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Generalized Cops and Robbers: A Multi-player Pursuit Game on Graphs

Ath. Kehagias[1]

Abstract

We introduce and study the *Generalized Cops and Robbers* (GCR) game, an N-player pursuit game in graphs. The two-player version is essentially equivalent to the classic *Cops and Robbers* (CR) game. The three-player version can be understood as two CR games played simultaneously on the same graph; a player can be at the same time both *pursuer and evader*. The same is true for four or more players. We formulate GCR as a *discounted stochastic game of perfect information* and prove that, for three or more players, it has at least two *Nash equilibria*: one in positional deterministic strategies and another in nonpositional ones. We also study the capturing properties of GCR Nash equilibria in connection with the *cop number* of a graph. Finally, we briefly discuss GCR as a member of a wider family of multi-player graph pursuit games with rather interesting properties.

Keywords N-player game · Pursuit · Cops and robbers

1 Introduction

We introduce and study *Generalized Cops and Robbers* (*GCR*); it is a *multi-player pursuit game* closely related to the classic two-player *Cops and Robbers* (*CR*) game [13,15].

GCR is played on a finite, simple, undirected graph G by N players $P_1, P_2, \ldots, P_N$ (with $N \geq 2$). The players start at given vertices of the graph, and at each turn, one player moves to a vertex in the closed neighbourhood of his current position; the other players stay at their current vertices. The game effectively terminates when, for some $n \in \{1, 2, \ldots, N-1\}$, P_n *captures* P_{n+1}, i.e., when they are located in the same vertex; if no capture ever takes place, the game continues *ad infinitum*.

Let us denote the GCR game with N players by Γ_N. Then Γ_2 is very similar to the classic CR game, where P_1 (the "cop") tries to capture P_2 (the "robber"). In Γ_3, P_1 tries to capture P_2 who tries to evade P_1 and capture P_3; P_1 can never be captured and P_3 can never capture.

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✉ Ath. Kehagias
kehagiat@ece.auth.gr

[1] Aristotle University of Thessaloniki, Thessaloniki, Greece

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Hence Γ_3 can be understood as two CR games played simultaneously on the same graph; a player is both pursuer and evader *at the same time*. The situation is extended similarly for higher N values.

As we will show, Γ_2 can be formulated as a *zero-sum stochastic game* which has a *value* (and both players have *optimal strategies*). On the other hand, for $N \geq 3$, Γ_N is a nonzero-sum game and the main question is the existence of Nash equilibria (NE). As we will show, more than one such equilibria always exists and they sometimes lead to surprising player behaviour. In this sense, GCR presents novel and (we hope) mathematically interesting problems.

There is a rich literature on pursuit games in graphs, Euclidean spaces and other more general structures, but it is generally confined to two-player games.

The seminal works on pursuit games in graphs are [13,15] in which the classic CR game was introduced. A great number of variations of the classic game have been studied; an extensive and recent review of the related literature appears in the book [1]. However, practically all of this literature concerns *two-player* games. Classic CR and its variants may involve more than one cops, but all of them are *tokens* controlled by a single *cop player*. A very interesting paper [2] deals with "Generalized Cops and Robbers games," but again the scope is restricted to two-player games. In fact, the only previous work (of which we are aware) dealing with *multi-player* games of pursuit in graphs is our own [11]. It is also remarkable that, while classic CR and many of its variants admit a natural game-theoretic formulation and study, this has not been exploited in the CR literature.

Regarding pursuit in Euclidean spaces, the predominant approach is in terms of *differential games* as introduced in the seminal book [9]. There is a flourishing literature on the subject, which contains many works involving multiple pursuers, but they are generally assumed to be *collaborating* [5,10,14,17,19]. The case of antagonistic pursuers has been studied in some papers [8,18], but the methods used in these works do not appear to be easily applicable to the study of pursuit/evasion on *graphs*.

This paper is organized as follows. Section 2 is preliminary: we introduce notation, define *states*, *histories* and *strategies* and give a general form of the *payoff* function. In Sect. 3, we prove that, for any graph and any number of players, GCR has a NE in *deterministic positional strategies*; this result is applicable not only to GCR but to a wider family of pursuit games, as will be discussed later. In Sect. 4, we show that in the two-player GCR game: (i) the *value* of the game exists (essentially it is the logarithm of the *optimal capture time*) and (ii) both players have optimal deterministic positional strategies. Because of the close connection of GCR to the classical CR game, these results also hold for CR; while they have been previously established by graph theoretic methods, we believe our proof is the first game-theoretic one. In Sect. 5, we study the three-player GCR game and prove: (i) the existence of a NE in deterministic positional strategies; (ii) the existence of an additional NE in deterministic but nonpositional strategies; (iii) various results connecting the classic *cop number* of a graph to *capturability*. In Sect. 6, we briefly discuss N-players GCR when $N \geq 4$. In Sect. 7, we show that the ideas behind GCR can be generalized to obtain a large family of multi-player pursuit games on graphs. Finally, in Sect. 8 we summarize, present our conclusions and discuss future research directions.

2 Preliminaries

The following notations will be used throughout the paper.

1. Given a graph $G = (V, E)$, for any $x \in V$, $N(x)$ is the *neighbourhood* of x: $N(x) = \{y : \{x, y\} \in E\}$; $N[x]$ is the *closed neighbourhood* of x: $N[x] = N(x) \cup \{x\}$.

2. The cardinality of set A is denoted by $|A|$; the set of elements of A which are not elements of B is denoted by $A \backslash B$.
3. $\mathbb{N}$ is the set of natural numbers $\{1, 2, 3, \ldots\}$ and $\mathbb{N}_0$ is $\{0, 1, 2, 3, \ldots\}$. For any $M \in \mathbb{N}$, we define $[M] = \{1, 2, \ldots, M\}$.
4. The *graph distance* (length of shortest path in G) between $x, y \in V$ is denoted by $d_G(x, y)$ or simply by $d(x, y)$.

In Sect. 1, we have described GCR informally; now, we define the elements of the game rigorously.

The game proceeds at discrete *turns* (time steps), and at every turn, all players except one must remain at their locations. In other words, at every turn $t \in \mathbb{N}$, for every player except one, the *action set* [see (2.1) below] is a singleton. This, in addition to the fact that all players are aware of all previously executed moves, means that GCR is a *perfect information game*.

Any player P_n can have the first move, but afterwards they move in the sequence implied by their numbering:

$$P_n \rightarrow P_{n+1} \rightarrow \cdots \rightarrow P_N \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots .$$

When a player has the move, he can either move to a vertex adjacent to his current one or stay in place. Hence the *game position* or *game state* has the form $s = \left(x^1, x^2, \ldots, x^N, p\right)$ where $x^n \in V$ is the position (vertex) of the nth player and $p \in [N]$ is the number of the player who has the next move. The set of *nonterminal* states is

$$S = \left\{\left(x^1, x^2, \ldots, x^N, p\right) : \left(x^1, x^2, \ldots, x^N\right) \in V \times V \times \cdots \times V \text{ and } p \in [N]\right\}.$$

We introduce an additional *terminal state* $\overline{s}$. Hence the full state set is

$$\overline{S} = S \cup \{\overline{s}\}.$$

We define S^n to be the set of states in which P_n has the next move:

$$\text{for each } n \in [N] : S^n = \left\{s : s = \left(x^1, x^2, \ldots, x^N, n\right) \in S\right\}.$$

Hence the set of nonterminal states can be partitioned as follows:

$$S = S^1 \cup S^2 \cup \cdots \cup S^N.$$

For any $n \in [N-1]$, we say that P_n *captures* P_{n+1} iff they are located in the same vertex; the set of P_n*-capture states*, i.e., those in which P_n captures P_{n+1} is $\widetilde{S}^n$:

$$\text{for each } n \in [N-1] : \widetilde{S}^n = \left\{s : s = (x^1, x^2, \ldots, x^N, p) \in S \text{ and } x^n = x^{n+1}\right\}.$$

Hence nonterminal states can be partitioned into two sets:

$$\begin{array}{ll} \text{capture states:} & S_C = \widetilde{S}^1 \cup \widetilde{S}^2 \cup \cdots \cup \widetilde{S}^{N-1}, \\ \text{noncapture states:} & S_{NC} = S \backslash S_C. \end{array}$$

As already mentioned, when P_n has the move, he can move to any vertex in the closed neighbourhood of x^n; when another player has the move, P_n can only stay in place; when the game is in a capture state or in the terminal state, every player has only the "null move" λ. Formally, when the game state is s, the nth player's *action set* is denoted by $A^n(s)$ and defined by

$$A^n(s) = \begin{cases} N\left[x^n\right] & \text{when } s = \left(x^1, x^2, \ldots, x^N, n\right) \in S^n \cap S_{NC}, \\ \{x^n\} & \text{when } s = \left(x^1, x^2, \ldots, x^N, m\right) \in S^m \cap S_{NC} \text{ with } m \neq n, \\ \{\lambda\} & \text{when } s \in S_C \cup \{\overline{s}\}. \end{cases} \quad (2.1)$$

The players' *actions* (i.e., *moves*) effect state-to-state transitions in the obvious manner. Suppose the game is at position $s \in S^n$ and P_n makes the move $a^n \in A_n(s)$; then, $\mathbf{T}(s, a^n)$ denotes the resulting game position. A capture state always transits to $\overline{s}$ and $\overline{s}$ always transits to itself:

$$\forall s \in S_C : \mathbf{T}(s, \lambda) = \overline{s} \quad \text{and} \quad \mathbf{T}(\overline{s}, \lambda) = \overline{s}.$$

In what follows x_t^n denotes P_n's position (vertex) at time t. Now we define the *capture time* to be

$$T_C = \min \left\{ t : x_t^1 = x_t^2 \text{ or } x_t^2 = x_t^3 \text{ or } \dots \text{or } x_t^{N-1} = x_t^N \right\}.$$

If no capture takes place, the *capture time* is $T_C = \infty$. Hence the game can evolve as follows.

1. If $T_C = 0$, then the initial state s_0 is a capture state and $s_t = \overline{s}$ for every $t \in \mathbb{N} = \{1, 2, \dots\}$.
2. If $0 < T_C < \infty$, then:
   (a) at the 0th turn the game starts at some preassigned state $s_0 \in S_{NC}$;
   (b) at the tth turn (for $0 < t < T_C$), the game moves to some state $s_t \in S_{NC}$;
   (c) at the T_Cth turn the game moves to some capture state $s_{T_C} \in S_C$ and
   (d) at $t = T_C + 1$ the game moves to the terminal state and stays there: for every $t > T_C$, $s_t = \overline{s}$.
3. Finally, if $T_C = \infty$ then $s_t \in S_{NC}$ for every $t \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

According to the above, the game starts at some preassigned state $s_0 = \left(x_0^1, x_0^2, x_0^3, p_0\right)$ and at the tth turn ($t \in \mathbb{N}$) is in the state $s_t = \left(x_t^1, x_t^2, \dots, x_t^N, p_t\right)$. This results in a *game history* $\mathbf{s} = s_0 s_1 s_2 \dots$. In other words, we assume each play of the game lasts an infinite number of turns; however, if $T_C < \infty$ then $s_t = \overline{s}$ for every $t > T_C$; hence, while the game lasts an infinite number of turns, it *effectively* ends at T_C. We define the following history sets.

1. Histories of length k : $H_k = \{\mathbf{s} = s_0 s_1 \dots s_k\}$;
2. Histories of finite length: $H_* = \cup_{k=1}^{\infty} H_k$;
3. Histories of infinite length: $H_\infty = \{\mathbf{s} = s_0 s_1 \dots s_k \dots\}$.

A *deterministic strategy* (also known as a *pure strategy*) is a function σ^n which assigns a move to each finite-length history:

$$\sigma^n : H_* \to V$$

At the start of the game, P_n selects a σ^n which determines all his subsequent moves. We will only consider *legal*[1] deterministic strategies.[2] A *strategy profile* is a tuple $\sigma = \left(\sigma^1, \sigma^2, \dots, \sigma^N\right)$, which specifies one strategy for each player. We are particularly interested in *positional* strategies, i.e., σ^n such that the next move depends only on the current state of the game (but not on previous states or current time):

$$\sigma^n (s_0 s_1 \dots s_t) = \sigma^n (s_t).$$

We define $\sigma^{-n} = \left(\sigma^j\right)_{j \in [N] \backslash \{n\}}$; for instance, if $\sigma = \left(\sigma^1, \sigma^2, \sigma^3\right)$ then $\sigma^{-1} = \left(\sigma^2, \sigma^3\right)$.

To complete the description of GCR, we must specify the players' *payoff functions*; we will do this in several steps. In this section, we give a general form of the payoff function,

1 That is, they never produce moves outside the player's action set.

2 As will be seen, since GCR is a game of perfect information, the player loses nothing by using only deterministic strategies.

which applies not only to GCR, but to a broader family of N-player games (with $N \geq 2$). In the next section, we will prove that any game of this family admits at least one NE in *positional deterministic* strategies. In subsequent sections, we will treat separately the cases of GCR with $N = 2$, $N = 3$ and $N \geq 4$ players; in each case, by completely specifying the payoff function, we will reach additional conclusions regarding the properties of the respective game.

For the time being, we only specify that the *total payoff* function of the nth player ($n \in [N]$) has the form

$$Q^n(s_0, \sigma) = \sum_{t=0}^{\infty} \gamma^t q^n(s_t), \tag{2.2}$$

where q^n is the *turn payoff* (it depends on s_t, the game state at time t) which is assumed to be bounded:

$$\exists M : \forall n \in [N], \forall s \in S : \left|q^n(s)\right| \leq M;$$

and $\gamma \in (0, 1)$ is the *discount factor*.

Since the total payoff is the sum of the discounted turn payoffs, GCR is a *multi-player discounted stochastic game* [6]. Recall that a stochastic game is one which consists of a sequence of one-shot games, each of which depends on the previous game played and the actions of the players. In GCR, the players can limit themselves to deterministic strategies; since the state transitions are also deterministic, while GCR is a "stochastic game" in the above sense, in all cases of interest it will actually evolve in a deterministic manner.

We will denote by $\Gamma_N(G|s_0)$ the GCR game played by N players on graph G, starting from state s_0. Our results hold for any $\gamma \in (0, 1)$ so, for simplicity of notation, we omit the γ dependence. In addition, γ will be omitted from statements of theorem, lemmas, etc. in the rest of the paper, since all the results presented hold for any $\gamma \in (0, 1)$.

3 Nash Equilibria for Perfect Information Discounted Games

The following theorem establishes that every $\Gamma_N(G|s_0)$ has a NE in *deterministic* (i.e., pure) *positional* strategies. The proof of the theorem is based on a more general theorem by Fink [7], which establishes the existence of a NE in *probabilistic* (i.e., mixed) *positional* strategies; the proof presented here essentially shows that in turn-based (and hence perfect information) games, deterministic strategies can be used without loss to the players. As will be explained a little later, the theorem actually applies to a broad family of games, which includes GCR.

Theorem 3.1 *For every graph G, every $N \geq 2$ and every initial state $s_0 \in S$, the game $\Gamma_N(G|s_0)$ admits a profile of deterministic positional strategies $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \ldots, \widehat{\sigma}^N\right)$ such that*

$$\forall n \in [N], \forall s_0 \in S, \forall \sigma^n : Q^n\left(s_0, \widehat{\sigma}^n, \widehat{\sigma}^{-n}\right) \geq Q^n\left(s_0, \sigma^n, \widehat{\sigma}^{-n}\right). \tag{3.1}$$

For every s and n, let $u^n(s) = Q^n(s, \widehat{\sigma})$. Then the following equations are satisfied

$$\forall n, \forall s \in S^n : \widehat{\sigma}^n(s) = \arg \max_{a^n \in A^n(s)} \left[q^n(s) + \gamma u^n\left(\mathbf{T}\left(s, a^n\right)\right)\right], \tag{3.2}$$

$$\forall n, m, \forall s \in S^n : u^m(s) = q^m(s) + \gamma u^m\left(\mathbf{T}\left(s, \widehat{\sigma}^n(s)\right)\right). \tag{3.3}$$

Proof Fink has proved in [7] that every N-player discounted stochastic game has a positional NE in *probabilistic* strategies; this result holds for the general game (i.e., with *concurrent* moves and probabilistic strategies and state transitions). According to [7], at equilibrium the following equations must be satisfied for all m and s:

$$\mathfrak{u}^m(s) = \max_{\mathbf{p}^m(s)} \sum_{a^1\in A^1(s)} \sum_{a^2\in A^2(s)} \cdots \sum_{a^N\in A^N(s)} p^1_{a^1}(s)\, p^2_{a^2}(s) \ldots p^N_{a^N}(s) \times \left[q^m(s) + \gamma \sum_{s'} \Pi\left(s'|s, a^1, a^2, \ldots, a^N\right) \mathfrak{u}^m\left(s'\right)\right], \tag{3.4}$$

where we have modified Fink's original notation to fit our own; in particular:

1. $\mathfrak{u}^m(s)$ is the expected value of $u^m(s)$;
2. $p^m_{a^m}(s)$ is the probability that, given the current game state is s, the mth player plays action a^m;
3. $\mathbf{p}^m(s) = \left(p^m_{a^m}(s)\right)_{a^m\in A^m(s)}$ is the vector of all such probabilities (one probability per available action);
4. $\Pi\left(s'|s, a^1, a^2, \ldots, a^N\right)$ is the probability that, given the current state is s and the player actions are $a^1, a^2, \ldots, a^N$, the next state is s'.

Now choose any n and any $s \in S^n$. For all $m \neq n$, the mth player has a single move, i.e., we have $A^m(s) = \{a^m\}$, and so $p^m_{a_m}(s) = 1$. Also, since transitions are deterministic,

$$\sum_{s'} \Pi\left(s'|s, a^1, a^2, \ldots, a^N\right) \mathfrak{u}^n\left(s'\right) = \mathfrak{u}^n\left(\mathbf{T}\left(s, a^n\right)\right).$$

Hence, for $m = n$, (3.4) becomes

$$\mathfrak{u}^n(s) = \max_{\mathbf{p}^n(s)} \sum_{a^n\in A^n(s)} p^n_{a^n}(s)\left[q^n(s) + \gamma\mathfrak{u}^n\left(\mathbf{T}\left(s, a^n\right)\right)\right]. \tag{3.5}$$

Furthermore, let us define $\widehat{\sigma}^n(s)$ (for the specific s and n) by

$$\widehat{\sigma}^n(s) = \arg\max_{a^n\in A^n(s)} \left[q^n(s) + \gamma\mathfrak{u}^n\left(\mathbf{T}\left(s, a^n\right)\right)\right]. \tag{3.6}$$

If (3.5) is satisfied by more than one a^n, we set $\widehat{\sigma}^n(s)$ to one of these arbitrarily. Then, to maximize the sum in (3.5) the nth player can set $p^n_{\widehat{\sigma}^n(s)}(s) = 1$ and $p^n_a(s) = 0$ for all $a \neq \widehat{\sigma}^n(s)$. Since this is true for all states and all players (i.e., every player can, without loss, use deterministic strategies), we also have $\mathfrak{u}^n(s) = u^n(s)$. Hence (3.5) becomes

$$u^n(s) = \max_{a^n\in A^n(s)} \left[q^n(s) + \gamma u^n\left(\mathbf{T}\left(s, a^n\right)\right)\right] = q^n(s) + \gamma u^n\left(\mathbf{T}\left(s, \widehat{\sigma}^n(s)\right)\right). \tag{3.7}$$

For $m \neq n$, the mth player has no choice of action and (3.5) becomes

$$u^m(s) = q^m(s) + \gamma u^m\left(\mathbf{T}\left(s, \widehat{\sigma}^n(s)\right)\right). \tag{3.8}$$

We recognize that (3.6)–(3.8) are (3.2)–(3.3). Also, (3.6) defines $\widehat{\sigma}^n(s)$ for every n and s and so we have obtained the required deterministic positional strategies $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \ldots, \widehat{\sigma}^N\right)$. □

Note that the initial state s_0 plays no special role in the system (3.2)–(3.3). In other words, using the notation $u(s) = \left(u^1(s), u^2(s), \ldots, u^N(s)\right)$ and $\mathbf{u} = (u(s))_{s\in S}$, we see that $\mathbf{u}$ and $\widehat{\sigma}$ are the same for every starting position s_0 and every game $\Gamma_N(G|s_0)$ (when N, G and γ are fixed).

Fink's proof requires that, for every n, the total payoff is $Q^n(s_0, \sigma) = \sum_{t=0}^{\infty} \gamma^t q^n(s_t)$, but does not place any restrictions (except boundedness) on q^n. The same is true of our proof;

hence, Theorem 3.1 applies not only to the GCR game, for which the form of q^n will be specified in Sects. 4, 5 and 6, but to a wider family of games, which will be discussed in Sect. 7.

4 GCR with Two Players and (Classic CR)

We now proceed to a more detailed study of $\Gamma_2 (G|s_0)$. To this end, we first specify the form of the turn payoff functions q^1 and q^2:

$$q^1(s) = -q^2(s) = \begin{cases} 1 & \text{iff } s \in \widetilde{S}^1 \\ 0 & \text{else.} \end{cases} \quad . \tag{4.1}$$

Recalling that T_C is the capture time (and letting $\gamma^{\infty} = 0$), for every s_0 and deterministic σ which result in capture at time T_C, we clearly have:

$$Q^1(s_0, \sigma) = -Q^2(s_0, \sigma) = \gamma^{T_C}.$$

So $\Gamma_2 (G|s_0)$ is a zero-sum game. Furthermore, since γ^{T_C} is a decreasing function of T_C it follows that P_1 (resp. P_2) will maximize his payoff by minimizing (resp. maximizing) capture time T_c. Hence we have the following simple description:

> $\Gamma_2 (G|s_0)$ is a two-player game in which, starting from an initial position $s_0 = \left(x^1, x^2, p\right)$, P_1 attempts to capture P_2 in the shortest possible time and P_2 attempts to delay capture as long as possible.

This is true whenever both players use deterministic strategies, which they can do without loss since $\Gamma_2 (G|s_0)$ is a perfect information game. In particular, according to Theorem 3.1 , this holds when they play optimally. In fact, according to Theorem 3.1 (for every G and s_0) $\Gamma_2 (G|s_0)$ has a NE $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2\right)$ in deterministic positional strategies. And, since $\Gamma_2 (G|s_0)$ is a zero-sum game, it follows that $\widehat{\sigma}^1, \widehat{\sigma}^2$ are *optimal* and yield the *value* of the game. More precisely, we have the following.

Theorem 4.1 *For every graph G and every initial state $s_0 \in S$, the profile of deterministic positional strategies $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2\right)$ specified by Theorem 3.1 satisfies*

$$\max_{\sigma^1} \min_{\sigma^2} Q^1\left(s_0, \sigma^1, \sigma^2\right) = Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2\right) = \min_{\sigma^2} \max_{\sigma^1} Q^1\left(s_0, \sigma^1, \sigma^2\right).$$

Furthermore, $\widehat{\sigma}^1, \widehat{\sigma}^2$ and $Q^n\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2\right)$ can be computed by a *value iteration algorithm* [16]. Hence $\Gamma_2 (G|s_0)$ is completely solved.

Let us now discuss the connection of $\Gamma_2 (G|s_0)$ to the classic CR game. Note that the above description of $\Gamma_2 (G|s_0)$ is almost identical to that of the *time optimal* version of the classic CR game (e.g., see [1, Section 8.6]). We only have the following differences.

1. In $\Gamma_2 (G|s_0)$, time is measured in turns; in classic CR, it is measured in *rounds*, where each round consists of one P_1 turn and one P_2 turn.
2. In $\Gamma_2 (G|s_0)$, the starting position s_0 is *given*; in classic CR, it is chosen by the players, in an initial "*placement*" round. In other words, classic CR starts with an "empty" graph; in the first turn of the 0th round, P_1 chooses his initial position; in the second turn P_2, having observed P_1's placement chooses his initial position (after placement, classic CR is played exactly as $\Gamma_2 (G|s_0)$).[3]

[3] It is worth noting that $\Gamma_2 (G|s_0)$ can be expanded to incorporate a placement round as well. This is done as follows. Using ξ to denote that a player is positioned "outside" the graph, we must introduce a starting

At any rate, the important points are the following.

1. Having computed the values $u(s_0)$ of $\Gamma_2(G|s_0)$ for every $s_0 \in S$, we can easily obtain the *optimal capture time* $\widehat{T}_C$ of the classic CR game[4] as follows:

$$\widehat{T}_C = \frac{\log\left(\max_{x^1}\min_{x^2} u^1\left(\left(x^1, x^2, 1\right)\right)\right)}{\log\gamma};$$

   furthermore, any $\widehat{x}^1, \widehat{x}^2$ which satisfy $\widehat{T}_C = \frac{\log u^1\left(\left(\widehat{x}^1, \widehat{x}^2, 1\right)\right)}{\log\gamma}$ are optimal initial placements for P_1 and P_2; and the optimal policies of $\Gamma_2(G|s_0)$ are time optimal policies (after placement) of the classic CR.
2. In the classic CR literature, a graph G is called *cop-win* iff a single cop can capture the robber when both cop and robber play optimally on G. In the more general case, where the cop player controls one or more cop *tokens*, the *cop number* of G is denoted by $c(G)$ and defined to be the smallest number of cop tokens which guarantees capture when CR is played optimally on G. Clearly a graph is cop-win iff $c(G) = 1$. It is easily seen that we can check whether G is cop-win by solving $\Gamma_2(G|s_0)$ (for all s_0) as indicated by the following equivalence:

$$c(G) = 1 \Leftrightarrow \max_{x^1}\min_{x^2} u^1\left(\left(\widehat{x}^1, \widehat{x}^2, 1\right)\right) > 0. \tag{4.2}$$

While the above questions regarding classic CR have been studied in the related literature and answered using graph theoretic methods, the connection to game theory appears to not have been previously exploited.

5 GCR with Three Players

By substituting $N = 3$ in the definitions of Sect. 2, we obtain the game $\Gamma_3(G|s_0)$; in particular, we get the sets of capture states

$$\widetilde{S}^1 = \left\{s : \left(x^1, x^2, x^3, p\right), x^1 = x^2\right\} \quad (P_1 \text{ captures } P_2),$$
$$\widetilde{S}^2 = \left\{s : \left(x^1, x^2, x^3, p\right), x^2 = x^3\right\} \quad (P_2 \text{ captures } P_3)$$

and we use these to define the turn payoffs q^n as follows

$$q^1(s) = \begin{cases} 1 & \text{iff} \quad s \in \widetilde{S}^1, \\ 0 & \text{else;} \end{cases} \quad q^2(s) = \begin{cases} -1 & \text{iff} \quad s \in \widetilde{S}^1, \\ 1 & \text{iff} \quad s \in \widetilde{S}^2 \backslash \widetilde{S}^1, \\ 0 & \text{else;} \end{cases}$$
$$q^3(s) = \begin{cases} -1 & \text{iff} \quad s \in \widetilde{S}^2 \backslash \widetilde{S}^1, \\ 0 & \text{else.} \end{cases} \quad . \tag{5.1}$$

Note that, according to previous remarks, P_2 (resp. P_3) is rewarded (resp. penalized) when P_2 captures P_3 *and is not simultaneously captured by* P_1. Also, recall that the total payoff function is, as usual,

Footnote 3 continued
state (ξ, ξ) (no player has played yet) and, for each $x \in V$, a state (x, ξ) (P_1 has already played his first move but not P_2). Actions, state transitions, payoffs, etc. can be similarly modified to represent the "classic" CR move sequence. Under this approach (which can also be applied to the $\Gamma_N(G|s_0)$ games introduced in later sections), all our essential results still hold. This route is eschewed in the current paper, for reasons of simplicity.

[4] Up to a time rescaling, due to the above-mentioned difference in time units.

$$\forall n \in [3] : Q^n (s_0, s_1, \ldots) = \sum_{t=0}^{\infty} \gamma^t q^n (s_t) .$$

We are now ready to study $\Gamma_3 (G|s_0)$.

5.1 Nash Equilibria: Positional and Nonpositional

By Theorem 3.1, we know that $\Gamma_3 (G|s_0)$ has, for every G and s_0, a NE in deterministic positional strategies. In addition, as we will now show, $\Gamma_3 (G|s_0)$ has at least one NE in *nonpositional* deterministic strategies.

To this end, we will introduce a family of *auxiliary games* and *threat strategies* [3,4,20]. For every $n \in [3]$, we define the game $\widetilde{\Gamma}_3^n (G|s_0)$ played on G (and starting at s_0) by P_n against a player P_{-n} who controls the remaining two entities. For example, in $\widetilde{\Gamma}_3^1 (G|s_0)$, P_1 plays against P_{-1} who controls P_2 and P_3. The $\widetilde{\Gamma}_3^n (G|s_0)$ elements (e.g., movement sequence, states, action sets and capturing conditions) are the same as in $\Gamma_3 (G|s_0)$. P_n uses a strategy σ^n and P_{-n} uses a strategy profile σ^{-n}; these form a strategy profile $\sigma = \left(\sigma^1, \sigma^2, \sigma^3\right)$ (which can also be used in $\Gamma_3 (G|s_0)$). The payoffs to P_n and P_{-n} in $\widetilde{\Gamma}_3^n (G|s_0)$ are

$$\widetilde{Q}^n (s_0, \sigma) = Q^n (s_0, \sigma) = \sum_{t=0}^{\infty} \gamma^t q^n (s_t) \quad \text{and} \quad \widetilde{Q}^{-n} (s_0, \sigma) = -\widetilde{Q}^n (s_0, \sigma) .$$

Since the capture rules of $\widetilde{\Gamma}_3^n (G|s_0)$ are those of $\Gamma_3 (G|s_0)$, P_{-n} can use one of his tokens to capture the other. For instance, in $\widetilde{\Gamma}_3^1 (G|s_0)$, P_{-1} can use P_2 to capture P_3. (As will be seen in a later example, in certain cases this can be an optimal move.) Note, however, that in this case P_1 receives zero payoff (since he did not capture) and P_{-1} also receives zero payoff (since, by construction, $\widetilde{\Gamma}_3^1 (G|s_0)$ is a zero-sum game).

In short, $\widetilde{\Gamma}_3^n (G|s_0)$ is a two-player *zero-sum* discounted stochastic game and the next lemma follows from the results of [6, Theorem 4.3.2].

Lemma 5.1 *For every n, G and s_0, the game $\widetilde{\Gamma}_3^n (G|s_0)$ has a value and the players have optimal deterministic positional strategies.*

Furthermore, the value and optimal strategies can be computed by Shapley's value iteration algorithm [16]. Let us denote by $\widehat{\phi}_n^n$ (resp. $\widehat{\phi}_n^{-n}$) the optimal strategy of P_n (resp. P_{-n}) in $\widetilde{\Gamma}_3^n (G|s_0)$. For example, in $\widetilde{\Gamma}_3^1 (G|s_0)$, P_1 has the optimal strategy $\widehat{\phi}_1^1$ and P_{-1} has the optimal strategy $\widehat{\phi}_1^{-1} = \left(\widehat{\phi}_1^2, \widehat{\phi}_1^3\right)$. In fact, the same $\widehat{\phi}_n^m$'s (for fixed n and any $m \in [3]$) are optimal in $\widetilde{\Gamma}_3^n (G|s_0)$ for every initial position s_0.

We return to $\Gamma_3 (G|s_0)$, and for each P_n, we introduce the *threat strategy* $\widehat{\pi}^n$ defined as follows:

1. as long as every player P_m (with $m \neq n$) follows $\widehat{\phi}_m^m$, P_n follows $\widehat{\phi}_n^n$;
2. as soon as some player P_m (with $m \neq n$) deviates from $\widehat{\phi}_m^m$, P_n switches to $\widehat{\phi}_m^n$ and uses it for the rest of the game.[5]

Note that the $\widehat{\pi}^n$ strategies are *not* positional. In particular, the action of a player at time t may be influenced by the action (deviation) performed by another player at time $t-2$. However, as we will now prove, $\left(\widehat{\pi}^1, \widehat{\pi}^2, \widehat{\pi}^3\right)$ is a (nonpositional) NE in $\Gamma_3 (G|s_0)$.

[5] Since $\Gamma_3 (G|s_0)$ is a perfect information game, the deviation will be detected immediately.

Theorem 5.2 *For every G, s_0 and γ, we have:*

$$\forall n \in \{1,2,3\}, \forall \pi^n : Q^n(s, \widehat{\pi}^1, \widehat{\pi}^2, \widehat{\pi}^3) \geq Q^n(s, \pi^n, \widehat{\pi}^{-n}). \tag{5.2}$$

Proof We choose some initial state s_0 and fix it for the rest of the proof. Now let us prove (5.2) for the case $n = 1$. In other words, we will show that

$$\forall \pi^1 : Q^1(s_0, \widehat{\pi}^1, \widehat{\pi}^2, \widehat{\pi}^3) \geq Q^1(s_0, \pi^1, \widehat{\pi}^2, \widehat{\pi}^3). \tag{5.3}$$

We take any π^1 and let

$$\text{the history produced by } (\widehat{\pi}^1, \widehat{\pi}^2, \widehat{\pi}^3) \text{ be } \widehat{\mathbf{s}} = \widehat{s}_0\widehat{s}_1\widehat{s}_2 \ldots,$$
$$\text{the history produced by } (\pi^1, \widehat{\pi}^2, \widehat{\pi}^3) \text{ be } \widetilde{\mathbf{s}} = \widetilde{s}_0\widetilde{s}_1\widetilde{s}_2 \ldots,$$

(where $\widehat{s}_0 = \widetilde{s}_0 = s_0$). We define T_1 as the earliest time in which π^1 and $\widehat{\pi}^1$ produce different states:

$$T_1 = \min\{t : \widetilde{s}_t \neq \widehat{s}_t\},$$

If $T_1 = \infty$, then $\widetilde{\mathbf{s}} = \widehat{\mathbf{s}}$ and

$$Q^1(s, \widehat{\pi}^1, \widehat{\pi}^2, \widehat{\pi}^3) = Q^1(s, \pi^1, \widehat{\pi}^2, \widehat{\pi}^3). \tag{5.4}$$

If $T_1 < \infty$, on the other hand, then $\widetilde{s}_t = \widehat{s}_t$ for every $t < T_1$ and we have

$$Q^1(s, \widehat{\pi}^1, \widehat{\pi}^2, \widehat{\pi}^3) = \sum_{t=0}^{T_1-2} \gamma^t q^1(\widehat{s}_t) + \sum_{t=T_1-1}^{\infty} \gamma^t q^1(\widehat{s}_t) = \sum_{t=0}^{T_1-2} \gamma^t q^1(\widetilde{s}_t) + \sum_{t=T_1-1}^{\infty} \gamma^t q^1(\widehat{s}_t), \tag{5.5}$$

$$Q^1(s, \pi^1, \widehat{\pi}^2, \widehat{\pi}^3) = \sum_{t=0}^{T_1-2} \gamma^t q^1(\widetilde{s}_t) + \sum_{t=T_1-1}^{\infty} \gamma^t q^1(\widetilde{s}_t). \tag{5.6}$$

We define $s^* = \widehat{s}_{T_1-1} = \widetilde{s}_{T_1-1}$ and proceed to compare the sums in (5.5) and (5.6).

First consider $\sum_{t=T_1-1}^{\infty} \gamma^t q^1(\widehat{s}_t)$. The history $\widehat{\mathbf{s}} = \widehat{s}_0\widehat{s}_1\widehat{s}_2 \ldots$ is produced by $(\widehat{\phi}_1^1, \widehat{\phi}_2^2, \widehat{\phi}_3^3)$ and, since the $\widehat{\phi}_n^n$'s are positional strategies, we have

$$\sum_{t=T_1-1}^{\infty} \gamma^t q^1(\widehat{s}_t) = \gamma^{T_1-1} \sum_{t=0}^{\infty} \gamma^t q^1\left(\widehat{s}_{T_1-1+t}\right) = \gamma^{T_1-1} \widetilde{Q}^1\left(s^*, \widehat{\phi}_1^1, \widehat{\phi}_2^2, \widehat{\phi}_3^3\right), \tag{5.7}$$

i.e., up to the multiplicative constant γ^{T_1-1}, the sum in (5.7) is the payoff to P_1 in $\widetilde{\Gamma}_3^1(G|s^*)$, under the strategies $\widehat{\phi}_1^1, \left(\widehat{\phi}_2^2, \widehat{\phi}_3^3\right)$. Since $\widetilde{\Gamma}_3^1(G|s^*)$ is a zero-sum game in which the optimal response to $\widehat{\phi}_1^1$ is $\left(\widehat{\phi}_1^2, \widehat{\phi}_1^3\right)$, we have

$$\gamma^{T_1-1} \widetilde{Q}^1\left(s^*, \widehat{\phi}_1^1, \widehat{\phi}_2^2, \widehat{\phi}_3^3\right) \geq \gamma^{T_1-1} \widetilde{Q}^1\left(s^*, \widehat{\phi}_1^1, \widehat{\phi}_1^2, \widehat{\phi}_1^3\right). \tag{5.8}$$

Next consider $\sum_{t=T_1-1}^{\infty} \gamma^t q^1(\widetilde{s}_t)$. The history $\widetilde{\mathbf{s}} = \widetilde{s}_0\widetilde{s}_1\widetilde{s}_2 \ldots$ is produced by $(\pi^1, \widehat{\phi}_1^2, \widehat{\phi}_1^3)$ and, since π^1 is not necessarily positional, $\widetilde{s}_{T_1}\widetilde{s}_{T_1+1}\widetilde{s}_{T_1+2} \ldots$ may depend on $\widetilde{s}_0\widetilde{s}_1 \ldots \widetilde{s}_{T_1-2}$. However, we can introduce a (not necessarily positional) strategy ρ^1 which will produce the same history $\widetilde{s}_{T_1}\widetilde{s}_{T_1+1}\widetilde{s}_{T_1+2} \ldots$ as σ^1.[6] Then, since in $\widetilde{\Gamma}_3^1(G|s^*)$ the optimal response to $\left(\widehat{\phi}_1^2, \widehat{\phi}_1^3\right)$ is $\widehat{\phi}_1^1$, we have

[6] We define ρ^1 such that, when combined with $\widetilde{s}_{T_1-1}, \widehat{\phi}_1^2, \widehat{\phi}_1^3$, will produce the same history $\widetilde{s}_{T_1}\widetilde{s}_{T_1+1}\widetilde{s}_{T_1+2} \ldots$ as σ^1. Note that ρ^1 will in general depend (in an indirect way) on $\widetilde{s}_0\widetilde{s}_1 \ldots \widetilde{s}_{T_1-2}$.

$$\gamma^{T_1-1}\widetilde{Q}^1\left(s^*,\widehat{\phi}_1^1,\widehat{\phi}_1^2,\widehat{\phi}_1^3\right) \geq \gamma^{T_1-1}\widetilde{Q}^1\left(s^*,\rho^1,\widehat{\phi}_1^2,\widehat{\phi}_1^3\right) = \sum_{t=T_1-1}^{\infty}\gamma^t q^1\left(\widetilde{s}_t\right). \tag{5.9}$$

Combining (5.5)–(5.9), we have:

$$\begin{aligned} Q^1(s_0,\widehat{\pi}^1,\widehat{\pi}^2,\widehat{\pi}^3) &= \sum_{t=0}^{T_1-2}\gamma^t q^1\left(\widetilde{s}_t\right) + \gamma^{T_1-1}\widetilde{Q}^1(s^*,\widehat{\phi}_1^1,\widehat{\phi}_2^2,\widehat{\phi}_3^3) \\ &\geq \sum_{t=0}^{T_1-2}\gamma^t q^1\left(\widetilde{s}_t\right) + \gamma^{T_1-1}\widetilde{Q}^1(s^*,\widehat{\phi}_1^1,\widehat{\phi}_1^2,\widehat{\phi}_1^3) \\ &\geq \sum_{t=0}^{T_1-2}\gamma^t q^1\left(\widetilde{s}_t\right) + \gamma^{T_1-1}\widetilde{Q}^1(s^*,\rho^1,\widehat{\phi}_1^2,\widehat{\phi}_1^3) = Q^1(s,\pi^1,\widehat{\pi}^2,\widehat{\pi}^3). \end{aligned}$$

and we have proved (5.3), which is (5.2) for $n = 1$. The proof for the cases $n = 2$ and $n = 3$ is similar and hence omitted. □

We have seen that every $\Gamma_3(G|s_0)$ has at least two deterministic NE (one in positional strategies and another in *non*positional ones); and in fact, as is well known, a stochastic game may possess any number of NE. On the other hand, we only know how to compute a single NE of $\Gamma_3(G|s_0)$, namely the nonpositional one of Theorem 5.2, which is constructed in terms of the two-player strategies of $\widetilde{\Gamma}_3^n(G|s_0)$. One may be tempted to construct additional NE of $\Gamma_3(G|s_0)$ using the optimal strategies of $\Gamma_2(G|s_0)$. For example, one may reason as follows: P_3's best chance to avoid capture in $\Gamma_3(G|s_0)$ is by ignoring P_1 and playing his best (in $\Gamma_2(G|s_0)$) evasion strategy against P_2. By a similar reasoning for the other players, one may conclude that $\left(\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)$ is a NE of $\Gamma_3(G|s_0)$ if (i) $\left(\widehat{\sigma}^1,\widehat{\sigma}^2\right)$ is a NE of $\Gamma_2(G|s_0)$ played between P_1 and P_2, and (ii) $\left(\widehat{\sigma}^2,\widehat{\sigma}^3\right)$ is a NE of $\Gamma_2(G|s_0)$ played between P_2 and P_3.[7] But this conclusion is wrong, as shown by the following example.

Example 5.3 Take the graph of Fig. 1 with the players positioned as indicated (i.e., $x_0^1 = 1, x_0^2 = 12, x_0^3 = 2$). Now consider $\Gamma_2(G|(12,2,3))$ played on G by P_2 (as pursuer) and P_3 (as evader); suppose for the time being that P_1 is not on the graph. In $\Gamma_2(G|(12,2,3))$, P_3's best strategy is to move towards vertex 10, postponing capture as long as possible; P_2's best strategy is to always move towards P_3. Next consider $\Gamma_3(G|(1,12,2,3))$ with P_1, P_2, P_3 having their usual roles. In this game, P_3's best strategy is to first move into vertex 1 and afterwards always keep P_1 between himself and P_2; he can always achieve this and thus avoid capture ad infinitum. And P_2's best strategy is to stay at vertex 12, keeping away from P_1 for as long as possible. So in this example P_2 and P_3's optimal $\Gamma_2(G|(12,2,3))$ strategies are not good (and certainly not in NE) in $\Gamma_3(G|(1,12,2,3))$.

[7] A clarification is needed here: the domain of $\Gamma_2(G|s_0)$ (positional) strategies is $V \times V \times \{1,2\}$, while the domain of $\Gamma_3(G|s_0)$ (positional) strategies is $V \times V \times V \times \{1,2\}$. However, we can "extend" a $\Gamma_2(G|s_0)$ strategy to use it in $\Gamma_3(G|s_0)$. For example, suppose $\sigma^1\left(x^1,x^2\right)$ is a P_1 strategy in $\Gamma_2(G|s_0)$; then, it can also be extended to a $\Gamma_3(G|s_0)$ strategy $\widetilde{\sigma}^1\left(x^1,x^2,x^3\right)$ by letting

$$\forall x^3 : \widetilde{\sigma}^1\left(x^1,x^2,x^3\right) = \sigma^1\left(x^1,x^2\right).$$

In other words, P_1 applies σ^1 in $\Gamma_3(G|s_0)$ by ignoring P_3's position. We will often use this and similar constructions in what follows, without further comment; and we will denote the $\Gamma_2(G|s_0)$ and $\Gamma_3(G|s_0)$ strategies by the same symbol, e.g., σ^n.

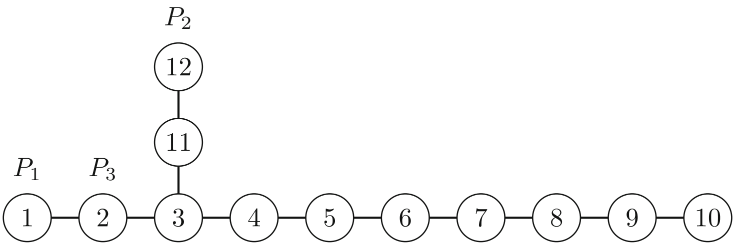


Fig. 1 A case in which the CR optimal strategies do not achieve NE in $\Gamma_3(G|s_0)$

5.2 Capturability

In Sect. 4, we have presented a connection between the cop number of G and "capturability" in $\Gamma_2(G|s_0)$; this was described by (4.2) which can be equivalently rewritten as

$$(\forall s_0 \text{ every optimal } \widehat{\sigma} \text{ of } \Gamma_2(G|s_0) \text{ results in capture}) \Leftrightarrow c(G) = 1. \tag{5.10}$$

The analogue of (5.10) in $\Gamma_3(G|s_0)$ would be:

$$(\forall s_0 \text{ every NE } \widehat{\sigma} \text{ of } \Gamma_3(G|s_0) \text{ results in capture}) \Leftrightarrow c(G) = 1. \tag{5.11}$$

As will be seen, (5.11) *is not true*. But connections between cop number and capturability exist, as will be established in the remainder of this section. To this end, we first define the capture function $\mathbf{K}_3(G|s_0, \sigma)$.

Definition 5.4 For the game $\Gamma_3(G|s_0)$ played with strategies $\sigma = \left(\sigma^1, \sigma^2, \sigma^3\right)$, we write

$$\mathbf{K}_3(G|s_0, \sigma) = \begin{cases} 0 & \text{when} \quad Q^1(s_0, \sigma) = Q^2(s_0, \sigma) = Q^3(s_0, \sigma) = 0, \\ 1 & \text{when} \quad Q^1(s_0, \sigma) > 0, \\ 2 & \text{when} \quad Q^2(s_0, \sigma) > 0. \end{cases}$$

Roughly, $\mathbf{K}_3(G|s_0, \sigma)$ tells us which player (if any) achieves a capture in $\Gamma_3(G|s_0)$ played with $\left(\sigma^1, \sigma^2, \sigma^3\right)$:

1. $\mathbf{K}_3(G|s_0, \sigma) = 0 \Leftrightarrow Q^1(s_0, \sigma) = Q^2(s_0, \sigma) = Q^3(s_0, \sigma) = 0 \Leftrightarrow$ no capture takes place;
2. $\mathbf{K}_3(G|s_0, \sigma) = 1 \Leftrightarrow Q^1(s_0, \sigma) > 0 \Leftrightarrow P_1$ captures P_2;
3. $\mathbf{K}_3(G|s_0, \sigma) = 2 \Leftrightarrow Q^2(s_0, \sigma) > 0 \Leftrightarrow P_2$ captures P_3 (and avoids being captured by P_1).

A weaker version of (5.11) is:

$$(\forall s_0 \text{ there exists a capturing NE } \widehat{\sigma} \text{ of } \Gamma_3(G|s_0)) \Rightarrow c(G) = 1$$

and this can be rewritten and proved in terms of $\mathbf{K}_3(G|s_0, \sigma)$, as follows.

Theorem 5.5 *The following holds for every G:*

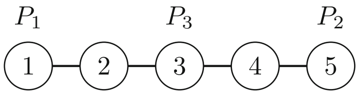
$$(\forall s_0 \textit{ there exists a NE } \widehat{\sigma} \textit{ of } \Gamma_3(G|s_0): \ \mathbf{K}_3(G|s_0, \widehat{\sigma}) > 0) \Rightarrow c(G) = 1. \tag{5.12}$$

Proof To prove the theorem, we will assume

$$(\forall s_0 \text{ there exists a NE } \widehat{\sigma} \text{ of } \Gamma_3(G|s_0): \ \mathbf{K}_3(G|s_0, \widehat{\sigma}) > 0) \text{ and } c(G) > 1 \tag{5.13}$$

and reach a contradiction. To this end, choose $s_0 = \left(x^1, x^2, x^3, 1\right)$ as follows.

Fig. 2 A graph G in which $\Gamma_3\left(G|s_0\right)$ has a noncapturing NE



1. Take arbitrary x^1.
2. Take some x^2 such that for $s_0' = \left(x^1, x^2, 1\right)$ there exists a $\overline{\sigma}^2$ which is escaping in $\Gamma_2\left(G|s_0'\right)$. (This is always possible, since $c\left(G\right) > 1$.)
3. Take some x^3 such that for $s_0'' = \left(x^2, x^3, 1\right)$ there exists a $\overline{\sigma}^3$ which is escaping in $\Gamma_2\left(G|s_0''\right)$. (This is always possible, since $c\left(G\right) > 1$.)

Now let $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ be a capturing NE of $\Gamma_3\left(G|s_0\right)$ and consider the following cases.

1. $\mathbf{K}_3\left(G|s_0, \widehat{\sigma}\right) = 1$; then, for some T_1 we will have
$$Q^2\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = -\gamma^{T_1} < 0 \leq Q^2\left(s_0, \widehat{\sigma}^1, \overline{\sigma}^2, \widehat{\sigma}^3\right).$$
Thus, when P_1 and P_2 play $\widehat{\sigma}^1$ and $\overline{\sigma}^2$, respectively, P_2 will always escape P_1. (Since P_3 can never influence, P_2 moves.) And furthermore, P_2 may in fact capture P_3, since $\widehat{\sigma}^3$ is not necessarily an escaping strategy.
2. $\mathbf{K}_3\left(G|s_0, \widehat{\sigma}\right) = 2$; then, for some T_2 we will have
$$Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = -\gamma^{T_2} < 0 = Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \overline{\sigma}^3\right)$$
Thus, when P_2 and P_3 play $\widehat{\sigma}^2$ and $\overline{\sigma}^3$, respectively, P_3 will always escape P_2. (Since P_1 can never influence, P_3 moves.)

Since in every case some P_n can unilaterally improve $Q^n\left(s_0, \widehat{\sigma}\right)$, $\widehat{\sigma}$ cannot be a NE of $\Gamma_3\left(G|s_0\right)$. □

Hence for every cop-win graph G and every starting state s_0, $\Gamma_3\left(G|s_0\right)$ has a capturing NE. However, perhaps surprisingly, there exist cop-win graphs and starting states for which $\Gamma_3\left(G|s_0\right)$ also has noncapturing NE, as the following example shows.

Example 5.6 Take a path with P_1 and P_2 at the endpoints and P_3 at the middle, as shown in Fig. 2.

The strategy profile $\overline{\sigma} = \left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)$ is defined as follows.

1. $\overline{\sigma}^1$: P_1 stays in place as long as P_2 does not move; if P_2 moves, P_1 chases him.
2. $\overline{\sigma}^2$: P_2 stays in place as long as P_3 does not move; if P_3 moves, P_2 chases him.
3. $\overline{\sigma}^3$: P_3 stays in place as long as nobody moves; if P_1 moves, P_3 goes towards P_2; if P_2 moves, P_3 goes towards P_1.

We will now show $\overline{\sigma}$ is a noncapturing NE of $\Gamma\left(G|s_0\right)$. Obviously we have
$$\forall n \in \{1, 2, 3\} : Q^n\left(s_0, \left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)\right) = 0.$$
We will show no player profits by unilaterally changing his strategy.

1. Say P_1 uses any strategy σ^1. If, by σ^1, he moves at some time, then P_3 goes towards P_2 and P_2 goes towards P_3 resulting in a capture of P_3 by P_2. Hence
$$Q^1\left(s_0, \left(\sigma^1, \overline{\sigma}^2, \overline{\sigma}^3\right)\right) = 0 = Q^1\left(s_0, \left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)\right). \tag{5.14}$$

2. Say P_2 uses any strategy σ^2. If, by σ^2, he moves at some time, then P_3 goes towards P_1 and P_1 goes towards P_2 resulting in a capture of P_2 by P_1. Hence

$$Q^2\left(s_0, \left(\overline{\sigma}^1, \sigma^2, \overline{\sigma}^3\right)\right) < 0 = Q^2\left(s_0, \left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)\right). \tag{5.15}$$

3. Say P_3 uses any strategy σ^3. If, by σ^3, he moves at some time, then P_2 goes towards P_3 and then P_1 goes towards P_2. Depending on P_3's moves, we may have a capture of P_2 by P_1 or of P_3 by P_2. In either case (by the upper bound on P_3's payoff),

$$Q^3\left(s_0, \left(\overline{\sigma}^1, \overline{\sigma}^2, \sigma^3\right)\right) \leq 0 = Q^3\left(s_0, \left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)\right). \tag{5.16}$$

Combining (5.14)–(5.16), we get

$$\forall n \in \{1, 2, 3\}, \forall \sigma^n : Q^n\left(s_0, \left(\sigma^n, \overline{\sigma}^{-n}\right)\right) \leq Q^n\left(s_0, \overline{\sigma}\right)$$

which shows that $\overline{\sigma}$ is a noncapturing NE of $\Gamma\left(G|s_0\right)$.

The above example shows that the converse of Theorem 5.5 does not hold, i.e., there exist cop-win graphs G and initial states s_0 such that $\Gamma_3\left(G|s_0\right)$ has noncapturing NE. However, we can prove a weaker result: the converse does hold when G is a tree.

The first step in our proof is to revisit the two-player game $\widetilde{\Gamma}_3^2\left(G|s_0\right)$ of Sect. 5.1. Recall that it is played between P_2 and P_{-2} who controls the *tokens* P_1 and P_3. We now prove the following.

Theorem 5.7 *If* $c(G) = 1$, *then every optimal profile* $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ *of* $\widetilde{\Gamma}_3^2\left(G|s_0\right)$ *is a NE of* $\Gamma_3\left(G|s_0\right)$.

Proof Let us choose some initial state s_0 and some optimal (in $\widetilde{\Gamma}_3^2\left(G|s_0\right)$) profile $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$, and keep them fixed for the rest of the proof.

For any $\sigma = \left(\sigma^1, \sigma^2, \sigma^3\right)$, the capture function $\mathbf{K}_3\left(G|s_0, \sigma\right)$ will take a value in $\{0, 1, 2\}$. The values correspond to three outcomes in $\Gamma_3\left(G|s_0\right)$ and the same outcomes are obtained in $\widetilde{\Gamma}_3^2\left(G|s_0\right)$ (the two games differ in their payoffs but are played by the same rules):

1. $\mathbf{K}_3\left(G|s_0, \sigma\right) = 1$ means P_1 captures P_2;
2. $\mathbf{K}_3\left(G|s_0, \sigma\right) = 2$ means P_2 captures P_3 (and is not captured by P_1);
3. $\mathbf{K}_3\left(G|s_0, \sigma\right) = 0$ means neither P_2 nor P_3 is captured.

So we will consider the three mutually exclusive cases separately.
I. $\mathbf{K}_3\left(G|s_0, \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)\right) = 1$. Let us examine each player's payoff.

1. For all $\left(\sigma^1, \sigma^3\right)$ such that $\mathbf{K}_3\left(G|s_0, \left(\sigma^1, \widehat{\sigma}^2, \sigma^3\right)\right) = 1$, $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ optimality in $\widetilde{\Gamma}_3^2\left(G|s_0\right)$ implies

$$\begin{aligned} Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = \widetilde{Q}^{-2}\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) &\geq \widetilde{Q}^{-2}\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right) \\ &= Q^1\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right). \end{aligned}$$

And for all $\left(\sigma^1, \sigma^3\right)$ such that $\mathbf{K}_3\left(G|s_0, \left(\sigma^1, \widehat{\sigma}^2, \sigma^3\right)\right) \neq 1$, we have

$$Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) > 0 = Q^1\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right).$$

Hence

$$\begin{aligned} &\forall \sigma^1, \sigma^3 : Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \geq Q^1\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right) \\ &\Rightarrow \forall \sigma^1 : Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \geq Q^1\left(s_0, \sigma^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \end{aligned} \tag{5.17}$$

2. For all $\left(\sigma^1, \sigma^2, \sigma^3\right)$, we have

$$Q^2\left(s_0, \sigma^1, \sigma^2, \sigma^3\right) = \widetilde{Q}^2\left(s_0, \sigma^1, \sigma^2, \sigma^3\right);$$

combining with optimality in $\widetilde{\Gamma}_3^2\left(G|s_0\right)$, we get

$$\begin{aligned}\forall \sigma^2 : Q^2\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) &= \widetilde{Q}^2\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \geq \widetilde{Q}^2\left(s_0, \widehat{\sigma}^1, \sigma^2, \widehat{\sigma}^3\right) \\ &= Q^2\left(s_0, \widehat{\sigma}^1, \sigma^2, \widehat{\sigma}^3\right). \end{aligned} \tag{5.18}$$

3. And finally

$$\begin{aligned}\forall \sigma^1, \sigma^3 : Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = 0 \geq Q^3\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right) \Rightarrow \forall \sigma^3 : Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \\ = 0 \geq Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \sigma^3\right). \end{aligned} \tag{5.19}$$

Combining (5.17)–(5.19), we see that

$$\mathbf{K}_3\left(G|s_0, \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)\right) = 1 \Rightarrow \forall n, \forall \sigma^n : Q^n\left(s_0, \widehat{\sigma}\right) \geq Q^n\left(s_0, \sigma^n, \widehat{\sigma}^{-n}\right). \tag{5.20}$$

II. $\mathbf{K}_3\left(G|s_0, \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)\right) = 2$. Then the following hold.

1. From $\left(\widehat{\sigma}^1, \widehat{\sigma}^3\right)$ optimality, we have

$$\forall \sigma^1, \sigma^3 : 0 = Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) > \widetilde{Q}^{-2}\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \geq \widetilde{Q}^{-2}\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right). \tag{5.21}$$

We cannot have $\mathbf{K}_3\left(G|s_0, \left(\sigma^1, \widehat{\sigma}^2, \sigma^3\right)\right) = 1$, because then we would also have $\widetilde{Q}^{-2}(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3) > 0$, which contradicts (5.21). If we have either $\mathbf{K}_3(G|s_0, (\sigma^1, \widehat{\sigma}^2, \sigma^3)) = 2$ or $\mathbf{K}_3(G|s_0, (\sigma^1, \widehat{\sigma}^2, \sigma^3)) = 0$, then

$$Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = 0 = Q^1\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right).$$

In short

$$\begin{aligned}\forall \sigma^1, \sigma^3 : Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = 0 = Q^1\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right) \Rightarrow \forall \sigma^1 : Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \\ = Q^1\left(s_0, \sigma^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \end{aligned} \tag{5.22}$$

2. By the same argument as in the previous case, we get

$$\forall \sigma^2 : Q^2\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \geq Q^2\left(s_0, \widehat{\sigma}^1, \sigma^2, \widehat{\sigma}^3\right). \tag{5.23}$$

3. Finally

$$\forall \sigma^1, \sigma^3 : 0 > Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = \widetilde{Q}^{-2}\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \geq \widetilde{Q}^{-2}\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right). \tag{5.24}$$

We cannot have $\mathbf{K}_3\left(G|s_0, \left(\sigma^1, \widehat{\sigma}^2, \sigma^3\right)\right) = 0$ or $\mathbf{K}_3\left(G|s_0, \left(\sigma^1, \widehat{\sigma}^2, \sigma^3\right)\right) = 1$, because then we would also have $\widetilde{Q}^{-2}\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right) \geq 0$ which contradicts (5.24). Hence $\mathbf{K}_3\left(G|s_0, \left(\sigma^1, \widehat{\sigma}^2, \sigma^3\right)\right) = 2$ and then

$$Q^3\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right) = \widetilde{Q}^{-2}\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right);$$

hence, from (5.24) we get

$$\forall \sigma^1, \sigma^3 : Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \geq Q^3\left(s_0, \sigma^1, \widehat{\sigma}^2, \sigma^3\right)$$

and then

$$\forall \sigma^3 : Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \sigma^3\right). \tag{5.25}$$

Combining (5.22)–(5.25),we see that

$$\mathbf{K}_3\left(G|s_0,\left(\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)\right)=2\Rightarrow\forall n,\forall\sigma^n:Q^n\left(s_0,\widehat{\sigma}\right)\geq Q^n\left(s_0,\sigma^n,\widehat{\sigma}^{-n}\right). \quad (5.26)$$

III. $\mathbf{K}_3\left(G|s_0,\left(\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)\right)=0$.

1. For all σ^1,σ^3, we have

$$Q^1\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)=0=\widetilde{Q}^{-2}\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)\geq\widetilde{Q}^{-2}\left(s_0,\sigma^1,\widehat{\sigma}^2,\sigma^3\right). \quad (5.27)$$

   We cannot have $\mathbf{K}_3\left(G|s_0,\left(\sigma^1,\widehat{\sigma}^2,\sigma^3\right)\right)=1$, because then we would also have $\widetilde{Q}^{-2}\left(s_0,\sigma^1,\widehat{\sigma}^2,\sigma^3\right)>0$, which would contradict (5.27). If $\mathbf{K}_3\left(G|s_0,\left(\sigma^1,\widehat{\sigma}^2,\sigma^3\right)\right)=2$ or $\mathbf{K}_3\left(G|s_0,\left(\sigma^1,\widehat{\sigma}^2,\sigma^3\right)\right)=0$ then $Q^1\left(s_0,\sigma^1,\widehat{\sigma}^2,\sigma^3\right)=0$ and so

$$\begin{aligned}\forall\sigma^1,\sigma^3:0=Q^1\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)=Q^1\left(s_0,\sigma^1,\widehat{\sigma}^2,\sigma^3\right)\Rightarrow\forall\sigma^1:Q^1\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)\\ =Q^1\left(s_0,\sigma^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right). \quad (5.28)\end{aligned}$$

2. By the same argument as in the previous cases, we get

$$\forall\sigma^2:Q^2\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)\geq Q^2\left(s_0,\widehat{\sigma}^1,\sigma^2,\widehat{\sigma}^3\right). \quad (5.29)$$

3. Finally, we have seen that, for all $\left(\sigma^1,\sigma^3\right)$, either $\mathbf{K}_3\left(G|s_0,\left(\sigma^1,\widehat{\sigma}^2,\sigma^3\right)\right)=2$ or $\mathbf{K}_3\left(G|s_0,\left(\sigma^1,\widehat{\sigma}^2,\sigma^3\right)\right)=0$; in both cases

$$\begin{aligned}\forall\sigma^1,\sigma^3:Q^3\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)=\widetilde{Q}^{-2}\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)\geq\widetilde{Q}^{-2}\left(s_0,\sigma^1,\widehat{\sigma}^2,\sigma^3\right)\\ =Q^3\left(s_0,\sigma^1,\widehat{\sigma}^2,\sigma^3\right)\end{aligned}$$

   and so

$$\forall\sigma^3:Q^3\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)\geq Q^3\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right). \quad (5.30)$$

Combining (5.28)–(5.30), we see that

$$\mathbf{K}_3\left(G|s_0,\left(\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)\right)=0\Rightarrow\forall n,\forall\sigma^n:Q^n\left(s_0,\widehat{\sigma}\right)\geq Q^n\left(s_0,\sigma^n,\widehat{\sigma}^{-n}\right). \quad (5.31)$$

In conclusion, combining (5.20), (5.26) and (5.31) we see that every profile $\left(\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)$ which is optimal in $\widetilde{\Gamma}_3^2\left(G|s_0\right)$ is also a NE of $\Gamma_3\left(G|s_0\right)$. □

Before we prove additional facts about $\widetilde{\Gamma}_3^2\left(G|s_0\right)$, we need the following.

Definition 5.8 A graph G is called *median* if for every three vertices x, y and z there exists a *unique* vertex $m\left(x,y,z\right)$ (the *median* vertex of x,y,z) which belongs to shortest paths between each pair of x,y,z.

The following facts are well known [12]. First, every tree is a median graph. Second, in a tree the union of the three (unique) shortest paths between the pairs of vertices x, y and z is

1. either a path, in which case the median $m\left(x,y,z\right)$ is equal to one of x, y or z;
2. or a subtree formed by three paths meeting at a single central node,which is the median of x, y and z.

Now we can prove some additional properties of $\widetilde{\Gamma}_3^2\left(G|s_0\right)$.

Theorem 5.9 *If G is a path then, for any s_0, every strategy profile $\widehat{\sigma}=\left(\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)$ which is optimal in $\widetilde{\Gamma}_3^2\left(G|s_0\right)$ is capturing.*

Proof If s_0 is a capture state, the theorem is obviously true. Take any noncapture starting state $s_0 = \left(x_0^1, x_0^2, x_0^3, p\right)$; since G is a path, it is a median graph and one of x_0^1, x_0^2, x_0^3 is the median of the other two (we will also say that either x_0^1, x_0^2, x_0^3 or P_1, P_2, P_3 are *collinear*). We define strategies $\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3$ for each case.

1. If x_0^1 is the median of x_0^2 and x_0^3, then: P_1 moves towards P_2, P_2 moves away from P_1 and P_3 stays in place; eventually, P_2 is captured.
2. If x_0^2 is the median of x_0^1 and x_0^3, then: P_1 moves towards P_2, P_2 moves towards P_3 and P_3 moves away from P_2; eventually, P_3 is captured.
3. If x_0^3 is the median of x_0^1 and x_0^2, then: P_1 moves towards P_2, P_2 moves away from P_1 and P_3 moves away from P_2; eventually, P_2 is captured.

In every case, $\overline{\sigma} = \left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)$ is capturing *and* optimal. Since $\overline{\sigma}$ is capturing, the same holds for *every* optimal profile $\widehat{\sigma}$, because they all yield the same payoff. □

The above defined $\overline{\sigma}$ will be called *path strategies* and will be used to prove the following lemma, needed to extend Theorem 5.9 to trees.

Lemma 5.10 *If G is a tree then there exists a positional profile $\widetilde{\sigma} = \left(\widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ for which the following hold in $\widetilde{\Gamma}_3^2\,(G|s_0)$.*

1. *For every s_0: $\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ results in capture.*
2. *If $\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ results in capture* of P_2 *then, for every σ^2, $\left(s_0, \widetilde{\sigma}^1, \sigma^2, \widetilde{\sigma}^3\right)$ results in capture* of P_2.
3. *If $\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ results in capture* by P_2 *then for every σ^1, σ^3, $\left(s_0, \sigma^1, \widetilde{\sigma}^2, \sigma^3\right)$ results in capture* by P_2.

Proof A rough description of $\widetilde{\sigma}$ is quite simple: each player tries to reach the median as fast as possible; as soon as this happens, the players are collinear and they start playing their path strategies. We next give a (straightforward but rather tedious) rigorous proof. In what follows, we denote the median of x_t^1, x_t^2, x_t^3 by m_t.

For Part 1 of the theorem, we distinguish two cases.

Case A Suppose that at some time t the game state is $\left(x_t^1, x_t^2, x_t^3, p\right)$ where one of x_t^1, x_t^2, x_t^3 is the median of the other two (they are collinear). In this case $\widetilde{\sigma} = \overline{\sigma}$, i.e., the players use the path strategies of Theorem 5.9. It is easily checked that the players remain collinear for the rest of the game and a capture results.

Case B Suppose that x_0^1, x_0^2, x_0^3 are not collinear. The initial part of $\widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3$ prescribes that every player moves directly towards the median m_t. As a result, let t_0 denote the first time when a (single) player P_n is at distance 1 from m_t, as depicted in Figs. 3 and 4 (in the figures we only show the subtree of G which is defined by the positions of P_1, P_2 and P_3; dotted lines indicate paths of length one or more). We will now define $\widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3$ depending on which P_n first reaches m_t; when some move is not specified, the respective strategy can be defined arbitrarily.

1. Suppose $P_n = P_1$, i.e., at t_0 we have $d\left(x_{t_0}^1, m_{t_0}\right) = 1$, as shown in Fig. 3a. P_3 stays in place at t_0+2, P_1 enters m_t at t_0+3 and the players become collinear. Now every player starts using his path strategy. It is easy to check that this results in capture of P_2.
2. Suppose $P_n = P_2$, i.e., at t_0 we have $d\left(x_{t_0}^2, m_{t_0}\right) = 1$. Now we must distinguish two subcases.

    (a) Say $d\left(x_{t_0}^1, m_{t_0}\right) = 2$, as in Fig. 3b. P_3 stays in place at t_0+1 and t_0+4, P_1 moves to a at t_0+2 and to m_t at t_0+5, when the players become collinear and start playing their path strategies; eventually, P_2 is captured.

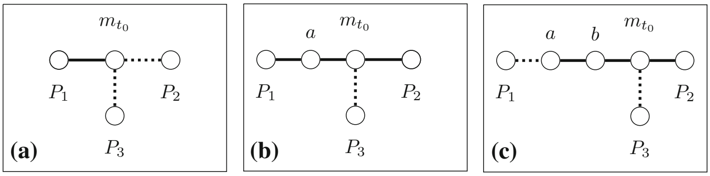


Fig. 3 Possible placements of P_1, P_2, P_3 in the proof of Theorem 5.10

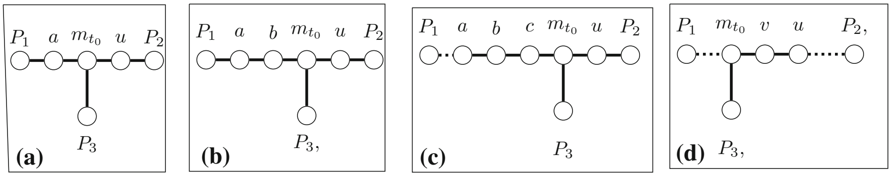


Fig. 4 Possible placements of P_1, P_2, P_3 in the proof of Theorem 5.10

(b) Say $d\left(x_{t_0}^1, m_{t_0}\right) \geq 3$, as in Fig. 3c. P_2 enters m_t at time t_0+3, the players become collinear and start playing their path strategies; eventually, P_3 is captured.

3. Suppose $P_n = P_3$, i.e., at t_0 we have $d\left(x_{t_0}^3, m_{t_0}\right) = 1$. Now we must distinguish four subcases.

   (a) Say $d\left(x_t^1, m_t\right) = 2$, $d\left(x_t^2, m_t\right) = 2$, as in Fig. 4a. P_1 enters a at t_0+1, P_2 enters u at t_0+2 and P_3 stays in place at t_0+3. At t_0+4 P_1 enters m_t, the players become collinear and start playing their path strategies; eventually, P_2 is captured.
   (b) Say $d\left(x_t^1, m_t\right) = 3$, $d\left(x_t^2, m_t\right) = 2$, as in Fig. 4b. P_1 enters a at t_0+1, P_2 enters u at t_0+2 and P_3 stays in place at t_0+3. At t_0+4 P_1 enters b and at t_0+7 he enters m_t, the players become collinear and start playing their path strategies; eventually, P_2 is captured.
   (c) Say $d\left(x_t^1, m_t\right) \geq 4$, $d\left(x_t^2, m_t\right) = 2$, as in Fig. 4c. P_2 enters u at t_0+2 and m_t at t_0+5, at which time the players become collinear and start playing their path strategies; eventually, P_3 is captured.
   (d) Say $d\left(x_t^2, m_t\right) \geq 3$, as in Fig. 4d. At t_0+3 P_3 enters m_t, the players become collinear and start playing their path strategies; eventually, P_2 is captured.

This completes the description of $\widetilde{\sigma} = \left(\widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ and it is readily seen that it always leads to capture; so the first part of the theorem has been proved.

For Part 2 of the theorem, assume that $\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ leads to P_2 capture; there are two ways for this to happen. Either the players are collinear in s_0 and P_2 is *not* in the middle; in this case, P_2 is captured for every σ^2 he uses, or the players are not collinear in s_0 but eventually reach one of cases 1, 2.a, 3.a, 3.b, 3.d; in this case, if P_2 uses a σ^2 which deviates from $\widetilde{\sigma}^2$, he will approach m_t no faster than if he used $\widetilde{\sigma}^2$ and a straightforward examination of cases 1, 2.a, 3.a, 3.b, 3.d shows that $\left(s_0, \widetilde{\sigma}^1, \sigma^2, \widetilde{\sigma}^3\right)$ will also lead to capture of P_2.

The proof of Part 3 is similar to that of Part 2 and hence omitted. □

Now we can expand Theorem 5.9 from paths to trees.

Theorem 5.11 *If G is a tree then, for any s_0, every strategy profile $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ which is optimal in $\widetilde{\Gamma}_3^2\left(G|s_0\right)$ is capturing.*

Proof Let $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ be an optimal (for any s_0) strategy profile and take some s_0 such that $\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ leads to capture of P_2. Then, by Part 2 of Lemma 5.10, we have

$$\forall \sigma^2 : \widetilde{Q}^2 \left(s_0, \widetilde{\sigma}^1, \sigma^2, \widetilde{\sigma}^3\right) < 0$$

and so

$$\widetilde{Q}^2 \left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = \min_{\sigma^1, \sigma^3} \max_{\sigma^2} \widetilde{Q}^2 \left(s_0, \sigma^1, \sigma^2, \sigma^3\right) \leq \max_{\sigma^2} \widetilde{Q}^2 \left(s_0, \widetilde{\sigma}^1, \sigma^2, \widetilde{\sigma}^3\right) < 0;$$

in other words, $\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ leads to capture of P_2. Similarly, by Part 3 of Lemma 5.10 we can prove that, for every s_0 such that $\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ leads to capture of P_3, the same holds for $\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$. Since, by Part 1 of Lemma 5.10, for every s_0, $\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ leads to capture (of either P_2 or P_3) we have proved the theorem. □

Now we return to the three-player game $\Gamma_3 (G|s_0)$ and show that: if G is a tree, then $\Gamma_3 (G|s_0)$ has a capturing NE for every initial state s_0. (Hence, while the converse of Theorem 5.5 does not hold for every cop-win graph, it holds for the special case of trees.)

Theorem 5.12 *If G is a tree, then*

$$\forall s_0 \text{ there exists a NE } \widehat{\sigma} \text{ of } \Gamma_3 (G|s_0) : \mathbf{K}_3 (G|s_0, \widehat{\sigma}) > 0.$$

Proof Since G is a tree, every optimal profile $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ of $\widetilde{\Gamma}_3^2 (G|s_0)$ is capturing in both $\widetilde{\Gamma}_3^2 (G|s_0)$ (by Theorem 5.11); and in $\Gamma_3 (G|s_0)$ (since the two games are played by the same rules). Hence $\mathbf{K}_3 (G|s_0, \widehat{\sigma}) > 0$. And $\widehat{\sigma}$ is a NE of $\Gamma_3 (G|s_0)$ by Theorem 5.7 (since every tree G has $c(G) = 1$). □

We conclude this section with a result for graphs which are not cop-win.

Theorem 5.13 $c(G) > 1 \Rightarrow (\exists s_0 : \Gamma_3 (G|s_0)$ *has a noncapturing NE* $\widehat{\sigma})$

Proof We will construct the required s_0 and $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$. Since $c(G) > 1$, there exist an $\widetilde{s}_0 = \left(x^1, x^2, p\right)$ and a $\Gamma_2 (G|\widetilde{s}_0)$-optimal noncapturing profile $\widetilde{\sigma} = \left(\widetilde{\sigma}^1, \widetilde{\sigma}^2\right)$. Now let $s_0 = \left(x^1, x^2, x^1, 1\right)$ and define the $\Gamma_3 (G|\widetilde{s}_0)$ strategies as follows: $\widehat{\sigma}^2$ is $\widetilde{\sigma}^2$ (expanded to work in $\Gamma_3 (G|\widetilde{s}_0)$) and, for $n \in \{1, 3\}$, $\widehat{\sigma}^n$ specifies that P_n always stays in place. Then, since $\widehat{\sigma}^2$ is an optimal evasion strategy we have

$$\forall \sigma^1 : Q^1 \left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = 0 = Q^1 \left(s_0, \sigma^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right).$$

Also, P_2 must enter x^1 to capture P_3, but then he would first be captured by P_1. Hence we have

$$\forall \sigma^2 : Q^2 \left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = 0 \geq Q^2 \left(s_0, \widehat{\sigma}^1, \sigma^2, \widehat{\sigma}^3\right).$$

Finally, since P_3 never receives positive payoff, we have

$$\forall \sigma^3 : Q^3 \left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = 0 \geq Q^3 \left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \sigma^3\right).$$

So s_0 is a noncapturing NE of $\Gamma (G|s_0)$. □

6 GCR with N Players, $N \geq 4$

We will now briefly examine $\Gamma_N(G|s_0)$ for $N \geq 4$. Most of the game elements have been defined in Sect. 2; we define the turn payoffs q^n by generalizing (5.1). Namely, at every turn P_n receives:

1. a payoff of -1 if he is captured by P_{n-1};
2. a payoff of 1 if he captures P_{n+1}, but is not simultaneously captured by P_{n-1};
3. a payoff of 0 in every other case.

The above turn payoffs and the total payoff Q^n of (2.2) complete the specification of $\Gamma_N(G|s_0)$.

Since it is a multi-player discounted stochastic game of perfect information, $\Gamma_N(G|s_0)$ has (by Theorem 3.1) a NE in deterministic positional strategies. The N-player analogue of Theorem 5.2 also holds.

Theorem 6.1 *For every G, s_0 and γ, $\Gamma_N(G|s_0)$ has a NE $\widehat{\pi} = (\widehat{\pi}^1, \widehat{\pi}^2, \ldots, \widehat{\pi}^N)$ in deterministic (generally nonpositional) strategies.*

Proof The proof involves the use of the auxiliary two-player, zero-sum games $\widetilde{\Gamma}_N^1(G|s_0), \ldots, \widetilde{\Gamma}_N^N(G|s_0)$. In $\widetilde{\Gamma}_N^n(G|s_0)$, P_n plays against P_{-n}, who controls the tokens $P_1, \ldots, P_{n-1}, P_{n+1}, \ldots, P_N$. The threat strategies $\widehat{\pi} = (\widehat{\pi}^1, \widehat{\pi}^2, \ldots, \widehat{\pi}^N)$ are defined in the same manner as in Sect. 5.1, in terms of the strategies $\left(\widehat{\phi}_n^m\right)_{m,n\in[N]}$ which are optimal in the corresponding $\widetilde{\Gamma}_N^n(G|s_0)$ games. The rest of the proof is omitted, since it follows closely that of Theorem 5.2. □

Similarly to $\Gamma_3(G|s_0)$, if $\Gamma_N(G|s_0)$ has a capturing NE for every initial state s_0, then G is cop-win. This is stated in the following theorem, where $\mathbf{K}_N(G|s_0, \sigma)$ is the obvious generalization of the capturability function $\mathbf{K}_3(G|s_0, \sigma)$. (The proof is omitted, since it is similar to that of Theorem 5.5.)

Theorem 6.2 *The following holds for every G:*

$$(\forall s_0 \text{ there exists a NE } \widehat{\sigma} \text{ of } \Gamma_N(G|s_0) : \mathbf{K}_N(G|s_0, \widehat{\sigma}) > 0) \Rightarrow c(G) = 1. \tag{6.1}$$

On the other hand, Theorem 5.7 does not generalize to the case $N \geq 4$. The following example shows that, even when G is a path, there may exist optimal profiles $\widehat{\sigma}$ of $\widetilde{\Gamma}_N^n(G|s_0)$ which are not NE of $\Gamma_N(G|s_0)$.

Example 6.3 In Fig. 5, G is a path, the tokens are positioned as depicted and P_4 has the starting move; in short, $s_0 = (1, 3, 4, 5, 4)$.

In the game $\widetilde{\Gamma}_4^2(G|s_0)$, P_2 plays against P_{-2} who controls P_1, P_3, P_4. Clearly the optimal P_{-2} move from s_0 is to move P_4 into vertex 4, since then the game ends and P_{-2} receives his maximum possible payoff of 0. (Otherwise, on his first move P_2 captures P_3 and P_{-2} receives negative payoff.) So every $\widehat{\sigma}^{-2} = \left(\widehat{\sigma}^1, \widehat{\sigma}^3, \widehat{\sigma}^4\right)$ which is optimal in $\widetilde{\Gamma}_4^2(G|s_0)$ must satisfy $\widehat{\sigma}^4(s_0) = 4$. But such a $\widehat{\sigma}^{-2}$ cannot be (part of) a NE of $\Gamma_4(G|s_0)$, because in this game P_4 can improve his payoff by moving from 5 to 6, rather than 4.

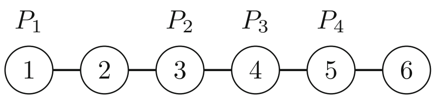


Fig. 5 A path G in which a $\widetilde{\Gamma}_4^2(G|s_0)$-optimal strategy profile is not a NE of $\Gamma_4^2(G|s_0)$

In Sect. 5, we have shown (Theorem 5.12) that, when G is a tree, for every s_0 there exists a capturing NE of $\Gamma_3(G|s_0)$; the proof depended on Theorem 5.7 which, as seen, does not generalize for $N \geq 4$. Hence we have not been able to generalize Theorem 5.12 either. On the other hand, we have not found a counterexample (i.e., a tree and some initial state for which no capturing NE exists); hence, the matter remains open.

The following generalizes Theorem 5.13 and is proved very similarly.

Theorem 6.4 *For every $N \geq 3$, we have*

$$c(G) > 1 \Rightarrow \left(\exists s_0 : \Gamma_N(G|s_0) \text{ has a noncapturing NE } \widehat{\sigma}\right).$$

7 More Multi-player Pursuit Games

In Sect. 2, we have developed a framework which we have used in Sects. 4, 5 and 6 to study the game $\Gamma_N(G|s_0)$, for various N values. As we will now explain, this framework applies to a wider family of graph pursuit games.

We have in mind games played by players $P_1, P_2, \ldots, P_N$ who take turns in moving tokens along the edges of a graph. For the time being, assume that each player controls one token and has, in general, two goals: (i) to capture some (other players') tokens and (ii) to avoid capture of his own token.

Any such situation can be described, by the formulation of Sect. 2, as a multi-player discounted stochastic game of perfect information. Assuming, without loss of generality, that the players move in the sequence implied by their numbering, the actual "capture relationship" will be encoded by the turn payoff functions q^n. To preserve the semantics of pursuit/evasion, they should have the form[8]

$$q^n\left(\left(x^1, \ldots, x^N, p\right)\right) = \begin{cases} 1 & \text{when for some } m: \quad x^n = x^m, m \in A^n, \\ -1 & \text{when for some } m: \quad x^n = x^m, m \in B^n, \\ 0 & \text{else.} \end{cases} \tag{7.1}$$

where

1. A^n is the set of P_n's "targets" (i.e., the players whom he can capture) and
2. B^n is the set of P_n's "pursuers" (i.e., the players who can capture him).

For example, in $\Gamma_2(G|s_0)$ we have players P_1 and P_2 with respective sets

$$A^1 = \{P_2\}, B^1 = \emptyset, \quad A^2 = \emptyset, B^2 = \{P_1\};$$

while in $\Gamma_3(G|s_0)$ we have players P_1, P_2 and P_3 with respective sets

$$A^1 = \{P_2\}, B^1 = \emptyset, \quad A^2 = \{P_3\}, B^2 = \{P_1\}, \quad A^3 = \emptyset, B^3 = \{P_2\}$$

and the additional condition of no simultaneous captures (which, as, requires a small modification of (7.1)).

As a final example, consider a game which we could call "*Cyclic Cops and Robbers*"; it involves players P_1, P_2 and P_3 in which: P_1 chases P_2 and avoids P_3; P_2 chases P_3 and avoids P_1; P_3 chases P_1 and avoids P_2. In this game, we will have

$$A^1 = \{P_2\}, \quad B^1 = \{P_3\}, \quad A^2 = \{P_3\}, \quad B^2 = \{P_1\}, \quad A^3 = \{P_1\}, \quad B^3 = \{P_2\}.$$

[8] The conditions in (7.1) encode "minimum" requirements, additional restrictions may be imposed, e.g., no simultaneous captures.

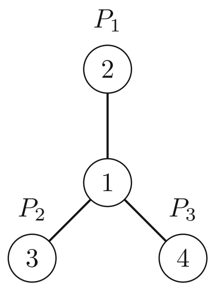


Fig. 6 In this graph, Cyclic Cops and Robbers have only noncapturing NE

This game has some interesting properties; they will be fully described in a separate publication, but as an example suppose it is played on the star graph of Fig. 6, with initial positions as indicated. It is easily checked that, even though the star graph is cop-win, the game has *only* noncapturing NE.

Many similar games can be constructed along these lines and the corresponding capture relationships can be quite involved. Let us represent the capture relationships by a directed graph, with vertices representing players and (P_k, P_m) being an arc iff P_k can capture P_m. Then our Γ_2 game corresponds to a directed path and "Cyclic Cops and Robbers" corresponds to a directed cycle. One can visualize more complex capture relationships, corresponding to directed graphs with multiple successors, cycles, etc.

At any rate, all such games (i) fall within the game-theoretic framework of Sect. 2 and hence (ii) by Theorem 3.1 possess a well-defined game-theoretic solution, namely a NE in deterministic positional strategies.

In fact, the framework of Sect. 2 can accommodate further generalizations for which Theorem 3.1 will still hold. We list some additional generalizations to the idea of graph pursuit game.

1. *Payoffs* The turn payoffs q^n can take values in $[-1, 1]$ rather than $\{-1, 1\}$. As an example, we have introduced and studied the game of *Selfish Cops and Robbers* [11], in which two cops pursue a robber but do not split the capture payoff equally; instead, the capturing (resp. noncapturing) cop receives payoff $(1-\varepsilon)$ (resp. ε), where $\varepsilon \in \left[0, \frac{1}{2}\right]$. Hence each cop has a motive to be the one who actually captures the robber; if this "selfishness" is sufficiently strong (this will depend on the ε value), it can be exploited by the robber to avoid capture ad infinitum.
2. *Teams* So far we have assumed that each player controls a single token. But we can also assume that a game is played by N players (with $N \geq 2$) with P_n controlling K_n tokens. An example of this is the classic CR game with more than one cop tokens (all of them controlled by a single *cop player*). Another example is the $\widetilde{\Gamma}_N^n\,(G|s_0)$ auxiliary games of Sects. 5.1 and 6. These are two-player games, but the idea can be applied to multi-player games as well. For example, we could have the three-player GCR game with P_1 controlling two pursuer tokens and each of P_2 and P_3 controlling one pursuer and one evader token.
3. *Game termination* So far we have assumed that the game terminates upon the first capture, but this can also be modified. For example, the game could end upon the elimination of all tokens of one player, or when no more captures are possible.

Since all of the above modifications can be accommodated by the formulation of Sect. 2, the respective games can be analysed by game-theoretic methods. At the very least, by

Theorem 3.1 they all possess NE; further results can be obtained by exploiting the special characteristics of each game.

Here is a final variation on pursuit games. Suppose that the game starts with the graph G being empty,[9] fixed move order $P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_N \rightarrow \cdots$ and fixed payoff functions q^n. However, at the very start of the game (i.e., even before token placement) each player can choose one of the available q^n's. In other words, P_n will play with payoff q^{m_n}, where $(m_1, m_2, \ldots, m_N)$ is a permutation of $(1, 2, \ldots, N)$. This amounts to each player choosing one of the available "roles." In certain cases, the choice is obvious; for instance, in classic CR and for a given cop-win (resp. robberwin) G, every player prefers to be the cop (resp. robber). However, depending on the payoff functions and the graph in which the game is played, situations will arise where the choice of role involves a quite complex and (perhaps) interesting "*meta-game*."

8 Conclusion

In this paper, we have introduced and studied the *Generalized Cops and Robbers* game $\Gamma_N\left(G|s_0\right)$, a multi-player pursuit game in graphs. The two-player version $\Gamma_2\left(G|s_0\right)$ is essentially equivalent to the classic CR game. The three-player version $\Gamma_3\left(G|s_0\right)$ can be understood as two CR games played simultaneously on the same graph; a player can simultaneously be pursuer *and* evader. This also holds for $\Gamma_N\left(G|s_0\right)$ when $N \geq 4$.

Using a formulation of $\Gamma_N\left(G|s_0\right)$ as a discounted stochastic game of perfect information, we have proved that it has at least one NE in positional deterministic strategies. Using auxiliary two-player games $\widetilde{\Gamma}_N^n\left(G|s_0\right)$, we have also proved the existence of an additional NE in nonpositional deterministic strategies. We have also studied the capturing properties of the $\Gamma_N\left(G|s_0\right)$ NE in connection with the cop number $c\left(G\right)$.

Both $\Gamma_N\left(G|s_0\right)$ and $\widetilde{\Gamma}_N^n\left(G|s_0\right)$ are members of a general family of graph pursuit games, which can be described by the framework of Sect. 2 and its generalizations, presented in Sect. 7. This family is a broad generalization of the two-player graph pursuit games previously studied to the multi-player case; it contains games with rather unexpected properties, and hence, we believe, it deserves additional study.

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[9] That is, the players have not yet placed their tokens; see footnote 3 for details.

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