CSE 211: Discrete Mathematics **(Due: 17/01/21)**

Homework #4

Instructor: Dr. Zafeirakis Zafeirakopoulos *Name: Ahmet Tuğkan Ayhan* *Student Id: 1901042692*
Assistant: Gizem Süngü

Course Policy: Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- It is not a group homework. Do not share your answers to anyone in any circumstance. Any cheating means at least -100 for both sides.
- Do not take any information from Internet.
- No late homework will be accepted.
- For any questions about the homework, send an email to gizemsungu@gtu.edu.tr
- The homeworks (both latex and pdf files in a zip file) will be submitted into the course page of Moodle.
- The latex, pdf and zip files of the homeworks should be saved as "Name_Surname_StudentId".{tex, pdf, zip}.
- If the answers of the homeworks have only calculations without any formula or any explanation -when needed- will get zero.
- Writing the homeworks on Latex is strongly suggested. However, hand-written paper is still accepted **IFF** hand writing of the student is clear and understandable to read, and the paper is well-organized. Otherwise, the assistant cannot grade the student's homework.

Problem 1 (15+15=30 points)

Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

(a) Show that whether $a_n = -2^{n+1}$ is a solution of the given recurrence relation or not. Show your work step by step.

(Solution)

---

* First, we find value for a_{n-1}.

$\rightarrow a_n = -2^{n+1}$
$\rightarrow a_{n-1} = -2^{(n-1)+1}$
$\rightarrow a_{n-1} = -2^n$

---

* Then, we insert a_{n-1} inside of nonhomogeneous linear recurrence relation.

$\rightarrow a_n = 3a_{n-1} + 2^n$
$\rightarrow a_n = 3(-2^n) + 2^n$
$\rightarrow a_n = 2^n(-3 + 1)$
$\rightarrow a_n = 2^n(-2)$
$\rightarrow a_n = -2^{n+1}$

---

* Answer is yes. we find same answer with a_{n-1}. The solution of the given recurrence relation is $a_n = -2^{n+1}$.

(b) Find the solution with $a_0 = 1$.

(Solution)

---

* First, we will try to find homogeneous part

$\rightarrow a_n - 3a_{n-1} = 2^n$ $\quad (a_n = a_n^h + a_n^p)$ h = homogeneous, p = particular

$\rightarrow a_n^h => a_n - 3a_{n-1} = 0$

$\rightarrow a_n^h => k^2 - 3k = 0$ (finding characteristic roots)

$\rightarrow a_n^h => k(k-3) = 0$ $(k_1 = 0, k_2 = 3)$ $(a_n^h = \alpha(k_1)^n + \beta(k_2)^n)$

$\rightarrow a_n^h = \alpha(3)^n + \beta(0)^n$

$\rightarrow a_n^h = \alpha(3)^n$ (we will hold this until a_n^p is found)

---

* Then, we find equation for particular part

$\rightarrow a_n^p => a_n - 3a_{n-1} = 2^n$ (we'll use Ar^n for this particular equation)

$\rightarrow a_n^p => A2^n - 3(A2^{n-1}) = 2^n$

$\rightarrow$ (for n=0) $=> A2^0 - 3(A2^{0-1}) = 2^0$

$\rightarrow$ (for n=0) $=> A$ - $\frac{3}{2}A = 1$

$\rightarrow$ (for n=0) $=> A = -2$

$\rightarrow a_n^p = Ar^n = -2(2)^n = -2^{n+1}$

---

* Finally, we sum these two equations

$\rightarrow a_n = a_n^h + a_n^p$

$\rightarrow a_n = \alpha(3)^n - 2^{n+1}$

$\rightarrow a_0 = \alpha(3)^0 - 2^{0+1}$

$\rightarrow 1 = \alpha - 2 \quad , \quad \alpha = 3 \quad , \quad hence \;\; a_n^h = 3(3)^n = 3^{n+1}$

$\rightarrow a_n = a_n^h + a_n^p$

$\rightarrow answer : a_n = 3^{n+1} - 2^{n+1}$

Problem 2 (35 points)

Solve the recurrence relation f(n) = 4f(n-1) - 4f(n-2) + n^2 for f(0) = 2 and f(1) = 5.

(Solution)

---

* First, we find homogeneous part equation
* Then, we find equation for particular part

$\rightarrow a_n = a_n^h + a_n^p$

<u>for a_n^h</u>

$\rightarrow f(n) - 4f(n-1) + 4f(n-2) = 0$

$\rightarrow k^2 - 4k + 4 = 0$ (finding characteristic roots)

$\rightarrow (k-2)^2 = 0$ (we will multiply β by n since we have two same roots)

$\rightarrow \alpha(2)^n + \beta n(2)^n$ we get this from $= (a_n^h = \alpha(k_1)^n + \beta n(k_2)^n)$

<u>for a_n^p</u>

$\rightarrow f(n) - 4f(n-1) + 4f(n-2) = n^2$ (since it equals to n^2 we use f(n) $= A_2n^2 + A_1n + A_0$)

$\rightarrow [(A_2n^2 + A_1n + A_0) - 4(A_2(n-1)^2 + A_1(n-1) + A_0) + 4(A_2(n-2)^2 + A_1(n-2) + A_0)] = n^2$

$\rightarrow A_0 + A_1(n-4) + A_2(n^2 - 8n + 12) = n^2$

$(for$ n $= 0) \rightarrow A_0 - 4A_1 + 12A_2 = 0$

$(for$ n $= 1) \rightarrow A_0 - 3A_1 + 5A_2 = 1$

$(for$ n $= 2) \rightarrow A_0 - 2A_1 + 0A_2 = 4$

$(for$ n $= 3) \rightarrow A_0 - 1A_1 - 3A_2 = 9$

$\rightarrow A_0 = 20$, $A_1 = 8$, $A_2 = 1$

<u>now $a_n^p + a_n^h$</u>

$\rightarrow f(n) = \alpha(2)^n + \beta n(2)^n + n^2 + 8n + 20$

$\rightarrow f(0) = \alpha(2)^0 + \beta 0(2)^0 + 0^2 + 8.0 + 20 = 2$

$\rightarrow f(0) = \alpha + 20 = 2$, hence $\alpha =$ -18

$\rightarrow f(1) = -18(2)^1 + \beta 1(2)^1 + 1^2 + 8.1 + 20 = 5$

$\rightarrow f(1) = -36 + 2\beta + 1 + 8 + 20 = 5$

$\rightarrow f(1) = 2\beta = 12$, hence $\beta = 6$

answer: $f(n) = -18(2)^n + 6n(2)^n + n^2 + 8n + 20$

Problem 3 (20+15 = 35 points)

Consider the linear homogeneous recurrence relation $a_n = 2a_{n-1}$ - $2a_{n-2}$.
(a) Find the characteristic roots of the recurrence relation.

(Solution)

---

* First we find determinant of the recurrence relation

$\rightarrow a_n - 2a_{n-1} + 2a_{n-2} = 0$

$\rightarrow k^2 - 2k + 2 = 0$

$\rightarrow \Delta = b^2 - 4ac$

$\rightarrow \Delta = (-2)^2 - 4.1.2$

$\rightarrow \Delta = 4 - 8$

$\rightarrow \Delta = -4$

---

* Then, we find roots (imaginary)

$\rightarrow r_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$

$\rightarrow r_1 = \frac{-(-2)+\sqrt{-4}}{2.1} = \frac{2+2i}{2} = 1 + i$

$\rightarrow r_2 = \frac{-(-2)-\sqrt{-4}}{2.1} = \frac{2-2i}{2} = 1 - i$

(b) Find the solution of the recurrence relation with $a_0 = 1$ and $a_1 = 2$.

(Solution)

---

* At last, we use $\alpha(r_1)^n + \beta(r_2)^n$ to find solution

$\rightarrow a_n = \alpha(1+i)^n + \beta(1-i)^n$

$\rightarrow a_0 = \alpha(1+i)^0 + \beta(1-i)^0 = 1$

$\rightarrow a_0 = \alpha + \beta = 1$

$\rightarrow a_1 = \alpha(1+i)^1 + \beta(1-i)^1 = 2$

we calculate $a_1 - a_0(1-i)$

$\rightarrow \alpha(1+i) + \beta(1-i) - \alpha(1-i) - \beta(1-i) = 1 + i$

$\rightarrow 2\alpha i = 1 + i$

$\rightarrow \alpha = \frac{1+i}{2i}$

$\rightarrow \beta = \frac{1-i}{2i}$

Finally, we insert α and β into a_n

---

answer: $a_n = \frac{1+i}{2i}\ (1+i)^n + \frac{1-i}{2i}\ (1-i)^n$