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Assignment 2
ME714: Computer Integrated Manufacturing

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1 Question 1

We can present point P on the interpolated curve parametrically as $P(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}$ such that $t \in [0, 1]$ & $P(t)$ is of the form $a_3t^3 + a_2t^2 + a_1t + a_0$.

Given information:

$$P(0) = A = (2, 5)$$

$$P(1) = B = (6, 0)$$

$$P'(0) = slope\ at\ A = (0, 1)$$

$$P'(1) = slope\ at\ B = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

$P(t)$ can be found as follows:

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{bmatrix}$$

$$\Rightarrow P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 6 & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -7.293 & 11.707 \\ 11.293 & -17.707 \\ 0 & 1 \\ 2 & 5 \end{bmatrix}$$

$$\Rightarrow P(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix} = \begin{bmatrix} -7.293t^3 + 11.293t^2 + 2 & 11.707t^3 - 17.707t^2 + t - 5 \end{bmatrix}$$

We need to find slope of this curve at $t = 0.5$. We proceed as follows:

$$P'(t) = \begin{bmatrix} x'(t) & y'(t) \end{bmatrix} = \begin{bmatrix} -21.879t^2 + 22.586t & 35.121t^2 - 35.414t + 1 \end{bmatrix}$$

Substituting $t = 0.5$, we get:

$$P'(t = 0.5) = \begin{bmatrix} 5.823 & -7.927 \end{bmatrix}$$

2 Question 2

Given 4 control points (V_0, V_1, V_2, V_3), the Bezier Curve $B(t)$ is given by $B(t) = \sum_{i=0}^{3} V_i B_{i,3}(t)$, where $B_{i,3}(t) = \binom{3}{i} t^i (1-t)^{3-i}$.

$$\therefore B(t) = t^3V_0 + 3t^2(1-t)V_1 + 3t(1-t)^2V_2 + (1-t)^3V_3$$

$$\Rightarrow B(t) = (-t^3 + 3t^2 - 3t + 1)V_0 + (3t^3 - 6t^2 + 3t)V_1 + (-3t^3 + 3t^2)V_2 + t^3V_3$$

Collecting the coefficients of each term & representing as matrices, we obtain:

$$B(t) = [t][M_b][V] = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

3 Question 3

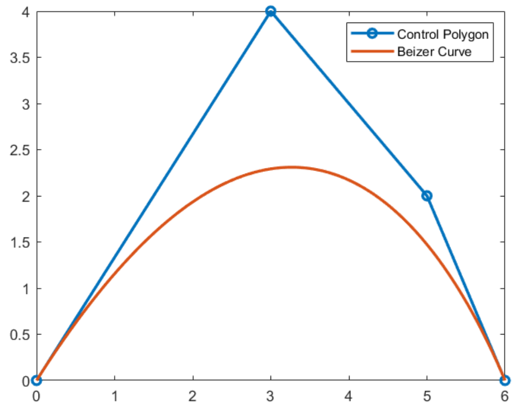
The MATLAB code to plot a Bezier curve given 4 control points (V_0, V_1, V_2, V_3):

```
function [] = bezier()
    % N*2 matrix containing 'N' control points [x1 y1;...;xN yN]
    V = input('Enter 4 Control Points:');
    N = size(V, 1);          % Number of control points

    % In our case, check that N = 4
    if (N ~= 4)
        disp('Number of control points not equal to 4. Exitting...');
        return;
    end

    num_pts = 100;           % Number of points to be used to plot the Bezier Curve
    t = (0:1/num_pts:1);     % Array of values of parameter 't' for plotting the curve

    M_B = [-1 3 -3 1; 3 -6 3 0; -3 3 0 0; 1 0 0 0];  % Hermite matrix equivalent for Bezier Curve with 4 Control Points
    T = [t.^3; t.^2; t; t.^0]';                         % Parameter Matrix

    B = T * M_B * V;                                    % Points on the Bezier Curve can be obtained using B(t) = [t][M_B][V]


    % Plot the Control Polygon
    plot(V(:,1), V(:,2), 'Marker', 'o', 'LineWidth', 2, 'DisplayName', 'Control Polygon');
    hold on;

    % Plot the Bezier Curve
    plot(B(:, 1), B(:, 2), 'LineWidth', 2, 'DisplayName', 'Bezier Curve');
    hold off;

    legend;
end
```

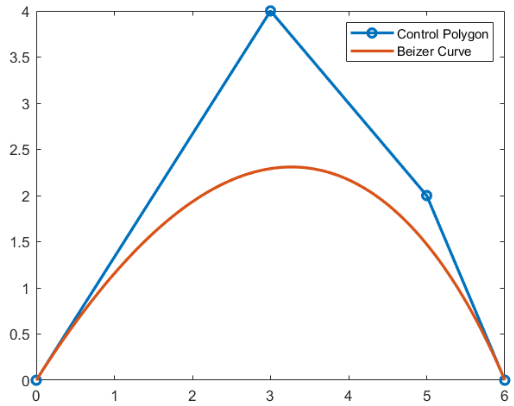
The same code can be found as a MATLAB file here.

4 Question 4

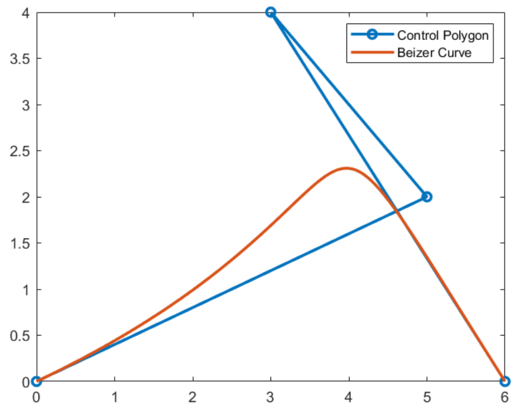
Plots for the 4 cases are shown below:



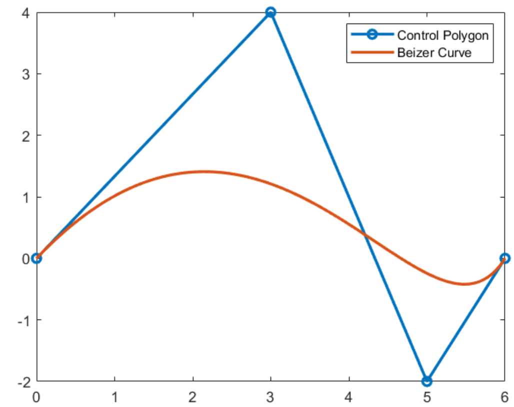
(a) $V_0 = (0, 0), V_1 = (3, 4), V_2 = (5, 2), V_3 = (6, 0)$



(b) $V_0 = (6, 0), V_1 = (5, 2), V_2 = (3, 4), V_3 = (6, 0)$



(c) $V_0 = (0, 0), V_1 = (5, 2), V_2 = (3, 4), V_3 = (6, 0)$



(d) $V_0 = (0, 0), V_1 = (3, 4), V_2 = (5, -2), V_3 = (6, 0)$

Comments:

(a)
- Reversing the order of control points does not affect the geometry of the Bezier Curve (because the equation of Bezier curve is symmetric w.r.t t & $1 - t$).
- Apart from reversing the order, any other change in the order of control points can completely change the geometry of the Bezier curve.
- If the order is changed such that the new V_0 & V_3 are not the same as original ones, the start & end points of the Bezier curve would change because Bezier curve passes through the first & last control points *(although this is not directly deduced from any of the 4 cases)*

(b) Moving one of the control points affects the Bezier curve globally & changes its overall geometry. Thus, Bezier curves are said to exhibit *global control*.

5 Question 5

Control points for 1st Bezier Curve B_1: $A = (2,3,4), B = (3,1,5), C = (x,y,z), D = (3,4,3)$

$$B_1(t) = [t][M_B][V_{B_1}] = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \\ V_D \end{bmatrix}$$

$$\Rightarrow B_1(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ x & y & z \\ 3 & 4 & 3 \end{bmatrix}$$

$$\Rightarrow B_1(t) = \begin{bmatrix} (10-3x)t^3 + (3x-12)t^2 + 3t + 2 \\ (4-3y)t^3 + (3y+3)t^2 - 6t + 3 \\ (14-3z)t^3 + (3z-18)t^2 + 3t + 4 \end{bmatrix}^T$$

Control points for 2nd Bezier Curve B_2: $D = (3,4,3), E = (2,6,0), F = (5,7,5), G = (5,2,3)$

$$B_2(t) = [t][M_B][V_{B_2}] = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_D \\ V_E \\ V_F \\ V_G \end{bmatrix}$$

$$\Rightarrow B_2(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 3 \\ 2 & 6 & 0 \\ 5 & 7 & 5 \\ 5 & 2 & 3 \end{bmatrix}$$

$$\Rightarrow B_2(t) = \begin{bmatrix} -7t^3 + 12t^2 - 3t + 3 \\ -5t^3 - 3t^2 + 6t + 4 \\ -15t^3 + 24t^2 - 9t + 3 \end{bmatrix}^T$$

B_1 & B_2 are C^1 continuous at common point D. Thus, their slopes at point D must be equal. For B_1, point D corresponds to $t = 1$ while for B_2, point D corresponds to $t = 0$.

Thus, we require:

$$B_1'(t=1) = B_2'(t=0)$$

$$\Rightarrow \begin{bmatrix} (9-3x) & (12-3y) & (9-3z) \end{bmatrix} = \begin{bmatrix} -3 & 6 & -9 \end{bmatrix}$$

Thus, $x = 4$, $y = 2$ & $z = 6$, i.e., $C \equiv (4,2,6)$. *[Answer]*