Lecture I213E – Class 2

Discrete Signal Processing

Sakriani Sakti

JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY 1990

Course Materials

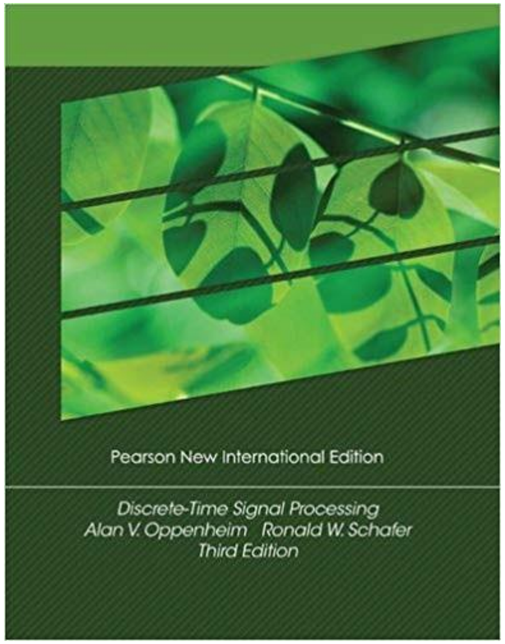
- **Materials**

→ Lecture notes will be uploaded before each lecture
https://jstorage-2018.jaist.ac.jp/s/PGXRrC7iFmN2FWo
Pass: dsp-i213e-2022
(Slide Courtesy of Prof. Nak Young Chong)

- **References**

→ Chi-Tsong Chen:
Linear System Theory and Design, 4th Ed., Oxford University Press, 2013.

→ Alan V. Oppenheim and Ronald W. Schafer:
Discrete-Time Signal Processing, 3rd Ed., Pearson New International Ed., 2013.



Related Courses & Prerequisite

- **Related Courses**
  - → I212 Analysis for Information Science
  - → I114 Fundamental Mathematics for Information Science

- **Prerequisite**
  - → None

Evaluation

▪ Viewpoint of evaluation

→ Students are able to understand:

- Basic principles in modeling and analysis of linear time-invariant systems
- Applications of mathematical methods and tools to different signal processing problems.

▪ Evaluation method

→ Homework, term project, midterm exam, and final exam

▪ Evaluation criteria

→ Homework/labs (30%), term project (30%)
midterm exam (15%), and final exam (25%)

Contact

- ## Lecturer

  → Sakriani Sakti

- ## TA

  Tutorial hours & Term project

  → WANG Lijun (s2010026)

  → TANG Bowen (s2110411)

  Homework

  → PUTRI Fanda Yuliana (s2110425)

- ## Contact Email

  → dsp-i213e-2022@ml.jaist.ac.jp

Schedule

- **December 8th, 2022 – February 9th, 2023**

- **Lecture Course Term 2-2**
  - → Tuesday 9:00 — 10:40
  - → Thursday 10:50 — 12:30

- **Tutorial Hours**
  - → Tuesday 13:30-15:10

Schedule

Dec

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Jan

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	~~5~~	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	~~24~~	25	26	27	28
29	30	31				

Feb

Sun	Mon	Tue	Wed	Thu	Fri	Sat
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28				

Lecture:
Tuesday 9:00 — 10:40
Thursday 10:50 — 12:30

Tutorial:
Tuesday 13:30 — 15:10

Course review &
term project evaluation
(on tutorial hours)

Midterm & final exam
Thursday 10:50 — 12:30

Syllabus

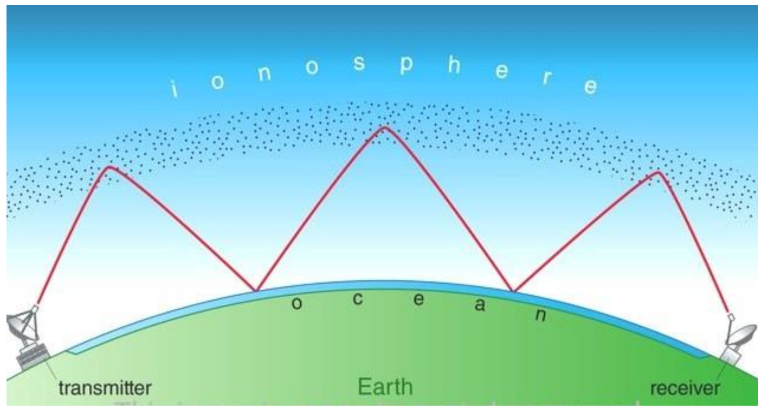
Class	Date	Lecture Course Tue 9:00 — 10:40 / Thr 10:50 — 12:30	Tutorial Hours Tue 13:30 — 15:10
1	12/08	Introduction to Linear Systems with Applications to Signal Processing	
2	12/13	State Space Description	○
3	12/15	Linear Algebra	
4	12/20	Quantitative Analysis (State Space Solutions) and Qualitative Analysis (Stability)	○
5	12/22	Discrete-time Signals and Systems	
X	01/05		
6	01/10	Discrete-time Fourier Analysis	
7	01/10*	Review of Discrete-time Linear Time-Invariant Signals and Systems (on Tutorial Hours)	
	01/12	Midterm Exam	
8	01/17	Sampling and Reconstruction of Analog Signals	○
9	01/19	z-Transform	
X	01/24		○
10	01/26	Discrete Fourier Transform	
11	01/31	FFT Algorithms	○
12	01/02	Implementation of Digital Filters	
13	02/07	Digital Signal Processors and Design of Digital Filters	
14	02/07*	Review of the Course and Term Project Evaluation (on Tutorial Hours)	
	02/09	Final exam	

Class 2
Mathematical Descriptions of the Systems

System
as a Series of Transformations of the Acoustic Signals

Mathematical Descriptions of Systems

A series of transformations of the acoustic signal



mic

Sound to Voltage → Voltage to Radio Waves → Transmission of Radio Waves → Radio Waves to Voltage → Voltage to Sound →

loudspeaker

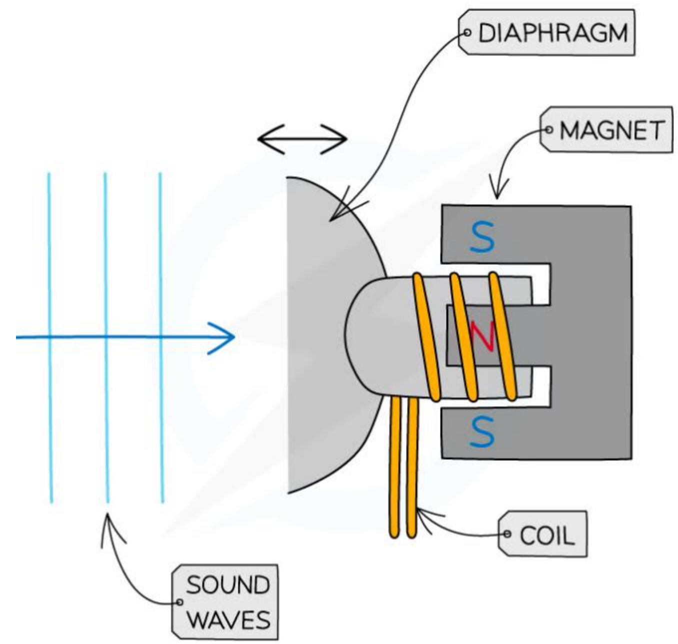
Electrical signals

How to model these transformations and to add additional boxes to correct the distortions? *filter*

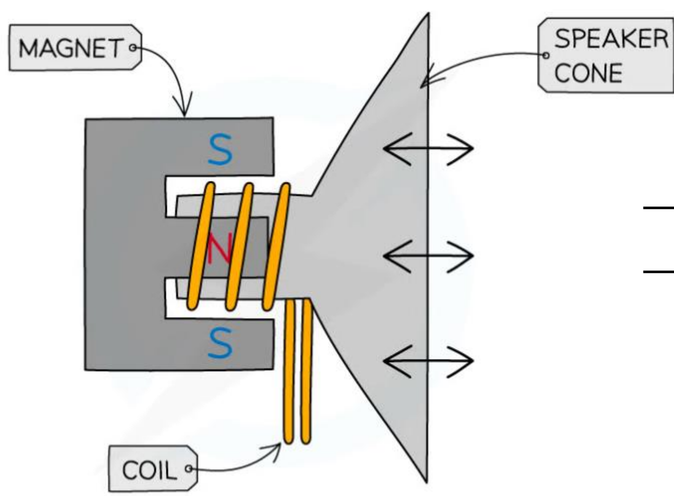
Mathematical Descriptions of Systems

▪ Microphone

→ Convert the sound waves into variations in current in electrical circuits
→ The pressure variations of sound waves cause the diaphragm to vibrate; then it causes the coil to move back and forth through the magnetic field



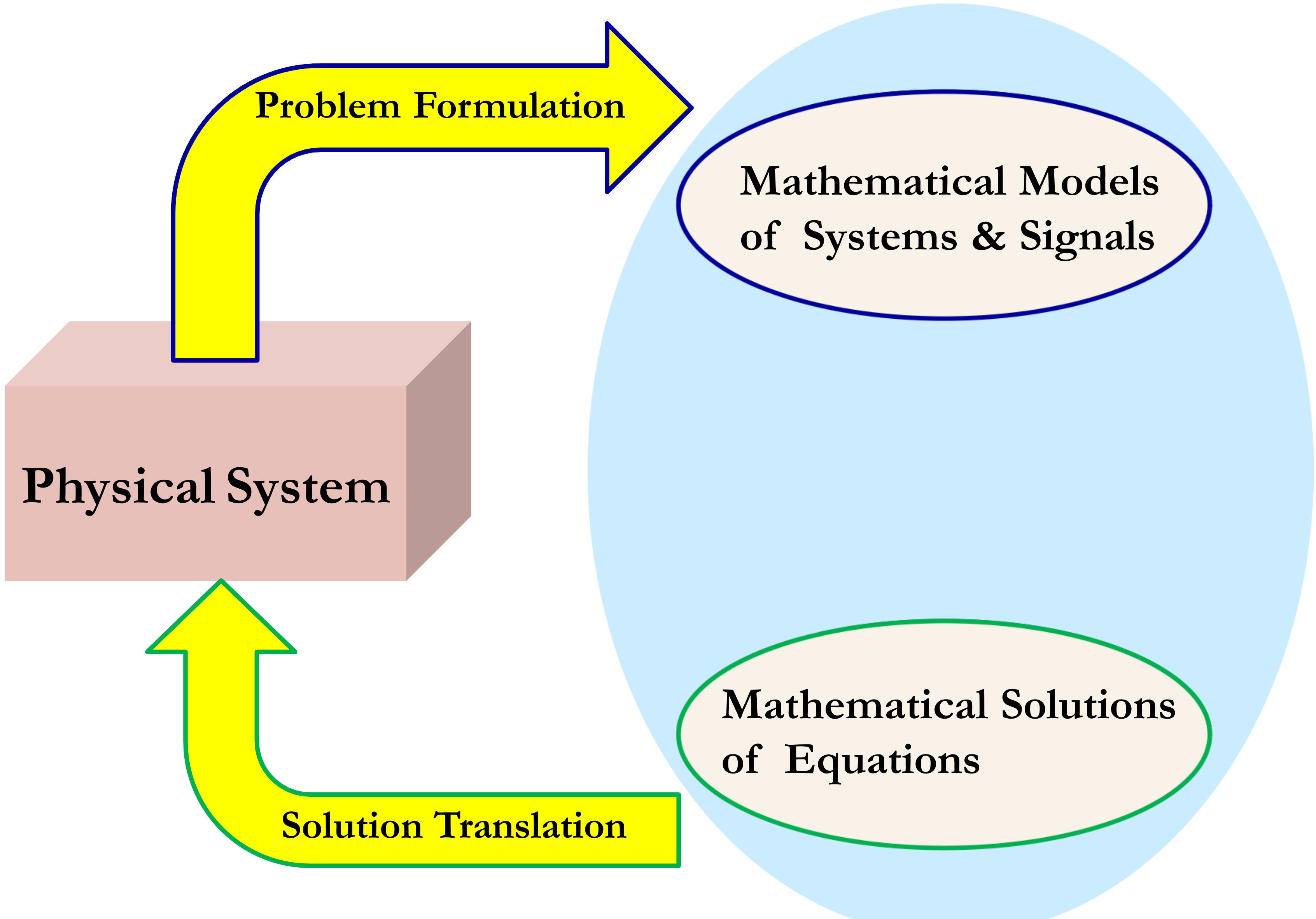
▪ Loudspeaker



→ Convert electrical signals into sound
→ The oscillating coil causes the speaker cone to oscillate; then this makes the air oscillate, creating sound waves

[Fig source: https://www.savemyexams.co.uk/]

Mathematical Descriptions of Systems



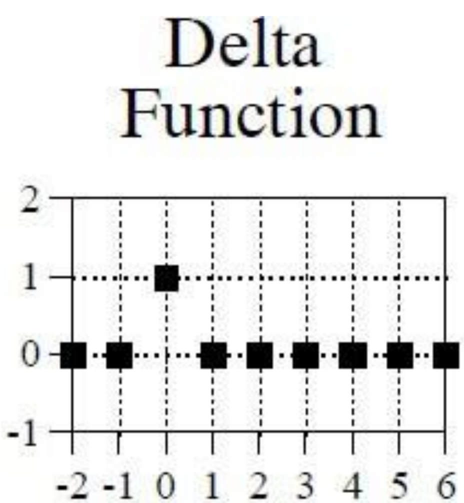
Input and System Response

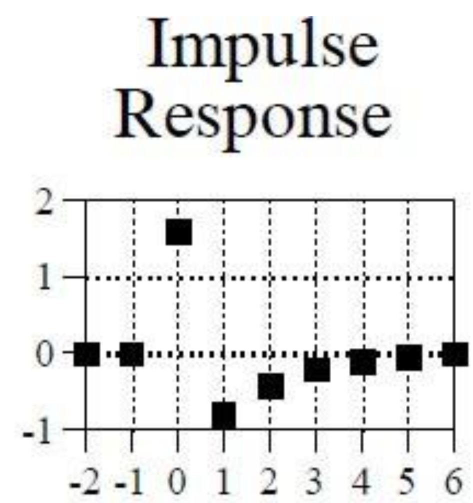
Delta Function and Impulse Response

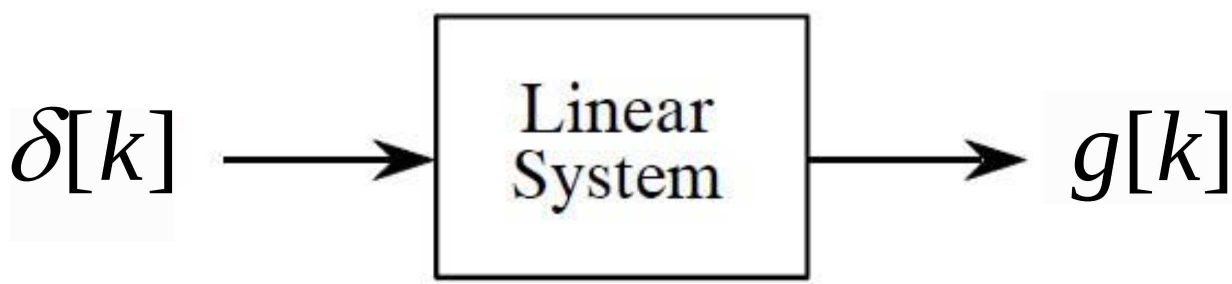
Two important terms used in DSP

$\rightarrow$ Delta Function

$\rightarrow$ Impulse Response



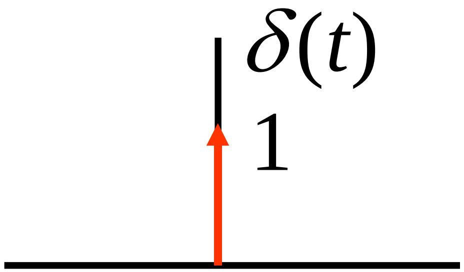




Delta Function

- **Definition of Delta function / Dirac Delta Function**

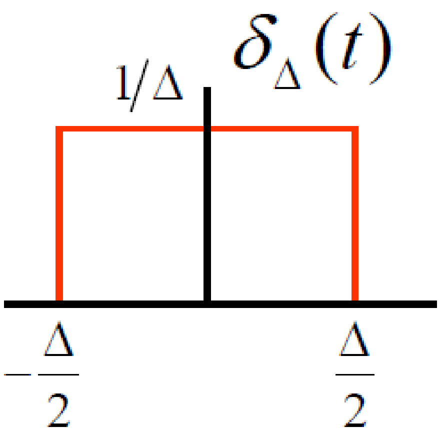
$\rightarrow$ Unit impulse / normalized impulse



$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$\rightarrow$ Unit pulse function: A brief rectangular pulse function with a duration



Delta function can be defined as
the limiting form of the unit pulse function
as the duration approaches zero

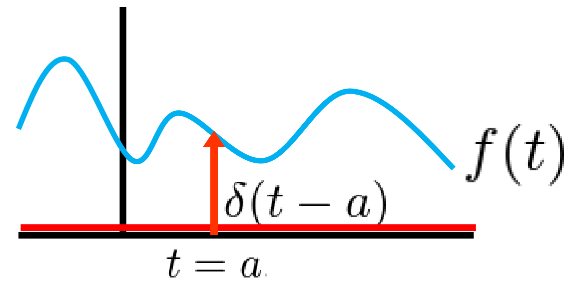
$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

Delta Function

▪ Properties of Delta function

→ **<u>Sifting</u>** property

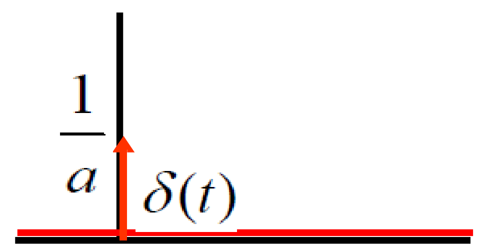
$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$



→ **<u>Scaling</u>** property

$$\delta(at) = \frac{1}{a}\delta(t), \quad a > 0$$

$$\int_{-\infty}^{\infty} \delta(at)\,\mathrm{d}t = \int_{-\infty}^{\infty} \delta(u)\,\frac{\mathrm{d}u}{|a|} = \frac{1}{|a|}$$



→ **<u>Convolution</u>** property

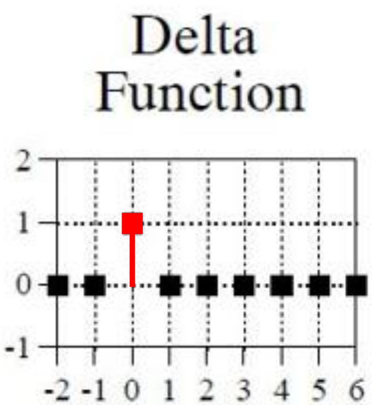
$$x(t) * \delta(t) = x(t)$$

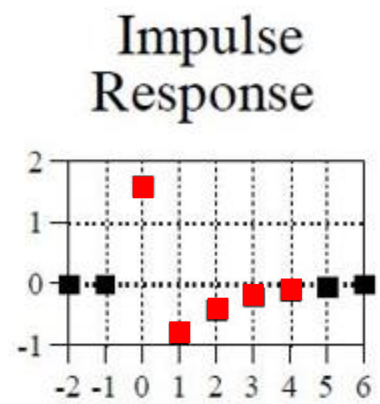
$$x(t) * a\delta(t) = ax(t)$$

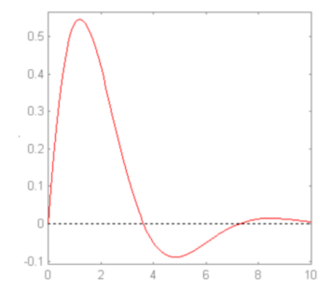
$$x(t) * \delta(t-a) = x(t-a)$$

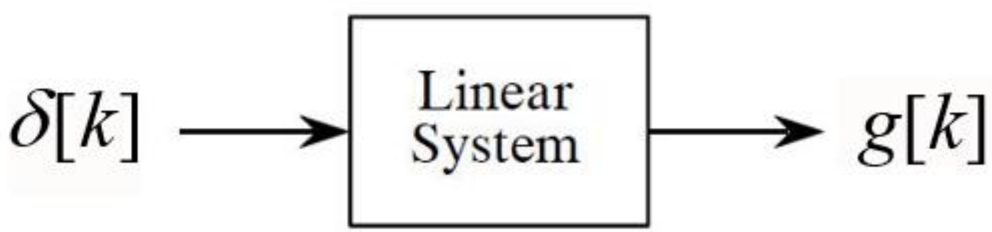
Impulse Response

- **Two important terms used in DSP**
  - → Delta Function
  - → Impulse Response









The impulse response is the signal that exits a system when a delta function (unit impulse) is the input

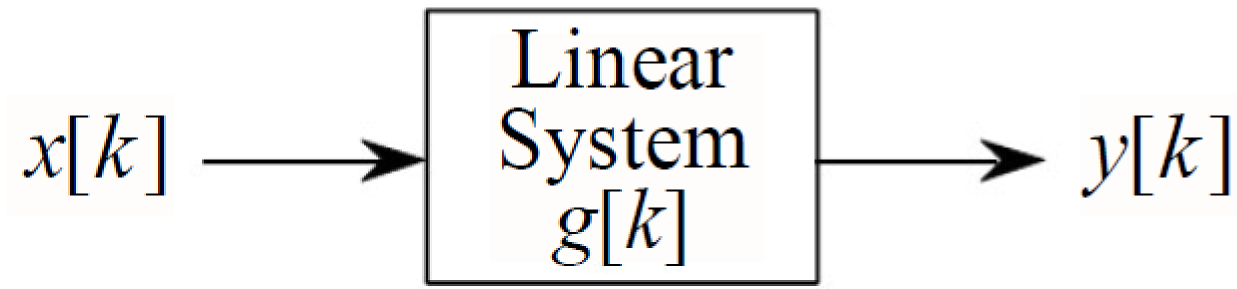
Convolution

Convolution

- **Convolution**

  → The most important technique in DSP

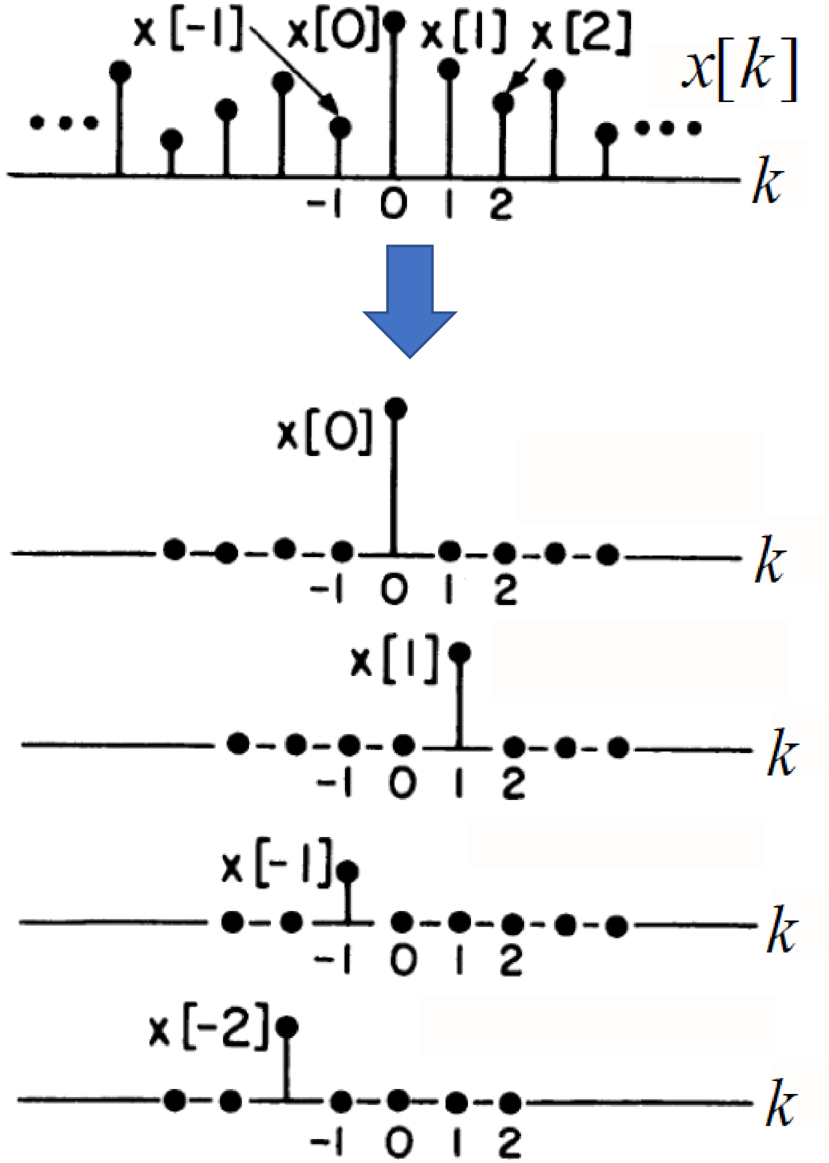
  → The mathematical way of combining two signals to form a third signal



$$x[k] * g[k] = y[k]$$

The output signal from a linear system is equal to the input signal convolved with the system's impulse response

Convolution with Delta Function



$$\boxed{\sum_{m=-\infty}^{\infty} x[m]\delta[k-m]}$$

$$x[0]\delta[k]$$

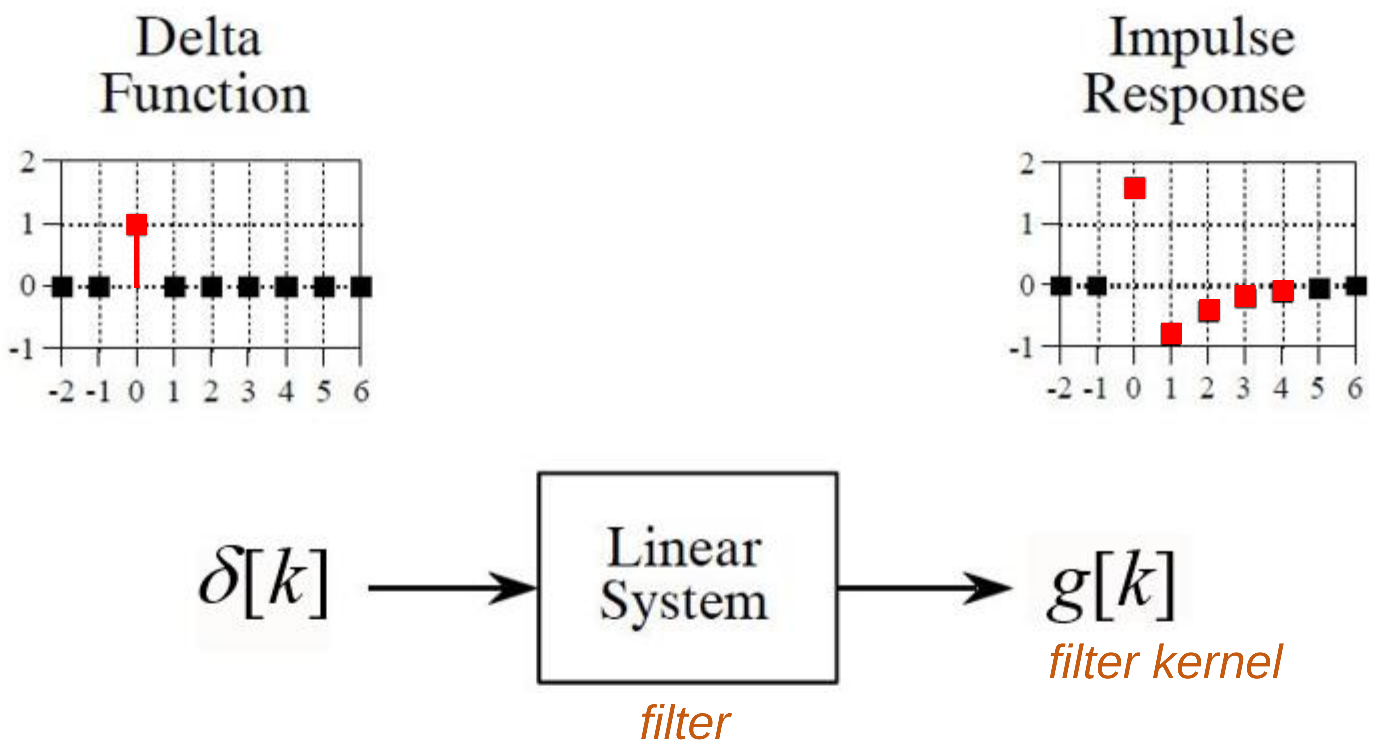
$$x[1]\delta[k-1]$$

$$x[-1]\delta[k+1]$$

$$x[-2]\delta[k+2]$$

Linear System

- ## Linear System



If the linear system is considered a <u>filter</u>, the impulse response is called the filter kernel, the convolution kernel, or the kernel

Linear System

- **An linear time-invariant (LTI) discrete system can be completely characterized by its impulse response**

$$y[k] = T\{\boxed{x[k]}\} = T\left\{\boxed{\sum_{m=-\infty}^{\infty} x[m]\delta[k-m]}\right\}$$

$$= \sum_{m=-\infty}^{\infty} x[m]T\{\delta[k-m]\}$$

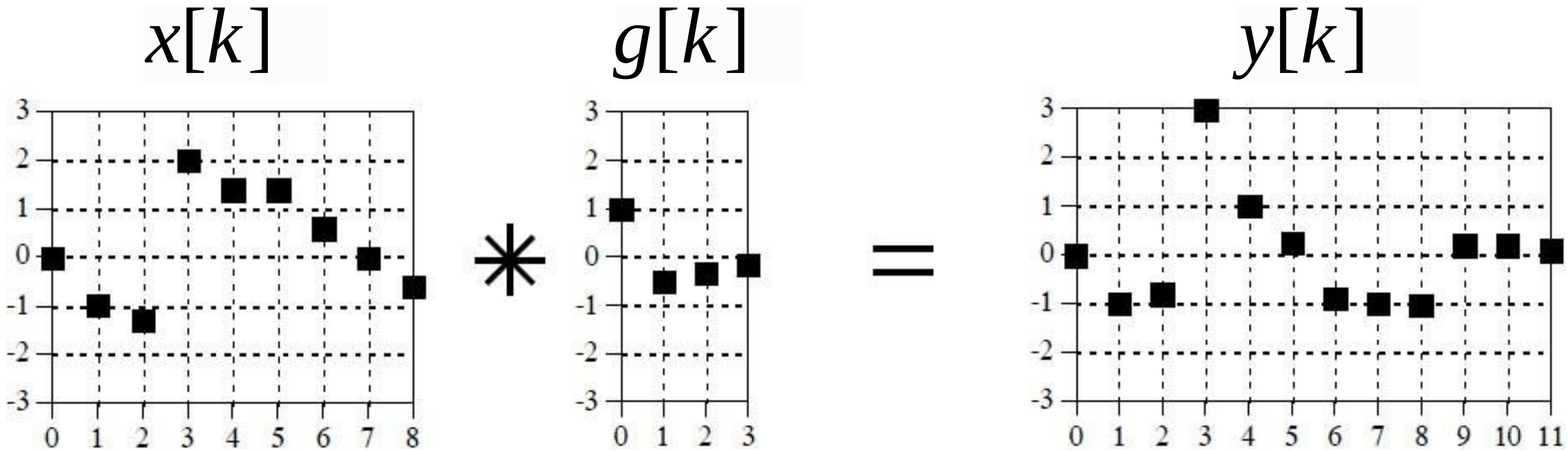
$$= \sum_{m=-\infty}^{\infty} x[m]g[k-m]$$

$$= x[k] * g[k]$$

A general discrete-time signal described as a superposition of weighted, delayed unit impulses

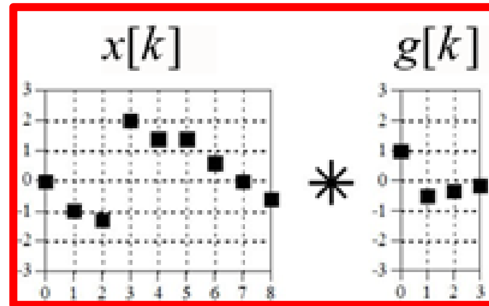
The best way to understand an LTI system is to study the frequency response function of the system (the Fourier transform of the impulse response)

Example Convolution Problem

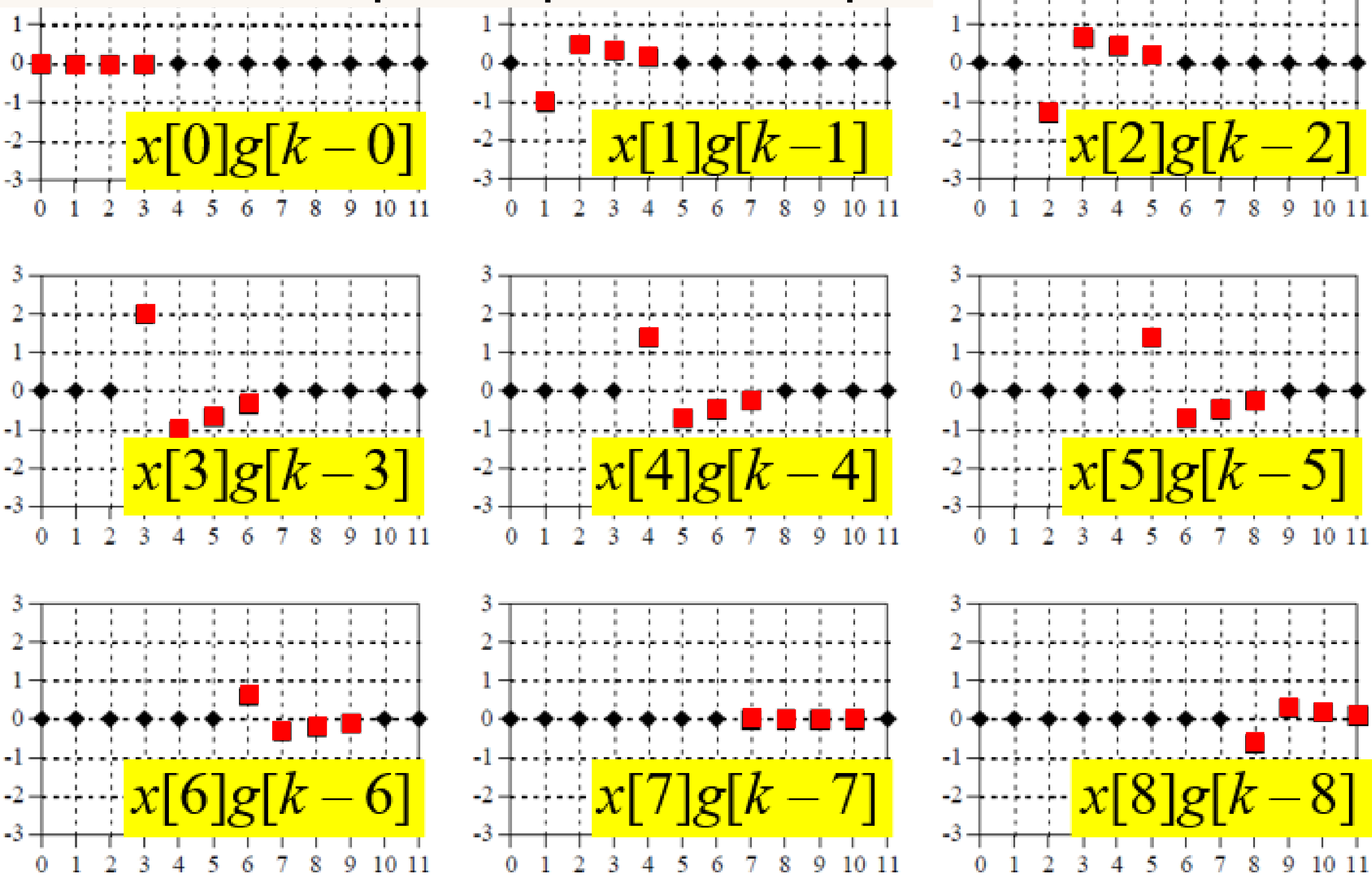


System output:
a sum of *weighted*, *delayed unit impulse* response

Input Side Algorithm

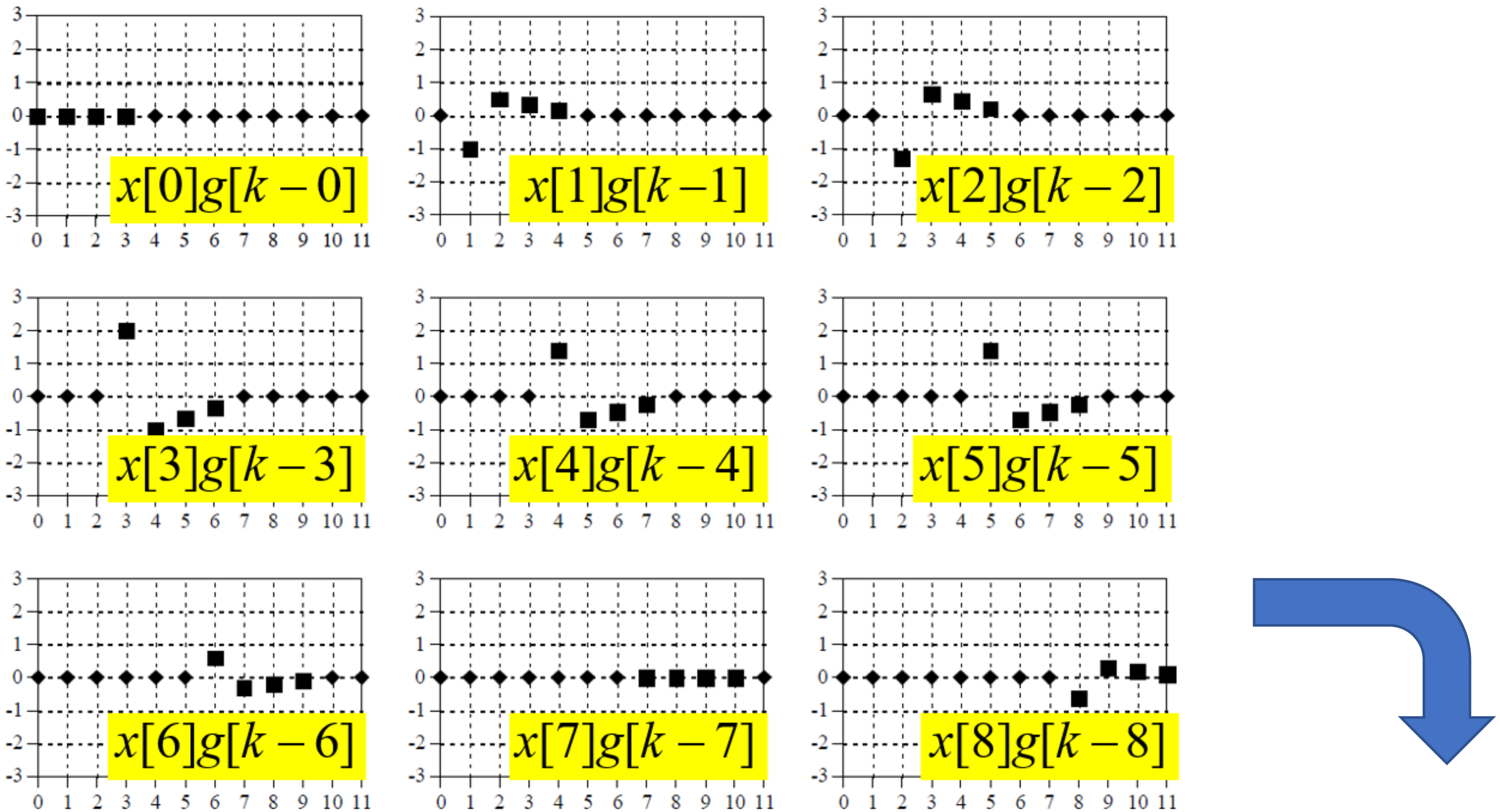


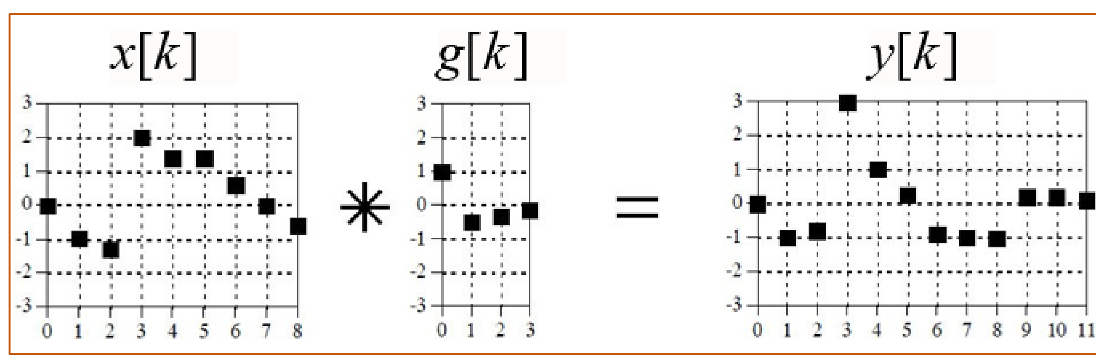
Each point in the input signal contributes
***a scaled and shifted* impulse response to the output.**



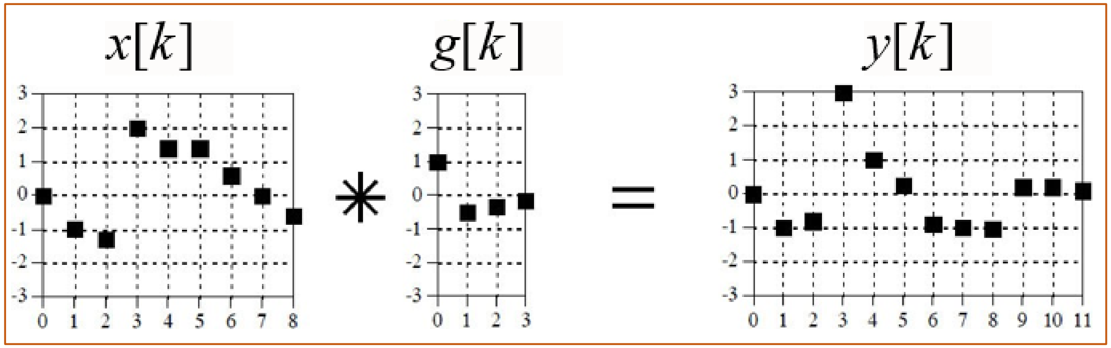
$$y[6] = x[0]g[6] + x[1]g[5] + x[2]g[4] + x[3]g[3] + x[4]g[2] + x[5]g[1] + x[6]g[0] + x[7]g[-1] + x[8]g[-2]$$

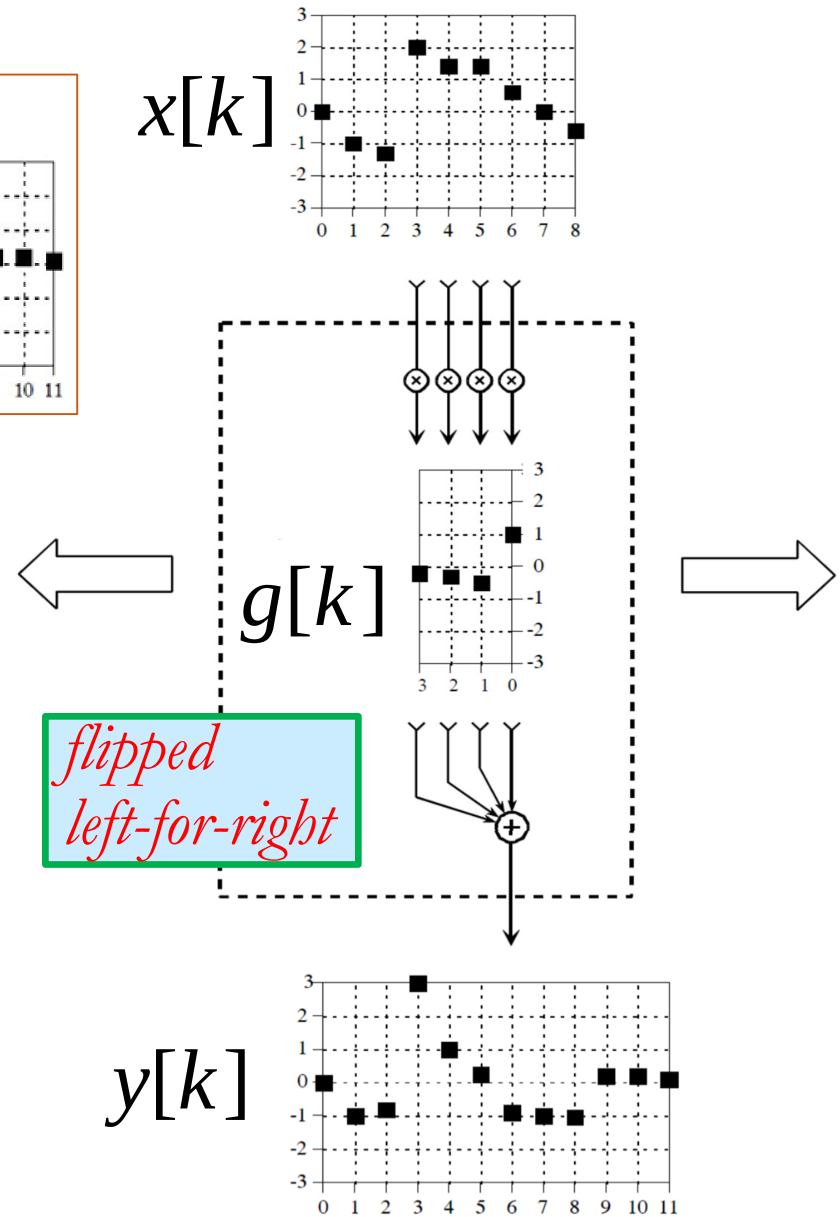
Details on Convolution Machine





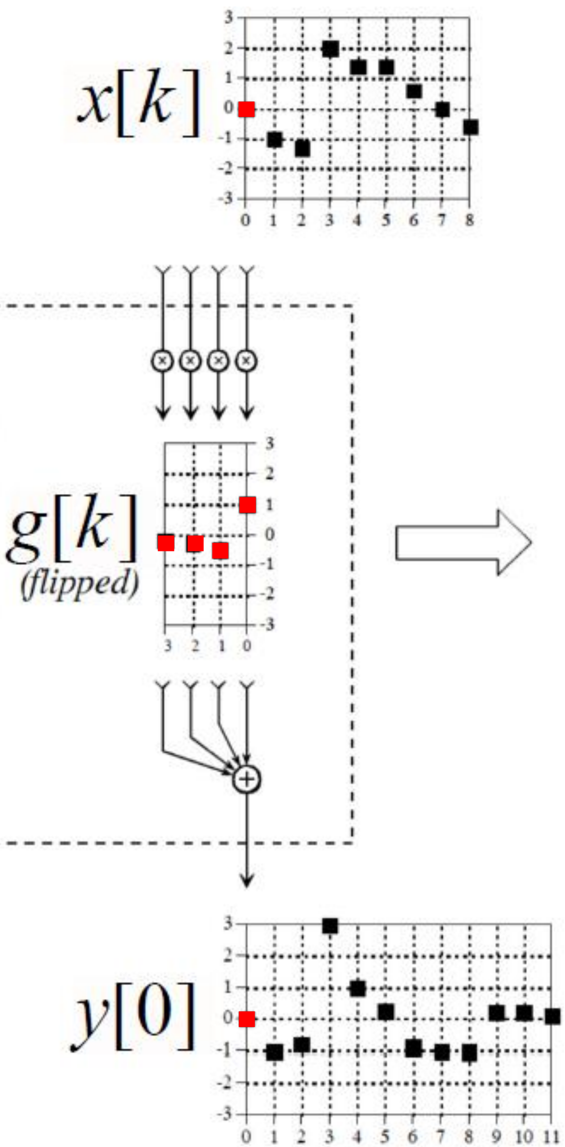
Output Side Algorithm

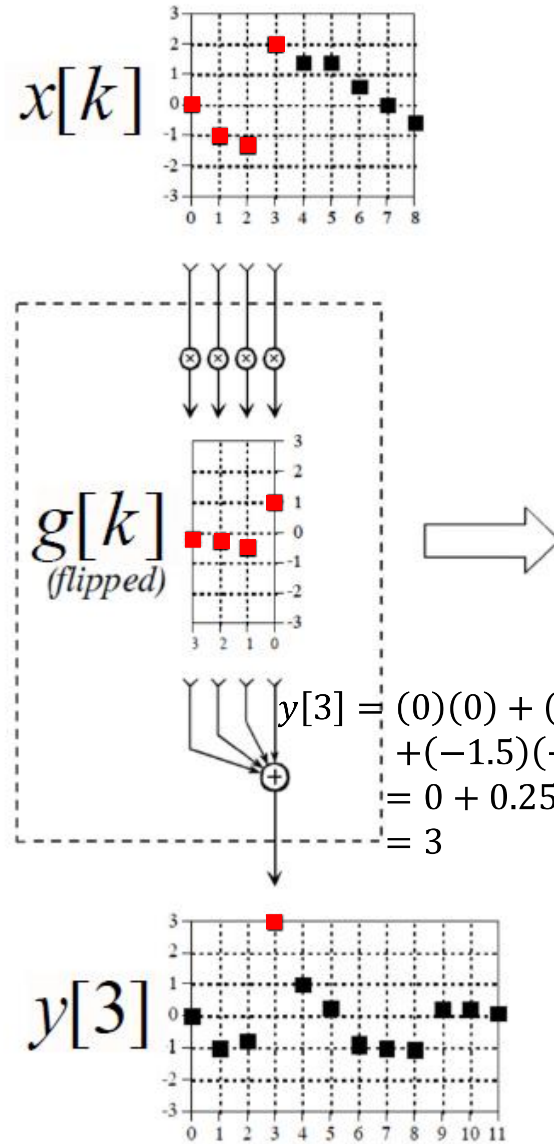




Showing how each sample in the output signal is influenced by the input signal and impulse response.

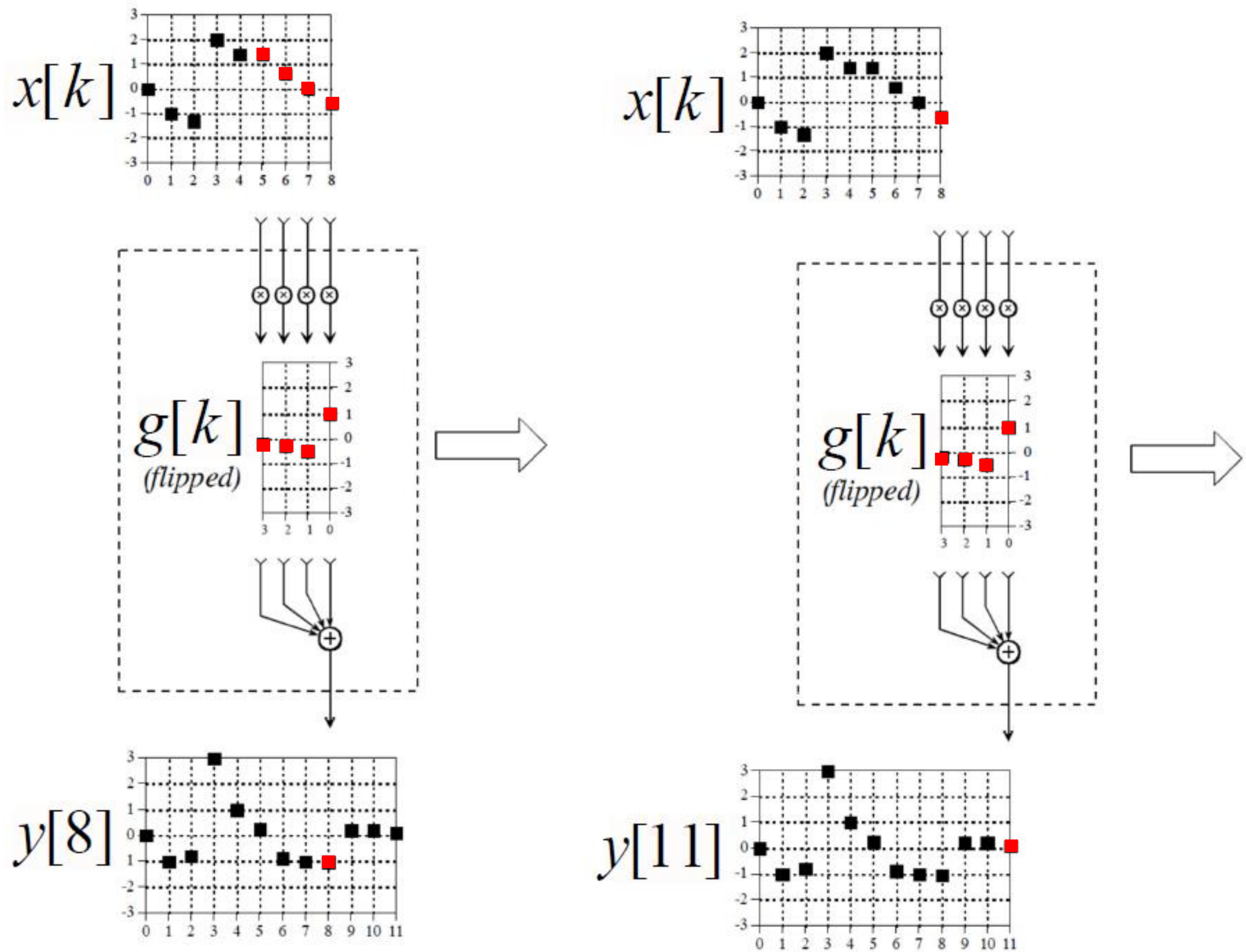
Output Side Algorithm





$$
\begin{aligned}
y[3] &= (0)(0) + (-1)(-0.25) \\
&\quad +(-1.5)(-0.5) + (2)(1) \\
&= 0 + 0.25 + 0.75 + 2 \\
&= 3
\end{aligned}
$$

Output Side Algorithm



Intuition behind Convolution

- **Sequence Multiplication**

$$
\begin{array}{rrrrrl}
 & & 2 & 1 & 1 & \\
 & & 1 & 3 & 2 & \times \\
\hline
 & & 4 & 2 & 2 & \\
 & 6 & 3 & 3 & & \\
2 & 1 & 1 & & & \\
\hline
2 & 7 & 8 & 5 & 2 & +
\end{array}
$$

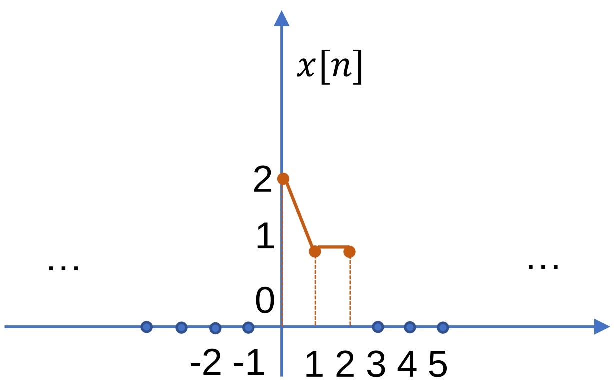
$$
\begin{array}{cccccl}
 & & g[0] & g[1] & g[2] & \\
 & & x[0] & x[1] & x[2] & \times \\
\hline
 & & g[0]x[2] & g[1]x[2] & g[2]x[2] & \\
 & g[0]x[1] & g[1]x[1] & g[2]x[1] & & \\
g[0]x[0] & g[1]x[0] & g[2]x[0] & & & \\
\hline
y[0] & y[1] & y[2] & y[3] & y[4] & +
\end{array}
$$

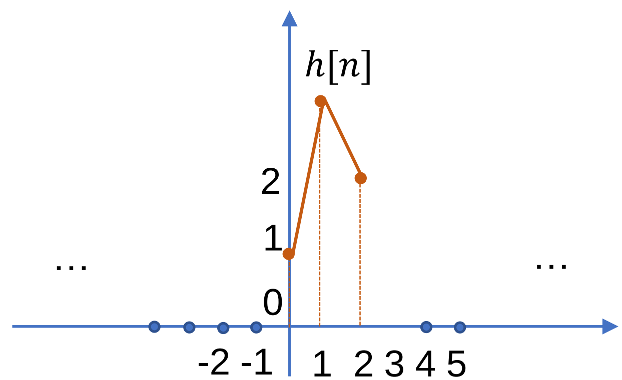
$$y[k] = \sum_{m=0}^{k} x[m]g[k-m]$$

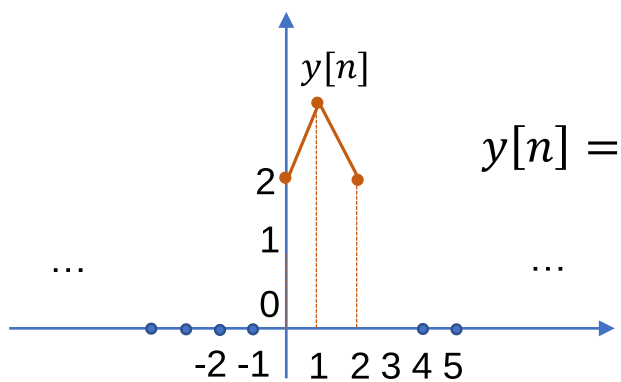
The difference with Signal Multiplication

- **Term-by-term Multiplication**

$$\left.\begin{array}{l} x[n] = \{2,1,1\} \\ g[n] = \{1,3,2\} \end{array}\right\} \longrightarrow \quad y[n] = x[n]g[n]$$





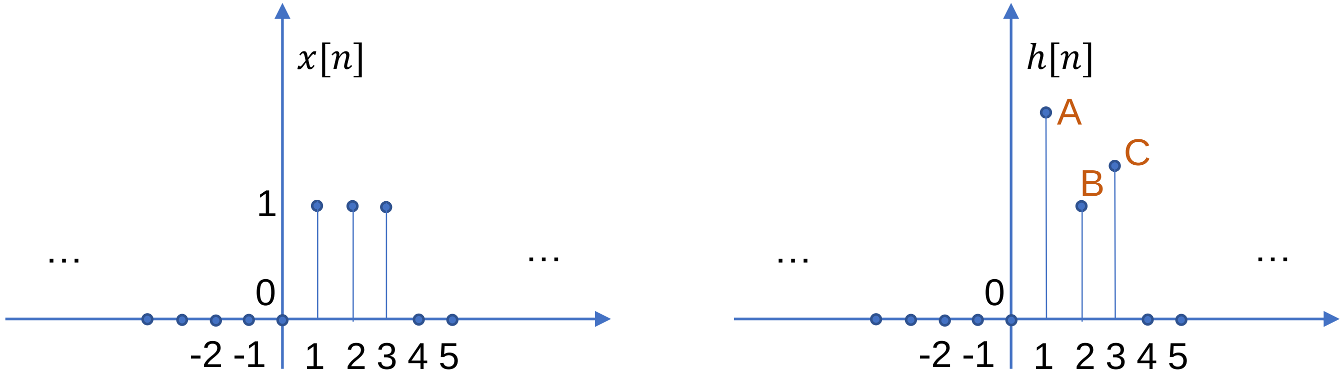


$$y[n] = \{2 \times 1, 1 \times 3, 1 \times 2\} = \{2,3,2\}$$

Homework #2.1 Convolution (1 pt.): Due Dec. 20

- **Convolution**

  → Compute the convolution and sketch the output $y[n]$ given the following input $x[n]$ and impulse response $h[n]$



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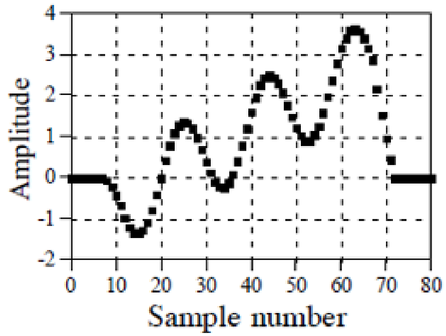
Examples:

Convolution for *low*-pass and *high*-pass filtering

three cycles of a sine wave, plus a slowly rising ramp

Low-pass Filter

$$g[k] = \frac{\sin(\omega_c k)}{\pi k}$$



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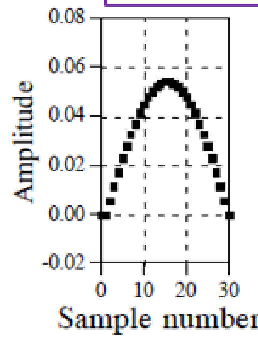

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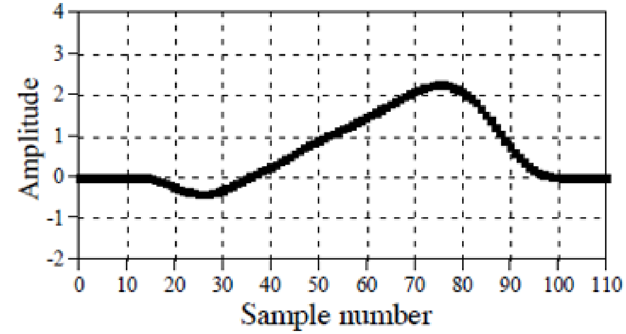
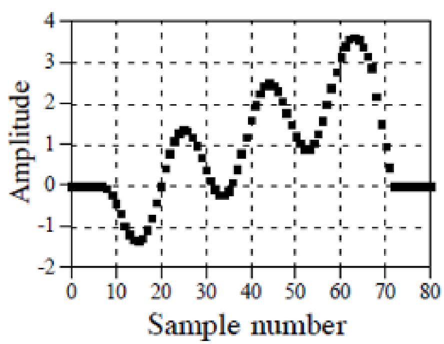


input signal *impulse response* *output signal*

High-pass Filter

$$g[k] = \delta[k] - \frac{\sin(\omega_c k)}{\pi k}$$



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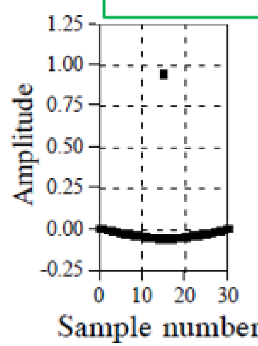

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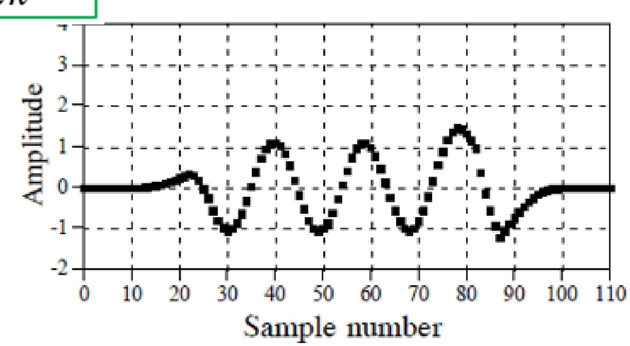
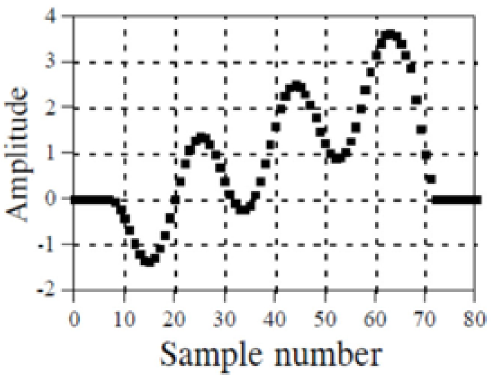


Examples:

Inverting Attenuator and Discrete Derivative

Inverting Attenuator

flipped top-for-bottom amplitude reduced



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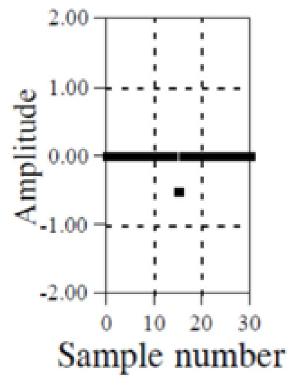

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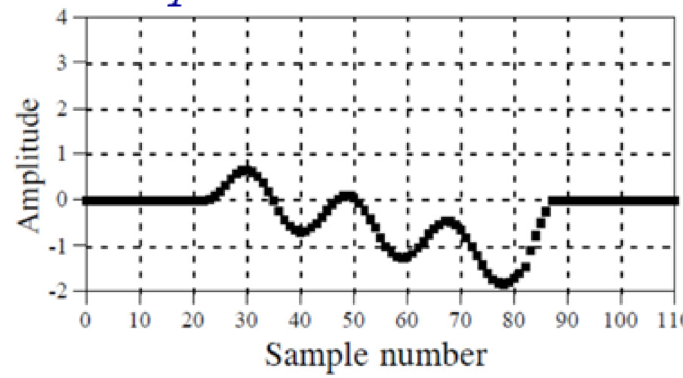
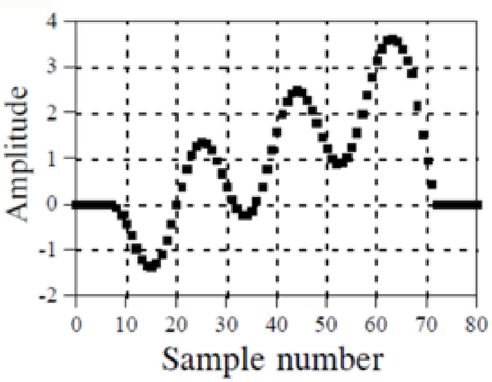


input signal *impulse response* *output signal*

Discrete Derivative (the first difference)

slope of the input signal



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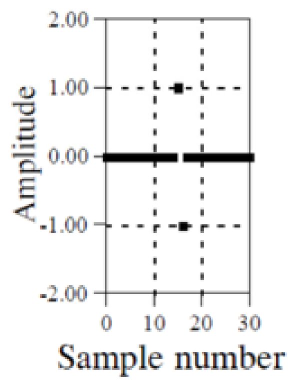

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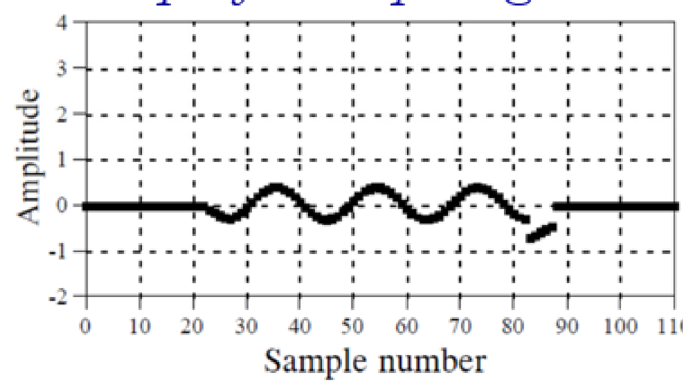
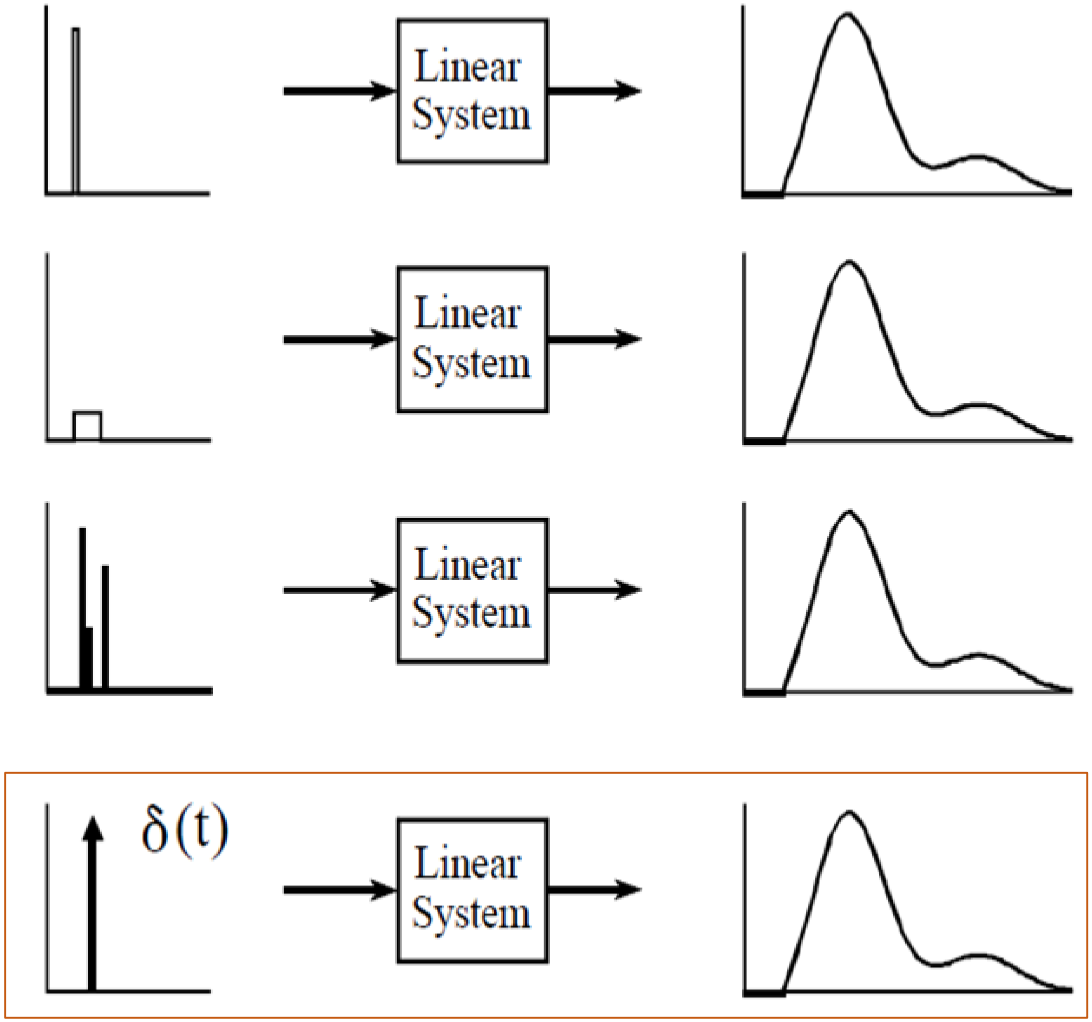


Examples:

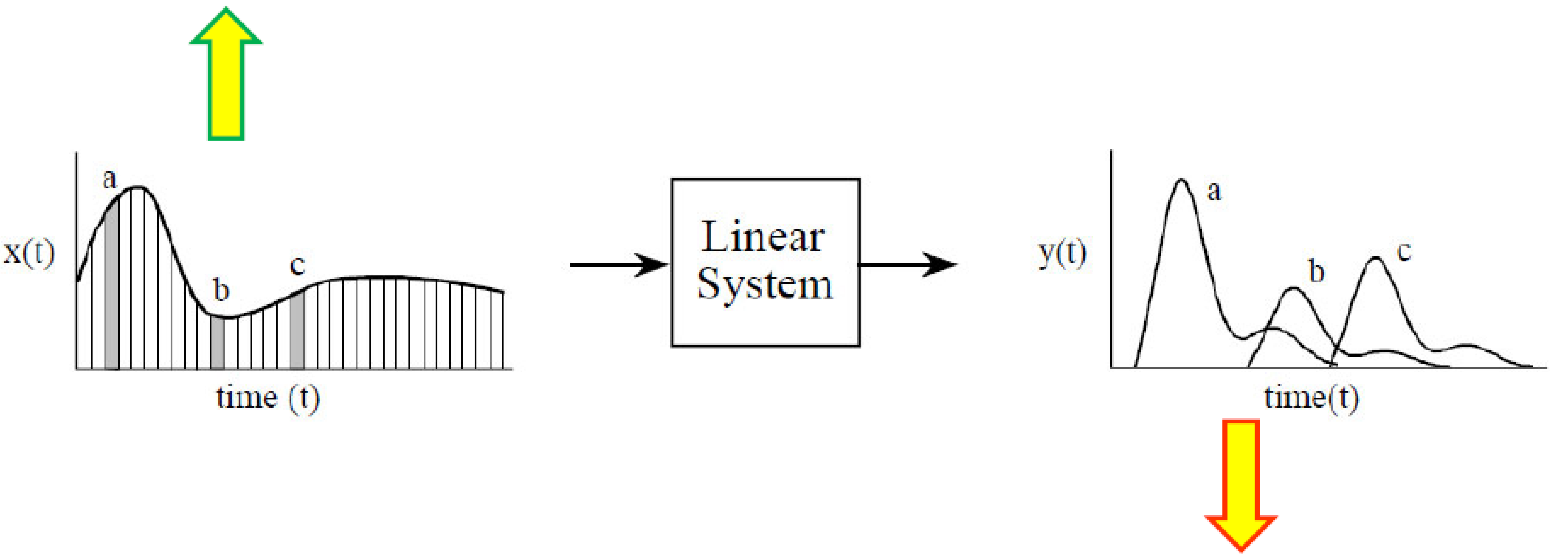


Impulses

If the input to a linear system is brief compared to the resulting output, the shape of the output depends only on the characteristics of the system and not the shape of the input

Convolution from Viewpoint of Input Signals

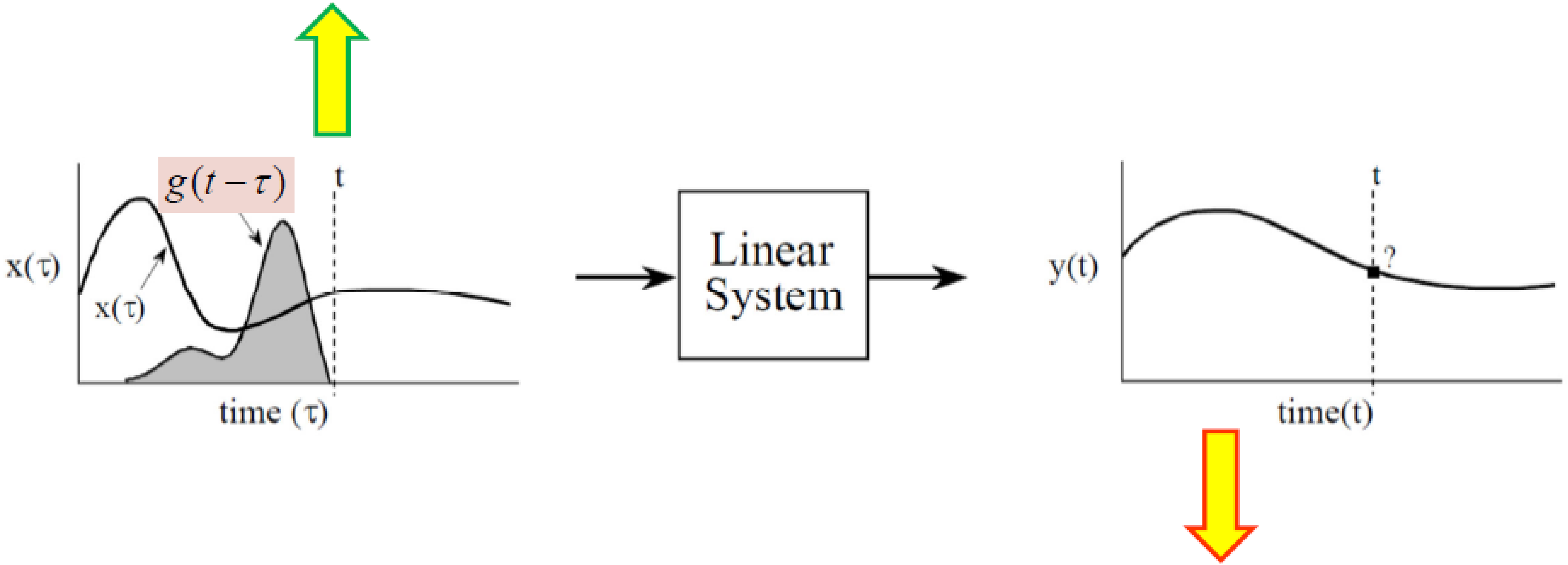
The input signal is divided into narrow segments, each acting as an impulse to the system.



The output is the sum of the resulting scaled and shifted impulse responses.

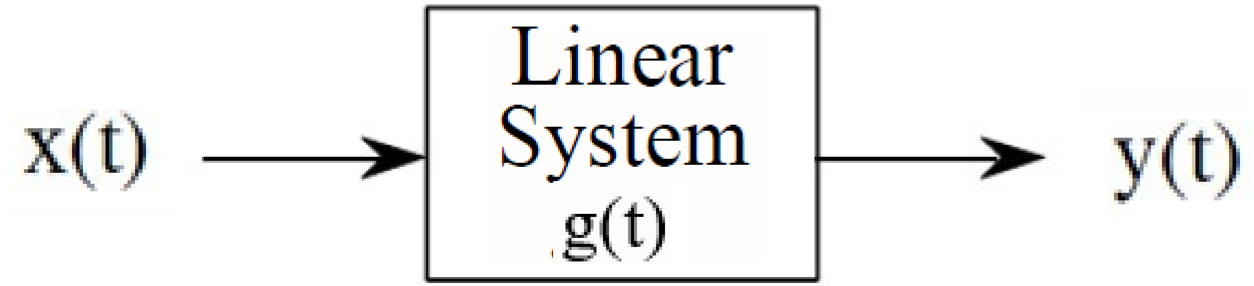
Convolution from Viewpoint of Output Signals

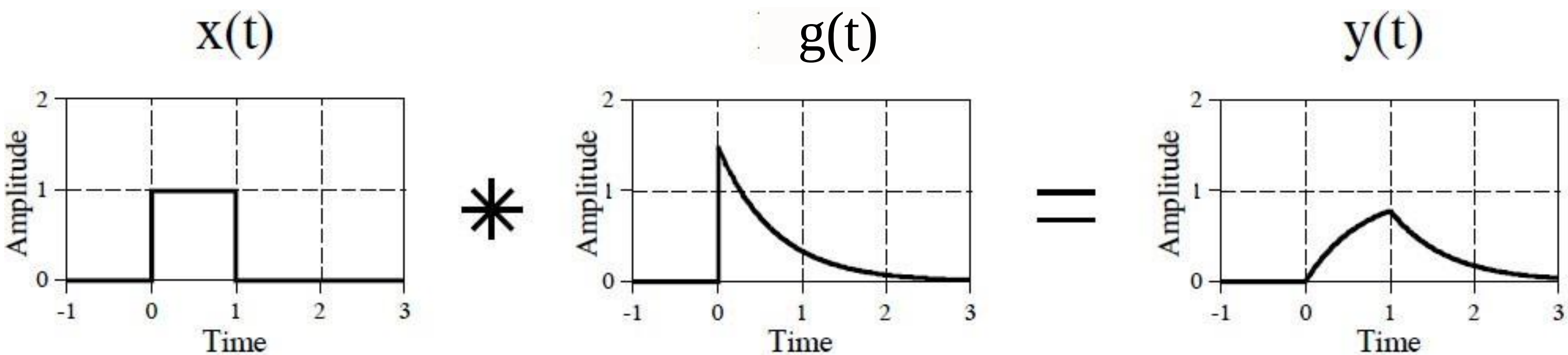
The input signal is weighted by
the flipped and shifted impulse response.



Integrating the weighted input signal produces
the value of the output point, $y(t)$.

Continuous Convolution

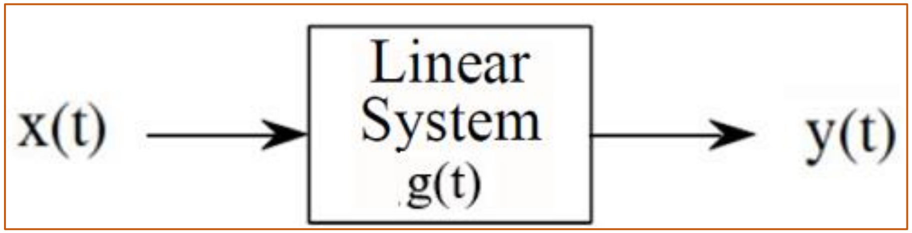




$$x(t) = 1, \quad 0 \le t \le 1$$
$$x(t) = 0, \quad otherwise$$

How the system will react to all possible signals?

Continuous Convolution



$$\delta(t) \xrightarrow{\text{Linear System } g(t)} g(t)$$

$$\delta(t-\tau) \xrightarrow{\text{Linear System } g(t)} g(t-\tau)$$

$$u(\tau)\delta(t-\tau) \xrightarrow{\text{Linear System } g(t)} u(\tau)g(t-\tau)$$

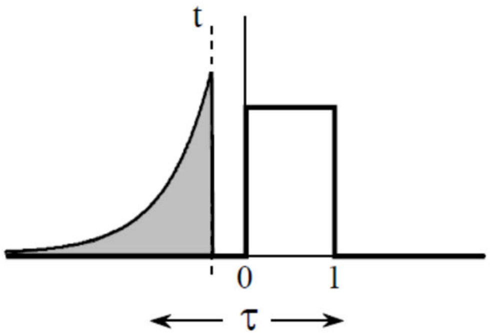
$$\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau \xrightarrow{\text{Linear System } g(t)} \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau$$

Continuous Convolution

Calculating Convolution by Segments

No overlap

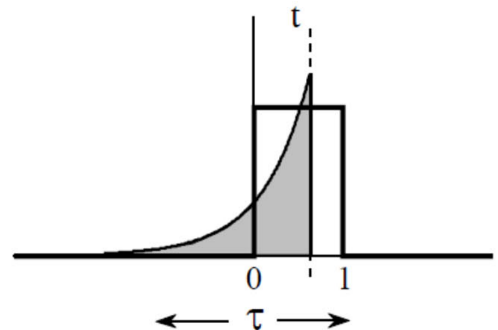
$(t < 0)$



$$y(t) = 0, \quad t < 0$$

Partial overlap

$(0 \le t \le 1)$



$$y(t) = \int_{-\infty}^{\infty} u(\tau) g(t-\tau) d\tau$$

$$= \int_0^t 1 \cdot \alpha e^{-\alpha(t-\tau)} d\tau$$

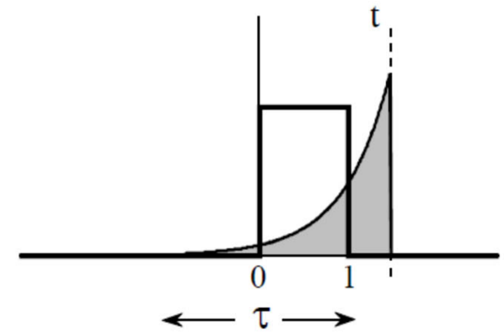
$$= e^{-\alpha t} [e^{\alpha\tau}]\Big|_0^t$$

$$= e^{-\alpha t} [e^{\alpha t} - 1]$$

$$= 1 - e^{-\alpha t}, \quad 0 \le t \le 1$$

Full overlap

$(t > 1)$



$$y(t) = \int_{-\infty}^{\infty} u(\tau) g(t-\tau) d\tau$$

$$= \int_0^1 1 \cdot \alpha e^{-\alpha(t-\tau)} d\tau$$

$$= e^{-\alpha t} [e^{\alpha\tau}]\Big|_0^1$$

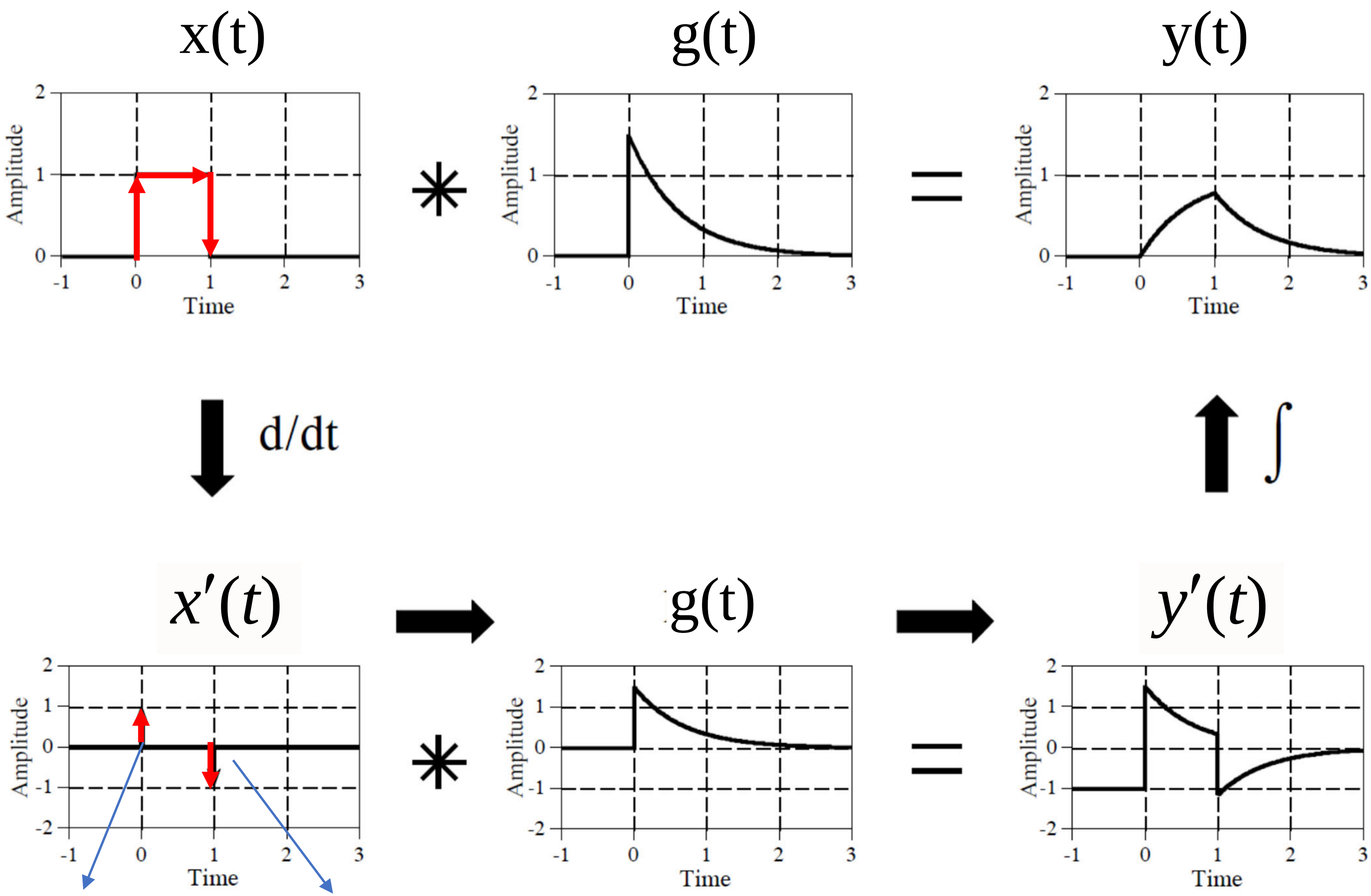
$$= [e^{\alpha} - 1] e^{-\alpha t}, \quad t > 1$$

$$g(t) = 0, \quad t < 0$$

$$g(t) = \alpha e^{-\alpha t}, \quad t \ge 0$$

Continuous Convolution

Another Strategy



$\delta(t)$ $\quad -\delta(t-1)$

$$\boxed{y'(t) = g(t) - g(t-1)}$$

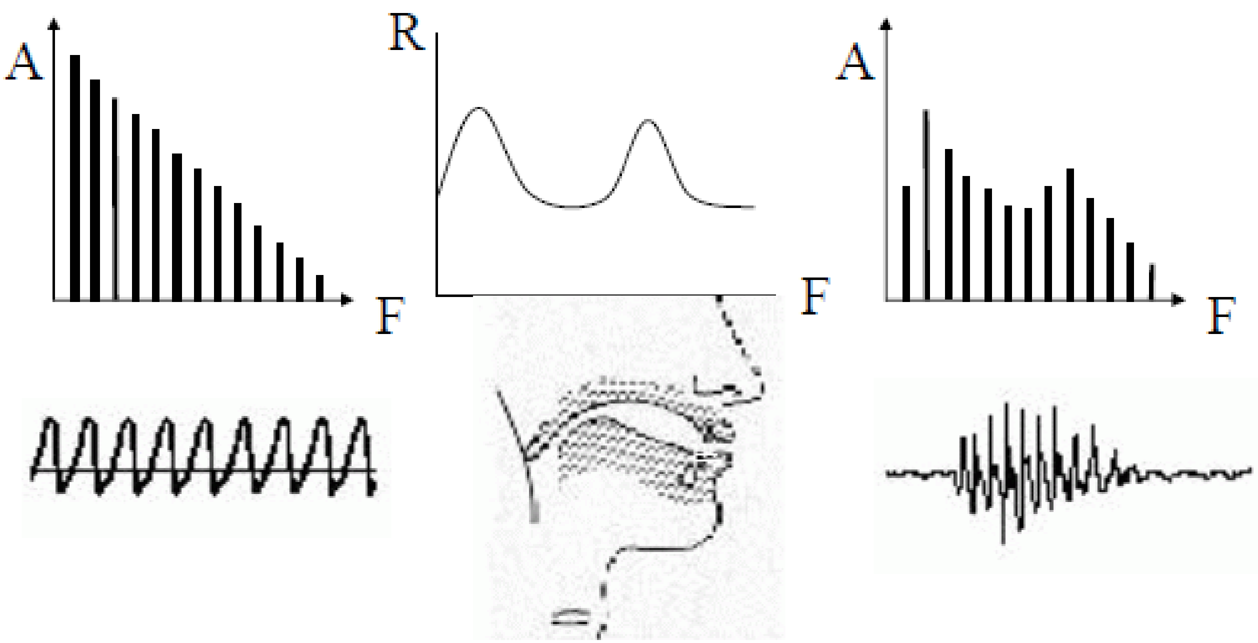
Human Speech

▪ Human Speech

→ **The combination of that the glottal source and vocal tract shape (filter)**

$h(t)$

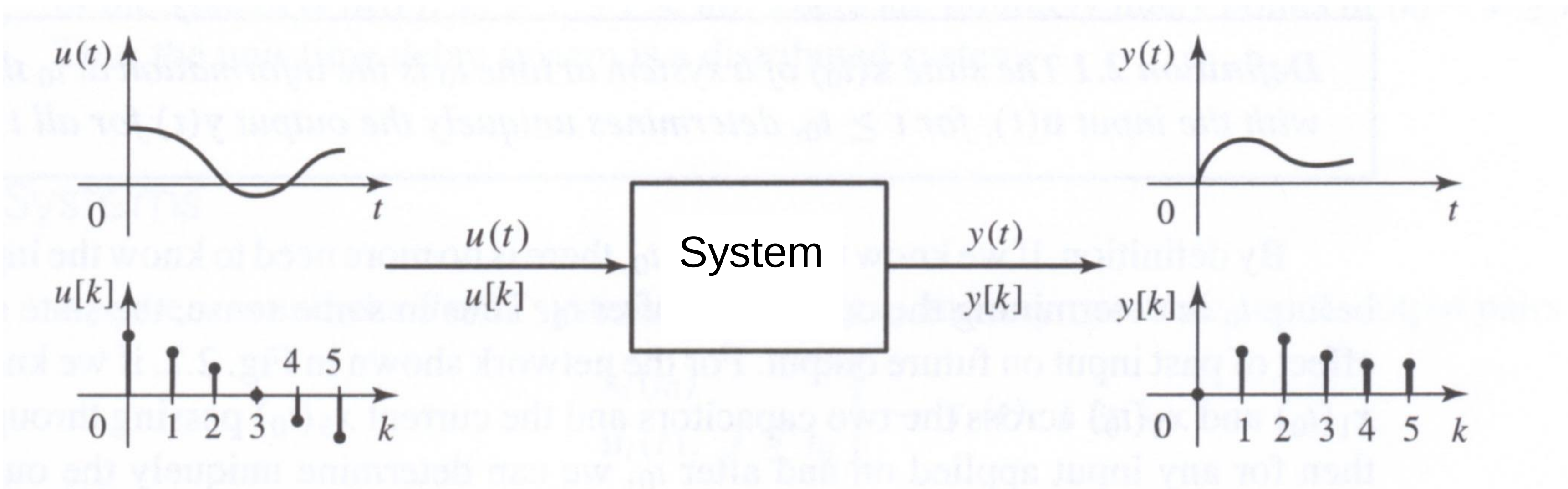
Source (Glotal Pulse) $x(t)$ → **Vocal tract filter** → **Speech Sound** $y(t) = x(t) * h(t)$



Mathematical Descriptions of Systems

Mathematical Descriptions of Systems

- **System**



input terminals

output terminals

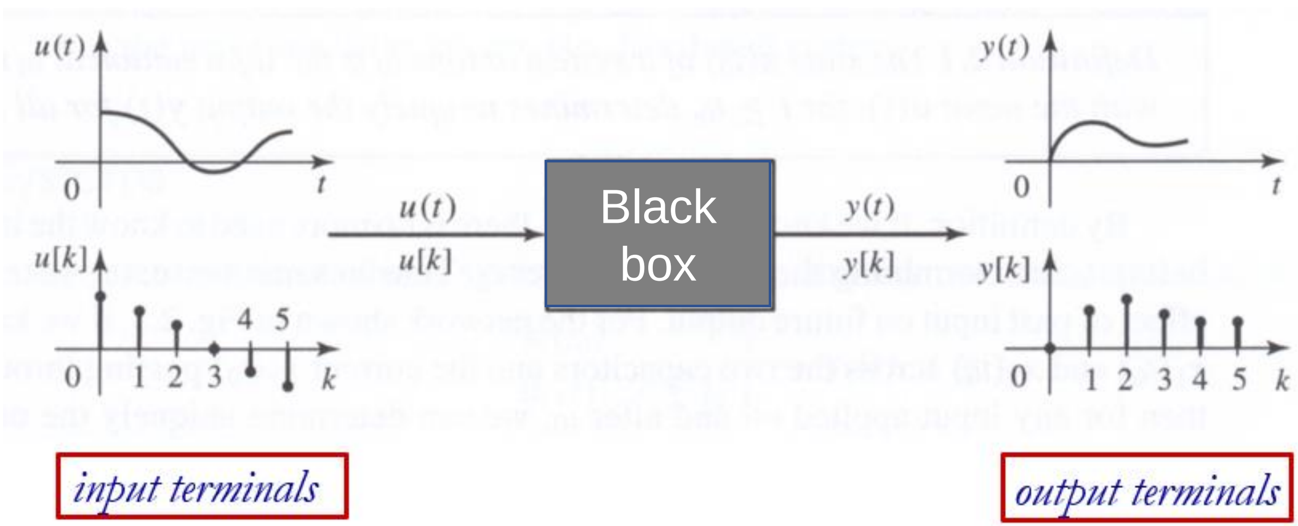
External Description

▪ External Description: Input-Output Description

→ View the system as a "black box" description:

no information on the internal details of the system

→ Characterize by the relation of input, output, and system response (impulse response)



$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

output

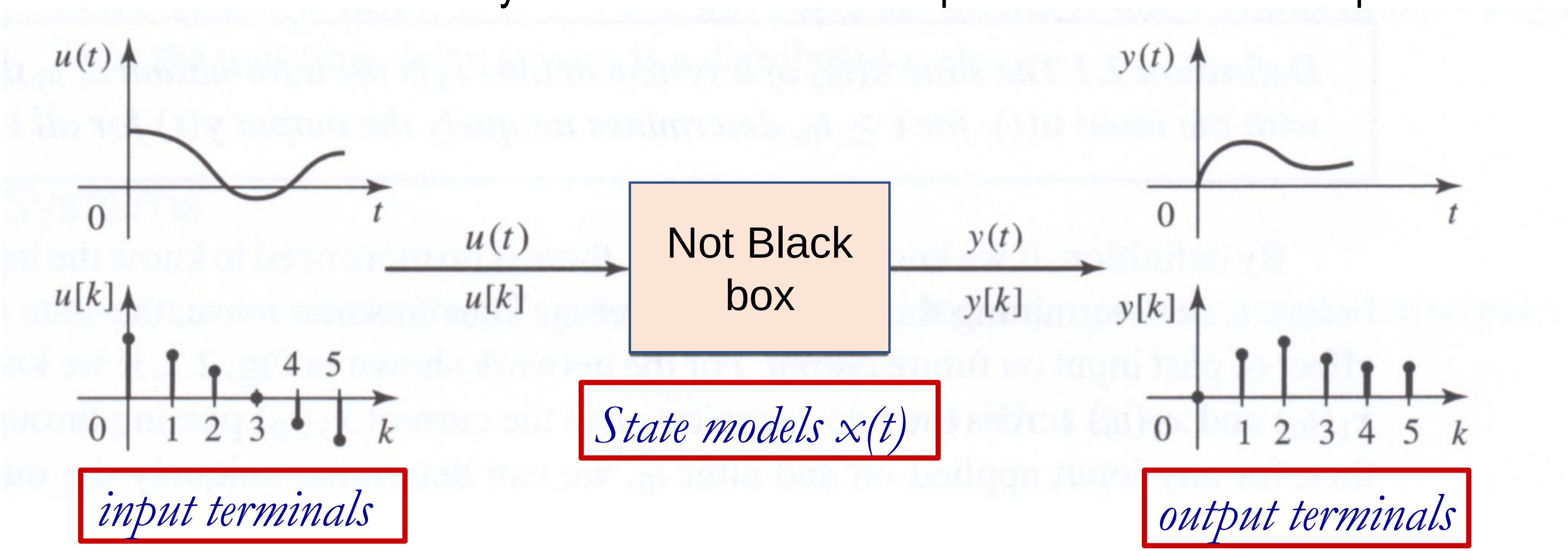
input

Internal Description

▪ Internal Description: State-Space Description

→ State-space representation:

a mathematical model of a physical system as a set of input, output, and state variables related by first-order differential equations or difference equations



$$\dot{x}(t) = Ax(t) + Bu(t)$$ *1st order DE*

$$y(t) = Cx(t) + Du(t)$$ *AE*

state *output* *input*

Input-Output Descriptions of Linear Systems

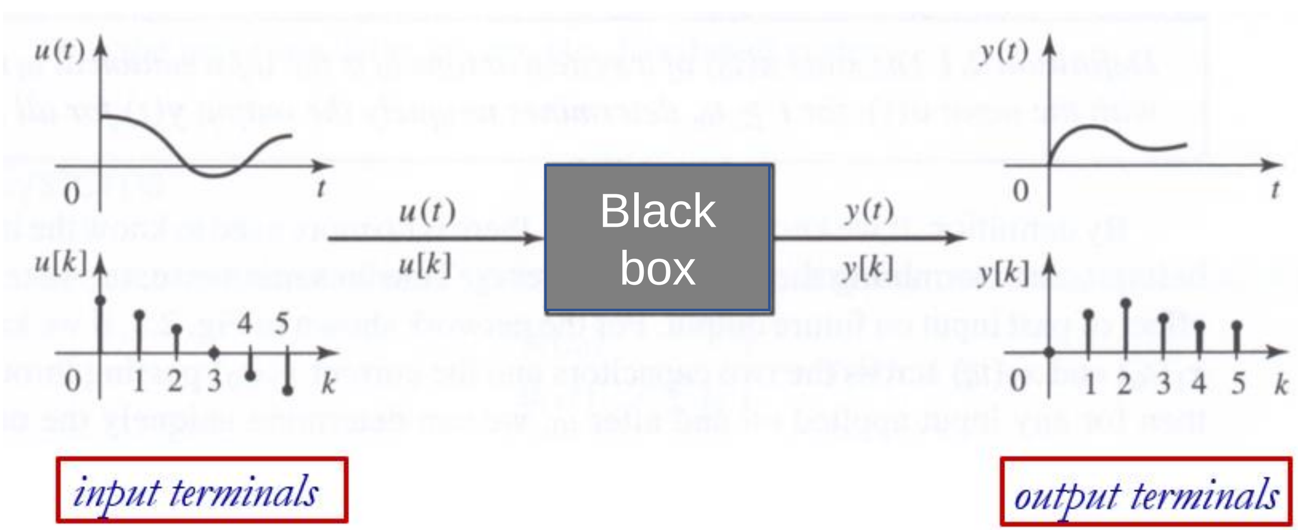
External Description

▪ External Description: Input-Output Description

→ View the system as a "black box" description:

no information on the internal details of the system

→ Characterize by the relation of input, output, and system response (impulse response)



$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

output *input*

Input-Output Description: Zero-state Response

a mathematical equation to describe the zero-state response of linear systems

→ *The initial state is assumed implicitly to be zero and the output is excited exclusively by the input*

$$u(t) \approx \sum_i u(t_i)\delta_\Delta(t - t_i)\Delta \quad : \textit{input}$$

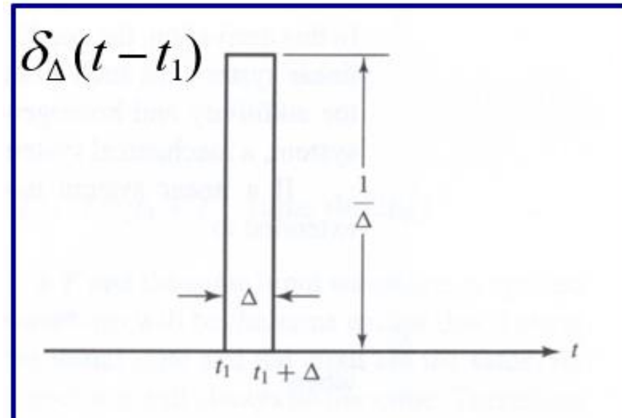
$$\delta_\Delta(t) \rightarrow g_\Delta(t) \quad : \textit{definition}$$

$$\delta_\Delta(t - t_i) \rightarrow g_\Delta(t - t_i) \quad : \textit{time shifting}$$

$$\delta_\Delta(t - t_i)u(t_i)\Delta \rightarrow g_\Delta(t - t_i)u(t_i)\Delta \quad : \textit{homogeneity}$$

$$\sum_i \delta_\Delta(t - t_i)u(t_i)\Delta \rightarrow \sum_i g_\Delta(t - t_i)u(t_i)\Delta \quad : \textit{additivity}$$

$$y(t) \approx \sum_i g_\Delta(t - t_i)u(t_i)\Delta \quad : \textit{output}$$

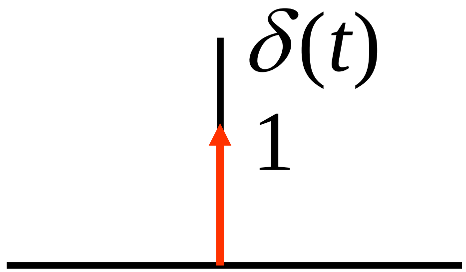


Pulse at t_1.

Delta Function

- **Definition of Delta function / Dirac Delta Function**

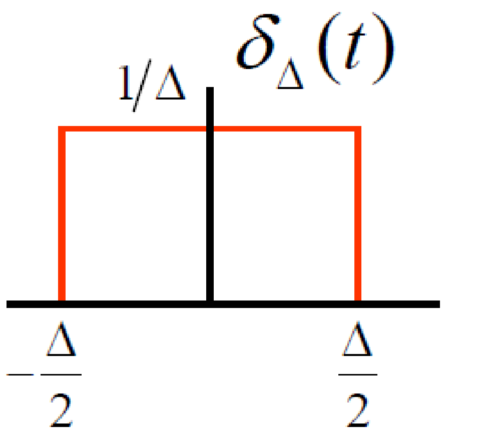
$\rightarrow$ Unit impulse / normalized impulse



$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

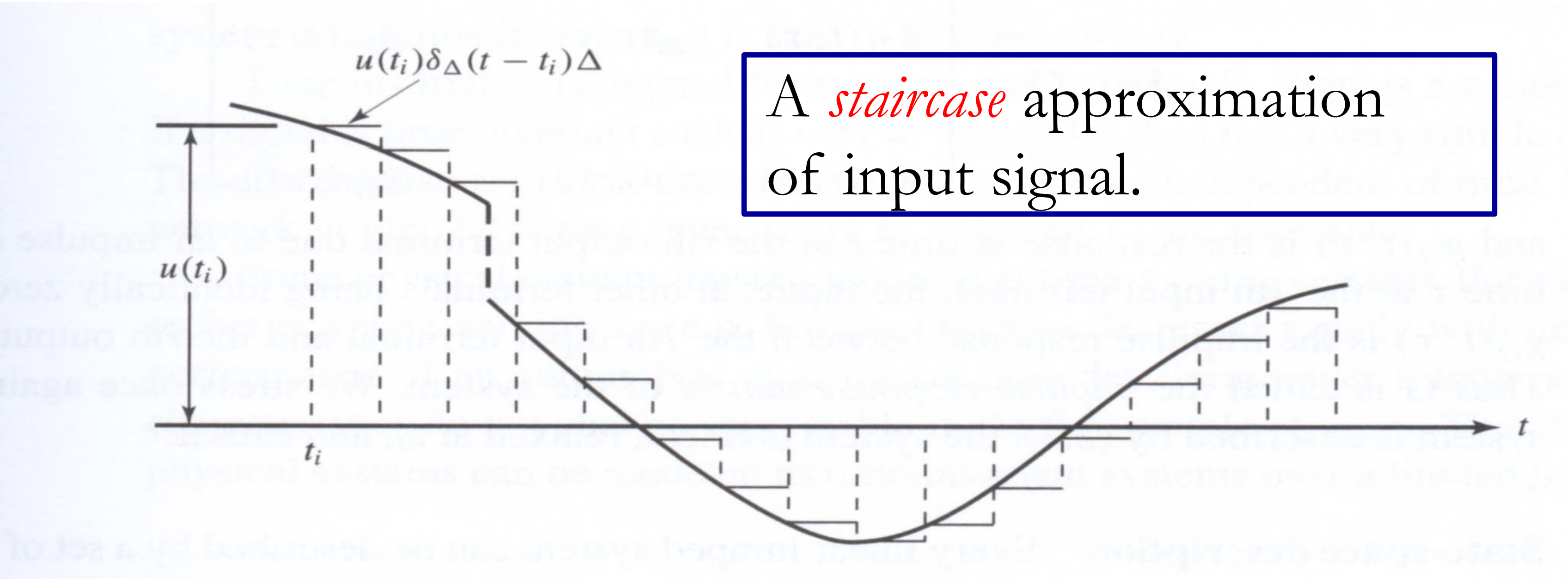
$\rightarrow$ Unit pulse function: A brief rectangular pulse function with a duration



$$\delta(t) = \lim_{\Delta \to 0} \delta_{\Delta}(t)$$

Delta function can be defined as
the limiting form of the unit pulse function
as the duration approaches zero

Input-Output Description: Zero-state Response



A *staircase* approximation of input signal.

As Δ approaches zero, the pulse $\delta_\Delta(t-t_i)$ becomes an *impulse* at t_i, denoted by $\delta(t-t_i)$, and the corresponding output will be denoted by $g(t-t_i)$. As $\Delta \to 0$,

1. *the approximation* $\to$ *an equality*
2. *the summation* $\to$ *an integration*
3. *the discrete* t_i $\to$ *a continuum*
4. $\Delta \to d\tau$

Input-Output Description of Linear System

$$\mathbf{y(t)=\int_{-\infty}^{\infty} g(t,\tau)u(\tau)d\tau}$$

the time at which the impulse input is applied

the time at which the output is observed

Because $g(t,\tau)$ is the responses excited by an impulse, it is called the *impulse response*.

Linear System: Causal

$$\mathbf{y(t)=\int_{-\infty}^{\infty} g(t,\tau)u(\tau)d\tau}$$

Causal systems are systems where the output only depend on past/present inputs.
If a system is causal,

$$\text{Causal} \Leftrightarrow g(t,\tau)=0 \qquad \text{for} \quad t<\tau$$

Every linear system that is *causal* and *relaxed at* t_0 can be described by

$$y(t)=\int_{t_0}^{t} g(t,\tau)u(\tau)d\tau$$

A system is said to be relaxed at t_0 if its initial state at t_0 is 0. In this case, the output $y(t)$, for $t \geq t_0$, is excited exclusively by the input $u(t)$, for $t \geq t_0$.

Linear System: Multi Input Multi Output

$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

where

$$\boxed{G(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & g_{12}(t,\tau) & \cdots & g_{1p}(t,\tau) \\ g_{21}(t,\tau) & g_{22}(t,\tau) & \cdots & g_{2p}(t,\tau) \\ \vdots & \vdots & & \vdots \\ g_{q1}(t,\tau) & g_{q2}(t,\tau) & & g_{qp}(t,\tau) \end{bmatrix}}$$

the impulse response matrix of the system

$g_{ij}(t,\tau)$: *the impulse response between the* j*th input terminal and the* i*th output terminal.*

Linear System: Time-Invariant

$$\boxed{y(t)=\int_{t_0}^{t} g(t,\tau)u(\tau)d\tau}$$

If the system is time invariant as well, then we have

$$g(t,\tau)=g(t+T,\tau+T)=g(t-\tau,0)=g(t-\tau)$$

for any *T*.

$$y(t)=\int_0^t g(t-\tau)u(\tau)d\tau=\int_0^t g(\tau)u(t-\tau)d\tau$$

This is called a convolution integral.

$$\begin{aligned}y(t)&=\int_{\tau=0}^{t} g(t-\tau)u(\tau)d\tau\\&=\int_{\bar\tau=t}^{0} g(\bar\tau)u(t-\bar\tau)(-d\bar\tau)\\&=\int_{\bar\tau=0}^{t} g(\bar\tau)u(t-\bar\tau)(d\bar\tau)\\&=\int_{\tau=0}^{t} g(\tau)u(t-\tau)d\tau\end{aligned}$$

By definition, $g(t)=g(t-0)$ is the output at time t due to an impulse input applied at time 0. The condition for a linear time-invariant system to be causal is $g(t)=0$ for $t<0$.

$$f*g=g*f,\ f*(g*h)=(f*g)*h,\ f*(g+h)=(f*g)+(f*h)$$

Transfer Function for Input-Output Descriptions of LTI Systems

Transfer Function

The *Laplace* transform is an important tool in the study of linear time-invariant (LTI) systems.

$$\hat{y}(s) = \int_0^\infty y(t) e^{-st} dt$$

For causal systems

$$\hat{y}(s) = \int_{t=0}^{\infty} \left(\int_{\tau=0}^{\infty} g(t-\tau) u(\tau) d\tau \right) e^{-s(t-\tau)} e^{-s\tau} dt$$

v

$$= \int_{\tau=0}^{\infty} \left(\int_{t=0}^{\infty} g(t-\tau) e^{-s(t-\tau)} dt \right) u(\tau) e^{-s\tau} d\tau$$

Changing the order of integration

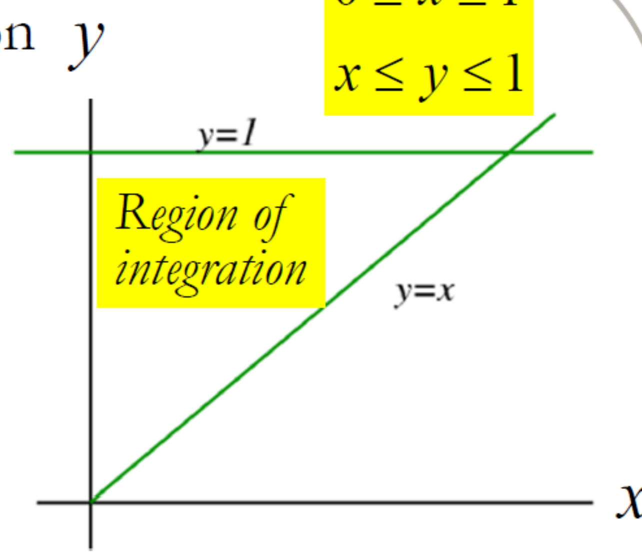
$$= \int_{\tau=0}^{\infty} \left(\int_{v=-\tau}^{\infty} g(v) e^{-sv} dv \right) u(\tau) e^{-s\tau} d\tau$$

Changing Order of Integration

Cf. Changing the order of integration

$0 \le x \le 1$

$x \le y \le 1$



$$\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx$$
$$= \int_0^1 \int_0^y \cos(y^2)\,dx\,dy$$
$$= \int_0^1 \left[\cos(y^2)x\right]_{x=0}^{x=y} dy$$
$$= \int_0^1 \cos(y^2)\,y\,dy$$

$$\boxed{\begin{aligned} u &= y^2 \\ du &= 2y\,dy \end{aligned}}$$

$$\int_0^1 \frac{1}{2}\cos(u)\,du$$
$$= \frac{1}{2}\sin(u)\Big|_0^1$$
$$= \frac{1}{2}\sin(1)$$

Transfer Function

$$\hat{y}(s) = \int_{v=0}^{\infty} g(v)e^{-sv}dv \int_{\tau=0}^{\infty} u(\tau)e^{-s\tau}d\tau$$

Causality condition

or

$$\hat{y}(s) = \hat{g}(s)\hat{u}(s)$$

where

$$\hat{g}(s) = \int_{0}^{\infty} g(t)e^{-st}dt$$

is called the *transfer function* of the system. Thus the transfer function is the *Laplace* transform of the impulse response and, conversely, the impulse response is the inverse *Laplace* transform of the transfer function.

Transfer Function: Multi Input Multi Output

For a p-input q-output system,

$$\begin{bmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \\ \vdots \\ \hat{y}_q(s) \end{bmatrix} = \begin{bmatrix} \hat{g}_{11}(s) & \hat{g}_{12}(s) & \cdots & \hat{g}_{1p}(s) \\ \hat{g}_{21}(s) & \hat{g}_{22}(s) & \cdots & \hat{g}_{2p}(s) \\ \vdots & \vdots & & \vdots \\ \hat{g}_{q1}(s) & \hat{g}_{q2}(s) & \cdots & \hat{g}_{qp}(s) \end{bmatrix} \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \\ \vdots \\ \hat{u}_p(s) \end{bmatrix}$$

$$\hat{y}(s) = \boxed{\hat{G}(s)}\hat{u}(s)$$

the transfer-(function) matrix

$\hat{g}_{ij}$: *the transfer function from the jth input to the ith output*

Rational Transfer Function

If a linear time-invariant system is lumped, then its transfer function will be a rational function of s.

Every rational transfer function can be expressed as $\hat{g}(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are polynomials of s. $\hat{g}(s)$ can be classified as follows:

$$\hat{g}(s) \ \textit{proper} \Leftrightarrow \deg D(s) \geq \deg N(s) \Leftrightarrow \hat{g}(\infty) = \textit{zero or nonzero constant.}$$

$$\hat{g}(s) \ \textit{strictly proper} \Leftrightarrow \deg D(s) > \deg N(s) \Leftrightarrow \hat{g}(\infty) = 0.$$

$$\hat{g}(s) \ \textit{biproper} \Leftrightarrow \deg D(s) = \deg N(s) \Leftrightarrow \hat{g}(\infty) = \textit{nonzero constant.}$$

$$\hat{g}(s) \ \textit{improper} \Leftrightarrow \deg D(s) < \deg N(s) \Leftrightarrow \left|\hat{g}(\infty)\right| = \infty.$$

Improper rational transfer functions will amplify high-frequency noise.

Zero-Pole Gain

A real or complex number λ is called a *pole* of the proper transfer function $\hat{g}(s) = N(s)/D(s)$ if $|\hat{g}(\lambda)| = \infty$; a *zero* if $\hat{g}(\lambda) = 0$.

If $N(s)$ and $D(s)$ are coprime, that is, have no common factors of degree 1 or higher, then all roots of $N(s)$ are the zeros of $\hat{g}(s)$, and all roots of $D(s)$ are the poles of $\hat{g}(s)$.

In terms of poles and zeros, the transfer function can be expressed as

$$\hat{g}(s) = k \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

This is called the *zero-pole-gain* form.

Example

$$\hat{g}(s) = \frac{2s+1}{s^2+5s+6} = 2\frac{s+\frac{1}{2}}{(s+3)(s+2)}$$

$$\boxed{\begin{array}{ll} & k = 2 \\ \textit{zeros} & s = -\frac{1}{2} \\ \textit{poles} & s = -3, s = -2 \end{array}}$$

$$\hat{g}(s) = \frac{6s^2+18s+12}{2s^3+10s^2+16s+12}$$

$$= \frac{6}{2}\frac{s^2+3s+2}{s^3+5s^2+8s+6}$$

$$= 3\frac{(s+1)(s+2)}{(s+1-j)(s+1+j)(s+3)}$$

$$\boxed{\begin{array}{ll} & k = 3 \\ \textit{zeros} & s = -1, s = -2 \\ \textit{poles} & s = -1+j, s = -1-j, s = -3 \end{array}}$$

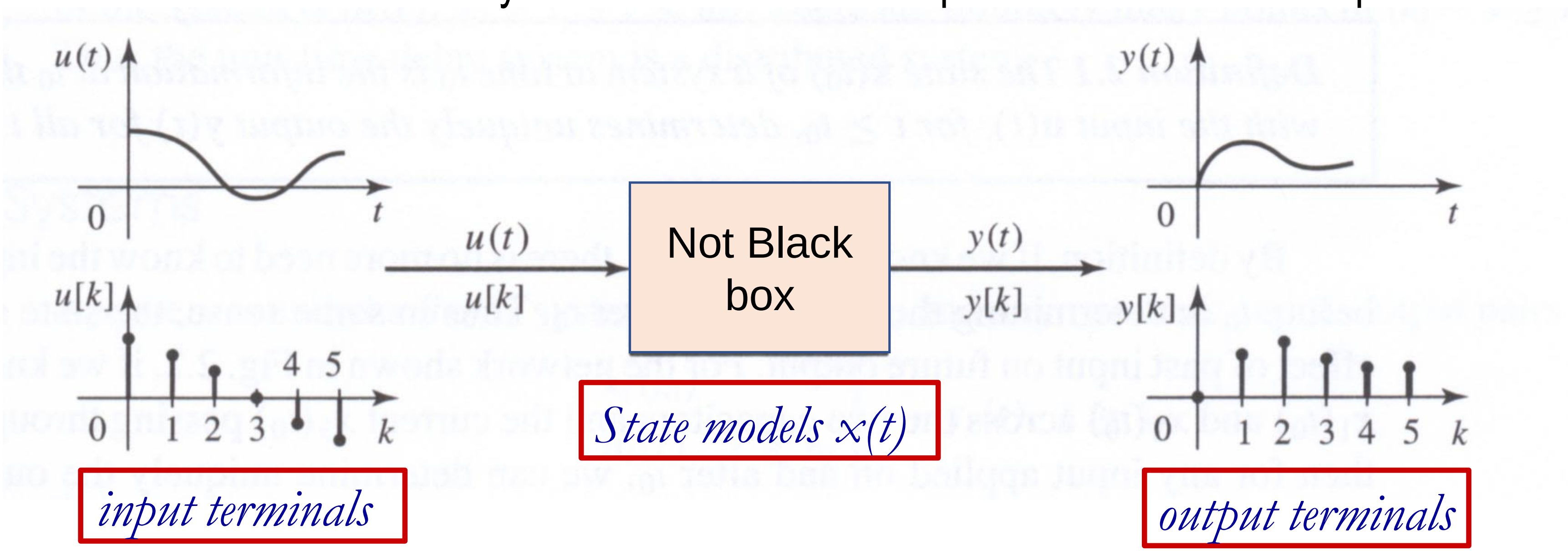
State-Space Descriptions of Linear Systems

Internal Description

▪ Internal Description: State-Space Description

→ State-space representation:

a mathematical model of a physical system as a set of input, output, and state variables related by first-order differential equations or difference equations



$$\dot{x}(t) = Ax(t) + Bu(t)$$ 1st order DE

state

$$y(t) = Cx(t) + Du(t)$$ AE

output

input

Internal Description

- ## Internal Description: State-Space Description

$$\dot{x}(t) = Ax(t) + Bu(t) \rightarrow$$ *A set of n first-order differential equations*

$$y(t) = Cx(t) + Du(t) \rightarrow$$ *q algebraic equations*

An n-dimensional state-(space) equation

For a p-input and q-output system,

u is a $p \times 1$ vector and y is a $q \times 1$ vector.

If the system has n state variables,

x is an $n \times 1$ vector,

A, B, C, and D must be $n \times n$, $n \times p$, $q \times n$, and $q \times p$ matrices.

Linear System: Time-Invariant

- **Definition**

A system is said to be time invariant if for every state input-output pair

$$\left.\begin{array}{c} x(t_0) \\ u(t),\ \ t \geq t_0 \end{array}\right\} \to y(t),\ \ t \geq t_0$$

Input & state Output

and any T, we have

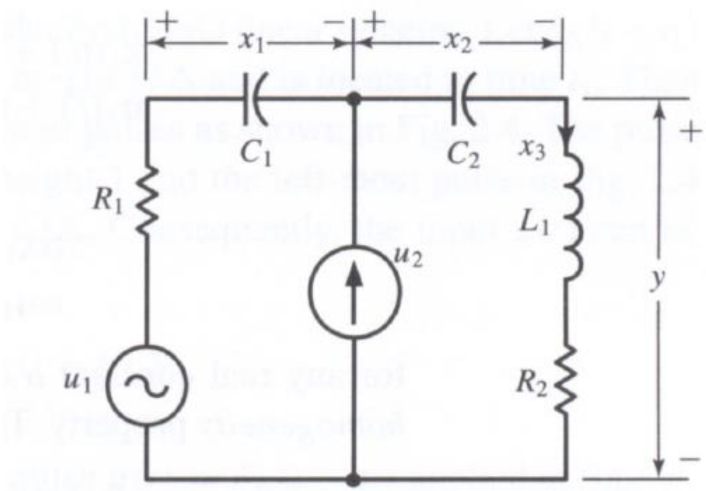
$$\left.\begin{array}{c} x(t_0+T) \\ u(t-T),\ \ t \geq t_0+T \end{array}\right\} \to y(t-T),\ \ t \geq t_0+T \ \textit{(time shifting)}$$

Input & state Output

If the initial state and the input are the same, no matter at what time they are applied, the output waveform will always be the same.

Example: Linear Time-Invariant Systems

For time-invariant systems, we can always assume, without loss of generality, that $t_0 = 0$.



This network is time invariant if R_i, C_i, and L_i are constants.

A large number of physical systems can be modeled as time-invariant systems over a limited time period.

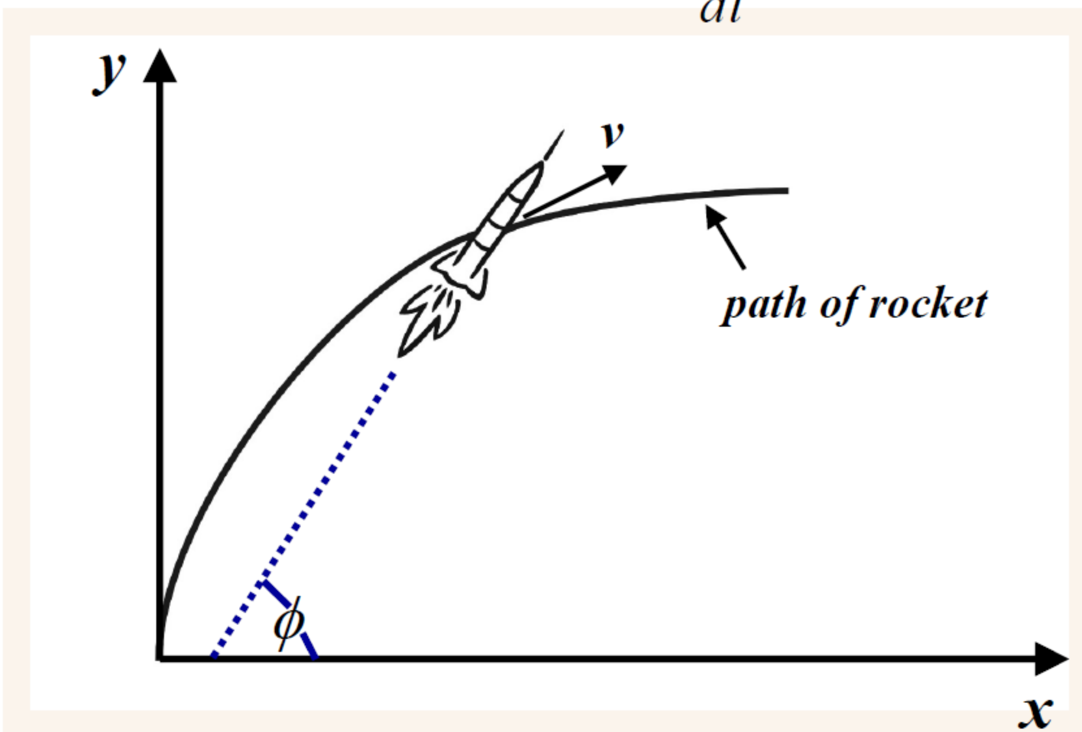
If a system is not time invariant, it is said to be time varying: *a burning rocket.*

Example: Linear Time-Variant Systems

Example: Rocket Trajectories

$$m(t)\frac{dv}{dt} = m(t)F - m(t)g,$$

$$|F| = \frac{c}{m(t)}\beta \rightarrow -\frac{dm}{dt}$$



Transfer Function for State-Space Descriptions of LTI Systems

State-Space Equations

Every linear time-invariant lumped system can be described by a set of equations of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Laplace Transform

$$s\hat{x}(s) - x(0) = A\hat{x}(s) + B\hat{u}(s)$$

$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$$

$$L[f'(t)](s) = sL[f(t)] - f(0)$$

$$L[f''(t)](s) = s^2 L[f(t)] - sf(0) - f'(0)$$

$$L[f^{(n)}(t)](s) = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

State-Space Equations

$$\hat{x}(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} B\hat{u}(s)$$
$$\hat{y}(s) = C(sI - A)^{-1} x(0) + C(sI - A)^{-1} B\hat{u}(s) + D\hat{u}(s)$$

Algebraic equations

Given $x(0)$ and $\hat{u}(s)$, $\hat{x}(s)$ and $\hat{y}(s)$ can be computed algebraically. Their inverse *Laplace* transforms yield the time responses $x(t)$ and $y(t)$.

If the initial state $x(0)$ is zero, then

$$\hat{y}(s) = \underbrace{\boxed{[C(sI - A)^{-1} B + D]}}_{\hat{G}(s)}\hat{u}(s)$$

This relates the input-output (or transfer matrix) and state-space descriptions.

The Laplace transfom is not used in studying linear time-varying systems.

Nonlinear Time-variant Systems

Nonlinear Time-Variant Systems

Most physical systems are nonlinear and time varying.

$$\dot{x}(t) = h(x(t), u(t), t)$$
$$y(t) = f(x(t), u(t), t)$$

Suppose for some input function $u_0(t)$ and some initial state, $x_0(t)$ is the solution of the above equation.

$$\dot{x}_0(t) = h(x_0(t), u_0(t), t)$$

Now suppose the input is perturbed slightly to become $u_0(t) + \bar{u}(t)$ and the initial state is also perturbed only slightly.

Approximate by Linear Equations

For some nonlinear equations, the corresponding solution may differ from $x_0(t)$ only slightly. In this case, the solution can be expressed as $x_0(t)+\bar{x}(t)$ with $\bar{x}(t)$ small for all t.

Under this assumption, we can expand the equation as

$$\begin{aligned}\dot{x}_0(t)+\dot{\bar{x}}(t) &= h(x_0(t)+\bar{x}(t), u_0(t)+\bar{u}(t), t) \\ &= h(x_0(t), u_0(t), t)+\frac{\partial h}{\partial x}\bar{x}+\frac{\partial h}{\partial u}\bar{u}+\cdots\end{aligned}$$

This is not true in general. For some nonlinear equations, a very small difference in initial states will generate completely different solutions, yielding the phenomenon of chaos.

Approximate by Linear Equations

$$\boxed{\dot{x}_0(t)} + \boxed{\dot{\bar{x}}(t)} = \boxed{h(x_0(t), u_0(t), t)} + \underbrace{\boxed{\frac{\partial h}{\partial x}\bar{x} + \frac{\partial h}{\partial u}\bar{u} + \cdots}}_{\text{Neglected}}$$

Neglecting higher powers of $\bar{x}$ and $\bar{u}$,

$$\boxed{\dot{\bar{x}}(t) = A(t)\bar{x}(t) + B(t)\bar{u}(t)}$$

This is a linear state-space equation.

Approximate by Linear Equations

For $h = [h_1 \quad h_2 \quad h_3]^T$, $x = [x_1 \quad x_2 \quad x_3]^T$, and $u = [u_1 \quad u_2]^T$,

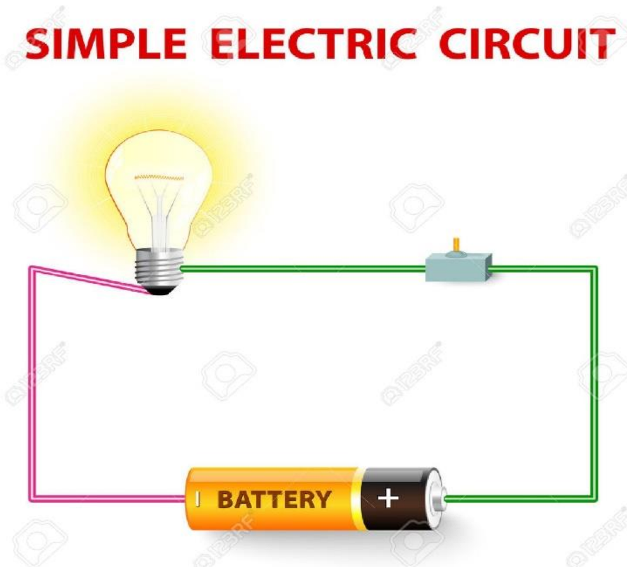
$$A(t) := \frac{\partial h}{\partial x} := \begin{bmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 & \partial h_1/\partial x_3 \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 & \partial h_2/\partial x_3 \\ \partial h_3/\partial x_1 & \partial h_3/\partial x_2 & \partial h_3/\partial x_3 \end{bmatrix}$$

$$B(t) := \frac{\partial h}{\partial u} := \begin{bmatrix} \partial h_1/\partial u_1 & \partial h_1/\partial u_2 \\ \partial h_2/\partial u_1 & \partial h_2/\partial u_2 \\ \partial h_3/\partial u_1 & \partial h_3/\partial u_2 \end{bmatrix}$$

They are called Jacobians.

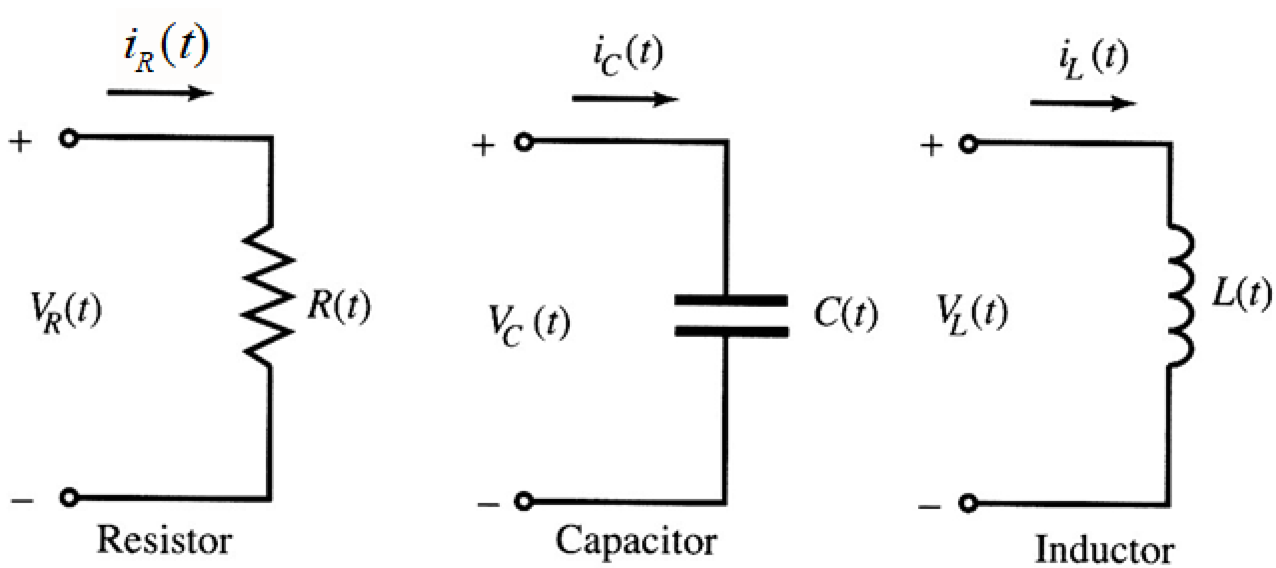
State-Space Equation in Electrical & Mechanical System

What is Electronic Circuits?



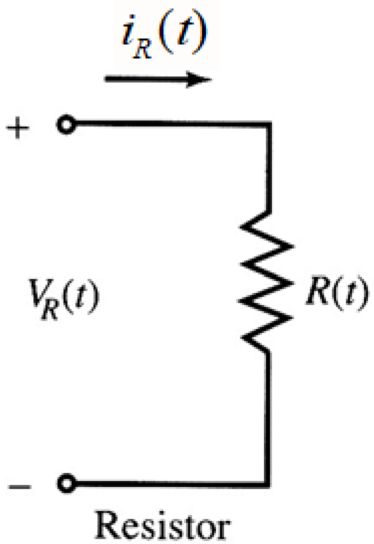
Electricity is the flow of electrons from an area of higher potential to one of lower potential

Basic Principle: Voltage (V) and current (I):
- Voltage pushes
- Current will flow as a result of voltage



RLC Components

▪ Resistor



Uses:
- Limit current flow
- Generating specific voltages
- Convert current to voltage
- Heating

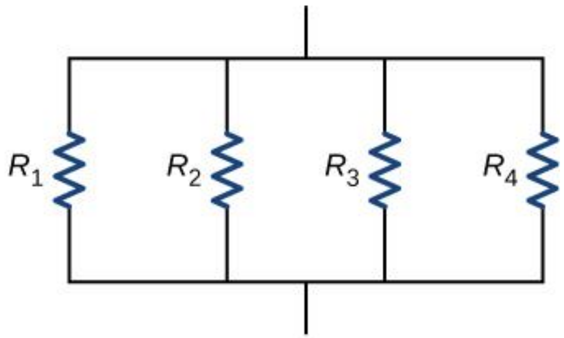
Voltage drop across resistor ~ Current flow with given voltage

$$\boxed{V = IR \sim I = \frac{V}{R}}$$

The resistor is a zero-memory element.

Resistors in series

Resistors in parallel

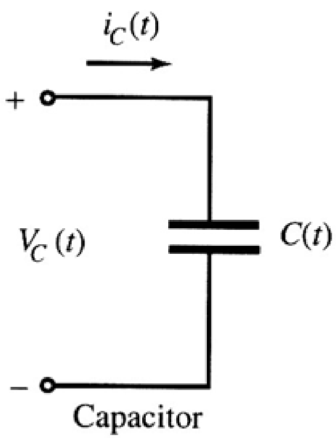


$$R_{series} = R_1 + R_2 \ldots + R_n$$

$$\frac{1}{R_{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} \ldots + \frac{1}{R_n}$$

RLC Components

▪ Capacitor

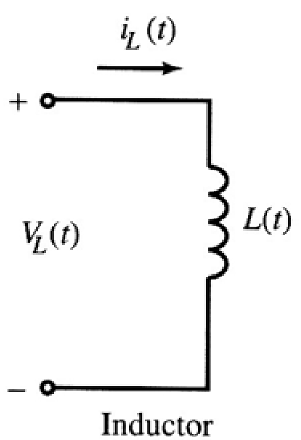


Uses:
- Stores charge
- Filtering/decoupling frequencies
- Oscillators

The energy is stored in the electrical field of the capacitor due to the charge

$$\boxed{I = C\frac{dV}{dt}}$$

▪ Inductor

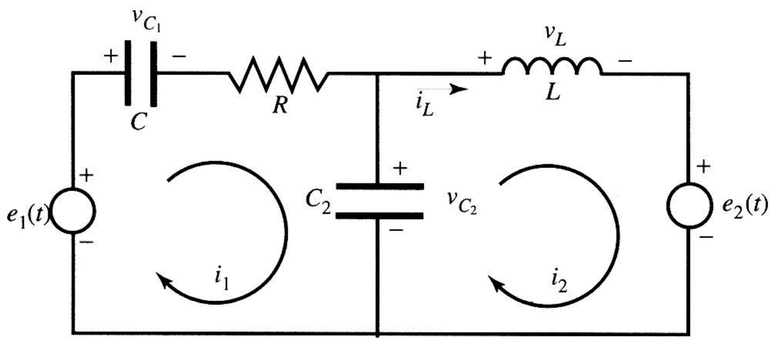


Uses:
- Stores energy
- Filtering frequencies

- The energy stored in magnetic field
- Inductor responds to changes in current

$$\boxed{V = L\frac{dI}{dt}}$$

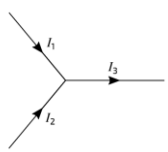
Kirchhoff Voltage and Current Law



The ***Kirchhoff voltage law*** (KVL) requires that the voltages around a closed loop sum to zero.

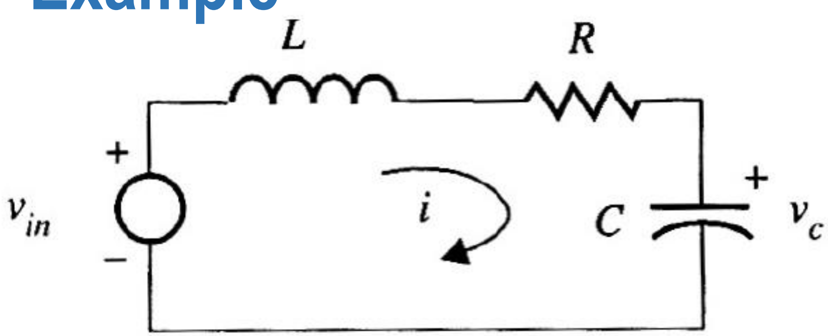
$$\sum_{k=1}^{n} V_k = 0$$

The ***Kirchhoff current law*** (KCL) requires that the algebraic sum of currents entering a node equals zero.



$$I_1 + I_2 = I_3$$

- **Example**



$\rightarrow$ **Kirchoff's voltage law:**

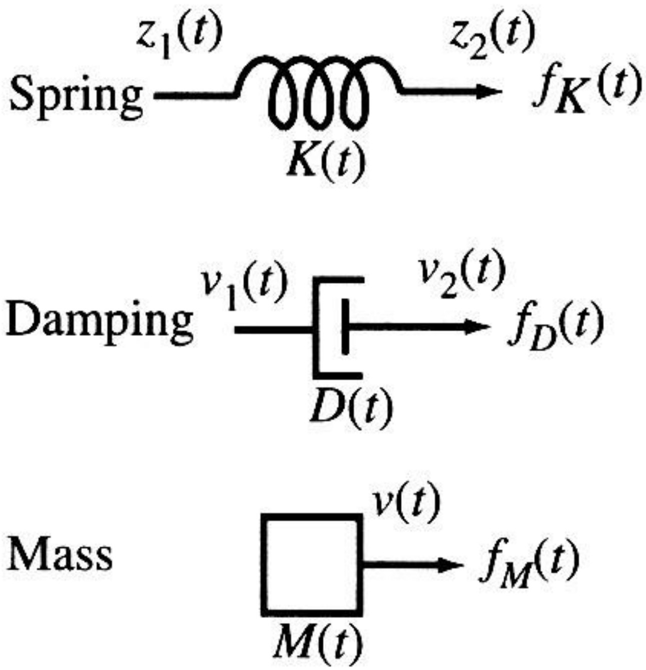
$$v_{in} = L\frac{di}{dt} + Ri + v_C$$

What is Mechanical Systems?

- **Mechanical System**

  A system that manages the power of forces and movements to accomplish a task.

- **Component**



(potential energy)

Natural initial conditions that describe the state of the system

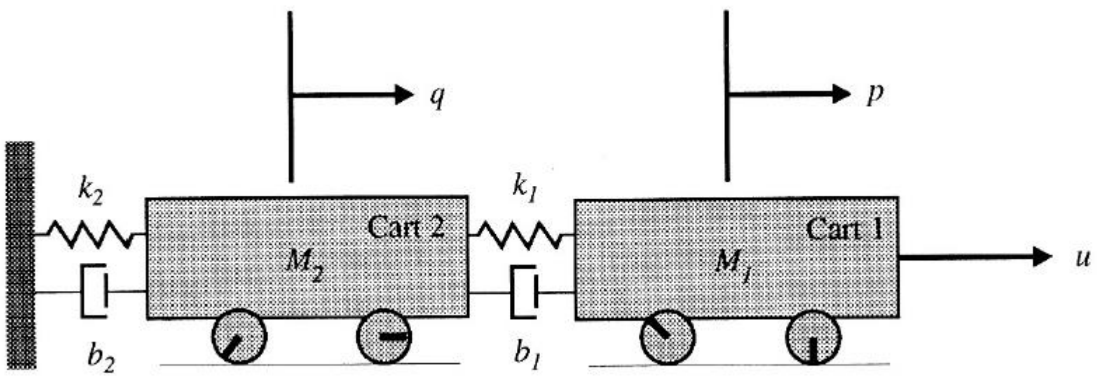
(kinetic energy)

Newton's Second Law

▪ Newton's Second Law

→ How force causes changes in motion.
→ Calculate what happens in situations involving forces and motion,
→ Mathematical relationship between force, mass, and *acceleration*.

▪ Example:



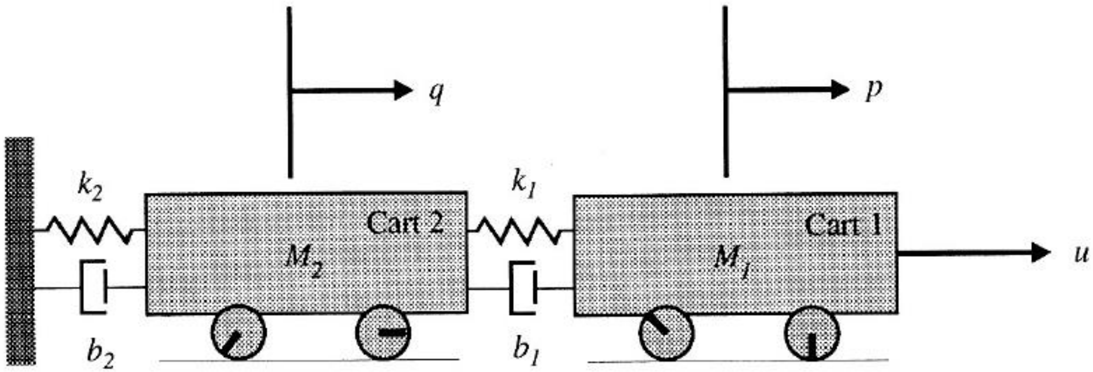
→ **Newton's second law:**

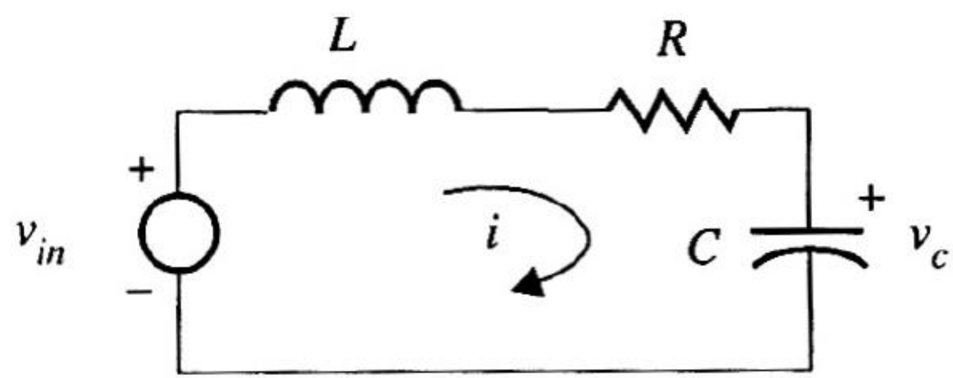
Sum of forced equals mass multiplied by acceleration

$$M_1\ddot{p} = u - k_1(p - q) - b_1(\dot{p} - \dot{q})$$

$$M_2\ddot{q} = k_1(p - q) + b_1(\dot{p} - \dot{q}) - k_2 q - b_2\dot{q}$$

Mechanical-Electrical Equivalency

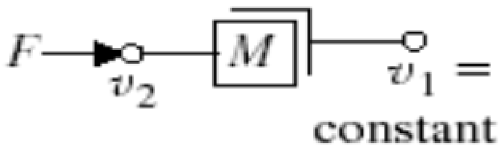




$$\left\{ \begin{array}{l} \text{voltage} \\ \text{velocity} \\ \text{pressure} \end{array} \right\} \text{ and } \left\{ \begin{array}{l} \text{current} \\ \text{force} \\ \text{volume flow rate} \end{array} \right\} \text{ in the } \left\{ \begin{array}{l} \text{electrical} \\ \text{mechanical} \\ \text{fluidic} \end{array} \right\} \text{ domains.}$$

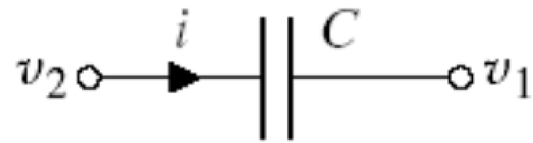
Mechanical-Electrical Equivalency

- **Translational Mass – Electrical Capacitance**



$$F = M\frac{dv_{12}}{dt}$$

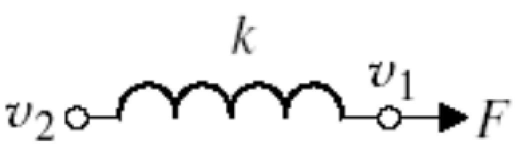
$$\frac{dv_{12}}{dt} = \frac{F}{M}$$



$$i = C\frac{dv_{12}}{dt}$$

$$\frac{dv_{12}}{dt} = \frac{i}{C}$$

- **Translational Spring – Electrical Inductance**



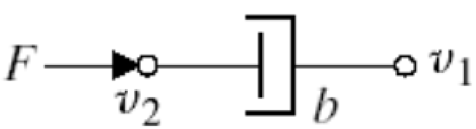
$$v_{12} = \frac{1}{k}\frac{dF}{dt}$$

$$\frac{dF}{dt} = kv_{12}$$

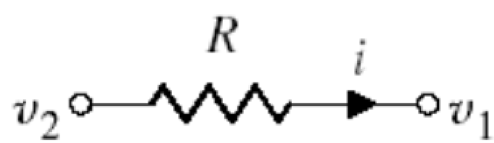
$$v_{12} = L\frac{di}{dt}$$

$$\frac{di}{dt} = \frac{1}{L}v_{12}$$

- **Translational Damper – Electrical Resistance**



$$F = bv_{12}$$



$$i = \frac{v_{12}}{R}$$

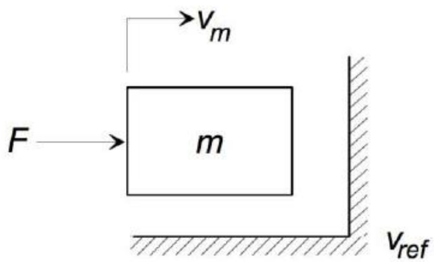
State-Space Model

▪ State Variables

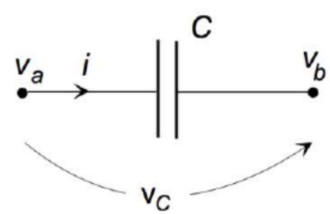
# of state variables = # of <u>independent</u> energy storage elements

▪ Energy Storage Elements

→ **Mass and Capacitor**: Stored energy as a function of Across-variable

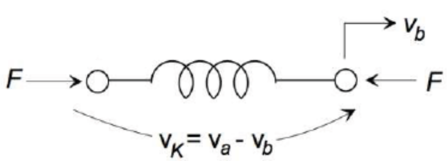


$$\begin{aligned} F &= m\frac{dv}{dt} \\ E &= \int_{-\infty}^{t} vF\,\mathrm{d}t = \int_0^t mv\,\mathrm{d}v \\ &= \frac{1}{2}mv^2 \end{aligned}$$

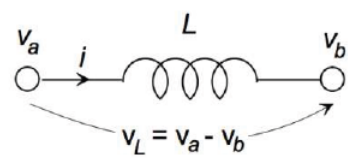


$$\begin{aligned} i &= C\frac{dv}{dt} \\ E &= \int_{-\infty}^{t} vi\,\mathrm{d}t = \int_0^t Cv\,\mathrm{d}v \\ &= \frac{1}{2}Cv^2 \end{aligned}$$

→ **Spring and Inductor**: Stored energy as a function of Through-variable



$$\begin{aligned} v &= \frac{1}{K}\frac{dF}{dt} \\ E &= \int_{-\infty}^{t} vF\,\mathrm{d}t = \frac{1}{K}\int_0^t F\,\mathrm{d}F \\ &= \frac{1}{2K}F^2 \end{aligned}$$

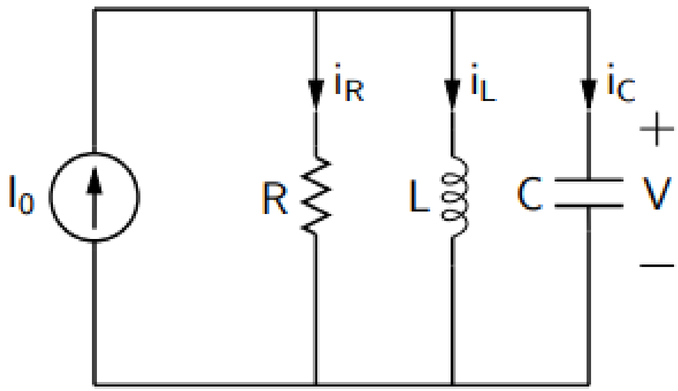


$$\begin{aligned} v &= L\frac{di}{dt} \\ E &= \int_{-\infty}^{t} vi\,\mathrm{d}t = \int_0^t Li\,\mathrm{d}i \\ &= \frac{1}{2}Li^2 \end{aligned}$$

Homework #2.2 Electrical System (1 pt.): Due Dec. 20

- **Electrical System**

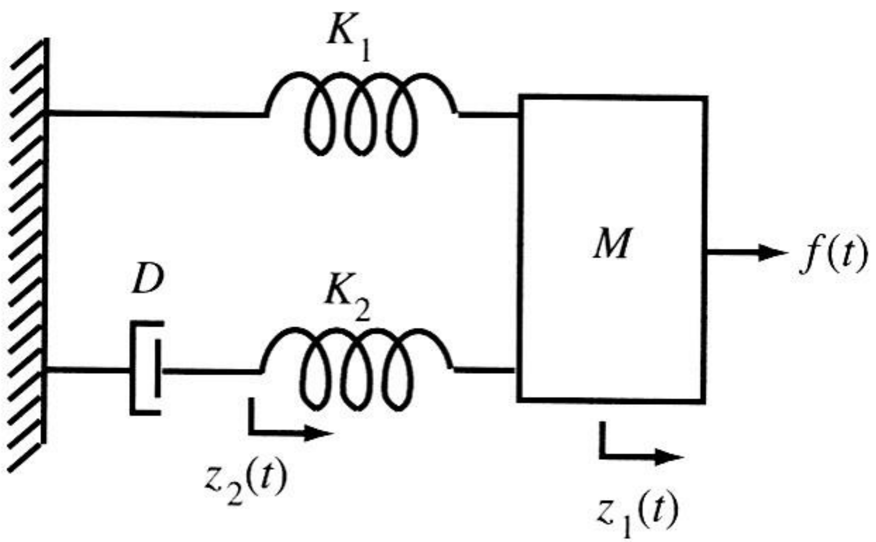
  → Derive *Kirchhoff current law*



Homework #2.2 Mechanical Syst. (1 pt.): Due Dec. 20

- **Mechanical System**

  → Derive *Newton's Second Law*



Discrete-time Systems

Discrete-Time Systems

Most concepts in continuous-time systems can be applied directly to the discrete-time systems.

A discrete-time system is linear if the additivity and homogeneity properties hold. The response of every linear discrete-time system can be decomposed as

$$\boxed{\textit{Response} = \textit{zero-state response} + \textit{zero-input response}}$$

and the zero-state responses satisfy the superposition property. So do the zero-input responses.

Input-Output Description

Let $\delta[k]$ be the *impulse sequence* defined as

time index

fixed integer

$$\delta_d[k-m] = \begin{cases} 1 & \text{if} \quad k = m \\ 0 & \text{if} \quad k \neq m \end{cases}$$

: It is the discrete counterpart of the impulse $\delta(t-t_1)$.

Let $u[k]$ be any input sequence.

$$u[k] = \sum_{m=-\infty}^{\infty} u[m]\delta_d[k-m]$$

Let $g[k,m]$ be the output at time instant k excited by the impulse sequence applied at time instant m.

Input-Output Description

$$\delta_d[k] \rightarrow g[k] \qquad : \textit{definition}$$

$$\delta_d[k-m] \rightarrow g[k-m] \qquad : \textit{time shifting}$$

$$\delta_d[k-m]u[m] \rightarrow g[k-m]u[m] \qquad : \textit{homogeneity}$$

$$\sum_m \delta_d[k-m]u[m] \rightarrow \sum_m g[k-m]u[m] \qquad : \textit{additivity}$$

Thus, the output $y[k]$ excited by the input $u[k]$ equals

$$y[k] = \sum_{m=-\infty}^{\infty} g[k-m]u[m]$$

The sequence $g[k,m]$ is called the *impulse response sequence*.

Input-Output Description

If a discrete-time system is causal, no output will appear before an input is applied.

$$\text{Causal} \;\Leftrightarrow g[k,m]=0, \text{ for } \; k<m$$

If a system is relaxed at k_0 and causal, then

$$y[k]=\sum_{m=k_0}^{k} g[k,m]u[m]$$

If a linear discrete-time system is time invariant as well, then the time shifting property holds. In this case, the initial time instant can always be chosen as $k_0=0$.

$$y[k]=\sum_{m=0}^{k} g[k-m]u[m]=\sum_{m=0}^{k} g[m]u[k-m]$$

This is called a discrete convolution.

Discrete Transform Function

The *z-transform* is an important tool in the study of LTI discrete-time systems. Let $\hat{y}(z)$ be the z-transform of $y[k]$ defined as

$$\hat{y}(z) := Z\big[y[k]\big] := \sum_{k=0}^{\infty} y[k] z^{-k}$$

$$\hat{y}(z) = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty} g[k-m]u[m]\right) z^{-(k-m)} z^{-m}$$

l

$$= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty} g[k-m] z^{-(k-m)}\right) u[m] z^{-m}$$

$$= \left(\sum_{l=0}^{\infty} g[l] z^{-l}\right)\left(\sum_{m=0}^{\infty} u[m] z^{-m}\right) =: \hat{g}(z)\hat{u}(z)$$

Discrete Transform Function

$$\hat{y}(z) = \hat{g}(z)\hat{u}(z)$$

The function $\hat{g}(z)$ is the z-transform of the impulse response sequence $g[k]$ and is called the *discrete transfer function*.

Both the discrete convolution and transfer function describe only zero-state responses.

State-Space Description

Every linear lumped discrete-time system can be described by

$$x[k+1] = A[k]x[k] + B[k]u[k]$$
$$y[k] = C[k]x[k] + D[k]u[k]$$

If the system is time invariant as well,

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

Discrete Transform Function

Let $\hat{x}(z)$ be the z-transform of $x[k]$ or

$$\hat{x}(z) = Z\left[x[k]\right] := \sum_{k=0}^{\infty} x[k] z^{-k}$$

Then we have

$$\begin{aligned}
Z\left[x[k+1]\right] &= \sum_{k=0}^{\infty} x[k+1] z^{-k} = z \sum_{k=0}^{\infty} x[k+1] z^{-(k+1)} \\
&= z\left[\sum_{l=1}^{\infty} x[l] z^{-l} + x[0] - x[0]\right] \qquad \boxed{l := k+1} \\
&= z\left[\sum_{l=0}^{\infty} x[l] z^{-l} - x[0]\right] \\
&= z\left(\hat{x}(z) - x[0]\right)
\end{aligned}$$

Discrete Transform Function

$$z\hat{x}(z) - zx[0] = A\hat{x}(z) + B\hat{u}(z)$$

$$\hat{y}(z) = C\hat{x}(z) + D\hat{u}(z)$$

$$\hat{x}(z) = (zI - A)^{-1} zx[0] + (zI - A)^{-1} B\hat{u}(z)$$

$$\hat{y}(z) = C(zI - A)^{-1} zx[0] + C(zI - A)^{-1} B\hat{u}(z) + D\hat{u}(z)$$

If $x[0]=0$, then

$$\hat{y}(z) = \underbrace{\boxed{[C(zI - A)^{-1} B + D]}}_{\hat{G}(z)}\hat{u}(z)$$

: The Laplace transform variable s is replaced by the z-transform variable z.

Thank you

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