The Duel Discounted Stochastic Game

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Abstract

A *duel* involves two *stationary* players who shoot at each other until at least one of them dies; a *truel* is similar but involves three players. In the past, the duel has been studied mainly as a component of the truel, which has received considerably more attention. However we believe that the duel is interesting in itself. In this paper we formulate the duel (with either simultaneous or sequential shooting) as a *discounted stochastic game*. We show that this game has a *unique* Nash equilibrium in *stationary strategies*; however, it also possesses *cooperation-promoting* Nash equilibria in *nonstationary strategies*. We show that these are also subgame perfect equilibria. Finally, we argue that the nature of the game and its equilibria is similar to that of the *repeated Prisoner's Dilemma*.

Keywords: Duel, stochastic game, Nash equilibrium, cooperation.

1 Introduction

1.1 The Game

In this paper we study a two-player *duel* game. The players P_1, P_2 are assumed to be standing in place and, in each turn, one or both *may* shoot at the "other" player. If P_n shoots at P_m ($m \neq n$), either he hits and kills him (with probability p_n, which we call P_n's *marksmanship*) or he misses him and P_m is unaffected (with probability $1 - p_n$). The game is continued until at least one player dies or, if nobody ever dies, ad infinitum. Each players' goal is to maximize his *total payoff*, which will be defined a little later.

In Sections 2 and 3 we will formulate the above as a *nonzero-sum discounted stochastic game* and present the game rules and payoff function in detail. At this point, the following are worth mentioning.

1. We will examine both the *simultaneous* duel (in which both players may shoot in the same turn) and the *sequential* duel (in which players may shoot in alternate turns).

2. In both the simultaneous and sequential variants, *a player can always choose to not shoot* ("abstain") in a particular turn.

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3. Most importantly, under our rules *both players can survive ad infinitum*; so the game is not zero-sum and does not necessarily have a unique "winner".

4. Additionally, each player receives a positive payoff for each turn in which he stays alive; hence each player has an incentive for abstaining from shooting, so as not to incur his opponent's retaliation. This is an important difference from previous models of the duel (and multiplayer analogs).

1.2 Related Work

The literature on duels is quite limited. In fact, the only studies of duels with which we are familiar, appear as a preliminary step in works (e.g. [4, 15, 16, 19, 29]) mainly concerned with the "*truel*", i.e., a game in which *three* stationary players shoot at each other.

Early works on the truel are [13, 18, 20, 21, 22]; the rules considered in these papers guarantee the existence of *exactly one* survivor ("winner") and can be used to define a *fixed-sum* game (either duel or truel). A more general but less rigorous analysis appears in [19] where the possibility of "cooperation" between the duelists is considered. The papers [15, 16, 17] and the less cited [29] present very detailed game theoretic treatments of both the truel and the duel. Recent additions to the literature include [2, 3, 4, 6, 7, 8, 9, 10, 25, 27, 28].

The major inspiration for our own work comes from [15, 16, 17, 19, 29]. While these papers focus on various forms of the truel, we believe that the duel is interesting in itself and has not received sufficient attention. Let us briefly discuss the above papers

Knuth is the first to introduce the idea of cooperation between duelists and, more generally, *nuelists* in [19]. Interestingly, he does not pay much attention to the duel; also, while we find his ideas very interesting (they have been a major inspiration for the current work) he does not formulate them in game theoretic terminology and his arguments are not always rigorous.

Of the three Kilgour papers, [15] deals with simultaneous truels in which abstention is not allowed and [17] studies an expanded class of truels, but presents no duel-related results. On the other hand, [16] deals with both cases (i.e., when abstention is allowed or disallowed) of sequential truels. When abstention is allowed and a player is killed in one turn, the truel devolves to a sequential duel with abstention which is analyzed by Kilgour. However, in his version payoffs are received only in terminal states; a consequence of this is that the payoff in all plays of Kilgour's sequential duel is bounded above by one. Kilgour proves that the sequential duel with abstention always has a stationary Nash equilibrium in which both players shoot in every turn; note that Kilgour only considers stationary strategies. Our approach is different in that we establish the existence of nonstationary Nash equilibria; a crucial factor in our proof is that, by letting the discount factor δ tend to one, the payoffs can become arbitrarily high.

In [29] Zeephongsekul formulates both duel and truel as *undiscounted* stochastic games in which the players shoot *simultaneously*. There is a terminal state s_0 and, under Zeephongsekul's formulation, for every state and *admissible* strategy, the probability of transiting to s_0 is bounded below by some $p^0 > 0$. Hence the game will terminate in finite time with probability one. It is known [24] that every stochastic game which satisfies the above condition (and has bounded stage payoffs) possesses a stationary NE. However, to ensure finite time termination, Zeephongsekul imposes the condition that x_1 and x_2 (the probabilities of each player shooting the other) satisfy $\min(x_1, x_2) > 0$. Because of this assumption, Zeephongsekul can establish the existence of a NE only when p_1 and p_2 are bounded below by a strictly positive number. It is still true (but not covered by the analysis of [29]) that $(\widehat{x}_1, \widehat{x}_2) = (0, 0)$ is *always* a Nash equilibrium in Zeephongsekul's formulation, because it yields *infinite* payoff to both duelists.

Finally, for the sake of completeness, let us mention that there is a very extensive literature on a quite different type of duel games, which can best be understood as *games of timing* [5, 11, 14]; this literature is not relevant to the game studied in this paper.

1.3 Our Results

In the current paper we model the duel (with both simultaneous and sequential shooting) as a discounted stochastic game, with discount factor δ. We then prove that the game has a *unique* Nash equilibrium in *stationary strategies*, but we also construct a "*cooperative*" Nash equilibrium in *nonstationary strategies*; these are also *subgame perfect equilibria* (SPE). We argue that these equilibria are similar to the ones occurring in the *repeated Prisoner's Dilemma* (PD).

The paper is structured as follows. In Section 2 we study the *simultaneous* duel: we prove that it possesses both stationary and nonstationary Nash equilibria and discuss its relationship to the repeated PD. In Section 3 we repeat this analysis for the *sequential* duel. Finally, in Section 4 we summarize our results and propose some future research directions.

2 The Simultaneous Duel

2.1 Notation and Rules

Let us formulate the simultaneous duel as a *discounted stochastic game* [12]. It involves players P_1, P_2 and proceeds at discrete time steps (*turns*) $t \in \{1, 2, ...\}$. The game *state* is $s(t) = (s_1(t), s_2(t))$; the *state space* contains all state vectors which may possibly appear in a play of the game and is

$$S = \{(1,1), (1,0), (0,1), (0,0), (\tau, \tau)\},$$

where $s_n(t)$ corresponds to player P_n at time $t \in \{0, 1, 2, ...\}$ and can be equal to

$$\begin{array}{rl} 1: & \text{when } P_n \text{ is alive,} \\ 0: & \text{when } P_n \text{ has died in the current turn,} \\ \tau: & \text{when one or both players have died in a previous turn.} \end{array}$$

Before proceeding, let us introduce the following standard game theoretic notation. Given the player P_n, we denote the "other" player by P_{-n} (for example P_{-1} is another notation for P_2). We use the same notation for states ($s_n(t)$ and $s_{-n}(t)$) and for any other quantities indexed by the player index n.

P_n's *action* at time t is $f_n(t)$, which can be 1 (P_n is shooting) or 0 (P_n is not shooting). If $s(t) = (1, 1)$ then $f_n(t)$ can be either 0 or 1; in any other case $f_n(t)$ must equal 0. When P_n shoots at P_{-n}, the probability of hitting (and killing) him is $p_n \in (0, 1)$ and the probability of missing him is $1 - p_n$; p_n is called the *marksmanship* of P_n. The actions of both players are always known to each other; in particular, if P_n shoots at P_{-n} at time t, P_{-n} will know this, whether he was hit or not. The *action vector* at time t is $f(t) = (f_1(t), f_2(t))$ and takes values in the *action set*

$$\widehat{F} = \{(1,1), (1,0), (0,1), (0,0)\}.$$

Note that this is the set of *all* actions which may be performed when the game is in *some* state; however the actions available to P_n at a given time t depend on the current state $s(t)$ (for example, a dead player cannot shoot). This limitation is enforced by requiring the use of *admissible* strategies, as will be seen presently.

The starting state $s(0)$ is given; obviously, the main case of interest is $s(0) = (1,1)$. At every $t \in \{1, 2, ...\}$ the players simultaneously choose actions $f_1(t)$ and $f_2(t)$ depending on the previous state $s(t-1)$; in other words a player's *current* action $f(t)$ depends on the *previous* state $s(t-1)$. Then the game transits to state $s(t)$ according to a (conditional) *state transition probability* $\Pr(s(t) | s(t-1), f(t))$. The state transition probabilities can be read from the state transition diagram of Figure 1, according to the following convention: states are represented by vertices and transitions are represented by edges, with the probability of each action / transition combination written next to the edge. For example, we see that the state transition $(1,1) \rightarrow (1,0)$ can be effected in two ways.

1. When the action vector is $f = (1,0)$ (P_1 shoots and P_2 abstains) the transition probability is p_1 (the probability that P_1 hit and killed P_2).

2. When the action vector is $f = (1,1)$ (both P_1 and P_2 shoot) the transition probability is $p_1(1-p_2)$ (the probability that P_1 hit and killed P_2, while P_2 missed).

The remaining probabilities, indicated in Figure 1, are obtained by similar reasoning.

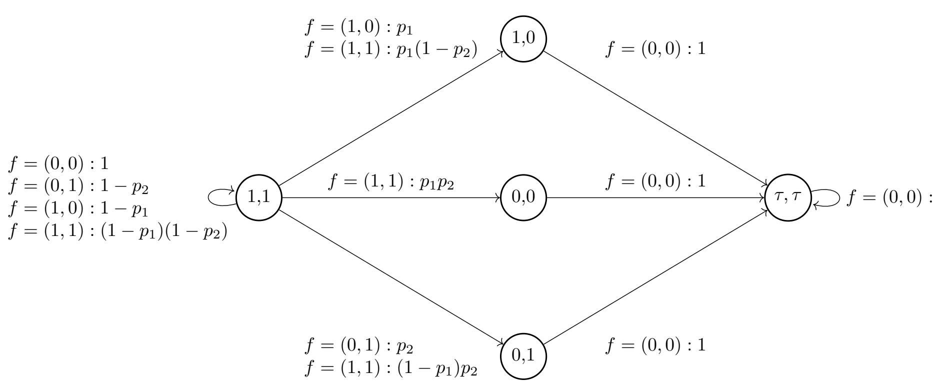

Figure 1: State transition diagram of simultaneous duel

According to the above diagram, the duel game lasts an infinite number of turns and we have two possibilities for the endgame.

1. If at some time t' at least one player is killed, the game transits to some $s(t') = (s'_1, s'_2)$ where at least one s'_n equals 0. By the transition rules of Figure 1, for all $t \geq t'+1$, $s(t) = s'' = (\tau, \tau)$ (the *terminal* state). While the time t keeps increasing, the game is *effectively* over at $t'+1$.

2. It is also possible that no player is ever killed: for all $t \in \{0, 1, 2, ...\}$, $s(t) = (1, 1)$.

A *finite history* is a sequence $h = s(0) f(1) s(1) ... f(T) s(T)$ (where the *length* of the history is T) and an *infinite history* is a sequence $h = s(0) f(1) s(1) ...$. We only consider *admissible* histories, i.e., those which can actually occur in a game played acording to the rules presented above. The set of all admissible finite (resp. infinite) histories will be denoted by H (resp. H^{∞}).

Roughly speaking, each players' goal is to maximize his *total expected payoff*. For every admissible history $h \in H^{\infty}$, P_n's *total payoff function* is

$$Q_n(h) = \sum_{t=0}^{\infty} \delta^t q_n(s(t))$$

where $\delta \in (0, 1)$ is the *discounting factor* and, for $n \in \{1, 2\}$, $q_n : S \to \mathbb{R}$ is P_n's *stage payoff function*. The q_n's are defined as follows.

1. No payoff accrues at the terminal state:

$$\forall n \in \{1, 2\} : q_n(\tau, \tau) = 0.$$

2. P_n receives a_n units for each turn in which he stays alive (the *survival payoff*), with $a_n > 0$. Hence we have

$$\forall n \in \{1, 2\} : q_n(1, 1) = a_n.$$

3. In a turn in which P_n kills P_{-n} (if such a turn occurs) P_n receives b_n units (the *killing payoff*), with $b_n > 0$, and P_{-n} receives 0 units. In addition P_n will receive a total of $\sum_{t=0}^{\infty} \delta^t a_n = \frac{a_n}{1-\delta}$ for the remaining turns in which he will stay alive. We write these as stage payoffs:

$$\begin{array}{ll} q_1(1,0) = c_1 = \frac{a_1}{1-\delta} + b_1, & q_2(1,0) = 0, \\ q_1(0,1) = 0, & q_2(0,1) = c_2 = \frac{a_2}{1-\delta} + b_2. \end{array}$$

Strictly speaking, $q_1(1, 0) = \frac{a_1}{1-\delta} + b_1$ is not a single turn payoff; as explained above it corresponds to P_1's visiting $(1, 0)$ an infinite number of times, each time receiving a_1 units. However, in order to keep the state transition mechanism simple, we postulate that the game stays in $(1, 0)$ for a single turn, in which P_1 receives his entire $\frac{a_1}{1-\delta} + b_1$ payoff. Similar remarks hold for $q_2(0, 1)$ and P_2.

4. In a turn in which both players are killed, their payoffs are

$$\forall n \in \{1, 2\} : q_n(0, 0) = b_n$$

(i.e., each player receives the payoff associated with killing the other player).

A *strategy* (for P_n) is a function σ_n which corresponds to every finite history h the probability $\sigma_n(h) = x_n \in [0,1]$ that P_n shoots at P_{-n}. A *stationary strategy* is a strategy σ_n which depends only on the current state s, hence we simply write $\sigma_n(s) = x_n \in [0,1]$. We assume the players use only *admissible strategies*, i.e., the ones which always produce shooting probabilities compatible with the game rules; we will denote the set of all admissible strategies by Σ and the set of all admissible stationary strategies by $\overline{\Sigma}$.

Given the initial state $s(0)$ and the strategies σ_1 and σ_2, used by P_1 and P_2 respectively, a probability measure is defined on the set of all infinite histories. Hence (since $\delta \in (0,1)$) the total expected payoffs are well defined:

$$\forall n \in \{1,2\} : \overline{Q}_n(s(0), \sigma_1, \sigma_2) = \mathbb{E}_{s(0),\sigma_1,\sigma_2}(Q_n(h)).$$

Hence we have given a complete description of the duel game starting at state $s(0)$; *we will henceforth denote this game as*

$$\Gamma(s(0), \delta; \mathbf{p}, \mathbf{a}, \mathbf{b})$$

where $\mathbf{p} = (p_1, p_2)$, $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$. The shorter form $\Gamma(s(0), \delta)$ will also be used, when the omitted parameters are clear from the context. This is a *stochastic game* with the following components.

1. State set: $S = \{(1,1), (1,0), (0,1), (0,0), (\tau,\tau)\}$.
2. Set of actions for each player in each state:

State	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$	(τ,τ)
P_1	$\{0,1\}$	$\{0,1\}$	$\{0\}$	$\{0\}$	$\{0\}$
P_2	$\{0,1\}$	$\{0\}$	$\{0,1\}$	$\{0\}$	$\{0\}$

3. Payoff functions: the previously defined $q_1(s)$ and $q_2(s)$.
4. Discount factor: γ.
5. Initial state $s(0)$.

In what follows our main interest is in the only nontrivial game, which starts with both players alive, namely $\Gamma((1,1), \delta)$. We will denote it by $\Gamma(\delta)$ and, similarly, we will write $\overline{Q}_n(\sigma_1, \sigma_2)$ instead of $\overline{Q}_n(s(0), \sigma_1, \sigma_2)$. Note that every $\Gamma(\delta)$ is a *nonzero-sum* game.

We have previously stated that each player's goal is to maximize his payoff, but we now modify this statement. Since $\Gamma(\delta)$ is a nonzero-sum game, we assume that the players will choose their strategies ($\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ respectively) so as to reach a *Nash equilibrium* (NE), i.e., $(\widehat{\sigma}_1, \widehat{\sigma}_2)$ must satisfy

$$\forall \sigma_1 \in \Sigma : \overline{Q}_1(\widehat{\sigma}_1, \widehat{\sigma}_2) \geq \overline{Q}_2(\sigma_1, \widehat{\sigma}_2),$$
$$\forall \sigma_2 \in \Sigma : \overline{Q}_2(\widehat{\sigma}_1, \widehat{\sigma}_2) \geq \overline{Q}_2(\widehat{\sigma}_1, \sigma_2).$$

A refinement of the Nash equilibrium is the *subgame perfect equilibrium* (SPE). A pair $(\widehat{\sigma}_1, \widehat{\sigma}_2)$ is said to be a SPE iff it is a NE in both the fulll game and in every subgame [26].

Finally, suppose P_1 and P_2 use *stationary* admissible strategies σ_1 and σ_2, respectively. Whenever the game state s belongs to the set $\{(1,0),(0,1),(0,0),(\tau,\tau)\}$ the strategies must satisfy $\sigma_n(s) = 0$ (i.e., each player must abstain); hence σ_n is fully specified by the value $\sigma_n(1,1) = x_n$. Consequently, when starting from $s(0) = (s_1, s_2)$ and using stationary strategies, we can write the total expected payoff as $\overline{Q}_n(x_1, x_2)$.

2.2 The Total Payoff Equations

Supposing for the time being that $s(0) = (s_1, s_2)$ and, for $n \in \{1,2\}$, P_n uses the *stationary* strategy x_n, we define

$$\forall n \in \{1,2\} : V^{(n)}_{s_1 s_2}(x_1, x_2) = \overline{Q}_n(s(0), x_1, x_2)$$

(the total *expected* payoff received by P_n). It is immediately seen that

$$V^{(1)}_{\tau\tau}(x_1,x_2) = 0, \quad V^{(1)}_{00}(x_1,x_2) = b_1, \quad V^{(1)}_{10}(x_1,x_2) = c_1 = b_1 + \frac{a_1}{1-\delta}, \quad V^{(1)}_{01}(x_1,x_2) = 0$$

and it remains to determine $V^{(1)}_{11}(x_1,x_2)$. From the rules of the game, the state transition diagram and the payoff function definitions, it follows that $V^{(1)}_{11}(x_1,x_2)$ must satisfy the following equation (for brevity's sake we have dropped the dependence on (x_1,x_2)):

$$\begin{aligned} V^{(1)}_{11} = \quad q_1(11) \quad & +\delta x_1 x_2 \left(p_1 p_2 V^{(1)}_{00} + p_1(1-p_2) V^{(1)}_{10} + (1-p_1) p_2 V^{(1)}_{01} + (1-p_1)(1-p_2) V^{(1)}_{11} \right) \\ & +\delta x_1 (1-x_2) \left(p_1 V^{(1)}_{10} + (1-p_1) V^{(1)}_{11} \right) \\ & +\delta (1-x_1) x_2 \left(p_2 V^{(1)}_{01} + (1-p_2) V^{(1)}_{11} \right) \\ & +\delta (1-x_1)(1-x_2) V^{(1)}_{11} \end{aligned}$$

which has solution

$$V^{(1)}_{11}(x_1,x_2) = \frac{a_1 + \delta\left(c_1 p_1 x_1 (1-p_2 x_2) + b_1 p_1 x_1 p_2 x_2\right)}{1 - \delta(1-p_1x_1)(1-p_2x_2)}.$$

Similar expressions hold for P_2's payoffs $\left(V^{(2)}_{s_1 s_2}(x_1,x_2)\right)_{(s_1,s_2)\in S}$.

2.3 Stationary Equilibrium

Proposition 2.1. For $n \in \{1,2\}$ and for every $x_{-n} \in [0,1]$, the function $V^{(n)}_{11}(x_1,x_2)$ is strictly increasing in x_n.

Proof. We will only prove the result for $n = 1$. Recall that

$$V^{(1)}_{11}(x_1,x_2) = \frac{a_1 + \delta\left(c_1 p_1 x_1 (1-p_2 x_2) + b_1 p_1 x_1 p_2 x_2\right)}{1 - \delta(1-p_1x_1)(1-p_2x_2)}.$$

Let

$$A = a_1, \quad B = \delta\left(c_1(1-p_2x_2) + b_1 p_2 x_2\right), \quad C = \delta(1-p_2x_2)$$

and define the function

$$F_1(z) = \frac{A + Bz}{1 - C(1 - z)}.$$

Clearly, for every fixed $x_2 \in [0,1]$ we have $F_1(p_1 x_1) = V_{11}^{(1)}(x_1, x_2)$. We also have

$$\frac{dF_1}{dz} = \frac{B - C(A + B)}{(Cz - C + 1)^2}.$$

After some algebra we find that

$$B - C(A + B) = \frac{\delta}{1 - \delta}\left(\delta a_1 p_2 x_2 (1 - p_2 x_2) + \delta b_1 p_2 x_2 (1 - \delta) + b_1 (1 - \delta)^2\right) > 0.$$

Consequently $F_1(z)$ is strictly increasing in z and, since $p_1 > 0$, $F_1(p_1 x_1) = V_{11}^{(1)}(x_1, x_2)$ is strictly increasing in x_1. ■

Proposition 2.2. For every $\delta \in (0,1)$, the unique stationary NE of the game $\Gamma(\delta)$ is $(1,1)$. Furthermore it is a SPE of $\Gamma(\delta)$.

Proof. Since $V^{(1)}(x_1, x_2)$ (resp. $V^{(2)}(x_1, x_2)$) is increasing in x_1 (resp. x_2) we have

$$\forall x_1 : V_{11}^{(1)}(1,1) \geq V_{11}^{(1)}(x_1, 1),$$
$$\forall x_2 : V_{11}^{(2)}(1,1) \geq V_{11}^{(2)}(1, x_2).$$

This shows that the stationary strategy $\widehat{x}_n = 1$ is a best response to the stationary strategy $\widehat{x}_{-n} = 1$, over all *stationary* strategies. As is well known [23], it follows that $\widehat{x}_n$ is also a best response to $\widehat{x}_{-n}$ over *all* (not necessarily stationary) strategies. Hence $(\widehat{x}_1, \widehat{x}_2) = (1,1)$ is a NE. Suppose there was another stationary NE $(\widetilde{x}_1, \widetilde{x}_2) \neq (1,1)$ and, without loss of generality, assume $\widetilde{x}_1 < 1$. Then we have

$$V_{11}^{(1)}(\widetilde{x}_1, \widetilde{x}_2) < V_{11}^{(1)}(1, \widetilde{x}_2).$$

Hence P_1 is better off by switching from $\widetilde{x}_1$ to $\widehat{x}_1 = 1$ and $(\widetilde{x}_1, \widetilde{x}_2)$ cannot be a NE. Finally, as is well known, a stationary NE is also a SPE. ■

2.4 Nonstationary Equilibrium with Improved Payoffs

From Proposition 2.1 we immediately see that, for every $\delta \in (0,1)$, we have

$$\forall x_2 \in [0,1] : \forall x_1 \in [0,1) : V_{11}^{(1)}(x_1, x_2) < V_{11}^{(1)}(1, x_2),$$
$$\forall x_1 \in [0,1] : \forall x_2 \in [0,1) : V_{11}^{(2)}(x_1, x_2) < V_{11}^{(2)}(x_1, 1).$$

In other words, *if the players only use stationary strategies*, whatever strategy P_{-n} uses, P_n is better off shooting with probability one. Hence, when the players are restricted to stationary strategies, their best strategy profile is one of "full mutual aggression". A natural question is whether there exist other (nonstationary) NE which promote "cooperation" and "pacifism". To this end we first prove the following.

Proposition 2.3. For $n \in \{1, 2\}$ we have

$$\forall \delta \in \left(\frac{p_n b_n - a_n p_{-n}}{p_n b_n}, 1 \right) \ : V_{11}^{(n)} (0,0) > V_{11}^{(n)} (1,1) \, .$$

Proof. We will prove the above for the case $n = 1$. We have

$$V_{11}^{(1)} (0,0) - V_{11}^{(1)} (1,1) = \frac{a_1}{1-\delta} - \frac{a_1 + \delta c_1 p_1 (1-p_2) + \delta b_1 p_1 p_2}{1 - \delta (1-p_1)(1-p_2)}.$$

Substituting $c_1 = \left(\frac{a_1}{1-\delta} + b_1 \right)$ and performing some algebra we obtain

$$V_{11}^{(1)} (0,0) - V_{11}^{(1)} (1,1) = \frac{\delta (a_1 p_2 - b_1 p_1 (1-\delta))}{(1-\delta)(1 - \delta (1-p_1)(1-p_2))}.$$

The inequality

$$a_1 p_2 - b_1 p_1 (1-\delta) > 0$$

is satisfied for all $\delta > \delta_1 = \frac{b_1 p_1 - a_1 p_2}{b_1 p_1}$. Hence

$$\forall \delta \in (\delta_1, 1) : V_{11}^{(1)} (0,0) - V_{11}^{(1)} (1,1) = \frac{\delta (a_1 p_2 - b_1 p_1 (1-\delta))}{(1-\delta)(1 - \delta (1-p_1)(1-p_2))} > 0.$$

which completes the proof. ■

The above proposition indicates that the discounted simultaneous duel has a certain similarity to the *repeated PD* [1]: on the one hand, if both players "cooperated", i.e., did not shoot, they would receive a higher payoff; on the other hand, given that one player does not shoot, the other player always has a motive to "defect", i.e., to shoot; hence $(x_1, x_2) = (0, 0)$ (mutual nonshooting) is not a NE and in fact, as we have already seen, the only NE in stationary strategies is $(x_1, x_2) = (1, 1)$ (mutual shooting). However, in analogy to well known results regarding the repeated PD, it is conceivable that, for δ sufficiently close to one, nonshooting and nonstationary NE may also exist.

Hence we borrow an idea from the PD literature and introduce the "*grim strategy*" σ_g, which can be described in words as follows:

> Grim Strategy: *As long as neither player has shot in previous turns, do not shoot; if at some turn a player shoots, then shoot at all subsequent turns with probability one.*

Obviously σ_g is not a stationary strategy (since its output depends on the previous history of the game).

Proposition 2.4. Suppose $p_1 \in (0, 1)$ and $p_2 \in (0, 1)$. Then, there exists some $\delta_0 \in (0, 1)$ such that

$$\forall \delta \in (\delta_0, 1) : (\sigma_g, \sigma_g) \text{ is a NE of } \Gamma(\delta) \, .$$

Furthermore, for these δ values, (σ_g, σ_g) is a SPE of $\Gamma(\delta)$.

Proof. If both P_1 and P_2 use σ_g then

$$\overline{Q}_1(\sigma_g,\sigma_g) = V_{11}^{(1)}(0,0) = q_1(1,1) + \delta V_{11}^{(1)}(0,0).$$

Now, if P_1 switches to another strategy σ_1, it suffices to examine σ_1's by which P_1 shoots at P_2 at the first turn and with probability one (because, as soon as P_1 shoots, P_2 will switch to the stationary strategy of always shooting, hence after this point, P_1's best response is a stationary strategy; and the best stationary strategy is to always shoot with probability one). In such a case, in the 0-th turn P_1 will receive payoff $q_1(0,0)$; in the first turn he will shoot at P_2 and then there exist two possibilities.

1. With probability p_1, P_2 will be killed and the game will transit to state $(1,0)$. Hence P_1 will receive payoff $q_1(1,0)$ and then the game will transit to state (τ,τ) which yields a further total payoff zero. Hence P_1's total expected payoff (from the first turn onward and conditioned on killing P_2) will be $q_1(1,0)$.

2. With probability $(1-p_1)$, P_2 will not be killed, the game will transit to state $(1,1)$ and in all subsequent turns P_2 will be shooting at P_1, who can do no better than always shoot back at P_2. But this is equivalent to a duel which starts at state $(1,1)$ and in which the players use the stationary strategies $x_1 = x_2 = 1$. Hence P_1's total expected payoff (from the first turn onward and conditioned on not killing P_2) will be $V_{11}^{(1)}(1,1)$.

In short, with (σ_1,σ_g), P_1's expected payoff will be

$$\overline{Q}_1(\sigma_1,\sigma_g) = q_1(1,1) + \delta\left(p_1 q_1(1,0) + (1-p_1)V_{11}^{(1)}(1,1)\right).$$

For (σ_g,σ_g) to be a NE, we must have $\overline{Q}_1(\sigma_g,\sigma_g) \geq \overline{Q}_1(\sigma_1,\sigma_g)$. Now

$$\begin{aligned}
&\overline{Q}_1(\sigma_g,\sigma_g) - \overline{Q}_1(\sigma_1,\sigma_g) \\
&= \delta\left(V_{11}^{(1)}(0,0) - \left(p_1 q_1(1,0) + (1-p_1)V_{11}^{(1)}(1,1)\right)\right) \\
&= \delta\left[\frac{a_1}{1-\delta} - \left(p_1\left(\frac{a_1}{1-\delta}+b_1\right) + (1-p_1)\frac{a_1 + \left(\frac{a_1}{1-\delta}+b_1\right)\delta p_1(1-p_2) + \delta b_1 p_1 p_2}{1-\delta(1-p_1)(1-p_2)}\right)\right] \\
&= \frac{\delta \Pi_1(\delta)}{(1-\delta)(1-\delta(1-p_1)(1-p_2))}
\end{aligned}$$

where

$$\Pi_1(\delta) = \left(-b_1p_1^2p_2 + b_1p_1p_2\right)\delta^2 + \left(-b_1p_1p_2 - a_1p_1p_2 + a_1p_2 + b_1p_1 + b_1p_1^2p_2\right)\delta - b_1p_1 \tag{1}$$

is a second degree polynomial. We can actually solve the inequality $\Pi_1(\delta) > 0$, but this yields a complicated expression. However, since $\Pi_1(1) = a_1p_2(1-p_1) > 0$, there exists a δ_1 such that, for all $\delta \in (\delta_1, 1)$, we have $\Pi_1(\delta) > 0$. Consequently

$$\forall \delta \in (\delta_1,1) : \overline{Q}_1(\sigma_g,\sigma_g) - \overline{Q}_1(\sigma_1,\sigma_g) = \frac{\delta \Pi_1(\delta)}{(1-\delta)(1-\delta(1-p_1)(1-p_2))} > 0.$$

By a similar argument we can establish the existence of a δ_2 such that

$$\forall \delta \in (\delta_2,1) : \overline{Q}_2(\sigma_g,\sigma_g) - \overline{Q}_2(\sigma_g,\sigma_1) > 0.$$

Letting $\delta_0 = \max(\delta_1, \delta_2)$, we have proved that (σ_g, σ_g) is a NE.

To prove that (σ_g, σ_g) is also a SPE, suppose that P_2 shoots at some t. By σ_g both players should shoot with probability one for all $t' \in \{t+1, t+2, ...\}$ (as long as they are alive), i.e., they adopt the stationary strategy $(x_1, x_2) = (1, 1)$ in the subgame starting at $t+1$, which is exactly $\Gamma(\delta)$. Since $(1, 1)$ is a NE of $\Gamma(\delta)$, it is also a NE of the subgame and hence (σ_g, σ_g) is a SPE of the full game. ■

Let us try to better understand the dependence of δ_0 on the game parameters (a_1, a_2, b_1, b_2, p_1 and p_2). Recall that δ_0 must be less than one and such that, when $\delta > \delta_0$, $\Pi_1(\delta)$ and $\Pi_2(\delta)$ (the analog of $\Pi_1(\delta)$ for P_2) are positive. We can solve $\Pi_1(\delta) = 0$, where $\Pi_1(\delta)$ is given by (1); it is a quadratic equation with two roots and, since the coefficient of δ^2 is

$$\left(-b_1 p_1^2 p_2 + b_1 p_1 p_2\right) = b_1 p_1 p_2 (1 - p_1) > 0,$$

the smallest admissible δ_0 is the largest root (we know it will be less than one). However, the expressions for the roots are complicated functions of the game parameters and do not yield an easy understanding of the dependence. Consequently, we resort to numerical computation and plotting. We still have to limit ourselves to simplified cases, because it is not easy to visualize the full $\delta_0(a_1, a_2, b_1, b_2, p_1, p_2)$ function. So we plot δ_0 for two cases. On the left side of Fig.2 we consider the symmetric case ($a_1 = a_2$, $b_1 = b_2$, $p_1 = p_2$) and plot δ_0 as a function of b_1, p_1, with fixed $a_1 = 1$. [1] On the right side we take $a_1 = a_2 = 1$, $b_1 = b_2 = 1$ and plot δ_0 as a function of p_1, p_2.

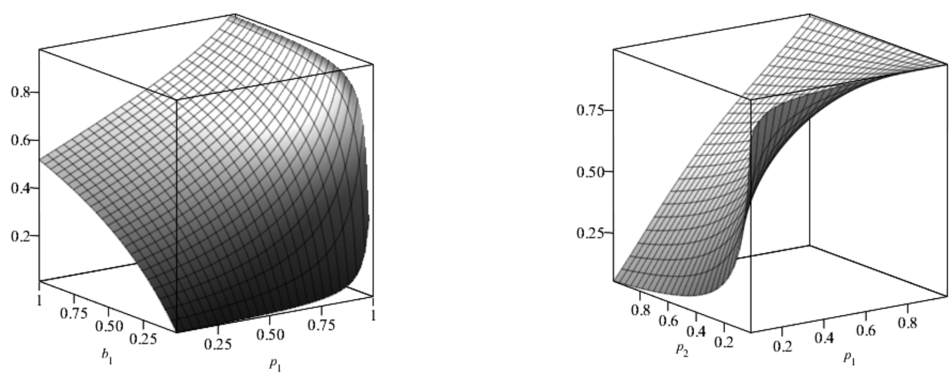

Figure 2: Dependence of δ_0 of Proposition 2.4 on game parameters. (a) Left: δ_0 as a function of p_1 and b_1, taking $a_1 = a_2 = 1$, $b_1 = b_2$, $p_1 = p_2$. (b) Right: δ_0 as a function of p_1 and p_2, taking $a_1 = a_2 = 1$, $b_1 = b_2 = 1$.

Let us interpret the above figures from P_1's point of view (similar considerations hold for P_2). We see that δ_0 is increasing with b_1 and p_1, and decreasing with p_2; this can be interpreted as follows.

1. When the killing payoff b_1 increases, P_1 needs a stronger motivation to cooperate; this requires a higher discount factor δ, so that "long-life" payoff $\frac{a_1}{1-\delta}$ becomes larger than b_1.

[1] There is no loss of generality in assuming $a_1 = 1$, because $\Pi_1(\delta)$ is homogeneous in a_1 and b_1.

2. When P_1's marksmanship p_1 increases, P_1 is encouraged to defect (since he has a higher probability to receive the killing payoff b_1) and hence needs a stronger motivation to cooperate; this again requires a higher δ.

3. Conversely, when P_2's marksmanship p_2 increases, P_1 is encouraged to cooperate (since he has a higher probability of getting killed) and this can be achieved with a lower δ.

3 The Sequential Duel

3.1 Notation and Rules

We will now formulate the sequential duel as a discounted stochastic game. Since the formulation is very similar to that of the simultaneous stochastic duel in Section 2, the presentation will now be briefer.

The game involves P_1, P_2 and proceeds at discrete time steps $t \in \{1, 2, ...\}$. The game state is $s(t) = (s_0(t), s_1(t), s_2(t))$, where $s_0(t) \in \{1, 2, \tau\}$ indicates the player who "has the shot" at time t, i.e., $s_0(t) = 1$ (resp. $s_0(t) = 2$) means that P_1 (resp. P_2) may shoot and $s_0(t) = \tau$ means that neither player can shoot; $s_1(t)$ and $s_2(t)$ belong to $\{0, 1\}$ and have the same meaning as in Section 2. Consequently the *state space* is

$$S = \{(1,1,1), (2,1,1), (2,1,0), (1,0,1), (\tau,\tau,\tau)\}.$$

P_n's *action* at time t is $f_n(t)$, same as in Section 2. The initial state $s(0)$ is given; the cases of interest are $s(0) = (2,1,1)$ and $s(0) = (1,1,1)$. At time t, the player P_n with $n = s_0(t)$ has the shot[2] and can choose his action between $f_n(t) = 0$ or $f_n(t) = 1$, while $f_{-n}(t) = 0$; the game transits to $s(t)$ according to $\Pr(s(t) | s(t-1), f(t))$. The state transition probabilities can be read from the state transition diagram of Figure 3, following the same conventions as for Figure 1. It can be seen that the "main" part of the game consists in cycling between $(1,1,1)$ and $(2,1,1)$; this corresponds to the fact that the players alternate in having the shot.

For the endgame, we have the following possibilities.

1. P_2 is killed: at some time, the game transits from $(1,1,1)$ to $(2,1,0)$ and then to (τ,τ,τ) where it stays ad infinitum.

2. P_1 is killed: at some time, the game transits from $(2,1,1)$ to $(1,0,1)$ and then to (τ,τ,τ) where it stays ad infinitum.

3. No player is ever killed: the game keeps cycling through states $(1,1,1)$ and $(2,1,1)$.

[2] Of course, when the game is in $(2,1,0)$, P_2 is supposed to have the shot, but he cannot actually shoot, since he is already dead; similarly for $(1,0,1)$ and P_1.

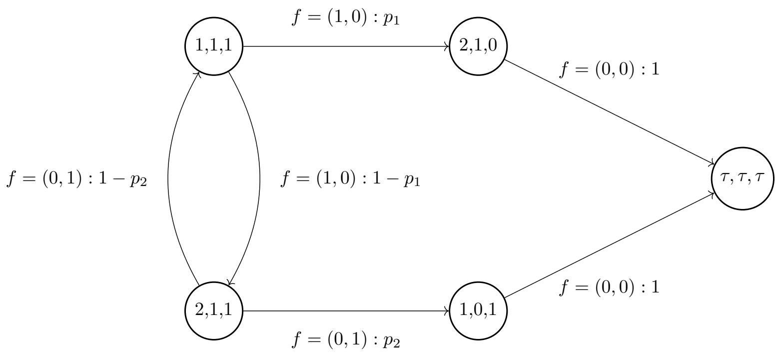

Figure 3: State transition diagram of sequential duel

Finite and infinite histories and strategies are defined as in the case of the simultaneous duel; the set of all admissible strategies is $\widehat{\Sigma}$ and that of all admissible stationary strategies is $\widetilde{\Sigma}$. The stage payoff functions q_n are defined as follows:

$$\begin{array}{ll} q_1(\tau,\tau,\tau) = 0, & q_2(\tau,\tau,\tau) = 0, \\ q_1(1,1,1) = a_1, & q_2(1,1,1) = a_2, \\ q_1(2,1,1) = a_1, & q_2(2,1,1) = a_2, \\ q_1(2,1,0) = \frac{a_1}{1-\delta} + b_1, & q_2(2,1,0) = 0, \\ q_1(1,0,1) = 0, & q_2(1,0,1) = \frac{a_2}{1-\delta} + b_2. \end{array}$$

P_n's total payoff and total expected payoffs are defined as in the case the simultaneous duel.

We will denote the sequential duel game by $\widetilde{\Gamma}(s(0), \delta; \mathbf{p}, \mathbf{a}, \mathbf{b})$, $\widetilde{\Gamma}(s(0), \delta)$ or $\widetilde{\Gamma}(\delta)$. It is a *stochastic game* with the following components.

1. State set: $S = \{(1,1,1), (2,1,1), (2,1,0), (1,0,1), (\tau,\tau,\tau)\}$.

2. Set of actions for each player in each state:

State	$(1,1,1)$	$(2,1,1)$	$(2,1,0)$	$(1,0,1)$	(τ,τ,τ)
P_1	$\{0,1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
P_2	$\{0\}$	$\{0,1\}$	$\{0\}$	$\{0\}$	$\{0\}$

3. Payoff functions: the previously defined $q_1(s)$ and $q_2(s)$.

4. Discount factor: γ.

5. Initial state $s(0)$.

The games of interest are $\widetilde{\Gamma}((1,1,1), \delta)$ and $\widetilde{\Gamma}((2,1,1), \delta)$. The NE conditions for both $\widetilde{\Gamma}((1,1,1), \delta)$ and $\widetilde{\Gamma}((2,1,1), \delta)$ are defined in the usual manner. Finally, as also remarked at the end of Section 2.1, for stationary strategies we can write the total expected payoff as $\overline{Q}_n(s(0), x_1, x_2)$.

3.2 The Total Payoff Equations

Supposing for the time being that $s(0) = (s_0, s_1, s_2)$ and, for $n \in \{1,2\}$, P_n uses the stationary strategy $(x_n, 1-x_n)$, we define

$$\forall n \in \{1,2\} : V^{(n)}_{s_0 s_1 s_2}(x_1, x_2) = \overline{Q}(s(0), x_1, x_2).$$

Similarly to the simultaneous duel, we immediately get

$$V^{(1)}_{\tau\tau\tau}(x_1, x_2) = 0, \qquad V^{(1)}_{210}(x_1, x_2) = c_1, \qquad V^{(1)}_{101}(x_1, x_2) = 0.$$

For $V^{(1)}_{111}(x_1, x_2)$ and $V^{(1)}_{211}(x_1, x_2)$ we have the system (dropping the (x_1, x_2) dependence):

$$\begin{aligned} V^{(1)}_{111} &= q_1(111) + \delta\left(x_1 p_1 V^{(1)}_{210} + x_1(1-p_1) V^{(1)}_{211} + (1-x_1) V^{(1)}_{211}\right) \\ V^{(1)}_{211} &= q_1(211) + \delta\left(x_2 p_2 V^{(1)}_{101} + x_2(1-p_2) V^{(1)}_{111} + (1-x_2) V^{(1)}_{111}\right) \end{aligned}$$

which has unique solution

$$\begin{aligned} V^{(1)}_{111}(x_1, x_2) &= \frac{a_1 + \delta a_1(1-p_1x_1) + \delta c_1 p_1 x_1}{1-\delta^2(1-p_1x_1)(1-p_2x_2)}, \\ V^{(1)}_{211}(x_1, x_2) &= \frac{a_1(1+\delta(1-p_2x_2)) + \delta^2 c_1 p_1 x_1 (1-p_2x_2)}{1-\delta^2(1-p_1x_1)(1-p_2x_2)}. \end{aligned}$$

Similar expressions can be obtained for $\left(V^{(n)}_{s_0s_1s_2}(x_1, x_2)\right)_{s_0s_1s_2 \in S}$.

3.3 Stationary Equilibrium

Proposition 3.1. For $n \in \{1,2\}$ and for every $x_{-n} \in [0,1]$, the functions $V^{(n)}_{111}(x_1, x_2)$ and $V^{(n)}_{211}(x_1, x_2)$ are strictly increasing in x_n.

Proof. We will only prove the result for $n = 1$. Recall that

$$V^{(1)}_{111}(x_1, x_2) = \frac{a_1 + \delta a_1(1-p_1x_1) + \delta c_1 p_1 x_1}{1-\delta^2(1-p_1x_1)(1-p_2x_2)}.$$

Let

$$A = a_1, \quad B = \delta a_1, \quad C = \delta c_1, \quad E = \delta^2(1-p_2x_2)$$

and define the function

$$F_1(z) = \frac{A + B(1-z) + Cz}{1 - E(1-z)}.$$

Clearly, for every fixed $x_2 \in [0,1]$, we have $F_1(p_1x_1) = V^{(1)}_{111}(x_1, x_2)$. Also

$$\frac{dF_1}{dz} = \frac{C - B - EC - EA}{(1-E+Ez)^2}.$$

After some algebra we find that

$$\frac{dF_1}{dz} = \frac{\delta}{1-\delta} \frac{\left(-\delta^3 b_1 p_2 x_2 + \delta^2 b_1 p_2 x_2 + \delta a_1 p_2 x_2 + \delta^3 b_1 - \delta^2 b_1 - \delta b_1 + b_1\right)}{\left(1 - \delta^2 (1-z)(1-p_2 x_2)\right)^2}.$$

But

$$\begin{aligned} & -\delta^3 b_1 p_2 x_2 + \delta^2 b_1 p_2 x_2 + \delta a_1 p_2 x_2 + \delta^3 b_1 - \delta^2 b_1 - \delta b_1 + b_1 \\ & = (1-\delta)\,\delta^2 b_1 p_2 x_2 + (1+\delta)(1-\delta)^2 b_1 + \delta a_1 p_2 x_2 > 0. \end{aligned}$$

Hence $\frac{dF_1}{dz} > 0$, i.e., $F_1(z)$ is strictly increasing in z and $F_1(p_1 x_1) = V_{111}^{(1)}(x_1, x_2)$ is strictly increasing in x_1. Next we recall that

$$V_{211}^{(1)}(x_1, x_2) = \frac{a_1 (1 + \delta (1 - p_2 x_2)) + c_1 \delta^2 p_1 x_1 (1 - p_2 x_2)}{1 - \delta^2 (1 - p_1 x_1)(1 - p_2 x_2)}.$$

Letting

$$A = a_1 (1 + \delta (1 - p_2 x_2)), \quad B = c_1 \delta^2 (1 - p_2 x_2), \quad C = \delta^2 (1 - p_2 x_2),$$

we define the function

$$F_2(z) = \frac{A + Bz}{1 - C(1-z)}.$$

Clearly, for every fixed $x_2 \in [0,1]$, we have $F_1(p_1 x_1) = V_{111}^{(1)}(x_1, x_2)$. Also

$$\frac{dF_2}{dz} = \frac{B - BC - CA}{(1 - C + zC)^2}.$$

After some algebra we find that

$$B - BC - CA = \frac{\delta^2 (1 - p_2 x_2)}{1 - \delta} \left(-\delta^3 b_1 p_2 x_2 + \delta^2 b_1 p_2 x_2 + \delta a_1 p_2 x_2 + \delta^3 b_1 - \delta^2 b_1 - \delta b_1 + b_1\right).$$

But

$$\begin{aligned} & -\delta^3 b_1 p_2 x_2 + \delta^2 b_1 p_2 x_2 + \delta a_1 p_2 x_2 + \delta^3 b_1 - \delta^2 b_1 - \delta b_1 + b_1 \\ & = a_1 \delta p_2 x_2 + b_1 (1-\delta)\left(1 - \delta^2 (1 - p_2 x_2)\right) > 0. \end{aligned}$$

Hence $\frac{dF_2}{dz} > 0$, i.e., $F_2(z)$ is strictly increasing in z and $F_2(p_1 x_1) = V_{211}^{(1)}(x_1, x_2)$ is strictly increasing in x_1. ■

Proposition 3.2. Assume $p_1 > 0$ and $p_2 > 0$. Then, for all $n \in \{1, 2\}$ and every $\delta \in (0,1)$, the unique stationary NE of $\widetilde{\Gamma}((n,1,1),\delta)$ is $(1,1)$. Furthermore, it is a SPE of $\widetilde{\Gamma}((n,1,1),\delta)$.

Proof. This is proved similarly to Proposition 2.2. ■

3.4 Nonstationary Equilibrium with Improved Payoffs

From Proposition 3.1 we immediately see that, when $p_1 > 0$ and $p_2 > 0$, for every $n \in \{1,2\}$ and every $\delta \in (0,1)$, we have

$$\forall x_2 \in [0,1] : \forall x_1 \in [0,1) : V_{n11}^{(1)}(x_1, x_2) < V_{n11}^{(1)}(1, x_2),$$
$$\forall x_1 \in [0,1] : \forall x_2 \in [0,1) : V_{n11}^{(2)}(x_1, x_2) < V_{n11}^{(2)}(x_1, 1).$$

In other words, whatever strategy P_{-n} uses, P_n is better off shooting with probability one. Hence, when the players are restricted to stationary strategies, their best strategy profile is one of "full mutual aggression". However, similarly to the simultaneous duel, we will show that there also exist (nonstationary) NE which promote "cooperation" and "pacifism".

Proposition 3.3. Assume $p_1 > 0$ and $p_2 > 0$. Then, for $n \in \{1,2\}$, there exists some δ_n such that

$$\forall \delta \in (\delta_n, 1) : V_{111}^{(n)}(0,0) > V_{111}^{(n)}(1,1),$$
$$\forall \delta \in (\delta_n, 1) : V_{211}^{(n)}(0,0) > V_{211}^{(n)}(1,1).$$

Proof. We will only prove the case $n = 1$. After some algebra we get

$$V_{111}^{(1)}(0,0) - V_{111}^{(1)}(1,1) = \frac{\delta^2 a_1 p_2 (1-p_1) - \delta(1-\delta) b_1 p_1}{(1-\delta^2(1-p_1)(1-p_2))(1-\delta)}.$$

Letting $\Pi_1(\delta) = \delta^2 a_1 p_2 (1-p_1) - \delta(1-\delta) b_1 p_1$, we have $\Pi_1(1) = a_1 p_2 (1-p_1) > 0$. Hence there exists some $\delta_1' \in (0,1)$ such that

$$\forall \delta \in (0, \delta_1') : V_{111}^{(1)}(0,0) > V_{111}^{(1)}(1,1).$$

Similarly, after some algebra we get

$$V_{211}^{(1)}(0,0) - V_{211}^{(1)}(1,1) = \frac{\delta(a_1 p_2 - \delta(1-\delta) p_1 b_1 (1-p_2))}{(1-\delta^2(1-p_1)(1-p_2))(1-\delta)}.$$

Letting $\Pi_2(\delta) = (a_1 p_2 - \delta(1-\delta) p_1 b_1 (1-p_2))$, we have $\Pi_2(1) = a_1 p_2 > 0$. Hence there exists some $\delta_1'' \in (0,1)$ such that

$$\forall \delta \in (0, \delta_1'') : V_{211}^{(1)}(0,0) > V_{211}^{(1)}(1,1).$$

Letting $\delta_1 = \max(\delta_1', \delta_1'')$ we have completed the proof. ■

From the above (as in the simultaneous case) we see a similarity of the sequential duel to the repeated PD. We now define a grim strategy $\widetilde{\sigma}_g$ for the sequential duel. Its description is exactly the same as that of σ_g given in Section 2.4 but $\widetilde{\sigma}_g$ is different from σ_g in the sense that it must respect the sequential shooting constraint. We will now prove that, under appropriate conditions, $(\widetilde{\sigma}_g, \widetilde{\sigma}_g)$ is a NE.

Proposition 3.4. Suppose $p_1 \in (0,1)$ and $p_2 \in (0,1)$. Then there exist some $\delta_0' \in (0,1)$ and $\delta_0'' \in (0,1)$ such that

$$\forall \delta \in (\delta_0', 1) : (\widetilde{\sigma}_g, \widetilde{\sigma}_g) \text{ is a NE of } \widetilde{\Gamma}\left((1,1,1), \delta\right),$$
$$\forall \delta \in (\delta_0'', 1) : (\widetilde{\sigma}_g, \widetilde{\sigma}_g) \text{ is a NE of } \widetilde{\Gamma}\left((2,1,1), \delta\right).$$

Furthermore, for these δ values, $(\widetilde{\sigma}_g, \widetilde{\sigma}_g)$ is a SPE of $\widetilde{\Gamma}\left((1,1,1), \delta\right)$ and $\widetilde{\Gamma}\left((2,1,1), \delta\right)$.

Proof. We will only prove that P_1 has no incentive to switch from σ_g to some shooting strategy σ_1 (the case for P_2 is treated identically). Let us first consider $\widetilde{\Gamma}\left((1,1,1), \delta\right)$. If both P_1 and P_2 use $\widetilde{\sigma}_g$ then

$$\overline{Q}_1\left((1,1,1), \widetilde{\sigma}_g, \widetilde{\sigma}_g\right) = V_{111}^{(1)}(0,0) = q_1(1,1,1) + \delta V_{211}^{(1)}(0,0).$$

Now, if P_1 switches to another strategy σ_1, it suffices to examine σ_1's by which P_1 shoots at P_2 at the first turn and with probability one. In such a case, in the 0-th turn P_1 will receive payoff $q_1(1,1)$; in the first turn he will shoot at P_2 and then there exist two possibilities.

1. With probability p_1, P_2 will be killed and the game will transit to $(2,1,0)$, where P_1 will receive payoff $q_1(2,1,0)$, and then the game will transit to state (τ,τ,τ) which yields a further total payoff zero. Hence P_1's total expected payoff (from the first turn onward and conditioned on killing P_2) will be $q_1(2,1,0)$.
2. With probability $(1-p_1)$, P_2 will not be killed, the game will transit to state $(2,1,1)$ and in all subsequent turns in which P_2 has the move, he will be shooting at P_1, who can do no better than always shooting back at P_2 (when he has the move). But this is equivalent to a game which starts at state $(2,1,1)$ and in which the players use the stationary strategies $x_1 = x_2 = 1$ and hence P_1's total expected payoff (from the first turn onward and conditioned on not killing P_2) will be $V_{211}^{(1)}(1,1)$.

In short, with $(\sigma_1, \widetilde{\sigma}_g)$, P_1's expected payoff will be

$$\overline{Q}_1\left((1,1,1), \sigma_1, \widetilde{\sigma}_g\right) = q_1(1,1,1) + \delta\left(p_1 q_1(2,1,0) + (1-p_1) V_{211}^{(1)}(1,1)\right).$$

For $(\widetilde{\sigma}_g, \widetilde{\sigma}_g)$ to be a NE, we must have $\overline{Q}_1\left((1,1,1), \widetilde{\sigma}_g, \widetilde{\sigma}_g\right) \geq \overline{Q}_1\left((1,1,1), \sigma_1, \widetilde{\sigma}_g\right)$. Now

$$\begin{aligned}
&\overline{Q}_1\left((1,1,1), \widetilde{\sigma}_g, \widetilde{\sigma}_g\right) - \overline{Q}_1\left((1,1,1), \sigma_1, \widetilde{\sigma}_g\right) \\
&= \delta\left(V_{211}^{(1)}(0,0) - \left(p_1 q_1(2,1,0) + (1-p_1) V_{211}^{(1)}(1,1)\right)\right) \\
&= \delta\left[\frac{a_1}{1-\delta} - \left(p_1\left(\frac{a_1}{1-\delta} + b_1\right) + (1-p_1)\frac{a_1\left(1+\delta(1-p_2)\right) + \delta^2\left(\frac{a_1}{1-\delta} + b_1\right)p_1(1-p_2)}{1-\delta^2(1-p_1)(1-p_2)}\right)\right] \\
&= \frac{\delta \widetilde{\Pi}_1(\delta)}{(1-\delta)\left(1-\delta^2(1-p_1)(1-p_2)\right)}.
\end{aligned}$$

where $\widetilde{\Pi}_1(\delta) = (b_1 p_1 + a_1 p_2 - a_1 p_1 p_2)\delta - b_1 p_1$. The inequality $\widetilde{\Pi}_1(\delta) > 0$ holds for all δ greater or equal to

$$\delta_1 = \frac{b_1 p_1}{b_1 p_1 + a_1 p_2 - a_1 p_1 p_2}. \tag{2}$$

It follows then that

$$\forall \delta \in (\delta_1, 1) : \overline{Q}_1 \left((1,1,1), \sigma_g, \sigma_g \right) - \overline{Q}_1 \left((1,1,1), \sigma_1, \sigma_g \right) = \frac{\delta \widetilde{\Pi}_1 (\delta)}{(1-\delta)(1-\delta(1-p_1)(1-p_2))} > 0.$$

By a similar argument we can show that, with $\delta_2 = \frac{b_2 p_2}{b_2 p_2 + a_2 p_1 - a_2 p_1 p_2}$, we have

$$\forall \delta \in (\delta_2, 1) : \overline{Q}_2 \left((1,1,1), \sigma_g, \sigma_g \right) - \overline{Q}_2 \left((1,1,1), \sigma_g, \sigma_1 \right) > 0.$$

Letting $\delta_0' = \max(\delta_1, \delta_2) \in (0,1)$ we have

$$\forall \delta \in (\delta_0', 1) : \begin{cases} \overline{Q}_1 \left((1,1,1), \sigma_g, \sigma_g \right) > \overline{Q}_1 \left((1,1,1), \sigma_1, \sigma_g \right) \\ \overline{Q}_2 \left((1,1,1), \sigma_g, \sigma_g \right) > \overline{Q}_2 \left((1,1,1), \sigma_g, \sigma_1 \right) \end{cases}.$$

Hence $(\widetilde{\sigma}_g, \widetilde{\sigma}_g)$ is a NE of $\widetilde{\Gamma}((1,1,1), \delta)$ for all $\delta \in (\delta_0', 1)$. The case of $\widetilde{\Gamma}((2,1,1), \delta)$ is essentially identical. Suppose, P_1 decides to use a shooting strategy σ_1. This will not influence the first turn of the game, since P_2 is using $\widetilde{\sigma}_g$ and will not shoot. Hence the game moves to state $(1,1,1)$ and now we can apply the previous analysis to show that P_1 has no incentive to switch from $\widetilde{\sigma}_g$ to σ_1.

Finally the argument for showing that $(\widetilde{\sigma}_g, \widetilde{\sigma}_g)$ is a SPE of $\widetilde{\Gamma}((1,1,1), \delta)$ and $\widetilde{\Gamma}((2,1,1), \delta)$ (for the appropriate δ values) is identical to the one used for the simultaneous duel. ■

Let us plot δ_0' as a function of a few game parameters. On the left side of Fig.4 we consider the symmetric case ($a_1 = a_2$, $b_1 = b_2$, $p_1 = p_2$) and plot δ_0' as a function of b_1, p_1, with fixed $a_1 = 1$. On the right side we take $a_1 = a_2 = 1$, $b_1 = b_2 = 1$ and plot δ_0' as a function of p_1, p_2.

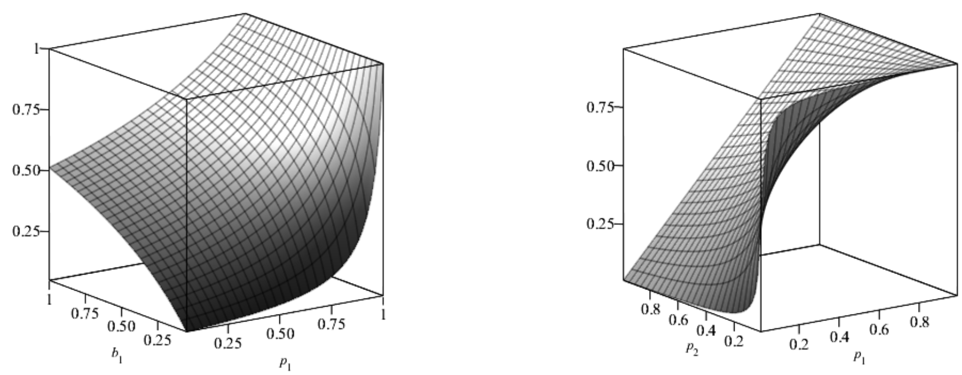

Figure 4: Dependence of δ_0' of Proposition 3.4 on game parameters. (a) Left: δ_0' as a function of p_1 and b_1, taking $a_1 = a_2 = 1$, $b_1 = b_2$, $p_1 = p_2$. (b) Right: δ_0' as a function of p_1 and p_2, taking $a_1 = a_2 = 1$, $b_1 = b_2 = 1$.

The plots of Fig. 4 are similar to those of Fig. 2 and admit the same interpretations. In this case we can actually obtain analytical corroboration, since δ_0' is given by (2) and we have

$$\frac{d}{db_1}\delta_0 = \frac{p_1 a_1 p_2 (1-p_1)}{(b_1 p_1 + a_1 p_2 - a_1 p_1 p_2)^2} > 0,$$
$$\frac{d}{dp_1}\delta_0 = \frac{b_1 a_1 p_2}{(b_1 p_1 + a_1 p_2 - a_1 p_1 p_2)^2} > 0,$$
$$\frac{d}{dp_2}\delta_0 = -\frac{b_1 p_1 a_1 (1-p_1)}{(b_1 p_1 + a_1 p_2 - a_1 p_1 p_2)^2} < 0.$$

This agrees with the plots: δ_0' is increasing with b_1 and p_1, and decreasing with p_2.

4 Conclusion

We have formulated the duel (with either simultaneous or sequential shooting) as a discounted stochastic game. We have shown that this game has a unique Nash equilibrium in stationary strategies (namely the "always-shooting" strategies) and an additional "cooperation-promoting" Nash equilibrium in nonstationary strategies; both of these are also SPE. In this sense our version of the duel is similar to the repeated Prisoner's Dilemma.

In the future we intend to extend the study of the duel in several directions. First, we conjecture that we can establish the existence of an infinity of nonstationary NE (a form of "*Folk Theorem*") by an approach similar to the one used for repeated bimatrix games. In addition, we want to formulate and study a version of the duel in which payoffs are only received at the terminal states (this is similar to [15, 16, 17]). Finally, we want to formulate and study a version of the duel in which each player wants to kill his opponent in the *shortest possible time*.

In addition, we intend to apply the approach of the current paper to formulate and study the *Nuel* as a discounted stochastic game.

Declarations

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