describe the brain concludes—on Penrose's understanding at least—that the mind is not merely the physical brain.

We shall look in more detail at Penrose's thesis below, first when we consider what an implication is and then, after we introduce the three acts of the mind, when we discuss the proof of an implication.

1.1.1 Logic

A *proposition* is a statement proposed as a truth. The proposition might be either true or false, but it represents a claim about what is true. For example, "the sky is cloudy" is a proposition. Below we shall consider the mind's role in determining whether a proposition be true.

1.1.2 Implication and Contrapositive Expression

An implication has the form of a conditional, "if H, then C", where each of H and C stands for a proposition. An implication is written compactly in the form, "$H \Rightarrow C$" and can be read, "H implies C". Each of the *hypothesis* H and the *conclusion* C stands for a statement

[1] In fact, each of the mind's three acts that we discuss in the text can on its own be taken as requiring the mind's immateriality.

[2] Scientism is popular because of the lack of appreciation for (a) the God of classical theism and (b) the qualitative distinction between the scientist and what the scientist studies.

(a) Modern scientific theory allows one reasonably to imagine how any ordinary physical thing could come to be without divine interference in the natural order. However, both classical theism and modern atheism oppose the idea of explaining ordinary things by way of divine interference. Even if all gaps in the natural explanation of ordinary things were closed, then only the gods of pantheism would be eliminated. The God of classical theism would not thus be eliminated. According to classical theism, logic requires God as the primary cause of those natural, ordinary, secondary causes which science explores.

(b) Beyond the secondary causes that science can explore are not only God's primary causality but also several aspects of the human person. While the human body might be the product of a natural process of evolution stretching over billions of years, the free will and the acts of the mind cannot even in principle be described in terms of science. Also, if subjective experience be real, then there is also the problem of the quale, the individual instance of subjective, conscious experience. A perfect scientific description of the human brain, its interface to the ear, the properties of the concert hall in which the body sits, and the properties of a grand piano could never allow a deaf person to know the experience of hearing the pitch of Concert A on the piano. Science will always fail to capture some aspect of subjective experience. If subjective experience be real, then the scientist is not entirely scrutable in terms of science.

$$
\begin{aligned}
2 + x = 7 \quad &\Rightarrow \quad x = 5 \\
x \neq 5 \quad &\Rightarrow \quad 2 + x \neq 7
\end{aligned}
$$

Figure 1.1: *The positive expression of an implication (top) and its contrapositive expression (bottom). In this example, $2 + x = 7$ is the hypothesis of the implication, and $x = 5$ is the conclusion. An implication is the assertion that if the hypothesis be true, then the conclusion must be true. The contrapositive expression of the implication is the assertion that if the conclusion be false, then the hypothesis must be false.*

that could be either true or false, but $H \Rightarrow C$ is the claim that C must be true when H be true, that H is a sufficient condition for C. Also, $H \Rightarrow C$ is equivalent to its *contrapositive* expression, that C is a necessary condition for H, or $[\neg C] \Rightarrow [\neg H]$, where the symbol "$\neg$" means "not". That is, H is false when C be false. See Figure 1.1. For an implication, a true hypothesis implies a true conclusion, and a false conclusion implies a false hypothesis.

Consider, for example, the Pythagorean Theorem. A theorem might not always be written in the form of an implication, but a theorem can always be rewritten as an implication. In it usual form, the Pythagorean Theorem is the claim that for a right triangle the square of the length of the hypotenuse is equal to the sum of (1) the square of the length of one side and (2) the square of the length of the other side. Here is the Pythagorean Theorem as an implication: If a be the length of one leg of a right triangle, and b be the length of the other leg, then the length of the hypotenuse is $\sqrt{a^2 + b^2}$. Here is the contrapositive expression of the same theorem: If the length of a right triangle's hypotenuse be not $\sqrt{a^2 + b^2}$, then it is not the case that both a is the length of one leg, and b is the length of the other leg. See Figure 1.2.

Consider, for another example, Penrose's Thesis: If the human brain operate according to general relativity and quantum mechanics, then the human brain is not the human mind. Here is the contrapositive expression: If the human brain be the human mind, then the human brain does not operate according to general relativity and quantum mechanics.

Exercise 1.1.1: *Invent two implications. For each implication, identify the hypothesis and the conclusion. Write each implication in its contrapositive expression.*

In mathematics, an implication that is widely useful is a *theorem*. An implication of limited use, typically as an intermediate step in the proof of a theorem, is a *lemma*. Proving an implication involves the construction of a sequence of *valid arguments* that, taken together, are equivalent to the implication. The importance of the contrapos-

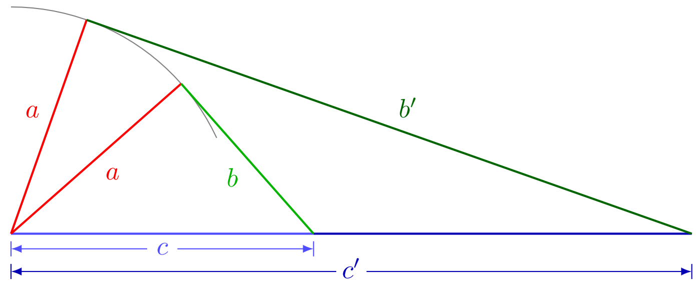


Figure 1.2: *Contrapositive expression of the Pythagorean Theorem. If the hypotenuse of a right triangle have some length other than $c = \sqrt{a^2 + b^2}$, then the triangle cannot have both a leg of length a and a leg of length b. In the figure, one regards first a right triangle that does have length c for its hypotenuse. By extending the hypotenuse to a length c' larger than c but holding one leg's length fixed at a, one finds that the other leg's length must grow to a length b' larger than b. Similar reasoning applies to the shortening of the hypotenuse.*

itive expression lies in its utility in a proof: Sometimes, proving an implication's contrapositive expression is easier than proving the positive expression. In any event, proving an implication is the essence of reasoning, which produces useful results in mathematics.

1.1.3 Proof

One must, in order to prove an implication, be able to grasp the meaning of each term mentioned in a proposition, to judge the proposed relationships among the terms, and to gather relevant propositions together in the right order to form a judgment about another proposition. St. Thomas Aquinas identifies these as the three acts of the mind.

1. The mind *simply apprehends* the concept behind a word or a phrase.[3]

[3] Coppens, *A Brief Text-Book of Logic and Mental Philosophy*, Chapter I, Article I: "9. *Simple apprehension* is the act of perceiving an object intellectually, without affirming or denying anything concerning it. . . . The mind cannot take an object physically into itself; but it knows an object by taking it in intellectually, in a manner suited to its own nature; forming to itself an intellectual image. . . . The act of forming this mental image is called a *conception*, and the fruit of it, the image itself, is the *concept*, *idea*, or *notion* of the object. The word 'simple' added to apprehension emphasizes the fact that the apprehension neither affirms nor denies the existence of the object. . . . 10. This intellectual image should not be confounded with the sensible image, or *phantasm*, which is a material representation of material objects, and which is formed by the imagination, by means of the material organ of the brain. . . . For instance, I intellectually *conceive* a triangle by apprehending a figure enclosed by three lines and thus having three angles. My notion or idea contains this and nothing more; it is very precise, and every one who conceives a triangle conceives it exactly the same way. But when I *imagine* a triangle, I cannot help imagining it with sensible material accidents, as being of such or such a size and shape, a foot long at one time, a mile long at another. The picture may be vague, various pictures of triangles may be blended together; but it can never be universal, representing all possible triangles, as my idea does. . . ."

Every man is mortal. Socrates is a man.
Socrates is mortal.

Figure 1.3: *A valid argument. The premises lie above the line, and the conclusion lies below the line. If the premises be true, then the conclusion must be true.*

2. The mind *judges* whether the two ideas connected by a proposition (like "modesty is praiseworthy") ought so to be connected.[4]

3. The mind *reasons* to derive some of its judgments from other judgments.[5]

In its third act, which operates on products of the other two acts, the mind reasons by arranging propositions into an valid argument.

An *argument* is a set of two or more propositions, one of which is the *conclusion* and the rest of which are *premises*. The mind may judge an argument by comparing (a) the claims made in the premises, taken together as though they were a single proposition, with (b) the claim made in the conclusion. If the meaning of the conclusion be contained in the premises, then the argument is *valid*; otherwise, it is *invalid*. Every argument purports to be an implication whose hypothesis comprises the premises.

For example: If every man be mortal, and Socrates be a man, then Socrates is mortal. This bit of reasoning is expressed in Figure 1.3. In comparing the premises to the conclusion, one recognizes the argument as valid. Every valid argument is obviously an implication.

Even if an argument's conclusion be not contained in the hypothesis, and so long as the argument contain no contradiction, then something additional might be provided in order for one to grasp that the argument is an implication. What might be provided is a chain of arguments, each of which can be recognized as valid, such that the chain is equivalent to the purported implication. The first valid argument in the chain has among its premises the hypothesis of the implication under examination, plus zero or more true statements brought in as necessary. The conclusion of that first valid argument then serves as a premise for the next valid argument in the chain. That valid argument may itself bring in additionally zero or more true statements as premises. This is repeated as necessary. The conclusion of the final valid argument in the chain is the conclusion of the implication under examination.

The human brain operates according to QM and GR.
The human brain is not the human mind.

Figure 1.4: *An invalid argument that might still be an implication. 'QM' stands for 'quantum mechanics', and 'GR' stands for 'general relativity'.*

The human brain operates according to QM and GR. QM and GR provide an algorithmic description.
The human brain operates according to an algorithm. No algorithm can do math like the human mind. (Gödel)
The human brain is not the human mind.

Figure 1.5: *A chain of two valid arguments that Penrose uses to establish that the argument in Figure 1.4 is an implication. The conclusion of the first is one of the premises of the second. The second premise of the second argument refers to the Incompleteness Theorem of Kurt Gödel.*

As an example, let us consider Penrose's thesis in some detail. Figure 1.4 shows the initial argument. Figure 1.5 shows a summary of Penrose's proof. There are two valid arguments in the chain of reasoning. The first takes the original hypothesis and adds a premise about the algorithmic nature of quantum mechanics and general relativity. The first conclusion is then combined with the Incompleteness Theorem of Gödel in order to form a valid argument whose conclusion is the same as the conclusion of the implication under examination. In a proof, any new premises brought in must be true. So an opponent of Penrose might try to attack each of the two new premises as not being true, but Penrose does seem to have provided deductively valid arguments in his chain. A chain of valid arguments is thus advanced to prove that the argument is an implication.

The conclusion of an implication that has been proved is *deduced* from the premises. A deduction is certain to be true if the premises

[4] Coppens, Chapter I, Article III: "17. A *judgment* may be defined as 'an act of the mind affirming or denying the agreement of two objective ideas'. The mind in judging compares two ideas … and affirms or denies that they agree with one another; e.g., 'modesty is praiseworthy'…. [If] the agreement or disagreement [be] seen to exist by the mere consideration or analysis of the ideas compared, the judgment is *analytic*; it is also styled *a priori*, i.e., formed antecedently to experience…. But if the agreement or disagreement [be] discovered consequently on experience, e.g., 'gold is malleable', the judgment receives the opposite appellations of *synthetic* [or] *a posteriori*…. 18. If a judgment of either kind [be] arrived at by reasoning, it is *mediately evident*; if the agreement or disagreement [be] seen without the aid of reasoning, the judgment is *immediately evident*. That 'ice is cold', is an immediate a posteriori judgment; that 'there is nothing without a reason for it', is immediately known a priori; that 'the sum of the angles of a triangle is equal to two right angles', is known mediately a priori; the physical laws are known mediately a posteriori. 19. A judgment expressed in words is called a *proposition*. The subject and predicate together are its *matter*, and the affirmation or negation its *form*; the *copula* is always the verb 'to be' in the present indicative, expressed or implied: 'I see' is equivalent to 'I am seeing', 'He said' to 'He is one who said', etc."

[5] Coppens, Chapter II: "22. *Reasoning* is the mental act or process of deriving judgments, called conclusions, from other judgments, called premises. The principle underlying all valid reasoning is that the conclusion is implicitly contained in the premises; therefore whoever grants the truth of the premises thereby really grants the truth of the conclusion."

be true.

The conclusion of an argument that is not an implication is *induced* from the premises. An induction is not certain to be true even if the premises be true. The *problem of induction*, for a given set of true premises, is to determine the probability of the conclusion's being true. An argument has an inductive strength proportional to the probability of its conclusion's truth, given the truth of the premises.

The word "sound" when applied to an argument indicates both that the argument is valid and that the premises are true.[6] One of the problems with reasoning is that it cannot operate in a vacuum. While the hypothesis of an implication might be the conclusion of another implication, any line of reasoning can be traced back to a point at which at least one premise is not itself the result of reasoning. So every application of reasoning depends ultimately on some truths that are known independently of any reasoning. We can know some truths directly by the second act of the mind, and the soundness of an argument must depend upon truths of that kind.

While a theorem in mathematics is a strict and certain implication, a theory in modern science is not. A scientific theory involves, in its chain of reasoning, one or more strong but invalid arguments. The Pythagorean Theorem takes the lengths of the legs and concludes that the length of the hypotenuse is a definite and certain value. However, even if the Sun is risen every morning since the beginning of history, the conclusion that the Sun will rise again tomorrow morning is *not* certainly true. The conclusion of the Pythagorean Theorem is a deduction, but the claim about the Sun is, of its scientific nature, a particular kind of induction taking the form of a prediction about what will be observed with the senses. A mathematical theorem may be proved true, but a scientific theory can *never* be proved true; there is always the chance that eventually a bit of evidence ruling out the scientific theory will turn up.

There is still a sense of proof in modern science, though. When a repeatable observation be inconsistent with the prediction of a scientific theory, the theory is proved false. After it has been ruled out by observation, the theory is no longer a candidate for a true description of the physical world. The theory may remain useful for engineering, but the theory has been shown inconsistent with what is known directly through sense experience. Modern science progresses by experimental proof of the *lack* of implication in the assertion of a scientific theory. Such progress ensures plenty of work for theorists, who must come up with new theories as the old ones are ruled out.

Mathematics proceeds positively, where modern science proceeds negatively. In modern science, the theorist proposes a theory that stands for a time until it is ruled out, after which a latter theory is proposed to take the place of the former. In mathematics, proofs are advanced to construct a body of theorems, which stand for all time. In what follows, we shall build up a body of theorems and see how the mathematics, which in itself stands forever, can be used in a scientific theory, which is destined to be ruled out.

1.2 Set

The idea of a *set* is foundational in mathematics. The development of *axiomatic set theory*, which can arguably serve as the basis for all of mathematics, is beyond the scope of this book. Nevertheless, a naive set theory can serve as the basis for the calculus.

Brady and Mansfield claim that the "concept of a *set of elements* is so fundamental" as not to warrant a definition "in terms of what would necessarily be less basic concepts."[7] It does seem desirable to define a word in terms of more basic concepts, and we do seem unable to make such a definition for "set". Still, we define "set" here to minimize the scope of the meaning.

One might occasionally see or hear the claim that a set is any arbitrary collection of things, such as the collection of deciduous trees in Kansas. Now the collection of deciduous trees in Kansas is *something*, but it is not what we shall call "a set". Our notion of a set must needs be purely mathematical to avoid a confusion between mathematics and the world of sense experience.[8] The collection of deciduous trees in Kansas combines the mathematical idea of a collection with something in nature: particular plants in a particular place. A combination of the mathematical and the natural is what one finds in a scientific theory, not in a purely mathematical idea. Even if one restrict one's conception of the set to the abstract, one might still run into trouble if one allow such a concept as *the set of all sets*. In fact, the inconsistent nature of a set theory that allows such a conception led to the recognition of the need for an axiomatic basis. For our purpose, it is sufficient to reject the notion of a set as an arbitrary collection of things, and we note, even in restricting elements to abstractions, that not every abstract idea of a set is permissible.

Definition 1.1: *A* set *is a coherent collection of mathematical abstractions in the mind.*

We have not defined "set" in terms of more basic concepts, for coherence, mathematical abstractions, and the mind seem hardly more basic than the set, but, in restricting the set and its elements to the mental, we have taken care from the outset not to confuse mathematics, which is constructed from mental abstractions, with nature, which is made of concrete things. Also, in referring here to coherence, we mean to discourage speculation about bizarre abstractions, such as the set of all sets.

1.2.1 Examples

Each of the following is an example of a set.

[6] The root "sund" in the German word "gesund", which in English means "healthy", derives from the same root as does one of the English words spelled as "sound". There is more than one English word with the same spelling. A sound that one hears is not related to "gesund", but a sound argument is indeed properly a healthy argument.

[7] Brady and Mansfield, *Calculus*, p. 1.

[8] Boyer, in *The History of the Calculus and Its Conceptual Development*, has pointed out that progress in mathematics has at several points in history been thwarted by the confusion between mathematics (geometry in particular) and the world of sense experience.

- In the plane, the set of all points equidistant from a central point.
- The set of all triangles.
- On the number line, the set of all points greater than 0.
- The set of all sets of two distinct real numbers.[9]

1.2.2 Membership

Definition 1.2: *A* member *or* element *of a set S is a mathematical abstraction that is collected by the mind into the set. We define the symbol "$\in$" so that $s \in S$ means that s is an element of the set S. We define the symbol "$\notin$" so that $n \notin S$ means that n is not an element of the set S.*

For example, if $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers, then $1 \in \mathbb{N}$, and $0 \notin \mathbb{N}$.

1.2.3 Equality of Sets

Definition 1.3: *For any two sets S and T, by "S and T are equal" we mean that both*

1. *for every $s \in S$, it is also true that $s \in T$, and*
2. *for every $t \in T$, it is also true that $t \in S$;*

and we write $S = T$. By "S and T are unequal" we mean that at least one of the above criteria fails, and we write $S \neq T$.

Three properties follow from this definition. The proof for one of them is left as an exercise.

Lemma 1.1: *If A be a set, then $A = A$.*

Proof. In many a case, to write a proof, one begins by looking at what is to be proved and by finding the relevant definition. In this case, we look at the Definition 1.3 of equality. In the text of the definition, S stands for the set on the left side of the equal sign, and T stands for the set on the right side. Two criteria must be satisfied in order for S to be equal to T.

1. We first show that for every element s of the set on the left side, s is also an element of the set on the right. The set on the left is A, and so $s \in A$ by hypothesis. The set on the right is A, and so we must show that $s \in A$. The hypothesis is the same as the conclusion; thus we have satisfied the first criterion.
2. We next show that for every element t of the set on the right side, t is also an element of the set on the left. The set on the right is A, and so $t \in A$ by hypothesis. The set on the left is A, and so we must show that $t \in A$. The hypothesis is the same as the conclusion; thus we have satisfied the second criterion.

[9]Note that this is not the set of all sets but a well defined set of sets. The qualification, "of two distinct real numbers", is crucial.

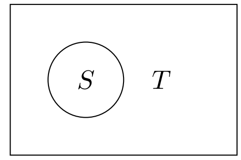


Figure 1.6: *Region S is a proper subset of Region T, which completely encloses Region S.*

□

Lemma 1.2: *If A and B be sets, and if $A = B$, then $B = A$.*

Lemma 1.3: *If A, B, and C be sets such that $A = B$, and $B = C$, then $A = C$.*

Proof. We must show both that every element of A is in C and that every element of C is in A.

1. Suppose that $x \in A$. Because $A = B$, we know that $x \in B$. Because $B = C$, we know that $x \in C$. So every element of A is also an element of C.
2. Suppose that $x \in C$. Because $B = C$, we know that $x \in B$. Because $A = B$, we known that $x \in A$. So every element of C is also an element of A.

□

1.2.4 Subset

Definition 1.4: *For any two sets S and T, by "S is a subset of T" we mean that for every $s \in S$, it is also true that $s \in T$, and we write $S \subset T$. By "S is a* proper subset *of T" we mean that both*

1. *$S \subset T$, and*
2. *$S \neq T$.*

For "S is not a subset of T" we write $S \not\subset T$.

In Figure 1.6, Region S inside the circle is drawn as a proper subset of Region T inside the rectangle. Several properties follow from the definition of subset. The proof of each is left as an exercise.

Lemma 1.4: *If A and B be sets, and if $A = B$, then $A \subset B$, and $B \subset A$.*

Lemma 1.5: *If A and B be sets such that $A \subset B$, and $B \subset A$, then $A = B$.*

Lemma 1.6: *If A, B, and C be sets such that $A \subset B$, and $B \subset C$, then $A \subset C$.*

Lemma 1.7: *If A be a set, then $A \subset A$.*

Note that although every set A is a subset of itself, A is not a proper subset of itself.

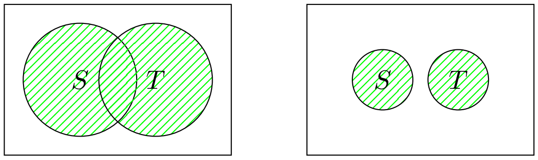


Figure 1.7: *The hatched region of points either in Region S or in Region T (or in both) represents $S \cup T$.*

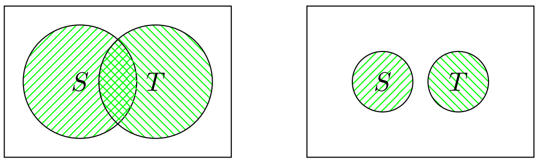


Figure 1.8: *The cross-hatched region in which Region S overlaps Region T represents $S \cap T$. When there be no overlap, $S \cap T = \emptyset$.*

1.2.5 Union

Definition 1.5: *By "the union of the sets S and T" we refer to the set whose every element is also an element of S, of T, or else of both S and T, and we write $S \cup T$.*

In Figure 1.7, Region S inside one circle and Region T inside another circle might or might not overlap. In either case, the union $S \cup T$ is indicated by the hatched region, every point of which is in at least one of S and T.

1.2.6 Intersection, Empty Set

Definition 1.6: *By "the intersection of the sets S and T" we refer to the set whose every element is also an element of both S and T, and we write $S \cap T$.*

In Figure 1.8, the case on the left depicts the situation in which Region S inside one circle and Region T inside another circle overlap. The intersection $S \cap T$ is indicated by the cross-hatched region, every point of which is in both S and T.

Definition 1.7: *By "the null set" or "the empty set" we refer to the set that has no members, and we write $\emptyset$ or $\{\}$.*

It is possible for two sets S and T not to have any elements in common. Such a case is depicted in the case on the right in Figure 1.8. In such a case, $S \cap T = \emptyset$. The null set has the interesting property of being a subset of every set.

Theorem 1.1: *If A be a set, then $\emptyset \subset A$.*

Proof. We must show that for any set A, every element of $\emptyset$ is an element of A. Because $\emptyset$ has by definition no elements, one might argue that we have nothing to do and that $\emptyset \subset A$ is true simply by definition of $\emptyset$.

There is, however, another approach. One can rephrase the definition of the subset: $P \subset Q$ means that for every set U and for every $u \in U$, if $u \in P$, then $u \in Q$. By applying modus tollens,[10] we find the equivalent assertion that if $u \notin Q$, then $u \notin P$. So we must show that, for any sets A and U, and for every $u \in U$, if $u \notin A$ then $u \notin \emptyset$. Because, by definition of the null set, no element is in $\emptyset$, every u not in A is also not in $\emptyset$. □

1.2.7 Brace Notation

A set may be constructed either by listing its elements explicitly or by referring to a property that every element shares. In either case, curly braces are used in writing such a construction. An explicit list may be finite or infinite. For example, $A = \{2, 4, 6\}$ is the set whose members are 2, 4, and 6, and $B = \{2, 4, 6, \ldots\}$ is the set of all positive, even integers. For an infinite set so constructed, enough of the members must be listed so that the pattern is clearly evident. An example of construction by property is $C = \{x \in B : x \text{ is a perfect square}\}$; C is then the set containing only every even number that is a perfect square.

We are now in a position to write a compact expression for each of the union and the intersecion. If for some set U, both $A \subset U$ and $B \subset U$, then

$$A \cup B = \{x \in U : x \in A \text{ or } x \in B\} \tag{1.1}$$
$$A \cap B = \{x \in U : x \in A \text{ and } x \in B\}. \tag{1.2}$$

1.2.8 Exercises

Exercise 1.2.1: *Give three examples of a set. Make sure that each example is not already included in the text above.*

Exercise 1.2.2: *Let $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ be the set of all integers. Find a set A and a set B such that $A \subset B \subset \mathbb{Z}$.*

Exercise 1.2.3: *List the eight subsets of $\{a, b, c\}$.*

Exercise 1.2.4: *Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4\}$, and $C = \{3, 5\}$. Find $A \cup B$, $(A \cup B) \cup C$, $(B \cup A) \cap C$, $(A \cap B) \cup C$, and $(A \cap B) \cap C$.*

Exercise 1.2.5: *Prove all of the unproved lemmas in the text above.*

Exercise 1.2.6: *Prove that $A \cap B \subset A$ and that $A \subset A \cup B$.*

[10] The Latin words "modus tollens" mean "the method of denying". Suppose that if some proposition S be true, then some other proposition T must also be true. In that case, whenever T be false, then one knows also that S is false. For example, suppose that if rain be falling, then there must be a cloud in the sky. If one observe that there is not a cloud in the sky, then, if one feel water drops falling upon one's skin, one knows that the drops are not drops of rain.

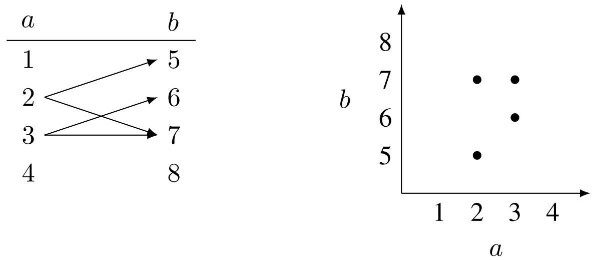


Figure 1.9: *Diagram (left) and graph (right) showing a relation from one set, whose every element is represented by a, to another set, whose every element is represented by b. In the diagram of a relation, each of zero or more elements in the left column may point to zero or more elements in the right column. Similarly, in the graph of a relation, each of zero or more columns may contain zero or more dots.*

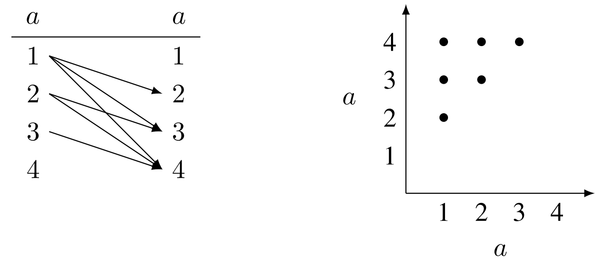


Figure 1.10: *Diagram (left) and graph (right) of the less-than relation on a small set of consecutive integers.*

1.3 Function

In the previous section, we explored some possibilities for the relationship between two sets. We considered whether an element of one set is also an element of the other. We defined the subset, the intersection, and the union.

In the present section, we explore some other possibilities. We now consider pairs of elements. We define terms involved in the consideration of pairs. We use these terms to talk about a generic *relation* from one set to another, a specific relation called a *function*, and three subspecies of function.

Definition 1.8: *An* ordered pair *is a set with two elements along with the ordering according to which one of the elements is the* first *element, and the other is the* second. *If the first element be a and the second be b, then we write the ordered pair as (a, b).*

Definition 1.9: *The* Cartesian product *$A \times B$ of any two sets A and B is the set of all ordered pairs (a, b), such that $a \in A$ and $b \in B$.*

For example, if $A = \{1, 2\}$ and $B = \{3, 4, 5\}$, then

$$A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}.$$

1.3.1 Relation

Definition 1.10: *A* relation *from a set A to a set B is a subset of $A \times B$.*

A relation from a set A to a set B is an association between each of zero or more members of the first set and zero or more members of the second set. Suppose that $A = \{1, 2, 3, 4\}$, and $B = \{5, 6, 7, 8\}$. An example of a relation from A to B is

$$R = \{(2, 5), (2, 7), (3, 6), (3, 7)\}.$$

Figure 1.9 shows the *diagram* and the *graph* for the relation. Each of the diagram and the graph expresses the idea of R. In the diagram, each ordered pair of the relation corresponds to an arrow that points from a member of A to a member of B. In the graph, each ordered pair of the relation corresponds to a black dot whose column is given by the first element of the ordered pair and whose row is given by the second element of the ordered pair.

A relation can also be defined on a single set; that is, from a set to itself. Consider, as an example of such a relation, the idea given by the words "less than" on the set A in the previous example. The relation is a subset of $A \times A$ and can be written as

$$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}.$$

The first element of each ordered pair is less than the second. The diagram and graph are shown in Figure 1.10. In the diagram, each arrow expresses the idea that the element at the tail is less than the element at the head. In the graph, the presence of a dot indicates that the element corresponding to the dot's column is less than the element corresponding to the dot's row.

1.3.2 Function

Definition 1.11: *A* function *f is a relation from a set A to a set B such that every element of A is associated with exactly one element of B. The set A is the* domain, *and B is the* codomain. *For every $a \in A$, the corresponding element in B is written $f(a)$.*

A mere relation from A to B might associate more than one element of B with an element of A, and it might not make an association for every element of A. A function, however, associates every element of A with exactly one element of B, though not every element of B need have an association, and an element of B might be associated with more than one element of A. Suppose that $A = \{1, 2, 3, 4\}$, and $B = \{5, 6, 7, 8\}$. An example of a function from A to B is

$$f = \{(1, 5), (2, 5), (3, 7), (4, 8)\}.$$

The diagram and graph for f are shown in Figure 1.11. In the diagram, every element of A lies at the tail of an arrow, and no more than

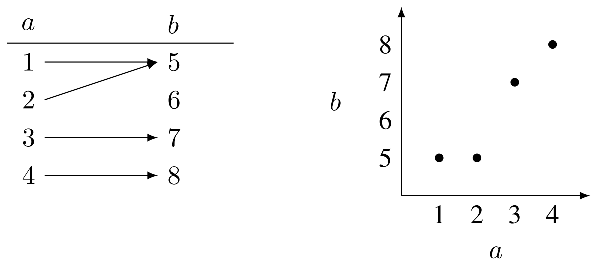


Figure 1.11: *Function from a domain, whose every element is represented by a, to a codomain, whose every element is represented by b. In the diagram of a function, every element in the left column points to exactly one element in the right column. Similarly, in the graph of a function, every column contains exactly one dot.*

one arrow originates at an element of A. In the graph, there is a dot for every element of A (for every column), and no column contains more than one dot. This is characteristic of a function.

A function may be defined by specifying a single rule that operates on every member of the domain. For example $f(x) = x^2$ defines a function that maps every element of the domain to the element's square.

1.3.3 Injection

Definition 1.12: *An* injection[11] *f from a set A to a set B is a function such that, for every $a_1 \in A$ and $a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$. An injection maps from its domain to its codomain in a* one-to-one *way; so an injection is sometimes called a "one-to-one function".*

Suppose that $A = \{1, 2, 3, 4\}$, and $B = \{-5, -6, -7, -8, -9\}$. An example of an injection from A to B is

$$f = \{(1, -5), (2, -7), (3, -8), (4, -9)\}.$$

The diagram and graph for f are shown in Figure 1.12. In the diagram, no more than one arrow terminates at any single element of B. In the graph, no row contains more than one dot. This is characteristic of an injection.

[11] By its etymology, "injection" indicates that which is thrown into something else. The word is used to name a function that might not map its domain completely to cover the codomain (so the domain is thrown somewhere inside the codomain) but does make sure that every element of the domain does map to its own element of the codomain.

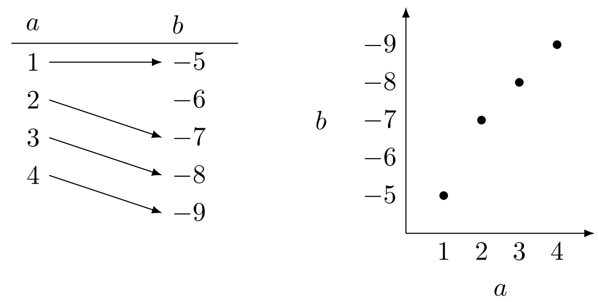


Figure 1.12: *Injection. In the diagram of an injective function, every element in the right column is pointed to by no more than one element in the left column. Similarly, in the graph of an injective function, every row contains no more than one dot.*

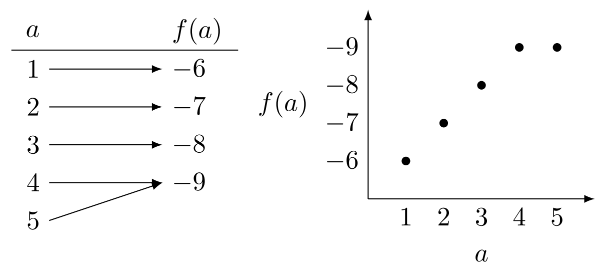


Figure 1.13: *Surjection. In the diagram of a surjective function, every element in the right column is pointed to by at least one element in the left column. Similarly, in the graph of a surjective function, every row contains at least one dot.*

1.3.4 Surjection

Definition 1.13: *A* surjection[12] *f from a set A to a set B is a function such that, for every element $b \in B$, there is at least one element $a \in A$ for which $b = f(a)$. A surjection maps its domain* onto *its codomain; so a surjection is sometimes called an "onto function".*

Suppose that $A = \{1, 2, 3, 4, 5\}$, and $B = \{-6, -7, -8, -9\}$. An example of an surjection from A to B is

$$f = \{(1, -6), (2, -7), (3, -8), (4, -9), (5, -9)\}.$$

For every element $b \in B$, there is at least one $a \in A$ such that $b = f(a)$. So, for a surjection, the diagram and graph may refer to $f(a)$ rather than to b. The diagram and graph for f are shown in Figure 1.13. In the diagram, every element of B has at least one arrow terminating on the element. In the graph, every row contains at least one dot. This is characteristic of a surjection.

[12] In coming to us from Latin through French, the prefix "super" changed to "sur". By its etymology, "surjection" indicates that which is thrown over or on top of something else. The word is used to name a function that throws its domain on top of its codomain so that the codomain is completely covered by the domain.

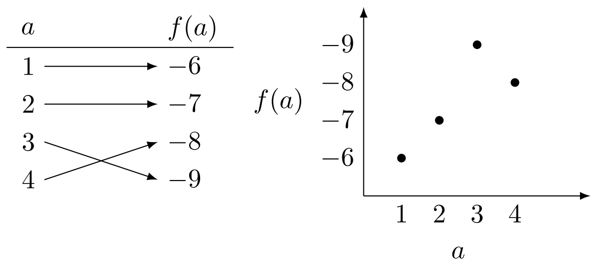


Figure 1.14: *Bijection. In the diagram of a bijective function, every element in the right column is pointed to by exactly one element in the left column. Similarly, in the graph of a bijective function, every row contains exactly one dot.*

Definition 1.14: *For any function f, the* image *of its domain A is the set $\{f(a) : a \in A\}$.*

By this definition, a surjection is just a function whose codomain is equal to the image of the domain.

1.3.5 Bijection

Definition 1.15: *A* bijection *f from a set A to a set B is both an injection and a surjection.*

Suppose that $A = \{1, 2, 3, 4\}$, and $B = \{-6, -7, -8, -9\}$. An example of an bijection from A to B is

$$f = \{(1, -6), (2, -7), (3, -9), (4, -8)\}.$$

For every element $b \in B$, there is exactly one $a \in A$ such that $b = f(a)$. The diagram and graph for f are shown in Figure 1.14. In the diagram, every element of B has exactly one arrow terminating on the element. In the graph of f, every row has exactly one dot in it. This is characteristic of a bijection. A bijection both

- pairs every element of the domain with exactly one element of the codomain and
- pairs every element of the codomain with exactly one element of the domain.

Theorem 1.2: *For every bijection f from a set A to a set B, there is an inverse function g from B to A such that for every $b \in B$, $f(g(b)) = b$, and for every $a \in A$, $g(f(a)) = a$.*

Definition 1.16: *For a bijection f, its inverse is written f^{-1}.*

1.3.6 Exercises

Exercise 1.3.1: *Let $A = \{0, 2, 4\}$ and $B = \{1, 3, 5\}$. What is $A \times B$?*

Exercise 1.3.2: *Pick a set of six numbers, and define on the set the relation expressing the idea given by the words "greater than". List the elements of the relation; draw the diagram of the relation; and draw the graph of the relation.*

Exercise 1.3.3: *Pick a set of six numbers, and define a function from the set to itself. List the elements of the function; draw the diagram of the function; and draw the graph of the function.*

Exercise 1.3.4: *Pick a set of six numbers, and define an injection from the set to itself. List the elements of the function; draw the diagram of the function; and draw the graph of the function.*

Exercise 1.3.5: *Pick a set of six numbers, and define a surjection from the set to itself such that the function is also an injection. List the elements of the function; draw the diagram of the function; and draw the graph of the function.*

Exercise 1.3.6: *Pick a set of six numbers, and define a surjection from the set to itself such that the function is not also an injection. List the elements of the function; draw the diagram of the function; and draw the graph of the function.*

Exercise 1.3.7: *Pick two sets of numbers, and define an injection from one set to the other such that the function is not also a surjection. List the elements of the function; draw the diagram of the function; and draw the graph of the function.*

Exercise 1.3.8: *Prove Theorem 1.2. Hint: Remember that a bijection f from a set A to a set B is a kind of function, which is a kind of relation, which is a subset of $A \times B$. Consider the elements of f. Each element is an ordered pair. Now consider from $B \times A$ the subset g that is obtained by reversing the order of every element of f. Show that the elements of g behave exactly as they would if $g = f^{-1}$.*

Chapter 2

Real Numbers

The calculus depends on a system of numbers. They are called "the continuum", "the real numbers", or simply "the reals". Geometry provides a way to visualize the reals: Imagine a straight line and associate a different number with every point on the line, so that the numbers increase in one direction and decrease in the opposite direction. See Figure 2.1, which illustrates a *real number line* or a *coordinate line*. For every *displacement* from the origin, there is a corresponding real number, which may be rational[1], irrational[2], or even non-computable[3]. The calculus refers to the *limit* of a sequence of numbers, and, even if every member of a sequence be rational, the limit of the sequence might be irrational or non-computable.[4] The set of real numbers contains every limit required by the calculus.

The name "real" was introduced in the 1600s by René Descartes.[5] Descartes distinguished the *real* square root of a positive number from the *imaginary* square root of a negative number. As Newton later did explicitly, Descartes implicitly treated every real number as the ratio of a length to a unit length.[6] The ancient Greeks, too, thought of number in terms of length. We shall develop the reals in a way that reflects this ancient tradition of regarding a number as a ratio of lengths.

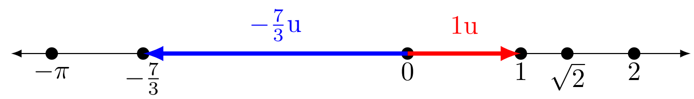


Figure 2.1: *A Cartesian coordinate system on a line allows one to visualize every real number r as the coordinate of a point. The coordinate gives the point's displacement from the origin, whose coordinate is 0. A displacement is not merely a distance but a directed distance. The unit u of displacement is the directed distance from the origin to the point whose coordinate is 1.*

The set of real numbers is denoted by the symbol "$\mathbb{R}$". We take the idea of displacement to be more fundamental than distance, and we introduce the idea of a dimensioned quantity, in which a fundamentally geometric unit is scaled by a real number. Considering the ideas of order and distance, we introduce the idea of open and closed intervals on $\mathbb{R}$. The development of some basic intuition about $\mathbb{R}$ prepares us for the discussion of the limit.

2.1 Coordinate

The real numbers are not simply the elements of the set $\mathbb{R}$; each real number is related to every other real number in certain, well defined ways. For any two distinct real numbers, one is *greater than* the other, and so the reals are ordered. Also, for any two real numbers, there is

[1] Equal to the ratio of two integers. The decimal representation of every rational number, like $\frac{1}{2} = 0.5 = 0.5\bar{0}$, repeats. Between every two distinct points on the number line, no matter how close together they are, there is an infinite number of rationals. So one might imagine that every point on the number line corresponds to a rational number. However, at least so far back as the Fourth Century, B.C., Greek mathematicians were aware that $\sqrt{2}$ is not rational. Despite the fact that the rationals are everywhere dense on the number line, there are also irrationals on the number line.

[2] Not equal to the ratio of two integers. The decimal representation of an irrational number, like $0.21221222122221\ldots$, never repeats and, like the decimal representation of $\pi = 3.14159265358979\ldots$, might not have an obvious pattern.

[3] A *non*-computable number has no algorithm that can be used to calculate its value so precisely as desired. Every irrational number (like $\sqrt{2}$ or π) that is used as an example in ordinary algebra, geometry, or calculus is a *computable number*. Among the computables are only every real number whose value can be approximated, by way of some algorithm, to arbitrary precision. On the one hand, because the set of algorithms is *countably* infinite, the set of computables is also countably infinite. On the other hand, the set of reals is *un*countably infinite because, as Cantor proved in 1874 and in a different way in 1891, there exists no bijection between the natural numbers and the real numbers.

In his *Shadows of the Mind: A Search for the Missing Science of Consciousness*, Roger Penrose provides a good discussion of what an algorithm is and why the strange non-computables lead to the uncountability of the reals.

[4] Consider the sequence, $(0.21, 0.21221, 0.212212221, \ldots)$. Every member of the sequence is rational, but the limit of the sequence is irrational. In this example, the limit is computable. There has been some interesting work on determining how much of the calculus can be retained on the replacement of the reals with the computables.

[5] Descartes was a mathematician and a philosopher. His contribution to mathematics is substantial. However, we view his departure from the hylemorphic metaphysics of Aristotle and St. Thomas Aquinas as a crucial error in the history of philosophy. For a good introduction to the hylemorphism, see *Aquinas: A Beginner's Guide* by Edward Feser.

[6] According to the encyclopedia entry on Descartes at `thefreedictionary.com`, "Descartes treated a real number as the ratio of any line segment to the unit segment, although such a definition for real numbers was explicitly stated much later by I. Newton; negative numbers were given a real interpretation in Descartes's work as directed ordinates."

a distance between them, and so $\mathbb{R}$ is a *metric space*.[7] In their ordered and metric nature, the reals are more than just a set of elements.

Some of the structure of the reals can be exposed by way of geometry; we can identify a deep correspondence between geometry and number. Recall Figure 2.1, which depicts the coordinate line. Considering order and distance together, we may define for any two distinct real numbers a *displacement*, the directed distance from the first to the second. Because the first is either less than or greater than the second, the displacement is either *positive* or *negative* in its direction. Noting the similarity between the coordinate line and a ruler, we use the coordinate line to visualize the nature given to $\mathbb{R}$ by ordinary arithmetic operations and comparisons.

2.1.1 Displacement

An abstract geometric displacement, which has both a length and a direction, is an example of a *vector*. In the one-dimensional metric space of a line, a symbol like "v", representing any vector, has no special notation to distinguish it from a symbol representing a number. (We shall in a later chapter see that in a two-dimensional or higher-dimensional space, the symbol for a vector will appear as "$\vec{v}$".) Although it can be scaled—multiplied—by a number, a displacement is not itself a number. Whenever we establish a fundamental unit of displacement, in terms of which most or all other displacements are expressed, we represent this unit without italics, with a symbol like "u". Thus we might write $v = -2.1\,\mathrm{u}$. The sign of the number indicates the direction along the line; the magnitude of the number scales the length of the unit; and the sense is that the number *multiplies* the unit of displacement even though the unit is not itself a number.

We develop, on the basis of geometric intuition, a feel for the relationships among the elements of $\mathbb{R}$. We distinguish among three different kinds of displacement:

1. the *number* that is the difference between two elements of $\mathbb{R}$,
2. the *abstract, directed distance* between two points on the coordinate line, and
3. the *concrete, directed distance* that can be measured by a physical device in the world of sense experience.

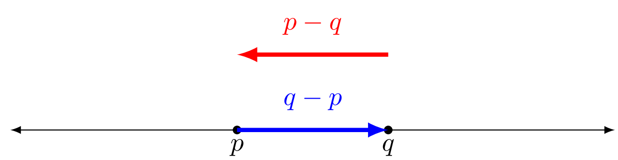
Displacement of the third kind does not yet concern us, for to speak of a relationship between $\mathbb{R}$ and the world of sense experience would be to invoke a scientific hypothesis (an hypothesis in physics), and we are concerned in the present section only with mathematics. We look now at the first two kinds of displacement and attach to the idea of numeric displacement our intuition about geometric displacement.

We regard any straight line L as a set of points. We assume that the reader knows what "straight", "line", and "point" mean, and so we do not attempt to define them here.

[7] A metric space is a space of measurement. The idea is that we can, at least in an abstract sense, *measure* the distance between any two elements of the space.

Definition 2.1: *For any line L and any two points p and q on L, the* displacement *d along L from p to q is the directed distance from p to q. We define the symbol "$-$" for points such that $d = q - p$.*

The difference between two numbers is another number, but we have defined the difference between two points *not* to be another point. Instead, the difference between two points is the directed distance from one point to the other. The following figure illustrates the idea.



Definition 2.2: *For any line L, any displacement d along L, and every point p on L, there is on L a point q displaced by d from p, and we define a sense of the symbol "$+$" such that $q = p + d = d + p$.*

The ordinary operation of addition combines a pair of numbers and produces a number, but the operation defined above combines a point and a displacement to produce a point. Definition 2.2 expresses the idea that although a displacement can be defined in terms of the directed distance between two particular points, the displacement is not located at a particular location. The same displacement applies equally well to every pair of points that share the same relationship to each other. The following example illustrates the idea.

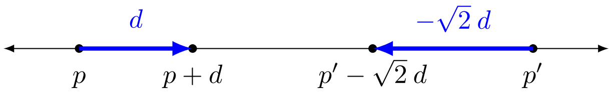
Sliding a displacement from one location to another does not cause the displacement to lose its identity.

Definition 2.3: Scaling *For any displacement d, a real number r* scales *d as by multiplication to form another displacement $e = rd = dr$, so that r is the ratio of the length of e to the length of d.*

- *If $r = 0$, then e is the* null displacement *0, which has neither length nor direction.*
- *If $r < 0$, then e points in the direction opposite to that of d.*
- *If $r = 1$, then $e = d$.*
- *We define the symbol "$-$" for the displacement such that if $r = -1$, then $e = -d$.*

Scaling the displacement, the real number is in this context called a scalar.

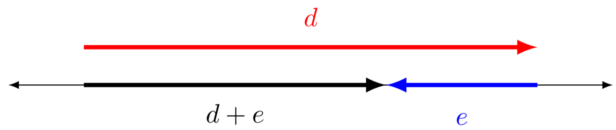
Definition 2.3 provides the fundamental basis for our geometric intuition about the nature of the reals. The following example illustrates the definition by way of $-\sqrt{2}$, an irrational scalar.[8]



The negative sign in $-\sqrt{2}$ causes $-\sqrt{2}\,d$ to point in the direction opposite to that of d. The size of $\sqrt{2}$ causes $-\sqrt{2}\,d$ to be a bit more than 1.41 times the length of d.

Definition 2.4: *For any two displacements d and e on a line, we define a sense of the symbol "$+$" such that the sum is written $f = d+e$, and, when the displacements be visualized with arrows, if the head of d be placed at the tail of e, and if the tail of f be placed at the tail of d, then the head of f is located at the head of e.*

Just as the ordinary operation of addition combines a pair of numbers and produces a number, so, too, the operation defined above combines a pair of displacements and produces another displacement. We saw that the difference of two points is a displacement and that the sum of a point and a displacement is another point. For the sum of two displacements to be another displacement makes displacements, considered by themselves under addition, like ordinary numbers. In fact, we shall see below that the relationship is deep, and we can sensibly define a one-to-one relationship between displacements along the line and real numbers. The illustration below shows that, according to Definition 2.4, displacements add together in a *head-to-tail* manner. In the sum $d+e$, the head of the augend d is attached to the tail of the addend e, and the sum points from the tail of the augend to the head of the addend.[9]

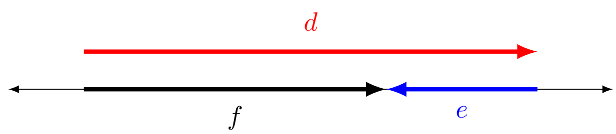


Theorem 2.1: *For any displacement d on a line and for any two real numbers r and s, $rd+sd = [r+s]d$.*

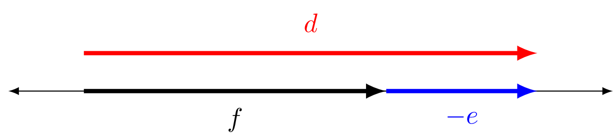
Definition 2.5: *For any two displacements d and e on a line L, we define the symbol "$-$" for the difference of displacements such that $e-d = e+[-d\,]$.*

Theorem 2.2: *For any sum $f = d+e$ of displacements on a line L, $d = f-e$, and $e = f-d$.*

Proof. By hypothesis we have $f = d+e$. By Definition 2.4, the displacements may be arranged head-to-tail.



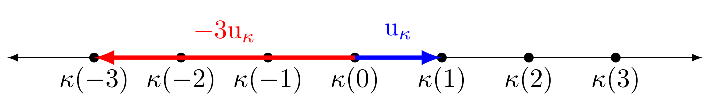
By Definition 2.3, $-e$ has the same length as e does and so will fit in its place, though with head and tail reversed.



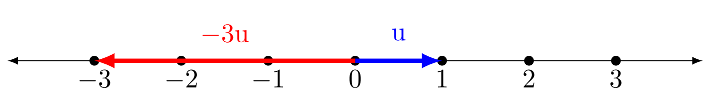
By Definition 2.4, $d = f+[-e\,]$. By Definition 2.5, $d = f-e$. The same process may be applied to reverse d (instead of e) in order to derive $e = f-d$. □

Definition 2.6: *A* Cartesian coordinate system κ *on a line is a bijection from the real numbers to the points on the line, such that for any real number r, there is on the line a point $\kappa(r) = \kappa(0) + r\,\mathrm{u}_\kappa$, where r is* the coordinate*; the point $\kappa(0)$ is the* origin*; and $\mathrm{u}_\kappa = \kappa(1)-\kappa(0)$ is the* unit displacement *under κ.*

The unit displacement u_κ can be visualized as follows:



When there is only one coordinate system under consideration, we draw the coordinate line as follows:



[8] When the proof for the irrationality of $\sqrt{2}$ was recognized by the ancient Greeks, the proof appears to have caused some controversy. At least for a time, the Greeks preferred to imagine that a unit length could be scaled only by a rational number, the ratio of two integers. The idea is that, for an integer n, a unit length could be subdivided into n equal subunits. Then an integer number m of subunits could be stacked end-to-end in order to make, by the appropriate choice of n and m, an overall length of any desired magnitude. That is equivalent to multiplying the original unit length by the rational number m/n. The side of a square, chosen as the unit length, would need to be scaled by $\sqrt{2}$ in order to equal the length of the diagonal. For $\sqrt{2}$ to be irrational means that there exists no n that can subdivide the side in such a way that m subunits would equal the length of the diagonal. The problem for us is that although the rational scaling is easy to visualize, we must somehow comfortably visualize an irrational scaling. In its decimal representation, an irrational number does not repeat, but by truncating or rounding the decimal expression at any desired level of accuracy, we may approximate the irrational number by a rational number. We can get an idea of the irrational scaling by thinking of a rationally scaled length that converges to the right, irrationally scaled length as we extend the decimal expression further and further out. The intuitive, geometric idea of convergence comes from recognizing that, with each successive digit in the decimal expression, the correction to the length becomes about ten times smaller than the previous correction.

[9] The augend is that which is to be augmented., and the addend is that which is to be added.

Theorem 2.3: *For a Cartesian coordinate system κ on a line L, and for any two real numbers r and s, the displacement between the corresponding points is $\kappa(s) - \kappa(r) = [s - r]\mathrm{u}_\kappa$.*

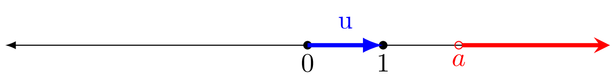
Theorem 2.3 expresses the relationship between a displacement of the first kind (the difference $s - r$) and a displacement of the second kind (the abstract, geometric displacement $\kappa(s) - \kappa(r)$).

2.1.2 Order

Definition 2.7: *For a Cartesian coordinate system κ on a line L, the* positive direction *on L is the direction of u_κ. The* negative direction *on L is the direction of $-\mathrm{u}_\kappa$.*

When there is a single coordinate line, it is almost always drawn horizontally with the positive direction to the right.

The set $S_{>a} = \{r \in R : r > a\}$ is depicted on the line as a positively directed ray emanating from the point labeled a (or labeled $\kappa(a)$ when the coordinate system is made explicit). Because $a \notin S_{>a}$, an open circle is placed around the point corresponding to a.



2.1.3 Distance

2.1.4 Exercises

Exercise 2.1.1: *Draw a diagram like the one used to illustrate Definition 2.4, but reverse the order of the addition. That is, show geometrically that the same result f obtains by adding the displacements head-to-tail in the order $e + d$.*

Exercise 2.1.2: *Pick any two displacements d and e that point in the same direction along a line. Draw a diagram showing that their sum is the same, regardless of the order in which they are added.*

Exercise 2.1.3: *Pick two displacements d and e that point in the same direction along a line, and let d be longer than e. Draw a diagram showing that their difference depends on the order in which they are subtracted.*

Exercise 2.1.4: *(Optional, difficult.) Prove Theorem 2.1.*

Exercise 2.1.5: *Prove Theorem 2.3. Avoid algebraic rearrangement of any expression involving points. Hint: Draw a diagram with points $\kappa(0)$, $\kappa(r)$, and $\kappa(s)$ on the line. Use Definition 2.4 and Theorem 2.1.*

Appendix A

Solutions to Selected Exercises

1.2.3 Remember that the empty set is a subset of every set and that every set is a subset of itself. So the subsets of $\{a, b, c\}$ are Ø, $\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$, and $\{a, b, c\}$

1.2.6 There are two statements to prove.

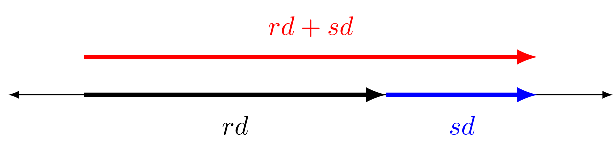
1. To prove that $A \cap B \subset A$, we must show that every element of $A \cap B$ is an element of A. By the definition of intersection, every element of $A \cap B$ is both an element of A and an element of B. Therefore, every element of $A \cap B$ is an element of A.

2. To prove that $A \subset A \cup B$, we must show that every element of A is an element of $A \cup B$. By the definition of union, $A \cup B$ contains every element that is either in A or in B. Therefore, every element of A is an element of $A \cup B$.

2.1.4 There are three cases to consider.

1. Either $r = 0$ or $s = 0$ (or $r = s = 0$). Suppose that $s = 0$. By Definition 2.3, $sd = 0d = 0$, which has zero length. By Definition 2.4, $rd + sd = rd$ because the head and the tail of sd are at the same point. Finally, we write $rd + sd = [r + 0]d = [r + s]d$. The same result obtains by a similar argument on the supposition that $r = 0$.

2. Either both $r < 0$ and $s < 0$ or both $r > 0$ and $s > 0$; that is, r and s have the same sign. In this case, by Definition 2.3, both rd and sd point in the same direction. Then by Definition 2.4, the length of $rd + sd$ is the sum of the lengths of rd and sd.

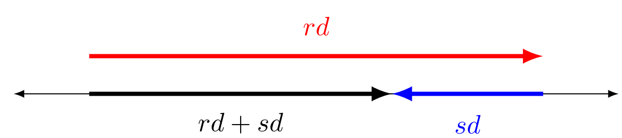
   

   By Definition 2.3, the ratio of the length of rd to the length of d is r; the ratio of the length of sd to the length of d is s; and, because the length of the sum of the displacements is in this case the sum of the lengths of the displacements, the ratio of the length of $rd + sd$ to the length of d is $|r + s|$. The sign of $r + s$ is the same as the sign of r and s. By Definition 2.4, $rd + sd$ points in the same direction as rd and sd, and, by Definition 2.3, the sign of the scalar directs the product; therefore $rd + sd = [r + s]d$.

3. Either both $r < 0$ and $s > 0$ or both $r > 0$ and $s < 0$; that is, r and s have opposite signs. In this case, by Definition 2.3, rd and sd point in opposite directions. There are three subcases to consider.

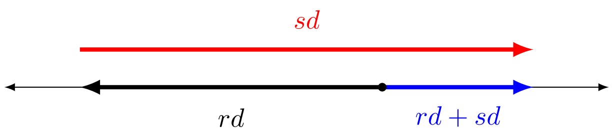
   (a) Suppose that $|r| = |s|$ so that rd is the same length as sd. Then by Definition 2.4, the length of $rd + sd$ is zero. We have by hypothesis, however, that $r + s = 0$, because r and s are of opposite signs but equal magnitude. So we can can write that $rd + sd = 0d = [r + s]d$.

   (b) Suppose that $|r| > |s|$ so that rd is longer than sd. Then by Definition 2.4, the length of $rd + sd$ is the length of rd minus the length of sd.

   By Definition 2.3, the ratio of the length of rd to the length of d is r; the ratio of the length of sd to the length of d is s; and, because the length of the sum of the displacements is in this case the difference of the lengths of the displacements, the ratio of the length of $rd + sd$ to the length of d is $|r| - |s|$. Because $|r| > |s|$ and because r and s have opposite signs, $|r| - |s| = |r + s|$. The sign of $r + s$ is the same as the sign of r. By Definition 2.4, $rd + sd$ points in the same direction as rd, and, by Definition 2.3, the sign of the scalar directs the product; therefore $rd + sd = [r + s]d$.

   (c) Suppose that $|s| > |r|$ so that sd is longer than rd. Then by Definition 2.4, the length of $rd + sd$ is the length of sd minus the length of rd.



By Definition 2.3, the ratio of the length of rd to the length of d is r; the ratio of the length of sd to the length of d is s; and, because the length of the sum of the displacements is in this case the difference of the lengths of the displacements, the ratio of the length of $rd + sd$ to the length of d is $|s| - |r|$. Because $|s| > |r|$ and because r and s have opposite signs, $|s| - |r| = |s + r| = |r + s|$. The sign of $r + s$ is the same as the sign of s. By Definition 2.4, $rd + sd$ points in the same direction as sd, and, by Definition 2.3, the sign of the scalar directs the product; therefore $rd + sd = [r + s]d$.

Appendix B

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