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Three-Gambler Ruin Game: A Game Theoretic Analysis

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Abstract
We study the following game. Three players start with initial capitals of s_1, s_2, s_3 dollars; in each round player P_m is selected with probability $\frac{1}{3}$; then *he* selects player P_n and they play a game in which P_m wins from (resp. loses to) P_n one dollar with probability p_{mn} (resp. $p_{nm} = 1 - p_{mn}$). When a player loses all his capital he drops out; the game continues until a single player wins by collecting everybody's money. This is a "strategic" version of the classical Gambler's Ruin game. It seems reasonable that a player may improve his winning probability by judicious selection of which opponent to engage in each round. We formulate the situation as a *stochastic game* and prove that it has at least one Nash equilibrium in stationary deterministic strategies.

Keywords Gambler's ruin · Multiple players · Game theory · Nash equilibrium

1 Introduction

In this paper we study a version of the three-gambler ruin problem in which, whenever a gambler is "activated", he *chooses* his opponent. More specifically, our version is played by the following rules.

1. The game is played in discrete time steps (rounds).
2. Three players start at the 0-th round with initial capitals, of s_1, s_2 and s_3 dollars.
3. In each round, one player is chosen with probability $\frac{1}{3}$ and then *he* chooses another player against whom he will play.
4. Suppose that the m-th player plays against the n-th one; with probability p_{mn} (resp. $p_{nm} = 1 - p_{mn}$) he wins from (resp. loses to) the n-th player one dollar.
5. The game continues until a single player wins (i.e. accumulates the total $s_1 + s_2 + s_3$ dollars); if this never happens, the game continues ad infinitum.

The above game is a "strategic" (in the *game-theoretic* sense) version of the "classical" Gambler's Ruin (GR). Intuitively, it appears reasonable that a player may improve his winning probability by judicious opponent selection in each round. As far as we know, this game theoretic approach to GR has not been explored previously.

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The "classic" Gambler's Ruin involves two gamblers, is one of the earliest-studied probability problems (for a historical review see [6, 29, 33]) and remains one of the most popular introductory examples in the theory of Markov Chains. For an overview of the basic results see [8].

An obvious generalization of GR is to have three or more gamblers. An early attempt in this direction is [23] but the first (as far as we know) major advance appeared in [27] and involved "*symmetric play*", i.e., in each round all players have equal probability of winning. Symmetric play was further studied in [7, 26, 32] and numerous other publications [1–5, 13, 30, 31]. The case of "*asymmetric play*", i.e., when players' winning probabilities are not necessarily equal, has also been studied, first in [24, 25] and more recently in [14–16]. Finally, versions played on graphs have also been studied, for example in [21].

In all of the above works, no player *strategy* is involved. In other words, the players cannot influence who participates in a given round of the game.[1]. Hence the evolution of the game is governed by purely probabilistic laws.

As already mentioned, in this paper we take a different approach. In Sect. 2 we provide a rigorous formulation of three-player GR as a stochastic game. In Sect. 3 we provide an introductory analysis of the simple case in which each player starts with initial capital of one dollar; for this case: (i) we compute the players' expected payoffs for *any stationary strategy* and (ii) we prove the existence of at least one *stationary deterministic Nash equilibrium* and show how it can be computed. This analysis is repeated in Sect. 4 for the general case (i.e., with arbitrary initial capitals). In Sect. 5 we present several numerical experiments. In Sect. 6 we summarize our results and present some future research directions.

It is worth emphasizing that our results can be generalized (rather easily) for the case of more than three players; we limit ourselves to the three-player case mainly for simplicity of presentation.

2 The Game

We denote our game by $\Gamma_3(\mathbf{p}, K)$. It involves three players (gamblers) P_1, P_2, P_3 and has parameters $\mathbf{p} = (p_1, p_2, p_3)$ and K, which satisfy:

$$\forall n \in \{1, 2, 3\} : p_n \in (0, 1) \text{ and } K \in \{3, 4, , ...\}.$$

$\Gamma_3(\mathbf{p}, K)$ is played as follows:

1. At times $t \in \{0, 1, 2, ..\}$ and for $n \in \{1, 2, 3\}$, P_n's *capital* is $s_n(t)$.
2. At $t = 0$, the capitals satisfy $\sum_{n=1}^{3} s_n(0) = K$.
3. At $t \in \{1, 2, ...\}$ a player P_n is selected equiprobably from the set $\{n : s_n(t) > 0\}$.
4. P_n selects another player P_m such that $s_m(t) > 0$.
5. With probability p_{nm} (resp. p_{mn}) P_n receives one unit from (resp. pays one unit to) P_m, where

   $$p_{12} = p_1, \quad p_{21} = 1 - p_1, \quad p_{23} = p_2, \quad p_{32} = 1 - p_2, \quad p_{31} = p_3, \quad p_{13} = 1 - p_3.$$

   Hence the players' capitals change as follows:

   (a) if P_n wins then $s_n(t) = s_n(t-1) + 1$ and $s_m(t) = s_m(t-1) - 1$;

[1] A notable exception appears in [26], where it is stated that "none of the preceding quantities [winning probability, expected game duration etc.] depend on the rule for choosing the players in each stage". Here we have a case in which strategies are implicitly considered, but they turn out to be irrelevant. This is the case because the author studies *symmetric play*; as we will see, things are different for asymmetric play.

(b) if P_n loses then $s_n(t) = s_n(t-1) - 1$ and $s_m(t) = s_m(t-1) + 1$;
(c) for $k \notin \{n, m\}$ we have $s_k(t) = s_k(t-1)$.

Obviously, at all t we have $\sum_{n=1}^{3} s_n(t) = K$.

6. If at some time t' one player is left with zero capital, the game continues between the two remaining players.
7. The game continues until at some time t'' there exists a single player P_m with $s_m(t'') = \sum_{n=1}^{3} s_n(0)$, in which case this player is the winner (he has collected all the available money). If such a player does not exist for any turn, the game continues ad infinitum.

The *game state* at time t is $\mathbf{s}(t) = (s_1(t)\, s_2(t)\, s_3(t))$. The *state set* is

$$S = \left\{ (s_1, s_2, s_3) : \forall n : s_n \geq 0 \text{ and } \sum_{n=1}^{3} s_n = K \right\}.$$

Note that each (s_1, s_2, s_3) can be rewritten as $(s_1, s_2, K - s_1 - s_2)$. For $s_1 = i$, we have $K - i + 1$ possible states of the form $(i, j, K - i - j)$ with $j \in \{0, 1, ..., K - i\}$; since i can take any value in $\{0, 1, ..., K\}$, the total number of states is

$$N_K = |S| = \frac{(K+1)(K+2)}{2}.$$

We define the following state sets:

$$S_1 = \{(K, 0, 0)\}, \quad S_2 = \{(0, K, 0)\}, \quad S_3 = \{(0, 0, K)\}, \quad S_i = \{(s_1, s_2, s_3) : \forall n : s_n > 0\}.$$

We will call the states $s \in S_\tau = S_1 \cup S_2 \cup S_3$ *terminal*, the states $\mathbf{s} \in S_i$ *interior* and the states $s \in S_b = S \backslash S_i$ *boundary*. It is easily checked that $|S_i| = \frac{(K-1)(K-2)}{2}$ and $|S_b| = 3K$. It will be convenient to number the states so that: the first state is $(K, 0, 0)$, the second is $(0, K, 0)$ and the third is $(0, 0, K)$; the remaining states can be numbered arbitrarily.

The *game history* at time t is $\mathbf{h}(t) = \mathbf{s}(0)\,\mathbf{s}(1) \ldots \mathbf{s}(t)$. The set of all *admissible* histories is denoted by H. *A terminal history* is an $\mathbf{h}(t) = \mathbf{s}(0)\,\mathbf{s}(1) \ldots \mathbf{s}(t)$ such that $\mathbf{s}(t) \in S_\tau$. For $n \in \{1, 2, 3\}$, P_n's *payoff* is defined for every terminal history $\mathbf{h} = \mathbf{s}(0)\,\mathbf{s}(1) \ldots \mathbf{s}(t)$ by

$$\mathsf{Q}_n(\mathbf{h}) = \begin{cases} 1 \text{ iff } \mathbf{s}_n(t) = K, \\ 0 \text{ else;} \end{cases}$$

the payoff of nonterminal histories is zero.
A *strategy* for P_n is a function $\sigma_n : H \to \Delta_3 \cup \{(0, 0, 0)\}$, where

$$\Delta_3 = \left\{ (x_{n1}, x_{n2}, x_{n3}) \text{ with } x_{nm} \geq 0 \text{ for } m \neq n, x_{nn} = 0, \sum_{m=1}^{3} x_{nm} = 1 \right\}.$$

The interpretation is that

$$\sigma_n(\mathbf{h}) = (x_{n1}, x_{n2}, x_{n3}) \Leftrightarrow (\forall m, n : \Pr(P_n \text{ selects } P_m \text{ at } t | \mathbf{h}) = x_{nm}).$$

We will only consider *admissible* strategies.[2] A strategy profile is a triple $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. A strategy σ_n is called *deterministic* iff

$$\forall \mathbf{h} = \mathbf{s}(0)\,\mathbf{s}(1) \ldots \mathbf{s}(t) : \sigma_n(\mathbf{h}) = (x_1(\mathbf{h}), x_2(\mathbf{h}), x_3(\mathbf{h})) \text{ has a single nonzero element.}$$

[2] For example, we exclude strategies which assign positive probability to selecting a player with zero capital, to a player selecting himself etc.

A strategy σ_n is called *stationary* iff, for every $\mathbf{h}$, $\sigma_n(\mathbf{h})$ depends only on the last state, i.e., iff

$$\forall \mathbf{h} = \mathbf{s}_0\mathbf{s}_1...\mathbf{s}_t : \sigma_n(\mathbf{h}) = \sigma_n(\mathbf{s}_t);$$

We will also use the following shorter notation:

$$x_1 = x_{12} \text{ (and } 1 - x_1 = x_{13}), \quad x_2 = x_{23} \text{ (and } 1 - x_2 = x_{21}), \quad x_3 = x_{31} \text{ (and } 1 - x_3 = x_{32}),$$

and we will denote a stationary strategy profile by $\mathbf{x} = (x_1, x_2, x_3)$.
An initial state $\mathbf{s}(0)$ and a strategy profile $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ define a probability measure on H. Hence the *expected payoff* to P_n is well defined by

$$Q_n(\mathbf{s}(0), \sigma) = \mathbb{E}_{\mathbf{s}(0),\sigma}(\mathrm{Q}_n(\mathbf{h})).$$

It is easily seen that $Q_n(\mathbf{s}(0), \sigma)$ is P_n's probability of winning when the starting state is $\mathbf{s}(0)$ and the players use the strategy profile σ.

An admissible strategy profile $\widehat{\sigma} = (\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3)$ is a *Nash equilibrium* (NE) of $\Gamma_3(\mathbf{p}, a, b)$ iff

$$\forall \mathbf{s}, \forall n, \forall \sigma_n : Q_n(\mathbf{s}, \widehat{\sigma}) \geq Q_n(\mathbf{s}, \sigma_n, \widehat{\sigma}_{-n}),$$

where, using standard game theoretic notation, $\widehat{\sigma}_{-1} = (\widehat{\sigma}_2, \widehat{\sigma}_3)$, $\widehat{\sigma}_{-2} = (\widehat{\sigma}_1, \widehat{\sigma}_3)$, $\widehat{\sigma}_{-3} = (\widehat{\sigma}_1, \widehat{\sigma}_2)$.

3 The Case $K = 3$

As a preliminary step in our analysis, let us consider the game when total capital is $K = 3$ (this is obviously the first nontrivial case) and, for $n \in \{1, 2, 3\}$, P_n uses the *stationary* strategy $\mathbf{x}_n = (x_n(\mathbf{s}))_{\mathbf{s}\in S}$. We will first compute the payoffs under *any* stationary strategy profile $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and then will use the obtained formulas to prove the existence of a stationary deterministic Nash equilibrium.

3.1 Computation of Payoffs Under Stationary Strategies

Suppressing the dependence on $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, we denote P_n's payoff, when the game starts at state $\mathbf{s}$, by

$$V_n(\mathbf{s}) = Q_n(\mathbf{s}, \mathbf{x}).$$

Let us compute $\mathbf{V}_1(\mathbf{s}) = (V_1(\mathbf{s}))_{\mathbf{s}\in S}$. We obviously have

$$V_1(3,0,0) = 1, \quad V_1(0,3,0) = V_1(0,0,3) = V_1(0,1,2) = V_1(0,2,1) = 0.$$

Also, when two players are left in the game, the only admissible strategy for each one is to select the other. For instance, in the state $(1, 2, 0)$, we must have

$$x_1(1,2,0) = x_{12}(1,2,0) = 1, \quad x_2(1,2,0) = x_{23}(1,2,0) = 0.$$

We have the following transition probabilites:

$$\Pr((2,1,0) \to (3,0,0)) = p_1, \quad \Pr((2,1,0) \to (1,2,0)) = 1 - p_1,$$
$$\Pr((1,2,0) \to (2,1,0)) = p_1, \quad \Pr((1,2,0) \to (0,3,0)) = 1 - p_1.$$

Consequently

$$V_1(2,1,0) = p_1 V_1(3,0,0) + (1 - p_1) V_1(1,2,0),$$

$$V_1(1,2,0) = p_1 V_1(2,1,0) + (1-p_1) V_1(0,3,0),$$

which becomes

$$\begin{aligned} V_1(2,1,0) &= p_1 + (1-p_1) V_1(1,2,0), \\ V_1(1,2,0) &= p_1 V_1(2,1,0). \end{aligned}$$

Solving the above system we get

$$V_1(2,1,0) = \frac{p_1}{1-p_1+p_1^2}, \quad V_1(1,2,0) = \frac{p_1^2}{1-p_1+p_1^2}.$$

Similarly, we have

$$\begin{aligned} V_1(2,0,1) &= (1-p_3) + p_3 V_1(1,0,2) \\ V_1(1,0,2) &= (1-p_3) V_1(2,0,1) \end{aligned}$$

and we get

$$V_1(2,0,1) = \frac{1-p_3}{1-p_3+p_3^2}, \quad V_1(1,0,2) = \frac{1-2p_3+p_3^2}{1-p_3+p_3^2}.$$

Finally, simplifying $x_n(1,1,1)$ to x_n (for $n \in \{1,2,3\}$) we have

$$\begin{aligned} V_1(1,1,1) = & \frac{1}{3}(x_1(p_1 V_1(2,0,1) + (1-p_1) V_1(0,2,1)) + (1-x_1)(p_3 V_1(0,1,2) + (1-p_3) V_1(2,1,0))) \\ & + \frac{1}{3}(x_2(p_2 V_1(1,2,0) + (1-p_2) V_1(1,0,2)) + (1-x_2)(p_1 V_1(2,0,1) + (1-p_1) V_1(0,2,1))) \\ & + \frac{1}{3}(x_3(p_3 V_1(0,1,2) + (1-p_3) V_1(2,1,0)) + (1-x_3)(p_2 V_1(1,2,0) + (1-p_2) V_1(1,0,2))). \end{aligned}$$

For example, $\frac{1}{3}x_1 p_1 V_1(2,0,1)$ is the expected payoff to P_1 with:

P_1 being selected and P_1 selecting P_2 and P_1 beating P_2.

Substituting the known right hand values, after a considerable amount of algebra we get

$$\begin{aligned} V_1(1,1,1) = & \frac{(1-p_3)(x_1+1-x_2)p_1}{3(p_3^2-p_3+1)} + \frac{p_1(x_3+1-x_1)(1-p_3)}{3(p_1^2-p_1+1)} \\ & + \frac{(1-p_3)^2(x_2+1-x_3)(1-p_2)}{3(p_3^2-p_3+1)} + \frac{p_1^2(x_2+1-x_3)p_2}{3(p_1^2-p_1+1)}. \end{aligned}$$

This can also be written as

$$V_1(1,1,1) = \frac{p_1(1-p_3)(p_1+p_3-1)(p_1-p_3)x_1 + \Pi_1(x_2,x_3)}{3(1-p_3+p_3^2)(1-p_1+p_1^2)} \tag{1}$$

where $\Pi_1(x_2,x_3)$ is a first degree polynomial in x_2 and x_3. Hence we have completed the computation of $\mathbf{V}_1 = (V_1(\mathbf{s}))_{\mathbf{s}\in S}$. For $n \in \{2,3\}$, a similar analysis yields P_n's payoff $\mathbf{V}_n$, as a function of the strategy profile $\mathbf{x} = (x_1, x_2, x_3)$.

3.2 Computation of a Stationary Nash Equilibrium

Before proceeding, we remind the reader of the following important fact [17, 28]: when checking whether a stationary strategy profile $\widehat{\sigma} = (\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3)$ is a NE, it suffices to consider for each player *only stationary strategy deviations*.

From (1) we can easily compute an *optimal* strategy for P_1. Since he wants to maximize his payoff by choice of $x_1 = x_1(1,1,1)$, he will use the following strategy:

$$\widehat{x}_1 = \begin{cases} 1 \text{ iff } p_1(1-p_3)(p_1+p_3-1)(p_1-p_3) \geq 0, \\ 0 \text{ else.} \end{cases}$$

In particular, when $p_1(1-p_3)(p_1+p_3-1)(p_1-p_3) > 0$ the optimal strategy will be given by

$$\widehat{x}_1 = \begin{cases} 1 \text{ iff: } (p_1+p_3-1>0 \text{ and } p_1-p_3>0) \text{ or } (p_1+p_3-1<0 \text{ and } p_1-p_3<0), \\ 0 \text{ iff: } (p_1+p_3-1>0 \text{ and } p_1-p_3<0) \text{ or } (p_1+p_3-1<0 \text{ and } p_1-p_3>0). \end{cases} \tag{2}$$

Reverting to the p_{mn} notation, we see that the condition (2) is actually symmetric:

$$\widehat{x}_1 = \begin{cases} 1 \text{ iff: } (p_{12}-p_{13}>0 \text{ and } p_{12}+p_{13}-1>0) \text{ or } (p_{12}-p_{13}<0 \text{ and } p_{12}+p_{13}-1<0), \\ 0 \text{ iff: } (p_{12}-p_{13}>0 \text{ and } p_{12}+p_{13}-1<0) \text{ or } (p_{12}-p_{13}<0 \text{ and } p_{12}+p_{13}-1>0). \end{cases}$$

Optimal strategies $\widehat{x}_2, \widehat{x}_3$ for P_2, P_3 can be expressed by similar conditions. Assuming that (for $n \in \{1,2,3\}$) P_n uses $\widehat{x}_n$, it is is easy to see that (for $n \in \{1,2,3\}$) P_n has no incentive to deviate from $\widehat{x}_n$, either to some other stationary strategy x_n or (by the previous remark) to some nonstationary strategy σ_n. Hence the triple $(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$ is a stationary deterministic NE of $\Gamma_3(\mathbf{p}, 3)$.

For certain combinations of the values p_1, p_2, p_3, additional NE may exist. For example, when $p_2 = p_3$, we get

$$V_1(1,1,1) = \frac{\Pi_1(x_2, x_3)}{3\left(1-p_3+p_3^2\right)\left(1-p_1+p_1^2\right)}$$

and $(x_1, \widehat{x}_2, \widehat{x}_3)$ is a stationary NE for any $x_1 \in [0,1]$; when $x_1 \in (0,1)$, this NE is stationary but *nondeterministic*. Additional examples can be easily constructed.

The existence of *nonstationary* NE is an open question; we conjecture that such equilibria can be obtained by the use of *trigger* (also known as *grim*) strategies [11].

4 The General Case

Let us now study the general case, i.e., when $K \in \{3,4,...\}$. We first derive the payoff equations and, using the theory of Markov chains, we establish that these equations always have a solution, which can be obtained by an iterative process. Then we establish the existence of a stationary deterministic NE.

4.1 Payoff Equations

For a given stationary strategy profile $\mathbf{x}$, for any $\mathbf{s} = (s_1, s_2, s_3) \in S$ and any $n \in \{1,2,3\}$, let $V_n(s_1, s_2, s_3) = Q_n(s_1, s_2, s_3, \mathbf{x})$ and $\mathbf{V}_n = (V_n(\mathbf{s}))_{\mathbf{s}\in S}$. We will now write the *payoff system*, i.e., the system of equations satisfied by $\mathbf{V}_n = (V_n(\mathbf{s}))_{\mathbf{s}\in S}$.

To this end we start by recognizing that, for a given stationary strategy profile $\mathbf{x}$, the process $(\mathbf{s}(t))_{t=0}^{\infty}$ is a Markov chain. Let us define for each $\mathbf{s} = (s_1, s_2, s_3) \in S$ the *neighborhood* $\mathbf{N}(s_1, s_2, s_3)$ of $\mathbf{s}$ to be the set of states reachable in one time step from $\mathbf{s}$. For example, when $\mathbf{s} \in S_i$, we have

$$\mathbf{N}(s_1, s_2, s_3) = \{(s_1+1, s_2-1, s_3), (s_1-1, s_2+1, s_3), ..., (s_1, s_2-1, s_3+1)\}.$$

For every $\mathbf{s}, \mathbf{s}' \in S$ we define the transition probabilities

$$\pi_{\mathbf{s}',\mathbf{s}} = \Pr\left(\mathbf{s}(t+1) = \mathbf{s} | \mathbf{s}(t) = \mathbf{s}'\right).$$

Then $\mathbf{V}_1$ satisfies the following equations.

1. For boundary states we have:

$$V_1(K,0,0) = 1; \tag{3}$$

for all $s_2 \in (0, 1, ..., K)$:

$$V_1(0, s_2, K - s_2) = 0; \tag{4}$$

and for all $s_1 \in (1, ..., K-1)$:

$$V_1(s_1, K - s_1, 0) = \sum_{\mathbf{s}' \in \mathbf{N}(s_1, K-s_1, 0)} \pi_{(s_1, K-s_1, 0), \mathbf{s}'} V_1\left(\mathbf{s}'\right),$$
$$V_1(s_1, 0, K - s_1) = \sum_{\mathbf{s}' \in \mathbf{N}(s_1, 0, K-s_1)} \pi_{(s_1, 0, K-s_1), \mathbf{s}'} V_1\left(\mathbf{s}'\right),$$

which becomes

$$V_1(s_1, K - s_1, 0) = p_1 V_1(s_1 + 1, K - s_1 - 1, 0) + (1 - p_1) V_1(s_1 - 1, K - s_1 + 1, 0) \tag{5}$$
$$V_1(s_1, 0, K - s_1) = (1 - p_3) V_1(s_1 + 1, 0, K - s_1 - 1) + p_1 V_1(s_1 - 1, 0, K - s_1 + 1). \tag{6}$$

2. For all interior states $(s_1, s_2, s_3) \in S_i$ we have

$$V_1(s_1, s_2, s_3) = \sum_{\mathbf{s}' \in \mathbf{N}(s_1, s_2, s_3)} \pi_{(s_1, s_2, s_3), \mathbf{s}'} V_1\left(\mathbf{s}'\right)$$

which can be written as either

$$\begin{aligned} V_1(s_1, s_2, s_3) = &\left(\frac{x_1(s_1, s_2, s_3) + (1 - x_2(s_1, s_2, s_3))}{3}\right) p_1 V_1(s_1 + 1, s_2 - 1, s_3) \\ &+ \left(\frac{x_1(s_1, s_2, s_3) + (1 - x_2(s_1, s_2, s_3))}{3}\right) (1 - p_1) V_1(s_1 - 1, s_2 + 1, s_3) \\ &+ \left(\frac{x_2(s_1, s_2, s_3) + (1 - x_3(s_1, s_2, s_3))}{3}\right) p_2 V_1(s_1, s_2 + 1, s_3 - 1) \\ &+ \left(\frac{x_2(s_1, s_2, s_3) + (1 - x_3(s_1, s_2, s_3))}{3}\right) (1 - p_2) V_1(s_1, s_2 - 1, s_3 + 1) \\ &+ \left(\frac{x_3(s_1, s_2, s_3) + (1 - x_1(s_1, s_2, s_3))}{3}\right) p_3 V_1(s_1 - 1, s_2, s_3 + 1) \\ &+ \left(\frac{x_3(s_1, s_2, s_3) + (1 - x_1(s_1, s_2, s_3))}{3}\right) (1 - p_3) V_1(s_1 + 1, s_2, s_3 - 1), \end{aligned} \tag{7}$$

or as

$$\begin{aligned} V_1(s_1, s_2, s_3) = &\frac{x_1(s_1, s_2, s_3)}{3} \left(p_1 V_1(s_1 + 1, s_2 - 1, s_3) + (1 - p_1) V_1(s_1 - 1, s_2 + 1, s_3)\right) \\ &+ \frac{(1 - x_1(s_1, s_2, s_3))}{3} \left((1 - p_3) V_1(s_1 + 1, s_2, s_3 - 1) + p_3 V_1(s_1 - 1, s_2, s_3 + 1)\right) \\ &+ \frac{x_2(s_1, s_2, s_3)}{3} \left(p_2 V_1(s_1, s_2 + 1, s_3 - 1) + (1 - p_2) V_1(s_1, s_2 - 1, s_3 + 1)\right) \\ &+ \frac{(1 - x_2(s_1, s_2, s_3))}{3} \left((1 - p_1) V_1(s_1 - 1, s_2 + 1, s_3) + p_1 V_1(s_1 + 1, s_2 - 1, s_3)\right) \\ &+ \frac{x_3(s_1, s_2, s_3)}{3} \left(p_3 V_1(s_1 - 1, s_2, s_3 + 1) + (1 - p_3) V_1(s_1 + 1, s_2, s_3 - 1)\right) \end{aligned}$$

$$+\frac{(1-x_3(s_1,s_2,s_3))}{3}\left((1-p_2)V_1(s_1,s_2-1,s_3+1)+p_2V_1(s_1,s_2+1,s_3-1)\right). \tag{8}$$

For every $(s_1, s_2, s_3) \in S_i$, the values of the $\pi_{(s_1,s_2,s_3),\mathbf{s}'}$ probabilities can be read from (7); for $(s_1, s_2, s_3) \in S_1$, the values of the $\pi_{(s_1,s_2,s_3),\mathbf{s}'}$ probabilities can be read from (5)-(6).

The equations (3)-(8) are the *payoff system* for $\mathbf{V}_1$. We have similar payoff systems for $\mathbf{V}_2$ and $\mathbf{V}_3$. In the rest of this section we will focus on $\mathbf{V}_1$. Let $\Pi(\mathbf{x})$ be the transition probability matrix with $\Pi_{\mathbf{s},\mathbf{s}'}(\mathbf{x}) = \pi_{\mathbf{s},\mathbf{s}'}(\mathbf{x})$ (for all $\mathbf{s}, \mathbf{s}' \in S$). Then the *payoff system* can be written as

$$V_1(K,0,0) = 1 \text{ and } V_1(0,K,0) = V_1(K,0,0) = 0 \text{ and } \mathbf{V}_1 = \Pi(\mathbf{x})\mathbf{V}_1. \tag{9}$$

This is a linear system and can be solved by the standard methods. An alternative but equivalent form of (9) is the following. Recall that we have numbered the states $\mathbf{s} \in S$ so that the first state is $(K, 0, 0)$, the second is $(0, K, 0)$ and the third is $(0, 0, K)$. Now define the matrix $\widetilde{\Pi}_1(\mathbf{x})$ and the vector $\mathbf{b}_1$ as follows:

$$\widetilde{\Pi}_1(\mathbf{x}) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \pi_{4,1} & \pi_{4,2} & \dots & \pi_{4,N_K} \\ \dots & \dots & \dots & \dots \\ \pi_{N_K,1} & \pi_{N_K,2} & \dots & \pi_{N_K,N_K} \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

Then (9) is equivalent to

$$\mathbf{V}_1 = \widetilde{\Pi}_1(\mathbf{x})\mathbf{V}_1 + \mathbf{b}_1. \tag{10}$$

If the payoff system (10) has exactly one solution $\overline{\mathbf{V}}_1$, then $\overline{V}_1(s)$ is P_1's payoff when the game starts with capitals $s = (s_1, s_2, s_3)$. The situation is similar for $\mathbf{V}_2$ and $\mathbf{V}_3$.

4.2 Existence of a Solution to the Payoff Equations

We wil now establish the existence of solutions to the equations (9) as well as (10) and we will provide a method for the computation of these solutions.

As already mentioned, for every stationary strategy profile $\mathbf{x}$, the process $(\mathbf{s}(t))_{t=0}^{\infty}$ is a Markov chain, with transition probability matrix $\Pi(\mathbf{x})$.

Proposition 4.1 *If for all $m, n \in \{1, 2, 3\}$ we have $p_{mn} \in (0, 1)$ then, for every $\mathbf{x}$, every interior state communicates with all terminal states.*

Proof Take any state $\mathbf{s} = (s_1, s_2, s_3) \in S_i$ and suppose that the game is in $\mathbf{s}$; let $\delta = \min_{m,n} p_{mn}$. Then we have

$$\begin{aligned} \Pr(\text{“}P_1 \text{ is selected and he loses the turn”}) &\geq \frac{1}{3}\left(x_{12}(\mathbf{s})p_{21} + x_{13}(\mathbf{s})p_{31}\right) \\ &\geq \frac{1}{3}\left(x_1(\mathbf{s})p_{21} + (1-x_1(\mathbf{s}))p_{31}\right) \geq \frac{\delta}{3}. \end{aligned}$$

In other words, there is a positive probability that the game moves to a state $\mathbf{s}' = \left(s_1 - 1, s_2', s_3'\right)$. Hence there is a positive probability that eventually the game will reach a boundary state $\mathbf{s}'' = \left(0, s_2', s_3'\right)$; either $\mathbf{s}''$ is a terminal state itself, or $\mathbf{s}'' \in S \backslash (S_i \cup S_\tau)$ and communicates with both $(0, K, 0)$ and $(0, 0, K)$. Hence $\mathbf{s}$ communicates with both $(0, K, 0)$ and $(0, 0, K)$. Repeating the argument for P_2 we see that $\mathbf{s}$ also communicates with $(K, 0, 0)$. $\square$

By Proposition 4.1 and standard Markov chain results [12, 18], every nonterminal state is transient and the game will reach *some* terminal state w.p. 1 and in finite expected time. Furthermore we have the following.

Proposition 4.2 *If for all* $m, n \in \{1, 2, 3\}$ *we have* $p_{mn} \in (0, 1)$*, then, for every* $\mathbf{x}$*, the limit* $\overline{\Pi}(\mathbf{x}) = \lim_{t\to\infty} \Pi^t(\mathbf{x})$ *exists.*

Proof Recall that the three terminal states are numbered as first, second and third. Since they are absorbing and all other states are transient, the probability transition matrix has the form

$$\Pi(\mathbf{x}) = \begin{bmatrix} I & 0 \\ W & U \end{bmatrix}. \tag{11}$$

Let $S_{abs} = \{1, 2, 3\}$ be the set of absorbing states and $S_{tr} = S \backslash S_{abs}$ the set of transient states. The 3×3 unit matrix I in the upper left block reflects the fact that each terminal state is absorbing and transits to itself with probability one. The matrix W (resp. U) contains the probabilities of transient states transiting to transient (resp. absorbing) states. That is:

$$\forall \mathbf{s}, \mathbf{s}' \in S_{tr} : W_{\mathbf{s},\mathbf{s}'} = \Pr\left(\text{“}\mathbf{s} \text{ transits to } \mathbf{s}'\text{”}\right),$$
$$\forall \mathbf{s}' \in S_{tr}, \mathbf{s}' \in S_{abs} : U_{\mathbf{s},\mathbf{s}'} = \Pr\left(\text{“}\mathbf{s} \text{ transits to } \mathbf{s}'\text{”}\right).$$

Then we have

$$\Pi^t(\mathbf{x}) = \begin{bmatrix} I & 0 \\ \left(\sum_{i=0}^{t} U^i\right) W & U^t \end{bmatrix}$$

By standard Markov chain results [12, 18] $\lim_{t\to\infty} U^t = 0$ and $\lim_{t\to\infty} \sum_{i=0}^{t} U^i$ exists and equals $(I - U)^{-1}$. Hence

$$\lim_{t\to\infty} \Pi^t(\mathbf{x}) = \begin{bmatrix} I & 0 \\ \lim_{t\to\infty} \left(\sum_{i=0}^{t} U^i\right) W & U^t \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I - U)^{-1} W & 0 \end{bmatrix}.$$

This completes the proof. □

Proposition 4.3 *If the limit* $\overline{\Pi}(\mathbf{x}) = \lim_{t\to\infty} \Pi^t(\mathbf{x})$ *exists then*

$$\forall (s_1, s_2, s_3) \in S : \overline{\Pi}_{(s_1,s_2,s_3),(K,0,0)}(\mathbf{x}) = V_1(s_1, s_2, s_3).$$

Proof Recall that

$$\forall (s_1, s_2, s_3) \in S : V_1(s_1, s_2, s_3) = \Pr(\text{“}P_1 \text{ wins when the game has started in } (s_1, s_2, s_3)\text{”})$$

We have the following possibilities.

1. If $(s_1, s_2, s_3) = (K, 0, 0)$, then $\overline{\Pi}_{(K,0,0),(K,0,0)}(\mathbf{x}) = 1 = V_1(K, 0, 0)$.
2. If $(s_1, s_2, s_3) = (0, K, 0)$, then $\overline{\Pi}_{(0,K,0),(K,0,0)}(\mathbf{x}) = 0 = V_1(0, K, 0)$.
3. If $(s_1, s_2, s_3) = (0, 0, K)$, then $\overline{\Pi}_{(0,0,K),(K,0,0)}(\mathbf{x}) = 0 = V_1(0, 0, K)$.
4. If $(s_1, s_2, s_3) \in S \backslash S_\tau$, then $\overline{\Pi}_{(s_1,s_2,s_3),(K,0,0)}(\mathbf{x}) = \left(\left(\sum_{i=0}^{\infty} U^i\right) W\right)_{(s_1,s_2,s_3),(K,0,0)}$. This is the probability that: $(s_1(t), s_2(t), s_3(t))$ stays in $S \backslash S_\tau$ for i steps (for some $i \in (0, 1, ...)$) and then moves into $(K, 0, 0)$. In every such case P_1 wins.

This completes the proof. □

Hence we can compute $\mathbf{V}_1$ as the limit of $\Pi^t(\mathbf{x})$. From standard MC analysis [12, 18] we can get estimates of the Π^t rate of convergence, mean absorption time etc. Another way to compute $\mathbf{V}_1$ is provided by the following.

Proposition 4.4 *Suppose that* $\mathbf{x}$ *is such that* $\overline{\Pi}(\mathbf{x}) = \lim_{t\to\infty} \Pi^t(\mathbf{x})$ *exists. Choose an arbitrary* $N_K \times 1$ *vector* $\mathbf{U}_1^{(0)}$ *and*

$$\text{For } t \in \{0, 1, 2...\} \text{ let: } \mathbf{U}_1^{(t+1)} = \widetilde{\Pi}_1(\mathbf{x})\,\mathbf{U}_1^{(t)} + \mathbf{b}_1. \tag{12}$$

The iteration (12) converges and $\overline{\mathbf{U}} = \lim_{t\to\infty} \mathbf{U}^{(t)}$ *is a solution of the payoff system (10).*

Proof The key fact here is that, for any solution $\mathbf{V}_1$ of the payoff system:

1. when $\mathbf{s} \in S_\tau$, $V_1(s)$ is fixed to be either 0 or 1;
2. when $\mathbf{s} \in S_\tau^c$, $V_1(\mathbf{s})$ is the weighted average of its neighbors:

$$V_1(\mathbf{s}) = \sum_{\mathbf{s}' \in N(\mathbf{s})} \pi_{\mathbf{s},\mathbf{s}'} V_1(\mathbf{s}').$$

Consequently, every solution of the payoff system is a harmonic function. Hence the payoff system has a unique solution $\mathbf{V}_1$ which can be obtained as the limit of (12). □

4.3 Existence of Stationary Equilibria

We want to show that $\Gamma_3(\mathbf{p}, K)$ posseses a Nash equilibrium (NE), i.e., a strategy profile $\widehat{\sigma}$ such that:

$$\forall n, \forall \sigma_n : Q_n(\mathbf{s}(0), \widehat{\sigma}_n, \widehat{\sigma}_{-n}) \geq Q_n(\mathbf{s}(0), \sigma_n, \widehat{\sigma}_{-n}) \tag{13}$$

To this end we introduce an auxiliary *discounted* game, denoted by $\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$, where $\gamma \in (0, 1)$. $\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$ is the same as $\Gamma_3(\mathbf{p}, K)$ except for the following.

1. In $\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$ the states $(K, 0, 0)$, $(0, K, 0)$, $(0, 0, K)$ are *preterminal* and we introduce a new terminal state $\overline{s}$; state transition probabilities remain the same except that every preterminal state $s \in S_\tau$ transits to $\overline{s}$ with probability one.
2. We define *turn payoff* functions $q_n(s)$:

$$\forall n : q_n(s) = \begin{cases} 1 \text{ iff } s \in S_n, \\ 0 \text{ else.} \end{cases}$$

3. The total payoff function $\widetilde{\mathsf{Q}}_n$ and the expected total payoff function $\widetilde{Q}_n$ are defined as follows:

$$\widetilde{\mathsf{Q}}_n(\mathbf{h}) = \sum_{t=0}^{\infty} \gamma^t q_n(s), \qquad \widetilde{Q}_n(s(0), \sigma) = \mathbb{E}_{\mathbf{s}(0),\sigma}\left(\widetilde{\mathsf{Q}}_n(\mathbf{h})\right)$$

$\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$ is a *discounted* stochastic game and, according to the following well known theorem, possesses a stationary NE.

Proposition 4.5 (Fink [10]) *Every* N*-player discounted stochastic game has a stationary Nash equilibrium.*

In the following proposition we strengthen Fink's theorem for the case of $\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$.

Proposition 4.6 *For every* $\mathbf{p}$, K *and* γ, $\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$ *has a deterministic stationary Nash equilibrium. In other words, there exists a* $\widehat{\mathbf{x}} = (\widehat{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \widehat{\mathbf{x}}_3)$ *such that*

$$\forall n \in \{1, 2, 3\}, \forall \mathbf{s}_0 \in S, \forall \mathbf{x}^n : \widetilde{Q}^n(\mathbf{s}_0, \widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) \geq \widetilde{Q}^n(\mathbf{s}_0, \mathbf{x}_n, \widehat{\mathbf{x}}_{-n}). \tag{14}$$

Proof According to [10], for a general N-player discounted stochastic game, the following equations must be satisfied for all n and $\mathbf{s}$ at a Nash equilibrium[3]:

$$\mathcal{V}_n(\mathbf{s}) = \max_{\sigma_n(\mathbf{s})} \sum_{a_1} \dots \sum_{a_N} \sigma_{1,a_1}(\mathbf{s}) \dots \sigma_{N,a_N}(\mathbf{s}) \left[q_n(\mathbf{s}) + \gamma \sum_{\mathbf{s}'} \mathcal{P}\left(\mathbf{s}'|\mathbf{s}, a_1, \dots, a_N\right) \mathcal{V}_n\left(\mathbf{s}'\right) \right], \tag{15}$$

where:

1. σ_n is P_n's strategy, $\sigma_n(\mathbf{s})$ is his action probability vector at state $\mathbf{s}$ and $\sigma_{n,a_n}(\mathbf{s})$ is the probability that, given the current game state is $\mathbf{s}$, P_n plays action a_n.
2. $\mathcal{V}_n(\mathbf{s})$ is the expected payoff to P_n obtained at equilibrium, when the game has started at state $\mathbf{s}$ and the strategy and the strategy profile $\sigma = (\sigma_1, \dots, \sigma_N)$ is used.
3. $q_n(\mathbf{s})$ is P_n's turn payoff at state $\mathbf{s}$.
4. $\mathcal{P}\left(\mathbf{s}'|\mathbf{s}, a_1, \dots, a_N\right)$ is the probability that, given that the current state is $\mathbf{s}$ and the player actions are $a_1, \dots, a_N$, the next state is $\mathbf{s}'$.

Let us now consider $\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$. Since no player has a choice of strategy on boundary states, we only need to consider interior states $\mathbf{s} = (s_1, s_2, s_3) \in S_i$. Letting $n = 1$ and taking into account (8), (15) becomes

$$V_1(s_1, s_2, s_3) = \max_{x_1(s_1,s_2,s_3)} \gamma G(x_1(s_1, s_2, s_3))$$

where

$$\begin{aligned} G(x_1(s_1, s_2, s_3)) = x_1(s_1, s_2, s_3) &\frac{p_1 V_1(s_1+1, s_2-1, s_3) + (1-p_1) V_1(s_1-1, s_2+1, s_3)}{3} \\ + (1 - x_1(s_1, s_2, s_3)) &\frac{(1-p_3) V_1(s_1+1, s_2, s_3-1) + p_3 V_1(s_1-1, s_2, s_3+1)}{3} \\ &+ \text{terms which do not involve } x_1(s_1 s_2 s_3) \end{aligned}$$

Clearly, the maximum is achieved at either $x_1(s_1, s_2, s_3) = 1$ or $x_1(s_1, s_2, s_3) = 0$. This holds for every interior state, hence P_1's strategy at equilibrium is deterministic (or can be substituted by an equivalent deterministic one).
The proof is completed by applyinjg the same analysis to P_2 and P_3. □

Corollary 4.7 *For every* $\mathbf{p}$ *and* K, $\Gamma_3(\mathbf{p}, K)$ *has a deterministic stationary NE.*

Proof Let $\widehat{\mathbf{x}} = (\widehat{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \widehat{\mathbf{x}}_3)$ be a a deterministic stationary NE of $\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$. Suppose that, for some $\mathbf{s}_0$ and t, we have

$$\widetilde{Q}(\mathbf{s}_0, \widehat{\mathbf{x}}) = \left(\widetilde{Q}_1(\mathbf{s}_0, \widehat{\mathbf{x}}), \widetilde{Q}_2(\mathbf{s}_0, \widehat{\mathbf{x}}), \widetilde{Q}_3(\mathbf{s}_0, \widehat{\mathbf{x}})\right) = \left(\gamma^t, 0, 0\right).$$

This means that, in $\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$, P_1 has won at time t. Applying the same $\widehat{\mathbf{x}}$ in $\Gamma_3(\mathbf{p}, K, \gamma)$, we have

$$Q(\mathbf{s}_0, \widehat{\mathbf{x}}) = (Q_1(\mathbf{s}_0, \widehat{\mathbf{x}}), Q_2(\mathbf{s}_0, \widehat{\mathbf{x}}), Q_3(\mathbf{s}_0, \widehat{\mathbf{x}})) = (1, 0, 0).$$

Clearly

$$\forall \mathbf{x}_1 : Q_1(\mathbf{s}_0, \widehat{\mathbf{x}}_1, \widehat{\mathbf{x}}_{-1}) \geq Q_1(\mathbf{s}_0, \mathbf{x}_1, \widehat{\mathbf{x}}_{-1}).$$

Also, if there was some $\mathbf{y}_n$ (with $n \in \{2, 3\}$) such that

$$Q_1(\mathbf{s}_0, \widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) < Q_1(\mathbf{s}_0, \mathbf{y}_n, \widehat{\mathbf{x}}_{-n})$$

[3] We have modified Fink's notation, so as to conform to our own.

Table 1 Values of $g(K)$ and $h(K)$ as functions of K

K	3	4	5	6	7
$g(K)$	1	3	6	10	15
$h(K)$	$2^3 = 8$	$2^9 = 512$	$2^{18} = 262\,144$	$2^{30} = 1073\,741\,824$	$2^{45} = 35\,184\,372\,088\,832$

then P_n could win $\Gamma_3(\mathbf{p}, K)$ (when starting at $\mathbf{s}_0$) by using $\mathbf{y}_n$ against $\widehat{\mathbf{x}}_{-n}$. But then he could also win $\widetilde{\Gamma}_3(\mathbf{p}, K, \gamma)$ (when starting at $\mathbf{s}_0$) by using the same $\mathbf{y}_n$ against $\widehat{\mathbf{x}}_{-n}$, so we would have

$$\widetilde{Q}_1(\mathbf{s}_0, \widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) < \widetilde{Q}_1(\mathbf{s}_0, \mathbf{y}_n, \widehat{\mathbf{x}}_{-n}).$$

But this is contrary to the hypothesis. The proof is similar for the case in which, for some $\mathbf{s}_0$ and t,

$$\widetilde{Q}(\mathbf{s}_0, \widehat{\mathbf{x}}) = \left(\widetilde{Q}_1(\mathbf{s}_0, \widehat{\mathbf{x}}), \widetilde{Q}_2(\mathbf{s}_0, \widehat{\mathbf{x}}), \widetilde{Q}_3(\mathbf{s}_0, \widehat{\mathbf{x}})\right) = (0, 0, 0).$$

We conclude that:

$$\forall n \in \{1, 2, 3\}, \forall \mathbf{s}_0 \in S, \forall \mathbf{x}^n : Q_n(\mathbf{s}_0, \widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) \geq Q_n(\mathbf{s}_0, \mathbf{x}_n, \widehat{\mathbf{x}}_{-n})$$

and the proof is complete. □

4.4 Computation of Stationary Nash Equilibria

We want to find a stationary strategy profile $\widehat{\mathbf{x}}$ such that:

$$\forall n, \mathbf{s}, \mathbf{x}_n : Q_n(\mathbf{s}, \widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) \geq Q_n(\mathbf{s}, \mathbf{x}_n, \widehat{\mathbf{x}}_{-n}) \tag{16}$$

or, equivalently,

$$\forall n, \mathbf{s}, \mathbf{x}_n : V_{n,\mathbf{s}}(\widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) \geq V_{n,\mathbf{s}}(\mathbf{x}_n, \widehat{\mathbf{x}}_{-n}). \tag{17}$$

Now, (17) can be rewritten as

$$\forall n, \mathbf{s} : V_{n,s}(\widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) = \max_{\mathbf{x}_n} V_{n,s}(\mathbf{x}_n, \widehat{\mathbf{x}}_{-n}) = \max_{\mathbf{x}_n}\left(\left[\widetilde{\Pi}(\mathbf{x}_n, \widehat{\mathbf{x}}_{-n})\mathbf{V}_n + \mathbf{u}_n\right]_s\right). \tag{18}$$

We define

$$\forall n, \mathbf{s} : F_{n,\mathbf{s}}(\mathbf{V}_1|\mathbf{x}) = \max_{\mathbf{x}_n}\left(\left[\widetilde{\Pi}(\mathbf{x})\mathbf{V}_n + \mathbf{u}_n\right]_s\right) \tag{19}$$

Then we can rewrite (18) as

$$\forall n : \mathbf{V}_n(\widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) = \mathbf{F}_n(\mathbf{V}_n|\widehat{\mathbf{x}}), \tag{20}$$

This system of nonlinear equations will, by the previous analysis, have at least one solution.

One possibility is to solve (20) by exhaustive enumeration. For a given K we have $g(K) = \frac{(K+1)(K+2)}{2} - 3K$ states in which there exist strategy choices and, if we limit ourselves to deterministic strategies, for each such state we have $2^3 = 8$ strategy outcome combinations. Hence there exist $h(K) = 2^{3\,g(K)}$ possible overall deterministic strategy combinations. In the following table we list the values of $g(K)$ and $h(K)$ as functions of K (Table 1).
It appears that finding a NE by exhaustive enumeration is feasible for $K < 6$; for larger K values the computational burden is probably unmanageable.

An alternative approach is to use an iterative approach, inspired from *Value Iteration* which provably yields a solution for *two-player zero-sum* games [9]. Based on this, we can use the *MultiValue Iteration* (MVI) defined as follows.

$$\forall n, s : V_n^{(t+1)}(s) = F_{n,s}\left(\mathbf{V}_n^{(t)}\left(s|\mathbf{x}^{(t)}\right)\right). \tag{21}$$

If (21) converges (which is not a priori guaranteed) it can be proved that the limit will be a solution of (20).

Finally, we can use some optimization algorithm to find the *global* minimum (which must be equal to zero) of the function

$$J(\widehat{\mathbf{x}}) = \sum_{n=1}^{3} \|\mathbf{V}_n(\widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) - \mathbf{F}_n(\mathbf{V}_n|\widehat{\mathbf{x}})\|^2. \tag{22}$$

We have compared the efficacy of the MVI and global optimization approaches by computational experiments, which will be reported in the next section.

5 Computational Experiments

As already mentioned, for large K values exhaustive enumeration is not viable computationally. Hence in this section we compare the MVI and global optimization approaches by computational experiments.

5.1 Convergence Issues

For the global minimization of $J(\widehat{\mathbf{x}})$ given by (22) we have tried several optimization algorithms provided by the Matlab software system. Most notably, we have used the following:

1. The Matlab function `fmincon` [19], which performs constrained gradient based optimization [22].
2. The Matlab function `particleswarm` [20], which implements *particle swarm optimization* [34].

Convergene of either MVI or the global optimization algorithms is not guaranteed. However, from preliminary experiments we have found out that the best performance is obtained by the MVI algorithm; hence we only report results obtained by this particular algorithm. We have run 100 repetitions of the following experiment.

After randomly selecting a probabilities vector $\mathbf{p} \in (0, 1)^3$ we run MVI for 100 runs for each $K \in \{3, 4, ..., 9\}$, always initializing with $\mathbf{x}^{(0)} = \mathbf{0}$. If a solution $\widehat{\mathbf{x}}$ of the optimality equations (i.e., an equilibrium strategy profile) is achieved we count this as one succesful run of the MVI algorithm. After the 100 runs are completed, we compute, for each K, the proportion of succesful runs. The results are plotted in Fig. 1.
It can be seen that MVI is "relatively" succesful. That is, the proportion of convergent runs is a decreasing function of K (i.e., it is harder to find a solution for larger K values) and we have a "reasonable" probability of success for $K \leq 7$; furthermore, even for $K \in \{8, 9\}$ we have a 50% or better probability of obtaining a solution. It must be emphasized that this is a lower bound on the performance of MVI. Running MVI repeatedly for a specific $\mathbf{p}$, with random initialization of $\mathbf{x}^{(0)}$, yields a higher sucess proportion, as seen in Fig. 2.

If we look at the MVI iteration (21) as a dynamical system, parametrized by $\mathbf{p}$ and K, a natural question is this: given some K, for which subset $\mathfrak{P}_K \subset [0, 1]^3$ does $\mathbf{p} \in \mathfrak{P}_K$ ensure

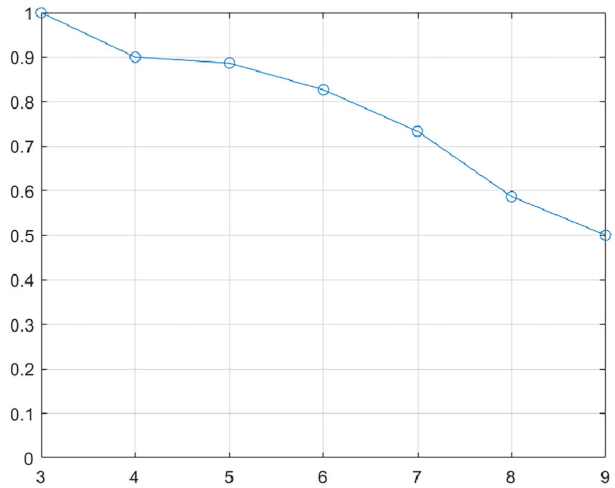

Fig. 1 Proportion of succesful MVI runs for 100 randomly selected $\mathbf{p}$ vectors, as a function of K. Initialization at $\mathbf{x}^{(0)} = \mathbf{0}$

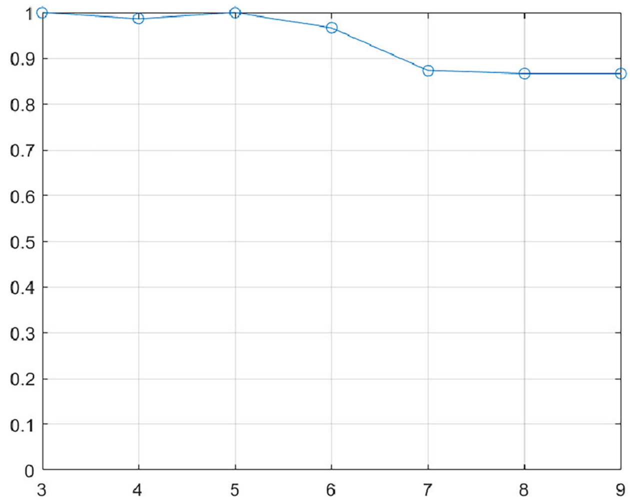

Fig. 2 Proportion of succesful MVI runs for 100 randomly selected $\mathbf{p}$ vectors, as a function of K. Random initialization of $\mathbf{x}^{(0)}$

that (21) converges? Roughly speaking, the results of Fig. 1 indicate that, for example, the volume of $\mathfrak{P}_3$ is 1.00, that of $\mathfrak{P}_4$ around 0.90 and so on. To further investigate this point, we repeat the above experiment for a restricted set of $\mathbf{p}$ values. For example, when we fix $p_1 = 1$ and choose p_2 and p_3 randomly, we obtain the results of Fig. 2. Obviously for $\mathbf{p} \in \{1\} \times [0, 1]^2$ the MVI algorithm has much higher probability of convergence.

5.2 The Advantage of Optimal Play

We conclude this section by an initial numerical exploration of the following question:

how much does a player benefit by playing optimally?

To answer this, we perform several experiments of the following type.

1. We fix a particular $\mathbf{p}$ vector and, for $K = 9$, use the MVI algorithm to obtain a NE $(\widehat{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \widehat{\mathbf{x}}_3)$ and the corresponding $V_1(\widehat{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \widehat{\mathbf{x}}_3)$.
2. We assign to P_1 a random strategy $\overline{\mathbf{x}}_1$ and compute his payoff $V_1(\overline{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \widehat{\mathbf{x}}_3)$.
3. We compute the difference $\overline{\delta V}_1 = V_1(\widehat{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \widehat{\mathbf{x}}_3) - V_1(\overline{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \widehat{\mathbf{x}}_3)$.
4. We repeat steps 2 and 3 by assigning the random strategy $\overline{\mathbf{x}}_n$ to P_n, for $n \in \{2, 3\}$.

For $n \in \{1, 2, 3\}$, $\overline{\delta V}_n$ is a measure of how much P_n benefits by using the equilibrium strategy $\widehat{\mathbf{x}}_n$ rather the random strategy $\overline{\mathbf{x}}_n$, given that the other two players use the equilibrium strategy. Because $(\widehat{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \widehat{\mathbf{x}}_3)$ is a NE, we know that $\overline{\delta V}_n$ will always be nonnegative; a large $\overline{\delta V}_n$ indicates that P_n has a large incentive for using $\widehat{\mathbf{x}}_n$.

We also repeat the above suite of experiments comparing the equilibrium strategy $\widehat{\mathbf{x}}_n$ to a uniform strategy $\widetilde{\mathbf{x}}_n$, by which, at every state $\mathbf{s}$, P_n selects equiprobably one of the surviving players; now we compute the quantities $\widetilde{\delta V}_n = V_1(\widehat{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n}) - V_1(\widetilde{\mathbf{x}}_n, \widehat{\mathbf{x}}_{-n})$.

We perform the above experiments for several $\mathbf{p}$ values and present our results in Table 2. Keeping in mind that the quantities $\overline{\delta V}_n$ and $\widetilde{\delta V}_n$ are differences between winning probabilities, the results of Table 2 indicate that in some cases at least one player has considerable incentive to use the equilibrium strategy; this is the case for the $\mathbf{p}$'s in rows one to four and eight to ten of the table. On the other hand, for the $\mathbf{p}$'s of rows five, six and seven no player loses much by deviating from equilibrium.

It is also worth noting that in rows seven to ten of Table 2 we have a $\mathbf{p}$ lying on the boundary of $[0, 1]^3$. While this case is not covered by our basic assumption (namely that $\mathbf{p} \in (0, 1)^3$) the MVI algorithm still converges to a NE.

6 Conclusion

We have presented a strategic game version of the Three Gamblers' Ruin, formulated it as a stochastic game and proved that it always has at least one stationary deterministic NE. We have also briefly investigated the computational aspects of the game. We believe there is scope for much additional research. First, we would like to improve the computational aspects of our analysis. Secondly, several variants of the presented game can be constructed and studied; we list a few such examples below (in every case we start with the original game and introduce one or more of the indicated modifications).

1. Having $N \geq 4$ (rather than 3) gamblers. As already mentioned, most of our results can be easily extended to this version.
2. Having N gamblers, each located on a vertex of a given *graph*, with each player engaging only his neighbors.
3. The payoff to each player is his *total discounted capital* $\sum_t \gamma^t s_n(t)$. In this case the following questions arise.

   (a) Is it possible that an eliminated player achieves higher payoff than the final survivor?
   (b) Even though a single player survives at the end of the game, each player has a motive to stay in the game as long as possible. Hence it may be profitable for the players

Table 2 Differences between winning probabilities in NE and deviations

p	$\overline{\delta V}_1$	$\overline{\delta V}_2$	$\overline{\delta V}_3$	$\widetilde{\delta V}_1$	$\widetilde{\delta V}_2$	$\widetilde{\delta V}_3$
(0.1000, 0.1000, 0.1000)	0.2357	0.2266	0.2072	0.1995	0.1995	0.1995
(0.8000, 0.8000, 0.5000)	0.1455	0.0339	0.0318	0.1177	0.0462	0.0308
(0.4144, 0.0879, 0.3494)	0.1456	0.0441	0.0826	0.1277	0.0423	0.1038
(0.3922, 0.8932, 0.6634)	0.0088	0.1421	0.0084	0.0091	0.1017	0.0069
(0.5300, 0.2000, 0.8000)	0.0025	0.0033	0.0948×10^{-12}	0.0032	0.0026	0.0963×10^{-12}
(0.9000, 0.8747, 0.2252)	0.0616	0.0032	0.0006	0.0519	0.0023	0.0009
(0.9000, 0.0000, 0.5000)	0.0656	0.0006	0.0851	0.0643	0.0005	0.0863
(1.0000, 0.5000, 0.5000)	0.1180	0.0003	0.6162	0.1083	0.0003	0.5729
(1.0000, 1.0000, 1.0000)	0.2897	0.2394	0.3082	0.2880	0.2880	0.2880
(0.0000, 0.0000, 0.0000)	0.1548	0.1291	0.1345	0.1568	0.1568	0.1568

to form “alliances”. These alliances can be enforced by the use of *nonstationary* strategies (for example grim trigger strategies) in which case it may be possible to establish the existence of nonstationary Nash equilibria.

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Declarations

Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

Ethical Approval This is not applicable to this article, because it does not include human and/ or animal studies.

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