Lecture I213E – Class 9

Discrete Signal Processing

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JAIST
JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY 1990

Course Materials

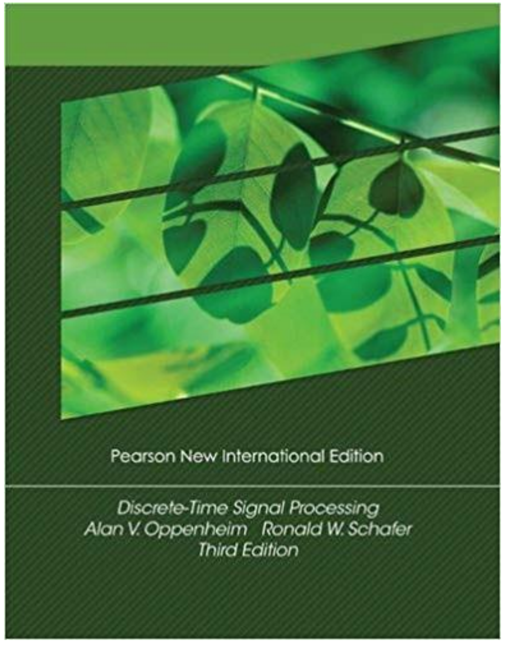
- ## Materials

→ Lecture notes will be uploaded before each lecture
https://jstorage-2018.jaist.ac.jp/s/PGXRrC7iFmN2FWo
Pass: dsp-i213e-2022
(Slide Courtesy of Prof. Nak Young Chong)

- ## References

→ Chi-Tsong Chen:
Linear System Theory and Design, 4th Ed.,
Oxford University Press, 2013.

→ Alan V. Oppenheim and Ronald W. Schafer:
Discrete-Time Signal Processing, 3rd Ed.,
Pearson New International Ed., 2013.



Related Courses & Prerequisite

- **Related Courses**
  - → I212 Analysis for Information Science
  - → I114 Fundamental Mathematics for Information Science

- **Prerequisite**
  - → None

Evaluation

- **Viewpoint of evaluation**

  → Students are able to understand:

  - Basic principles in modeling and analysis of linear time-invariant systems
  - Applications of mathematical methods and tools to different signal processing problems.

- **Evaluation method**

  → Homework, term project, midterm exam, and final exam

- **Evaluation criteria**

  → Homework/labs (30%), term project (30%) midterm exam (15%), and final exam (25%)

Contact

- **Lecturer**

  → Sakriani Sakti

- **TA**

  Tutorial hours & Term project
  → WANG Lijun (s2010026)
  → TANG Bowen (s2110411)
  Homework
  → PUTRI Fanda Yuliana (s2110425)

- **Contact Email**

  → dsp-i213e-2022@ml.jaist.ac.jp

Schedule

- **December 8th, 2022 – February 9th, 2023**

- **Lecture Course Term 2-2**
  - → Tuesday 9:00 — 10:40
  - → Thursday 10:50 — 12:30

- **Tutorial Hours**
  - → Tuesday 13:30-15:10

Schedule

Dec

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Jan

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	~~5~~	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	~~24~~	25	26	27	28
29	30	31				

Feb

Sun	Mon	Tue	Wed	Thu	Fri	Sat
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28				

Lecture:
Tuesday 9:00 — 10:40
Thursday 10:50 — 12:30

Tutorial:
Tuesday 13:30 — 15:10

Course review &
term project evaluation
(on tutorial hours)

Midterm & final exam
Thursday 10:50 — 12:30

Syllabus

Class	Date	Lecture Course Tue 9:00 — 10:40 / Thr 10:50 — 12:30	Tutorial Hours Tue 13:30 — 15:10
1	12/08	Introduction to Linear Systems with Applications to Signal Processing	
2	12/13	State Space Description	○
3	12/15	Linear Algebra	
4	12/20	Quantitative Analysis (State Space Solutions) and Qualitative Analysis (Stability)	○
5	12/22	Discrete-time Signals and Systems	
X	01/05		
6	01/10	Discrete-time Fourier Analysis	
7	01/10*	Review of Discrete-time Linear Time-Invariant Signals and Systems (on Tutorial Hours)	
	01/12	Midterm Exam	
8	01/17	Sampling and Reconstruction of Analog Signals	○
9	01/19	z-Transform	
X	01/24		○
10	01/26	Discrete Fourier Transform	
11	01/31	FFT Algorithms	○
12	01/02	Implementation of Digital Filters	
13	02/07	Digital Signal Processors and Design of Digital Filters	
14	02/07*	Review of the Course and Term Project Evaluation (on Tutorial Hours)	
	02/09	Final exam	

Class 9
Z-Transform

Function Transformation

	Complex Freq.	Real Freq.	
		Periodic	Not Periodic
Continuous Time	LT	**CTFS**	**CTFT**
Discrete Time	**ZT**	**DFT**	**DTFT**

Z-Transform

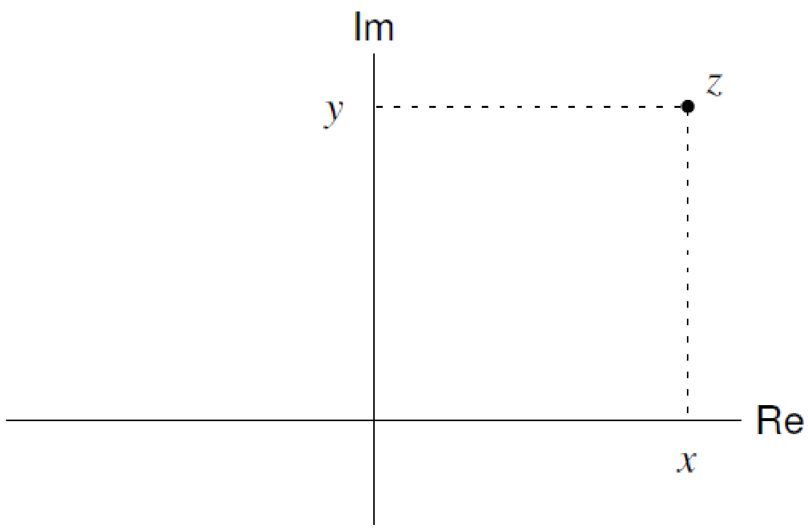
- *An extension of the discrete-time Fourier transform*
- *Analysis of discrete-time signals and LTI systems*

- *cf. The Laplace transform plays the same role in the analysis of continuous-time signals and LTI systems*

Review of Complex Analysis

Complex Numbers

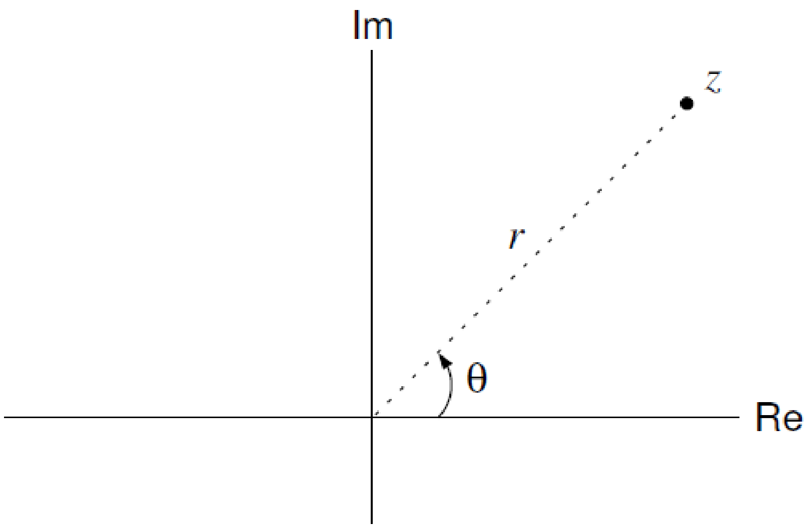
- ## Cartesian (Rectangular) Form



$$\boxed{z = x + jy}$$

where $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$

- ## Polar Form



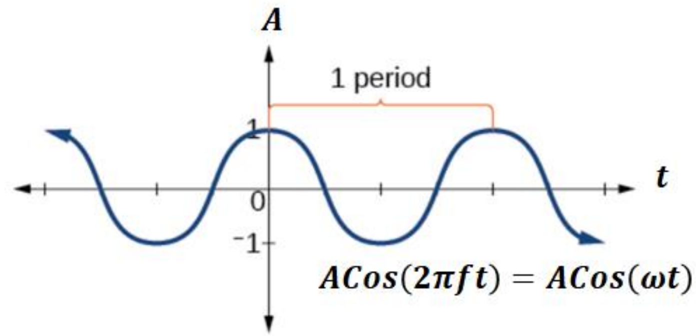
$$\boxed{z = r(\cos\theta + j\sin\theta) = re^{j\theta}}$$

where $r = |z|$ and $\theta = \arg z$

Review of Laplace Transform

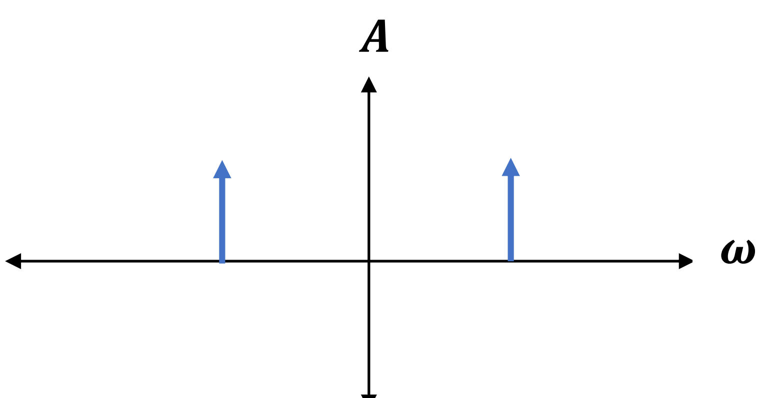
Function Transformation

▪ Fourier versus Laplace Transform



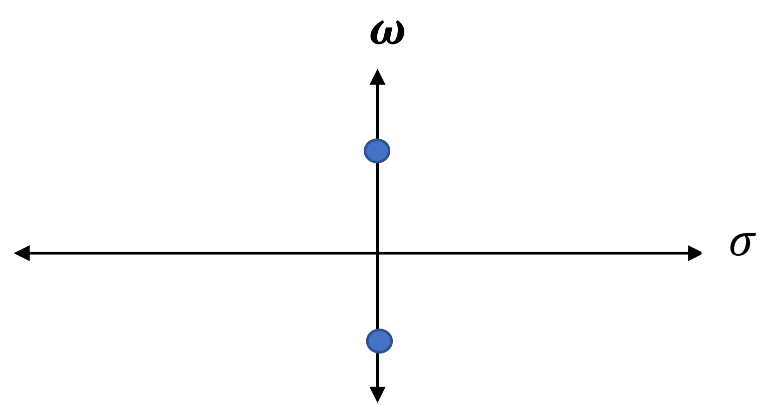
Fourier transform:
Map to the frequency domain

$$Y(\omega) = \mathcal{F}\big(y(t)\big) = \int_{-\infty}^{\infty} y(t)\boldsymbol{e^{-j\omega t}}dt$$



Laplace transform:
Map to the S-domain (freq+exponential)

$$Y(s) = \mathcal{L}\big(y(t)\big) = \int_{0}^{\infty} y(t)\boldsymbol{e^{-st}}dt$$

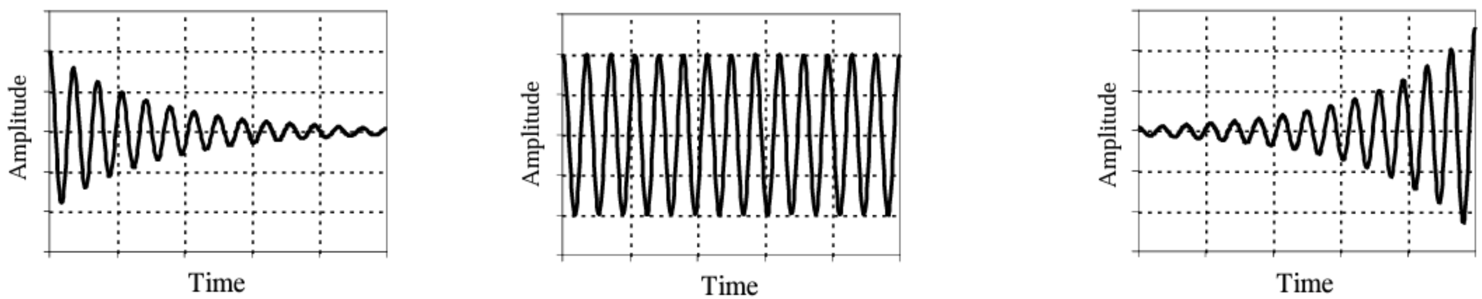


Function Transformation

▪ Fourier versus Laplace Transform

Fourier $\boxed{Y(\omega)} = \int_{-\infty}^{\infty} \boxed{y(t)} e^{-j\omega t} dt$

Which frequencies or sinusoids are present in a function?
A slice of the Lapalce transform.



Laplace $\hat{y}(s) = \int_{0}^{\infty} y(t) e^{-st} dt, \quad s = \sigma + j\omega$

$$= \int_{0}^{\infty} \boxed{y(t) e^{-\sigma t}} e^{-j\omega t} dt$$

Which sinusoids and exponentials are present in a function?

Laplace Transform

time domain signal

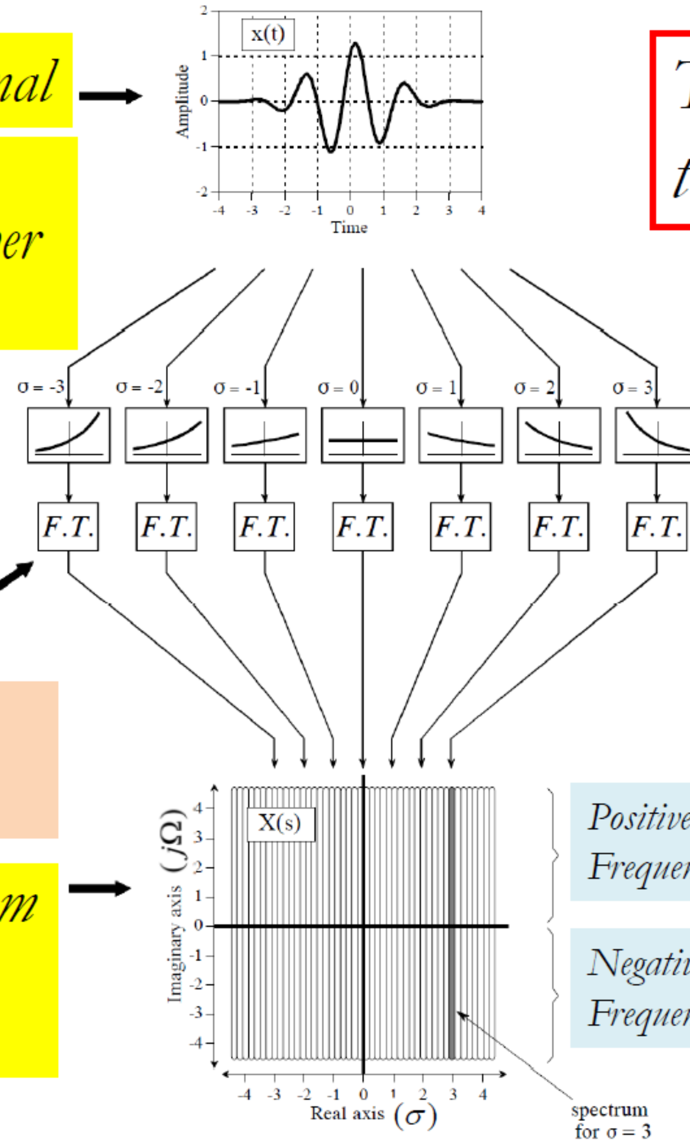
Multiply the time domain signal by an infinite number of exponential curves

$x(t)e^{-\sigma t}$

Take the complex Fourier Transform

$$\int_{-\infty}^{\infty}\left[x(t)e^{-\sigma t}\right]e^{-j\Omega t}\,dt$$

Arrange each spectrum along a vertical line in the s-plane



The Laplace transform

Positive Frequencies

Negative Frequencies

Increasing Exponentials *Decreasing Exponentials*

Laplace Transform

$$X(\sigma,\Omega) = \boxed{\int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t} \right] e^{-j\Omega t} dt} \Rightarrow \boxed{\sum_{k=-\infty}^{\infty} x[k] e^{-\sigma k} e^{-j\omega k}}$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\Omega)t} dt$$

$$s = \sigma + j\Omega$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

complex exponential

Laplace Transform

Forward Laplace Transform (*from CTFT*)

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt \qquad \Leftarrow \qquad \int_{-\infty}^{\infty} |x_a(t)| dt < \infty$$

Absolutely integrable

Being forced to be integrable: *<u>for certain values of</u>* σ

$$\int_{-\infty}^{\infty} x_a(t) e^{-\sigma t} e^{-j\Omega t} dt = \int_{-\infty}^{\infty} x_a(t) e^{-st} dt$$

$$s = \sigma + j\Omega$$

Inverse Laplace Transform

$$X(s) = X(\sigma + j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-(\sigma + j\Omega)t} dt = \int_{-\infty}^{\infty} \left[x_a(t) e^{-\sigma t} \right] e^{-j\Omega t} dt$$

$$x_a(t) e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\sigma + j\Omega) e^{j\Omega t} d\Omega$$

The integral w.r.t. Ω *from* $-\infty$ *to* ∞

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\sigma + j\Omega) e^{(\sigma + j\Omega)t} d\Omega$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X_a(s) e^{st} ds$$

An integral in the complex s-plane along a vertical line from $z = \sigma - j\infty$ *to* $z = \sigma + j\infty$ *with* σ *fixed.*

$$\frac{ds}{d\Omega} = \frac{d(\sigma + j\Omega)}{d\Omega} = j$$

$$d\Omega = \frac{ds}{j}$$

Review
Discrete-time Fourier Transform

Discrete-time Fourier Transform (DTFT)

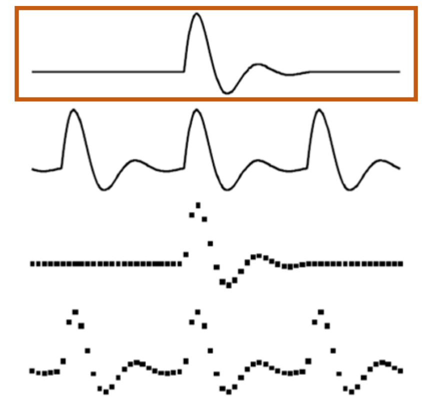
- **Continuous-time Fourier Transform (CTFT)**

<u>Aperiodic Signal</u>

$$x(t) = x(t + nT), \quad n \in Z \quad (integers)$$
$$T \rightarrow \infty$$

<u>Fourier Transform</u>

$$\boxed{X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt}$$



- **Discrete-time Fourier Transform (DTFT)**

<u>Aperiodic Signal</u>

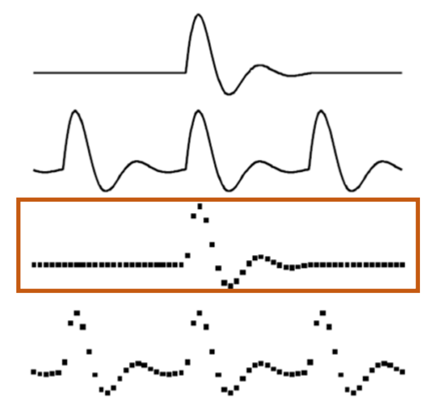
$$t = nT_s, \quad n \in Z$$
$$x(t) \xrightarrow{t = nT_s} x(nT_s) = x[n]$$

$$\Omega T_s = 2\pi \frac{F}{F_s} = \omega$$
normalized frequency

<u>Fourier Transform</u>

$$\int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \xrightarrow{t = nT_s} \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n T_s}$$

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}}$$



Discrete-time Fourier Transform (DTFT)

If $x(k)$ *is absolutely summable, that is,* $\sum_{k=-\infty}^{\infty} |x(k)| < \infty$, *then its* DTFT *is given by*

$$X(e^{j\omega}) \equiv F[x(k)] = \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k}$$

The inverse discrete-time Fourier transform (IDTFT) *of* $X(e^{j\omega})$ *is given by*

$$x(k) \equiv F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega k} d\omega$$

Z-Transform

Z-Transform

The Fourier transform of a sequence $x[n]$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Fourier Transform & Z-Transform

- **Complex variable z in the polar form**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \longrightarrow z = re^{j\omega}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$

$$X(re^{j\omega}) = \underbrace{\sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}}_{\text{Fourier Transform of } (x[n]r^{-n})}$$

Fourier Transform & Z-Transform

- Complex variable z in the polar form

$$r^{-k} = \left[e^{\ln(r)}\right]^{-k} = e^{-k\ln(r)} = e^{-\sigma k}$$

$$\sigma = \ln(r)$$

$$\boxed{X(r,\omega) = \sum_{k=-\infty}^{\infty} x[k] r^{-k} e^{-j\omega k}}$$

$$= \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

$$z = re^{j\omega}$$

Forward z-Transform (from *DTFT*)

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \quad \Leftarrow \quad \sum_{k=-\infty}^{\infty} |x(k)| < \infty$$

Absolutely summable

Being forced to be summable: for certain values of σ

$$\sum_{k=-\infty}^{\infty} \left[x(k)e^{-\sigma k}\right] e^{-j\omega k} = \sum_{k=-\infty}^{\infty} x(k)e^{-(\sigma + j\omega)k}$$

$$= \sum_{k=-\infty}^{\infty} x(k)e^{-sk} = \sum_{k=-\infty}^{\infty} x(k)z^{-k} = X(z)$$

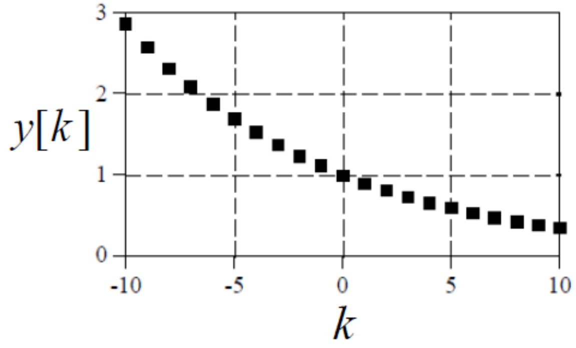
$$z = e^{s} = e^{\sigma + j\omega}$$

Fourier Transform & Z-Transform

Exponential signals

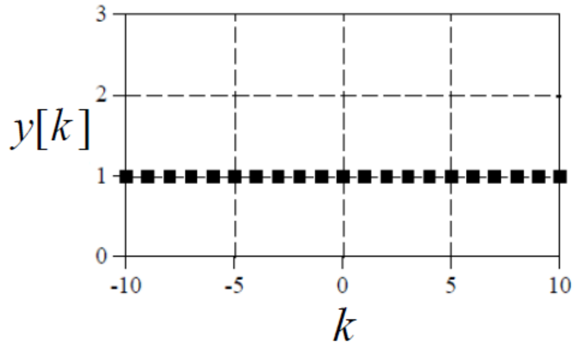
Decreasing

$$y[k] = e^{-\sigma k}, \quad \sigma = 0.105$$
$$y[k] = r^{-k}, \quad r = 1.1$$



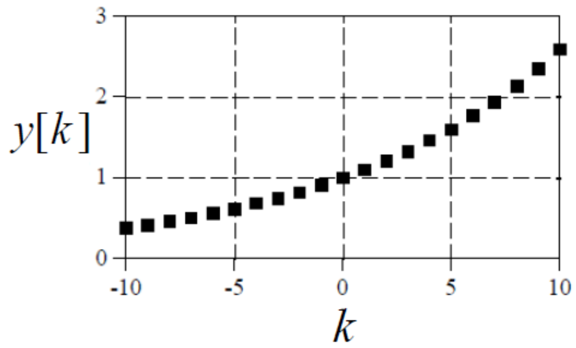
Constant

$$y[k] = e^{-\sigma k}, \quad \sigma = 0.000$$
$$y[k] = r^{-k}, \quad r = 1.0$$

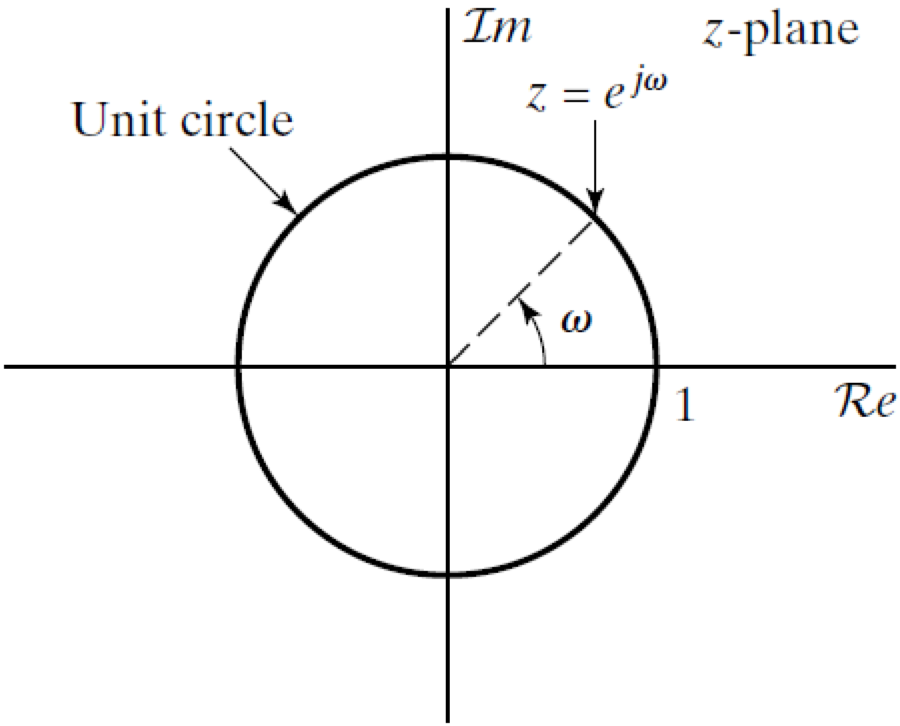


Increasing

$$y[k] = e^{-\sigma k}, \quad \sigma = -0.090$$
$$y[k] = r^{-k}, \quad r = 0.9$$



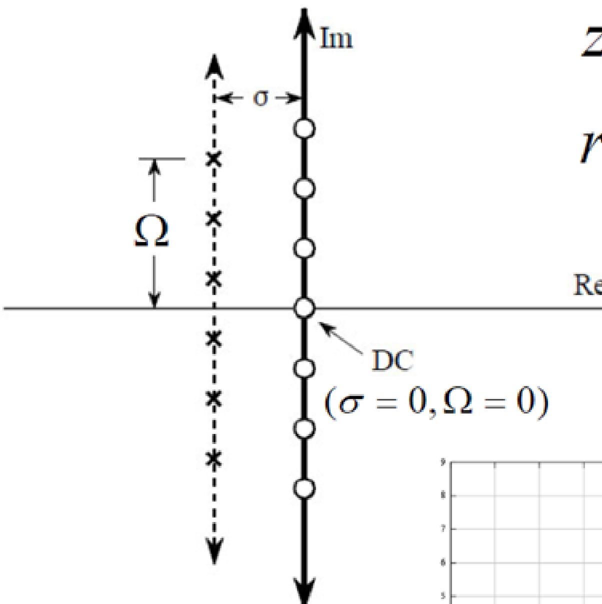
Complex Z-Plane



$$z = 1 \text{ (i.e., } \omega = 0)$$
$$z = j \text{ (i.e., } \omega = \pi/2)$$
$$z = -1 \text{ (i.e., } \omega = \pi)$$

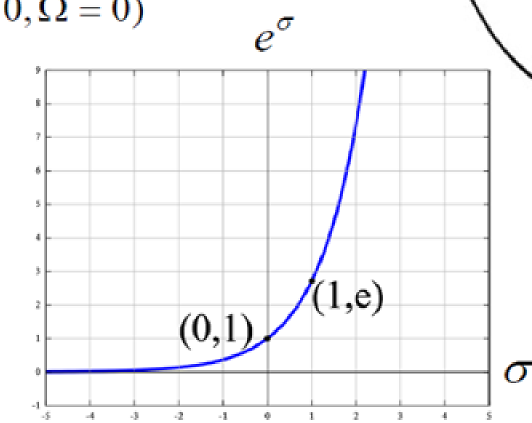
Relationship between S-Plane & Z-Plane

s - Plane

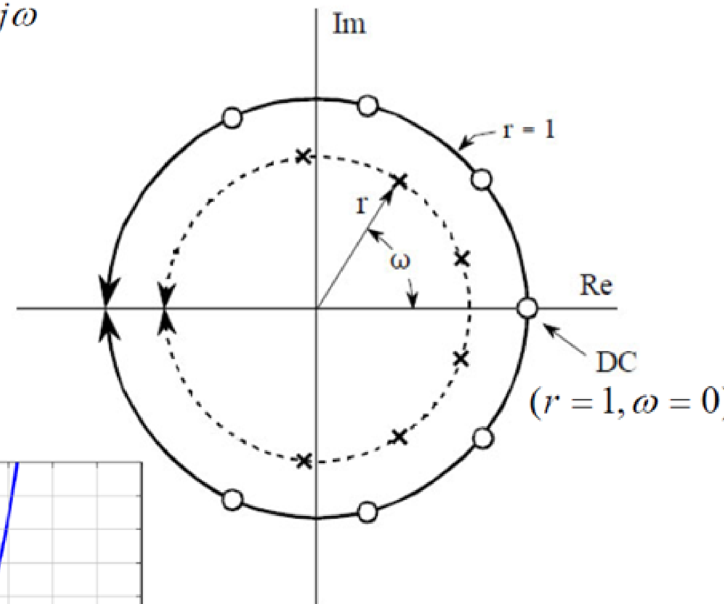


$$z = re^{j\omega}$$

$$r = e^{\sigma}$$



z - Plane



Example

- ANY SIGNAL has a z-Transform:

$$\boxed{X(z) = \sum_{n} x[n] z^{-n}}$$

POLYNOMIAL in z^{-1}

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ?$$

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Some Common Z Transform Pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$

Some Common Z Transform Pairs

Sequence	Transform	ROC
$\left[a^k \sin \omega_0 k\right] u(k)$	$\dfrac{(a \sin \omega_0) z^{-1}}{1-(2a \cos \omega_0) z^{-1} + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$\left[a^k \cos \omega_0 k\right] u(k)$	$\dfrac{1-(a \cos \omega_0) z^{-1}}{1-(2a \cos \omega_0) z^{-1} + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$k a^k u(k)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-k b^k u(-k-1)$	$\dfrac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert > \lvert b\rvert$

Homework #9.1 Z-Transform (2 pt.): Due Jan. 26

Determine the *z*-transforms of the following *finite-duration* signals.

(1) $x_1(k) = \{\underset{\uparrow}{1},2,5,7,0,1\}$

(2) $x_2(k) = \{1,2,\underset{\uparrow}{5},7,0,1\}$

(3) $x_3(k) = \{\underset{\uparrow}{0},0,1,2,5,7,0,1\}$

(4) $x_4(k) = \{2,4,\underset{\uparrow}{5},7,0,1\}$

(5) $x_5(k) = \delta(k)$

(6) $x_6(k) = \delta(k-m), m > 0$

(7) $x_7(k) = \delta(k+m), m > 0$

ROC of Z-Transform

Region of Convergence

▪ Convergence

The power series representing the Fourier transform does not converge for all sequences

Similarly, the z-transform does not converge for all sequences or for all values of z

▪ Example:

$x[n] = u[n]$ is not absolutely summable

However, $r^{-n}u[n]$ is absolutely summable if $r > 1$

the z-transform for the unit step exists with an ROC $r = |z| > 1$

Region of Convergence

Properties of the ROC:

- *The* ROC *is always bounded by a circle.*
- *The* ROC *for right-sided sequences is always outside of a circle of radius* R_{x-}.

  $x(k)$ *that is zero for some* $k < k_0;\ \ k_0 \geq 0,$ *causal*
- *The* ROC *for left-sided sequences is always inside of a circle of radius* R_{x+}.

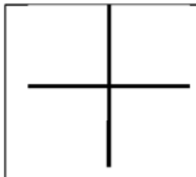
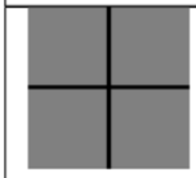
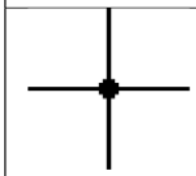
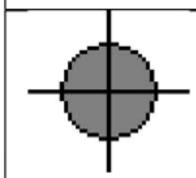
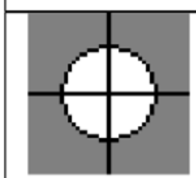
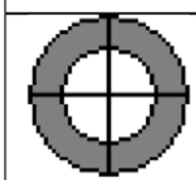
  $x(k)$ *that is zero for some* $k > k_0;\ \ k_0 \leq 0,$ *anticausal*
- *The* ROC *for two-sided sequences is always an open ring* $R_{x-} < |z| < R_{x+}$ *if it exists.*
- *The* ROC *for finite-duration sequences is the entire* z*-plane.*

  $x(k)$ *that is zero for some* $k < k_1;\ \ k > k_2.$

  $k_1 < 0 \Rightarrow z = \infty \notin ROC, k_2 > 0 \Rightarrow z = 0 \notin ROC$
- *The* ROC *cannot include a pole since* $X(z)$ *converges uniformly in there.*
- *There is at least one pole on the boundary of a* ROC *of a rational* $X(z)$.
- *The* ROC *is one contiguous region. does not come in pieces.*

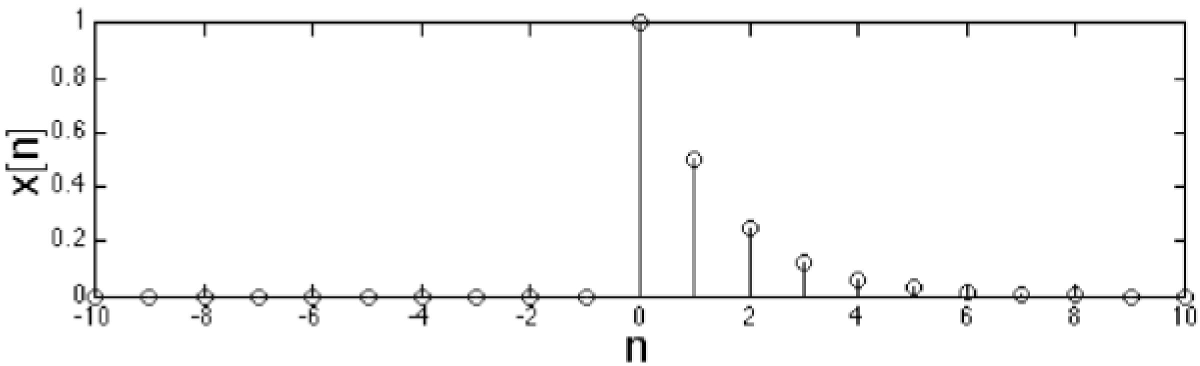
Region of Convergence

Possibilities for
ROC *shape:*

	Null set
	Everywhere
	Point at zero or infinity
	Inside a circle
	Outside a circle
	Inside two circles/Annulus

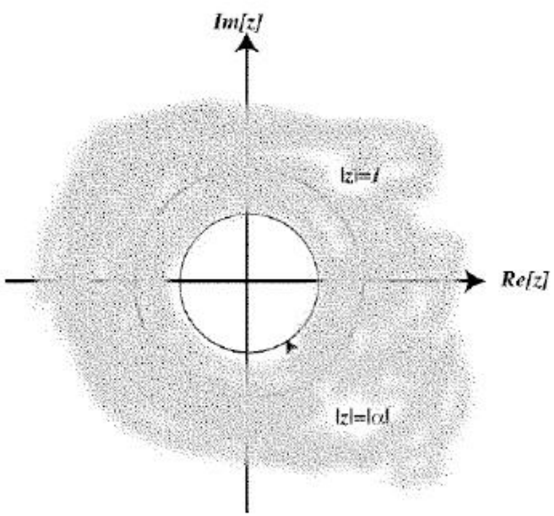
Right-side Sequences

Example 1: Consider the time function $x[n] = \alpha^n u[n]$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

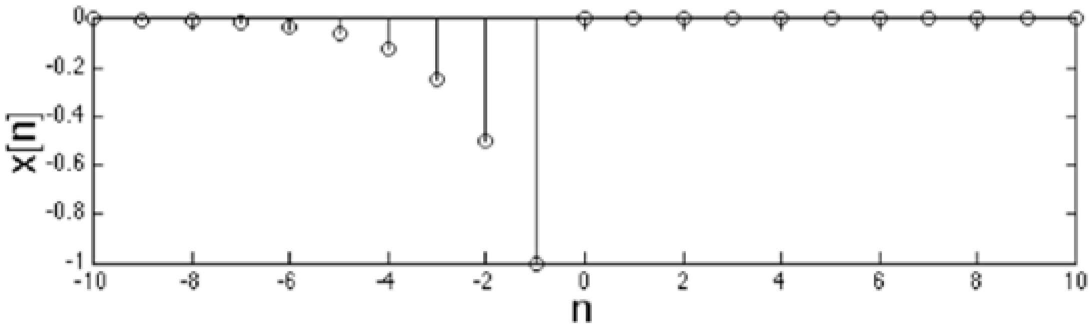
$$= \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$



the ROC was $|z| > |\alpha|$

Left-side Sequences

Example 2: Now consider the time function $x[n] = -\alpha^n u[-n-1]$

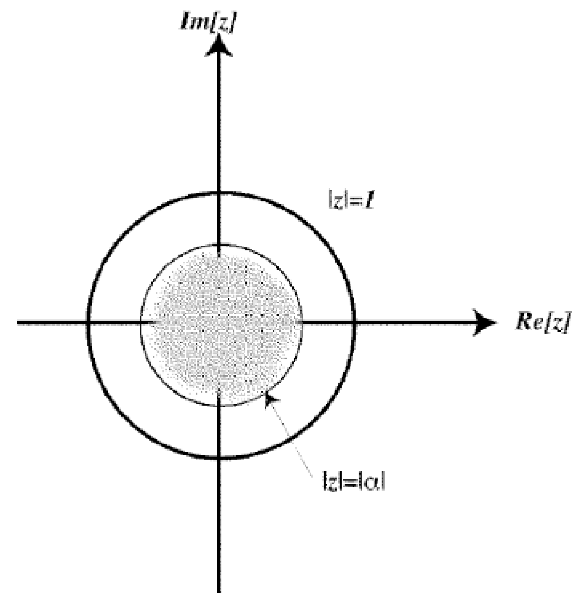


$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = \sum_{n=-\infty}^{-1} (\alpha z^{-1})^n$$

Let $l = -n; n = -\infty \Rightarrow l = \infty; n = -1 \Rightarrow l = 1$

Then, $\displaystyle\sum_{n=-\infty}^{-1} (\alpha z^{-1})^n = \sum_{l=1}^{\infty} -(z\alpha^{-1})^l = 1 - \sum_{l=0}^{\infty} (z\alpha^{-1})^l$

$$= 1 - \frac{1}{1 - z\alpha^{-1}} = \frac{1}{1 - \alpha z^{-1}}$$



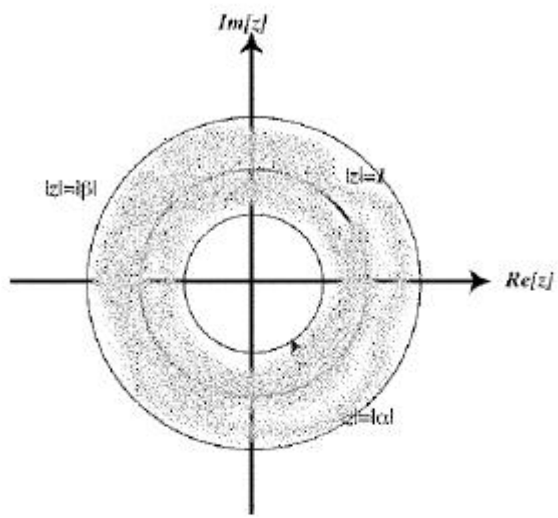
the ROC was $|z| < |\alpha|$

Both-sides Sequences

Consider the function $\alpha^n u[n] - \beta^n u[-n-1]$ **with** $|\alpha| < |\beta|$

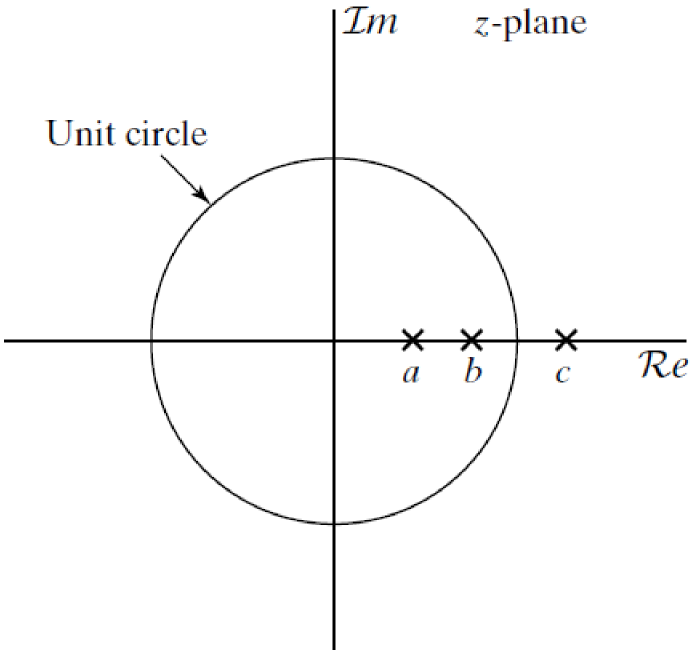
Using the results of Examples 1 and 2, we note that

$$X(z) = \frac{-1}{1-\beta z^{-1}} - \frac{1}{1-\alpha z^{-1}} = \frac{z(\alpha-\beta)}{(z-\alpha)(z-\alpha)}$$



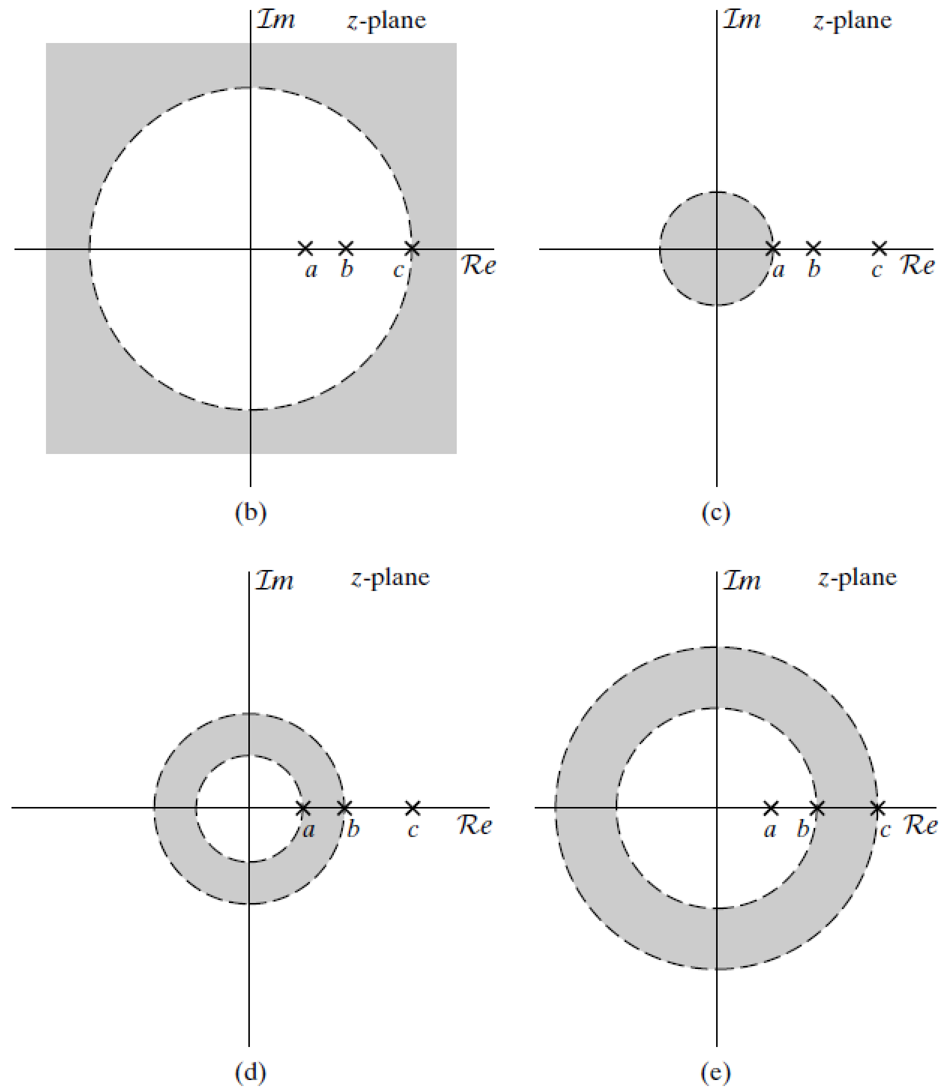
ROCs are of the form $|\alpha| < |z| < |\beta|$

ROC



(a)

(b) to a right-sided sequence
(c) to a left-sided sequence
(d) to a two-sided sequence.
(e) to a two-sided sequence



Properties of Z-Transform

Properties of Z Transform

1. Linearity:

$$Z\left[a_1x_1(k)+a_2x_2(k)\right]=a_1X_1(z)+a_2X_2(z); \quad ROC: ROC_{x_1}\bigcap ROC_{x_2}$$

$$X_s(z)=\sum_{k=-\infty}^{\infty}\left\{a_1x_1(k)+a_2x_2(k)\right\}z^{-k}$$

$$=a_1\sum_{k=-\infty}^{\infty}x_1(k)z^{-k}+a_2\sum_{k=-\infty}^{\infty}x_2(k)z^{-k}$$

$$=a_1X_1(z)+a_2X_2(z)$$

Properties of Z Transform

2. Sample shifting:

$$Z\left[x(k-k_0)\right] = z^{-k_0}X(z); \quad ROC : ROC_x$$

$$X(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k}$$

$$X_s(z) = \sum_{k=-\infty}^{\infty} x(k-k_0)z^{-k}$$

$$k - k_0 = m, \quad k = k_0 + m$$

$$X_s(z) = \sum_{m=-\infty}^{\infty} x(m)z^{-(k_0+m)}$$

$$= \sum_{m=-\infty}^{\infty} x(m)z^{-k_0}z^{-m}$$

$$= z^{-k_0}\sum_{m=-\infty}^{\infty} x(m)z^{-m}$$

$$= z^{-k_0}X(z)$$

Properties of Z Transform

3. Frequency shifting:

$$Z\left[a^k x(k)\right] = X\left(\frac{z}{a}\right); \quad ROC: ROC_x \text{ scaled by } |a|$$

$$X_s(z) = \sum_{k=-\infty}^{\infty} a^k x(k) z^{-k} = \sum_{k=-\infty}^{\infty} x(k)(a^{-1}z)^{-k} = X(a^{-1}z) = X\left(\frac{z}{a}\right)$$

$$\boxed{Z\left[e^{j\omega_0 k} x(k)\right] = X(e^{-j\omega_0} z)}$$

Properties of Z Transform

4. Folding:

$$Z[x(-k)] = X\left(\frac{1}{z}\right); \quad ROC : Inverted\ ROC_x$$

$$x(-n)$$

$$\sum x(-n)z^{-n} = \sum x(n)z^{n} = X(\frac{1}{z})$$

Properties of Z Transform

5. Complex conjugation:

$$Z[x^*(k)] = X^*(z^*); \quad ROC : ROC_x$$

$$Z\left[x^*(k)\right] = X^*(z^*); \quad ROC : ROC_x$$

$$Z\left[x^*(k)\right] = \sum_{k=-\infty}^{\infty} x^*(k) z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} \left[x(k)(z^*)^{-k}\right]^* = \left[\sum_{k=-\infty}^{\infty} x(k)(z^*)^{-k}\right]^* = X^*(z^*)$$

Properties of Z Transform

6. Differentiation in the z-domain:

$$Z[kx(k)] = -z\frac{dX(z)}{dz}; \quad ROC : ROC_x$$

multiplication by a ramp property

$$X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

$$\frac{dX(z)}{dz} = \sum_{k=-\infty}^{\infty} x(k) \frac{dz^{-k}}{dz} = \sum_{k=-\infty}^{\infty} x(k)(-kz^{-k-1})$$

$$= \sum_{k=-\infty}^{\infty} kx(k)\left(\frac{z^{-k}}{-z}\right)$$

$$\boxed{-z\frac{dX(z)}{dz} = \sum_{k=-\infty}^{\infty} kx(k) z^{-k}}$$

Properties of Z Transform

7. Multiplication

$$Z[x_1(k)x_2(k)] = \frac{1}{2\pi j}\oint_C X_1(\nu)X_2(z/\nu)\nu^{-1}d\nu;$$

$$ROC : ROC_{x_1} \bigcap \textit{Inverted } ROC_{x_2}$$

$$Z\left[x_1(k)x_2(k)\right] = \sum_{k=-\infty}^{\infty} x_1(k)x_2(k)z^{-k}$$

$$x_1(k) = \frac{1}{2\pi j}\oint_C X_1(z)z^{k-1}dz, \quad z \to \nu, \quad x_1(k) = \frac{1}{2\pi j}\oint_C X_1(\nu)\nu^{k-1}d\nu$$

$$Z\left[x_1(k)x_2(k)\right] = \frac{1}{2\pi j}\oint_C X_1(\nu)\left[\underbrace{\boxed{\sum_{k=-\infty}^{\infty} x_2(k)\left(\frac{z}{\nu}\right)^{-k}}}_{\boxed{X_2\left(\frac{z}{\nu}\right)}}\right]\nu^{-1}d\nu$$

Properties of Z Transform

8. Convolution:

$$Z\left[x_1(k) * x_2(k)\right] = X_1(z)X_2(z); \quad ROC: ROC_{x_1} \bigcap ROC_{x_2}$$

$$Z\left[x_1(k) * x_2(k)\right] = \sum_{k=-\infty}^{\infty} \{x_1(k) * x_2(k)\} z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_1(n)x_2(k-n)z^{-k}, \quad k-n=m, \quad k=n+m$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_1(n)x_2(m)z^{-(n+m)} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_1(n)x_2(m)z^{-n}z^{-m}$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)z^{-n} \sum_{m=-\infty}^{\infty} x_2(m)z^{-m} = X_1(z)X_2(z)$$

Example:

$$X_1(z) = 2 + 3z^{-1} + 4z^{-2},\ X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$$

$$X_3(z) = X_1(z)X_2(z)$$

$$x_1(k) = \{\underset{\uparrow}{2},3,4\},\ x_2(k) = \{\underset{\uparrow}{3},4,5,6\}$$

```
>> x1 = [2,3,4]; x2 = [3,4,5,6]; x3 = conv(x1,x2)
x3 =          6      17      34      43      38      24
```

$$X_3(z) = 6 + 17z^{-1} + 34z^{-2} + 43z^{-3} + 38z^{-4} + 24z^{-5}$$

Example:

[p,r] = deconv(b,a) *computes the result of dividing* b *by* a *in a polynomial part* p *and a remainder* r.

```
>> x3 = [6,17,34,43,38,24]; x1 = [2,3,4]; [x2,r] = deconv(x3,x1)
x2 =
     3     4     5     6
r =
     0     0     0     0     0     0
```

Example:

$$X_1(z) = z + 2 + 3z^{-1}, X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$$

$$X_3(z) = X_1(z)X_2(z)$$

$$x_1(k) = \{1, \underset{\uparrow}{2}, 3\}, x_2(k) = \{2, 4, \underset{\uparrow}{3}, 5\}$$

```
>> x1 = [1,2,3]; k1 = [-1:1]; x2 = [2,4,3,5]; k2 = [-2:1];
>> [x3,k3] = conv_m(x1,k1,x2,k2)
x3 =
      2     8     17     23     19     15
k3  =
     -3    -2     -1      0      1      2
```

$$X_3(z) = 2z^3 + 8z^2 + 17z + 23 + 19z^{-1} + 15z^{-2}$$

Inverse Z-Transform

Inverse z-Transform

$$X(z) = X(e^{\sigma + j\omega}) = \sum_{k=-\infty}^{\infty} \left[x(k) e^{-\sigma k} \right] e^{-j\omega k}$$

$$x(k)e^{-\sigma k} = \frac{1}{2\pi} \int_0^{2\pi} X(e^{\sigma + j\omega}) e^{j\omega k} d\omega$$

The integral w.r.t. ω *from* 0 *to* 2π

$$x(k) = \frac{1}{2\pi} \int_0^{2\pi} X(e^{\sigma + j\omega}) e^{(\sigma + j\omega)k} d\omega = \frac{1}{2\pi} \int_0^{2\pi} X(z) z^k d\omega$$

$$= \frac{1}{2\pi j} \oint X(z) z^{k-1} dz$$

An integral w.r.t. $z = e^{\sigma + j\omega}$ *in the complex z-plane, along a circle with a fixed radius* e^{σ} *and a varying angle* ω *from* 0 *to* 2π

$$\frac{dz}{d\omega} = \frac{d(e^{\sigma + j\omega})}{d\omega} = e^{\sigma} j e^{j\omega} = jz$$

$$d\omega = \frac{dz}{jz}$$

Inverse Z Transform

- As with other transforms, inverse z-transform is used to derive x[n] from X[z], and is formally defined as:

$$x[n] = \frac{1}{2\pi j} \oint X[z] z^{n-1}\, dz$$

- Here the symbol $\oint$ indicates an integration in counterclockwise direction around a closed path in the complex z-plane (known as contour integral).
- Such contour integral is difficult to evaluate (but could be done using Cauchy's residue theorem), therefore we often use other techniques to obtain the inverse z-tranform.
- One such technique is to use the z-transform pair table shown in the last two slides with partial fraction.

Partial Fraction Decomposition

$$\frac{2}{x-2} + \frac{3}{x+1} = \frac{2(x+1) + 3(x-2)}{(x-2)(x+1)}$$

Which can be simplified using Rational Expressions to:

$$= \frac{2x+2+3x-6}{x^2+x-2x-2}$$

$$= \frac{5x-4}{x^2-x-2}$$

... but how do we go in the opposite direction?

$$\frac{2}{x-2} + \frac{3}{x+1} \; ? \; \frac{5x-4}{x^2-x-2}$$

Partial Fractions

Partial Fraction Decomposition

The method is called *"Partial Fraction Decomposition"*, and goes like this:

Step 1: Factor the bottom

$$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x-2)(x+1)}$$

Step 2: Write one partial fraction for each of those factors

$$\frac{5x-4}{(x-2)(x+1)} = \frac{A_1}{x-2} + \frac{A_2}{x+1}$$

Step 3: Multiply through by the bottom so we no longer have fractions

$$5x-4 = A_1(x+1) + A_2(x-2)$$

Partial Fraction Decomposition

Step 4: Now find the constants A_1 and A_2

Substituting the roots, or "zeros", of (x−2)(x+1) can help:

Root for (x+1) is x = -1

$$
\begin{aligned}
5(-1) - 4 &= A_1(-1+1) + A_2(-1-2) \\
-9 &= 0 + A_2(-3) \\
A_2 &= 3
\end{aligned}
$$

Root for (x-2) is x = 2

$$
\begin{aligned}
5(2) - 4 &= A_1(2+1) + A_2(2-2) \\
6 &= A_1(3) + 0 \\
A_1 &= 2
\end{aligned}
$$

And we have our answer:

$$\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}$$

Example (1): Inverse Z Transform

$$X(z) = \frac{10z+5}{(z-1)(z-1/5)}$$

$$\Downarrow$$

$$\frac{X(z)}{z} = \frac{10z+5}{z(z-1)(z-1/5)} = 25\frac{1}{z} + \frac{75}{4}\frac{1}{z-1} - \frac{175}{4}\frac{1}{z-1/5}$$

$$\Downarrow$$

$$X(z) = 25 + \frac{75}{4}\frac{z}{z-1} - \frac{175}{4}\frac{z}{z-1/5} = 25 + \frac{75}{4}\frac{1}{1-z^{-1}} - \frac{175}{4}\frac{1}{1-1/5z^{-1}}$$

$$\Downarrow$$

$$x(k) = 25\delta(k) + \frac{75}{4}1^k - \frac{175}{4}(1/5)^k \qquad k \geq 0$$

Example (2): Inverse Z Transform

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

$$X(z) = \frac{z}{3\left(z^2 - \frac{4}{3}z + \frac{1}{3}\right)} = \frac{\frac{1}{3}z^{-1}}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}}$$

$$= \frac{\frac{1}{3}z^{-1}}{\left(1 - z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} = \frac{\frac{1}{2}}{1 - z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{3}z^{-1}}$$

$$X(z) = \frac{1}{2}\left(\frac{1}{1 - z^{-1}}\right) - \frac{1}{2}\left(\frac{1}{1 - \frac{1}{3}z^{-1}}\right)$$

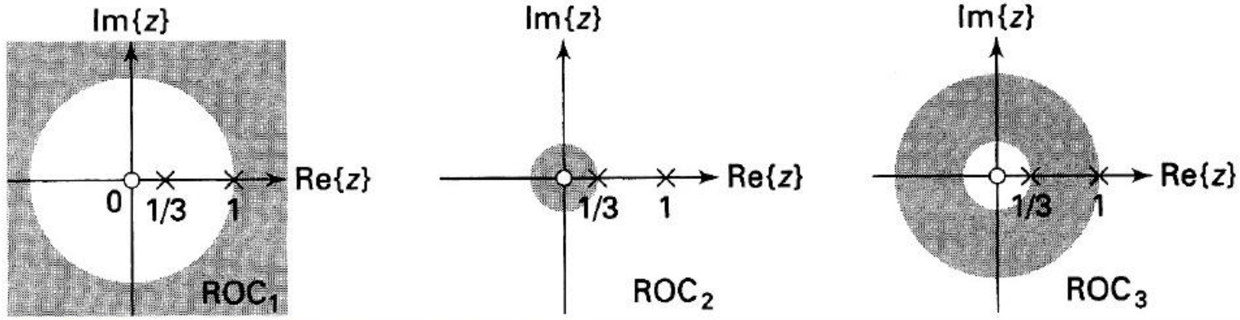
Example (2): Inverse Z Transform

$$X(z)=\frac{\frac{1}{3}z^{-1}}{\left(1-z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} \rightarrow X(z)=\frac{R_1}{1-z^{-1}}+\frac{R_2}{1-\frac{1}{3}z^{-1}}$$

$$R_1=(1-z^{-1})X(z)\Big|_{z^{-1}=1}=\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{2}$$

$$R_2=\left(1-\frac{1}{3}z^{-1}\right)X(z)\Big|_{z^{-1}=3}=\frac{1}{1-3}=-\frac{1}{2}$$

Example (2): Inverse Z Transform



$$ROC_1 : 1 < |z| < \infty, \quad |z_1| \le R_{x-} = 1, |z_2| \le 1$$

$$x_1(k) = \frac{1}{2}u(k) - \frac{1}{2}\left(\frac{1}{3}\right)^k u(k)$$

$$ROC_2 : 0 < |z| < \frac{1}{3}, \quad |z_1| \ge R_{x+} = \frac{1}{3}, |z_2| \ge \frac{1}{3}$$

$$x_2(k) = \frac{1}{2}\{-u(-k-1)\} - \frac{1}{2}\left\{-\left(\frac{1}{3}\right)^k u(-k-1)\right\} = \frac{1}{2}\left(\frac{1}{3}\right)^k u(-k-1) - \frac{1}{2}u(-k-1)$$

$$ROC_3 : \frac{1}{3} < |z| < 1, \quad |z_1| \ge R_{x+} = 1, |z_2| \le \frac{1}{3}$$

$$x_3(k) = -\frac{1}{2}u(-k-1) - \frac{1}{2}\left(\frac{1}{3}\right)^k u(k)$$

Example (3): Inverse Z Transform

If a pole p_r has multiplicity m, then its expansion is given by

$$\sum_{l=1}^{m} \frac{R_{r,l}\boxed{z^{-(l-1)}}}{(1-p_r z^{-1})^l} = \frac{R_{r,1}}{1-p_r z^{-1}} + \frac{R_{r,2}\boxed{z^{-1}}}{(1-p_r z^{-1})^2} + \cdots + \frac{R_{r,m}\boxed{z^{-(m-1)}}}{(1-p_r z^{-1})^m}$$

where the residues $R_{r,l}$ are computed using a more general formula.

Example (3): Inverse Z Transform

$$X(z)=\frac{1}{1-0.7z^{-1}+0.16z^{-2}-0.012z^{-3}}=\frac{1}{\left(1-0.3z^{-1}\right)\left(1-0.2z^{-1}\right)^{2}}$$

$$X(z)=\frac{R_1}{1-0.3z^{-1}}+\frac{R_2}{1-0.2z^{-1}}+\frac{R_3z^{-1}}{(1-0.2z^{-1})^{2}}$$

$$R_1=(1-0.3z^{-1})X(z)\Big|_{z^{-1}=\frac{1}{0.3}}=\frac{1}{(1-0.2z^{-1})^{2}}\Bigg|_{z^{-1}=\frac{1}{0.3}}=\frac{1}{\left(1-\frac{0.2}{0.3}\right)^{2}}=9$$

$$R_3=\left(1-0.2z^{-1}\right)^{2}zX(z)\Big|_{z^{-1}=5}=\frac{z}{(1-0.3z^{-1})}\Bigg|_{z^{-1}=5}=\frac{0.2}{1-0.3(5)}=-0.4$$

$$R_2=\frac{d}{dz}\left(1-0.2z^{-1}\right)^{2}zX(z)\Big|_{z^{-1}=5}=\frac{d}{dz}\frac{z}{(1-0.3z^{-1})}\Bigg|_{z^{-1}=5}$$

$$=\frac{(1-0.3z^{-1})-z(0.3)z^{-2}}{(1-0.3z^{-1})^{2}}\Bigg|_{z^{-1}=5}=\frac{1-0.6z^{-1}}{(1-0.3z^{-1})^{2}}\Bigg|_{z^{-1}=5}=-8$$

Example (3): Inverse Z Transform

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{B(z)}{A(z)}$$

$$= \sum_{r=1}^{N} \frac{R_r}{1 - p_r z^{-1}} + \underbrace{\sum_{r=0}^{M-N} C_r z^{-r}}_{M \geq N}$$

[R,p,C]=residuez(b,a)
[b,a]=residuez(R,p,C)

$p_r = \ldots = p_{r+m-1}$ *a pole of multiplicity m*

$$\frac{R_r}{1 - p_r z^{-1}} + \frac{R_{r+1}\boxed{}}{(1 - p_r z^{-1})^2} + \cdots + \frac{R_{r+m-1}\boxed{}}{(1 - p_r z^{-1})^m}$$

Example (3): Inverse Z Transform

$$X(z) = \frac{z}{3z^2 - 4z + 1} = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{0 + z^{-1}}{3 - 4z^{-1} + z^{-2}}$$

```
>> b = [0,1]; a = [3,-4,1]; [R,p,C] = residuez(b,a)
R =
    0.5000
   -0.5000
p =
    1.0000
    0.3333
C =
     []
```

$$X(z) = \frac{\frac{1}{2}}{1 - z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{3}z^{-1}}$$

Example (4): Inverse Z Transform

Example $X(z) = \frac{1}{(1-0.9z^{-1})^2(1+0.9z^{-1})}, \quad |z| > 0.9$

```
>> b = 1; a = poly([0.9,0.9,-0.9])
a =
      1.0000      -0.9000      -0.8100       0.7290
>> [R,p,C]=residuez(b,a)
R =
    0.2500
    0.5000
    0.2500
p =
    0.9000
    0.9000
   -0.9000
C =
     []
```

roots

$$X(z) = \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{(1-0.9z^{-1})^2} + \frac{0.25}{1+0.9z^{-1}}, \quad |z| > 0.9$$

$$= \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{0.9} z \frac{(0.9z^{-1})}{(1-0.9z^{-1})^2} + \frac{0.25}{1+0.9z^{-1}}, \quad |z| > 0.9$$

Example (4): Inverse Z Transform

$$x(k) = 0.25(0.9)^k u(k) + \frac{5}{9}(k+1)(0.9)^{k+1}u(k) + 0.25(-0.9)^k u(k)$$

$$= 0.75(0.9)^k u(k) + 0.5k(0.9)^k u(k) + 0.25(-0.9)^k u(k)$$

```
>>  [delta,k] = impseq(0,0,7); x = filter(b,a,delta) % check sequence
x =
   Columns 1 through 4
    1.0000000000    0.9000000000   1.6200000000   1.4580000000
   Columns 5 through 8
    1.9683000000    1.7714700000   2.1257640000   1.9131876000
>> x = (0.75)*(0.9).^k + (0.5)*k.*(0.9).^k + (0.25)*(-0.9).^k % answer sequ
x =
   Columns 1 through 4
    1.0000000000   0.9000000000   1.6200000000   1.4580000000
   Columns 5 through 8
    1.9683000000   1.7714700000   2.1257640000   1.9131876000
```

Homework #9.2 Inverse Z Transform (1 pt.): Due Jan. 26

Find the inverse z-transform of:

$$X[z] = \frac{8z - 19}{(z-2)(z-3)}$$

LTI System & Transfer Function

System Representation in Z Domain

Recall the response of a DT LTI system to the input $x[n]$ is

$$y[n] = (x * h)[n]$$

where h is the impulse response of the system.

If x and h have z-transforms, the convolution property implies

$$Y(z) = X(z)H(z)$$

in their common ROC.

Transfer Function

If the ROC has a nonempty interior point, the system function (aka transfer function) $H(z)$ uniquely determines h and hence system properties through the Laurent series expansion

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

Causality

An LTI system with system function $H(z)$ is causal iff

1. the ROC is the exterior of a circle
2. $\lim_{z \to \infty} H(z)$ exists and is finite

$$\boxed{\textbf{causal} \iff \textbf{ROC is the exterior of a circle including } \infty}$$

An LTI system with rational system function $H(z) = \frac{N(z)}{D(z)}$ is causal iff

1. the ROC is $|z| > |p|$, where p is the outermost pole
2. $\deg D \geq \deg N$

Stability

Recall an LTI system is stable iff its impulse response $h \in \ell_1$, i.e.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

i.e. $H(z)$ converges absolutely on the unit circle $|z| = 1$, so its ROC $R_1 < |z| < R_2$ must satisfy $R_1 < 1 < R_2$.

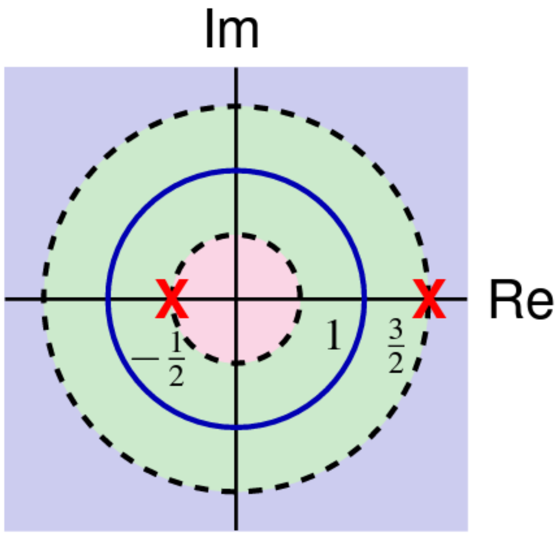
$$\boxed{\textbf{stable} \iff \textbf{ROC includes the unit circle } |z| = 1}$$

A **causal** LTI system with rational system function $H(z)$ is stable iff all its poles are inside the unit circle.

Example

$$H(z) = \frac{1}{2}\left[\frac{z}{z-\frac{3}{2}} - \frac{z}{z+\frac{1}{2}}\right]$$

There are two poles $p_1 = -\frac{1}{2}$ and $p_2 = \frac{3}{2}$.



1. $|z| > \frac{3}{2}$, causal, unstable

$$h_1[n] = \frac{1}{2}\left(\frac{3}{2}\right)^n u[n] - \frac{1}{2}\left(-\frac{1}{2}\right)^n u[n]$$

2. $\frac{1}{2} < |z| < \frac{3}{2}$, noncausal, stable

$$h_2[n] = -\frac{1}{2}\left(\frac{3}{2}\right)^n u[-n-1] - \frac{1}{2}\left(-\frac{1}{2}\right)^n u[n]$$

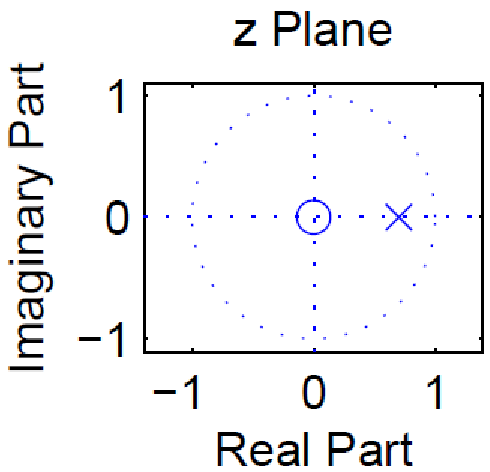
3. $|z| < \frac{1}{2}$, noncausal, unstable

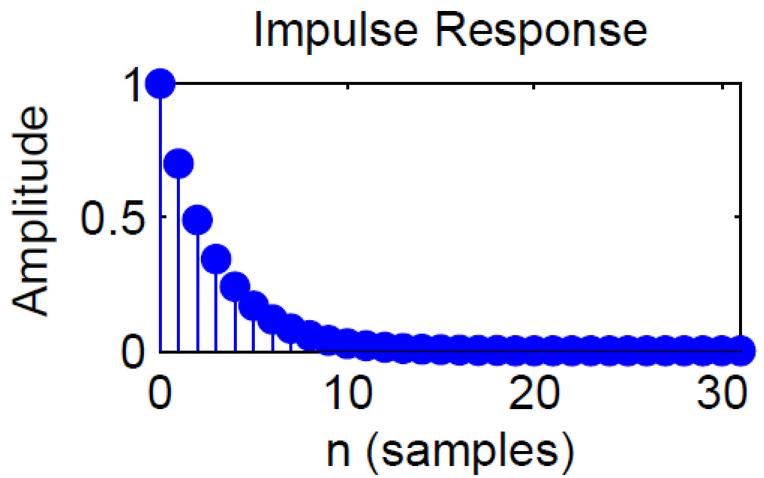
$$h_3[n] = -\frac{1}{2}\left(\frac{3}{2}\right)^n u[-n-1] + \frac{1}{2}\left(-\frac{1}{2}\right)^n u[-n-1]$$

Example

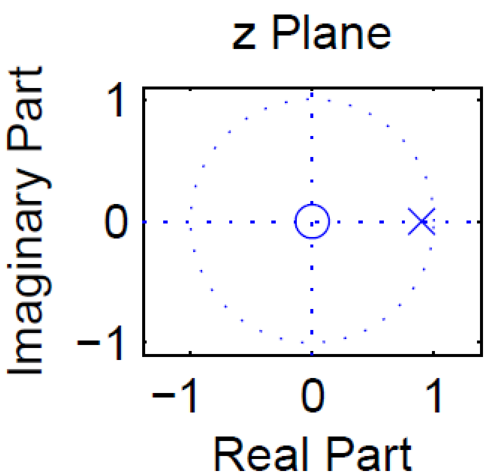
Example

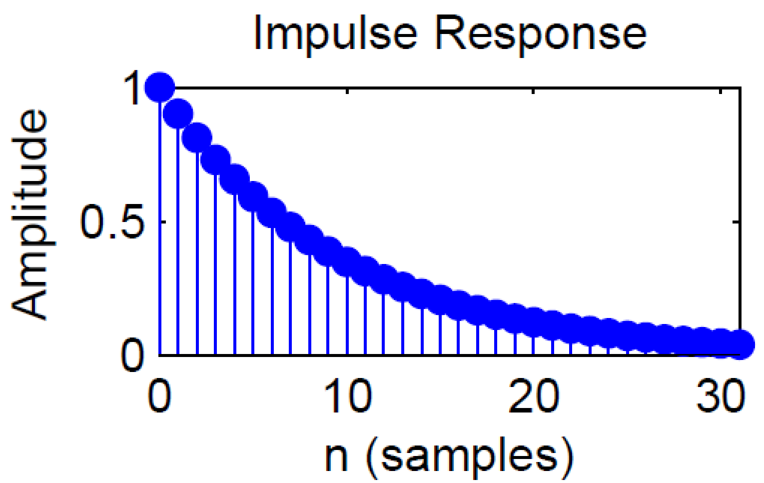
$$H(z) = \frac{z}{z-0.7} = \frac{1}{1-0.7 \cdot z^{-1}}$$





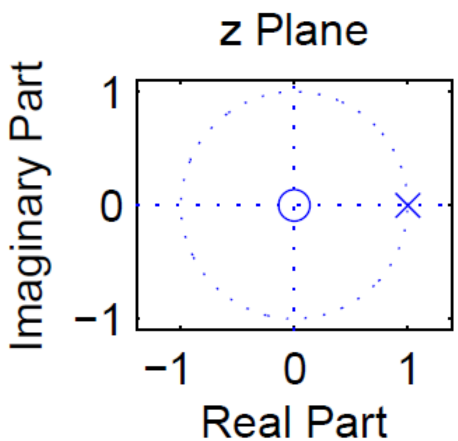
$$H(z) = \frac{z}{z-0.9} = \frac{1}{1-0.9 \cdot z^{-1}}$$

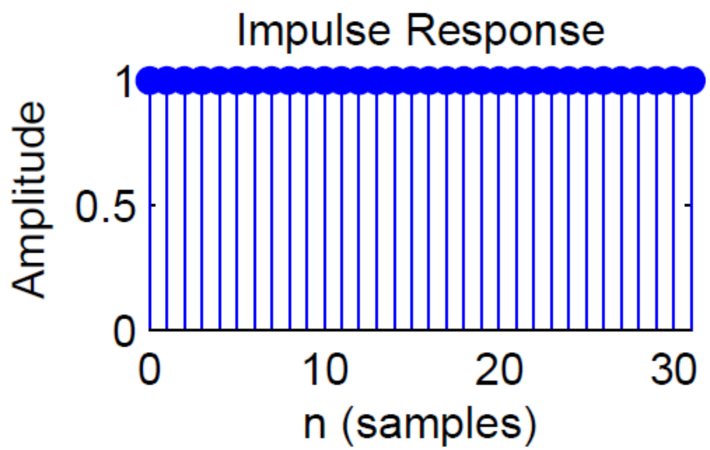




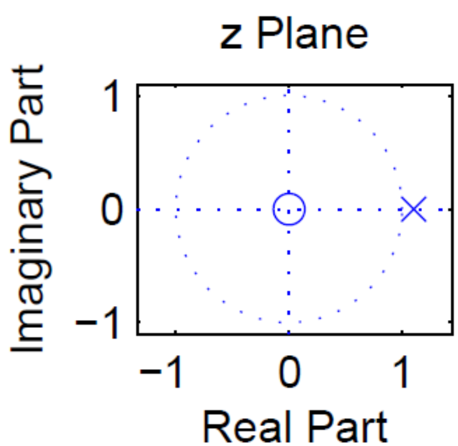
Example

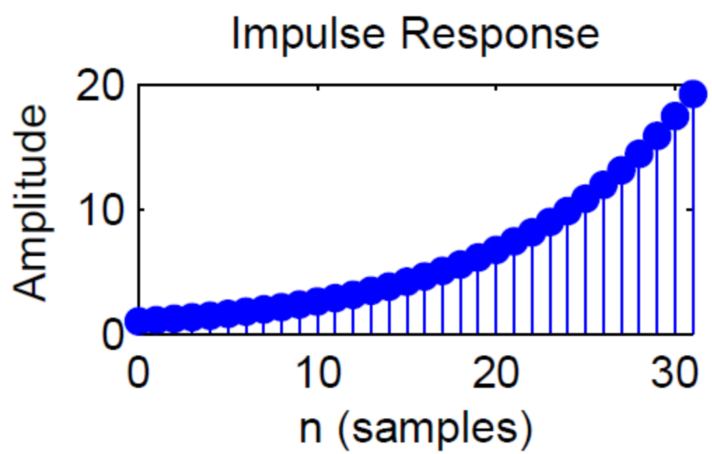
$$H(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$





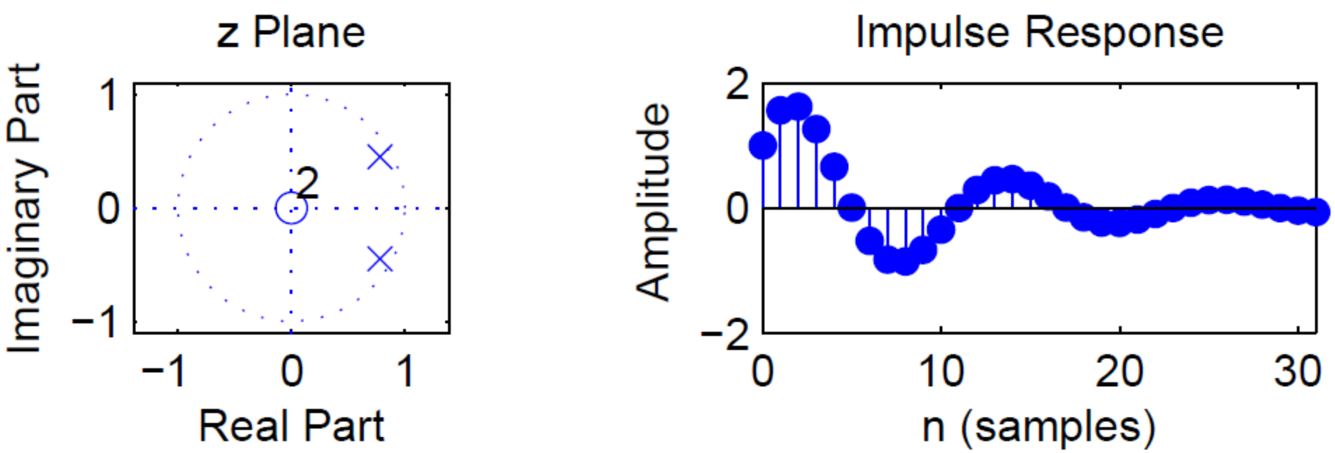
$$H(z) = \frac{z}{z-1.1} = \frac{1}{1-1.1 \cdot z^{-1}}$$



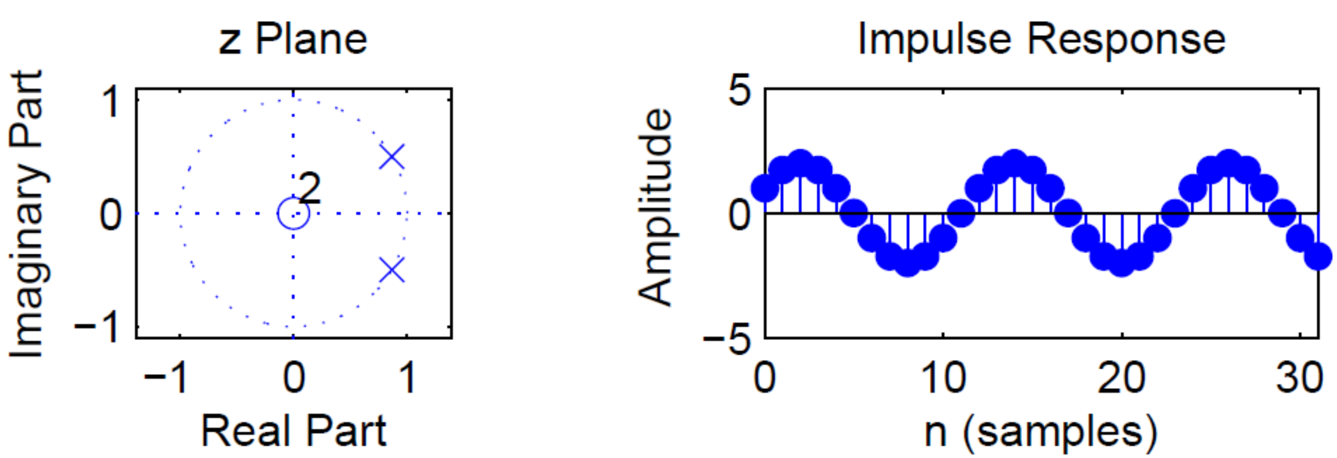


Example

$$H(z) = \frac{z^2}{(z-0.9\cdot e^{j\pi/6})\cdot(z-0.9\cdot e^{-j\pi/6})} = \frac{1}{1-1.8\cos(\pi/6)z^{-1}+0.9^2\cdot z^{-2}}$$

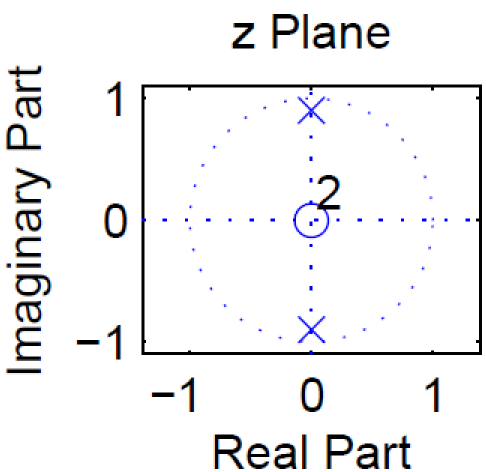


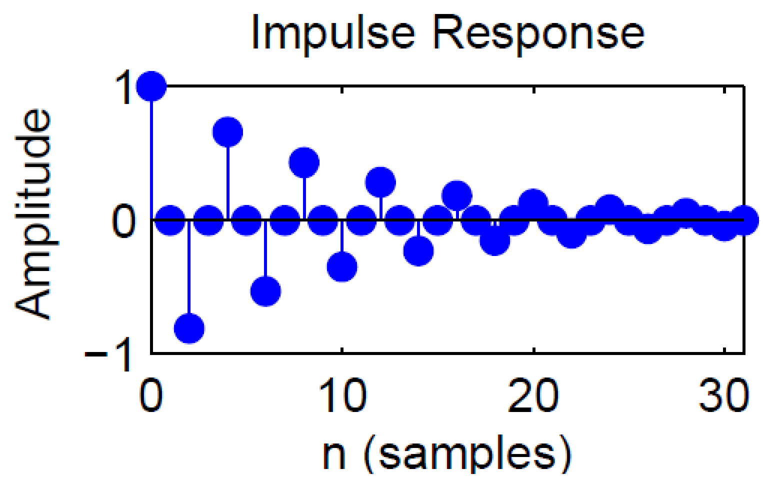
$$H(z) = \frac{z^2}{(z-e^{j\pi/6})\cdot(z-e^{-j\pi/6})} = \frac{1}{1-2\cos(\pi/6)z^{-1}+z^{-2}}$$



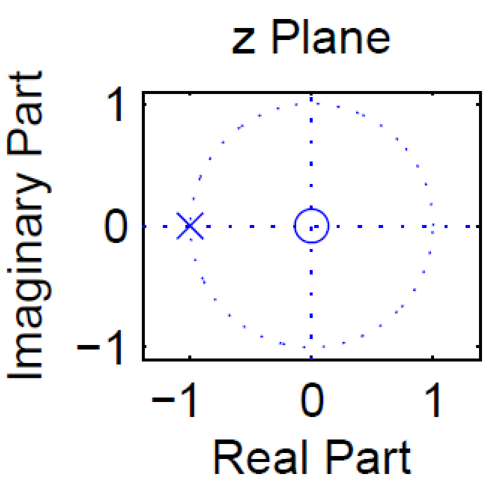
Example

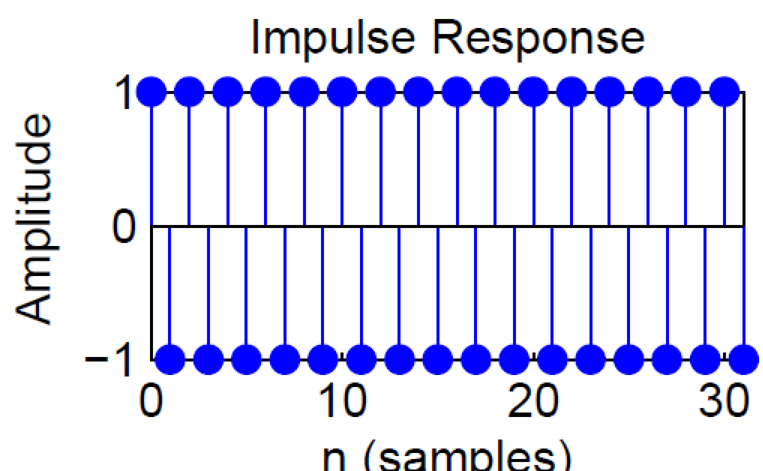
$$H(z) = \frac{z^2}{(z-0.9\cdot e^{j\pi/2})\cdot(z-0.9\cdot e^{-j\pi/2})} = \frac{1}{1-1.8\cos(\pi/2)z^{-1}+0.9^2\cdot z^{-2}} = \frac{1}{1+0.9^2\cdot z^{-2}}$$





$$H(z) = \frac{z}{z+1} = \frac{1}{1+z^{-1}}$$

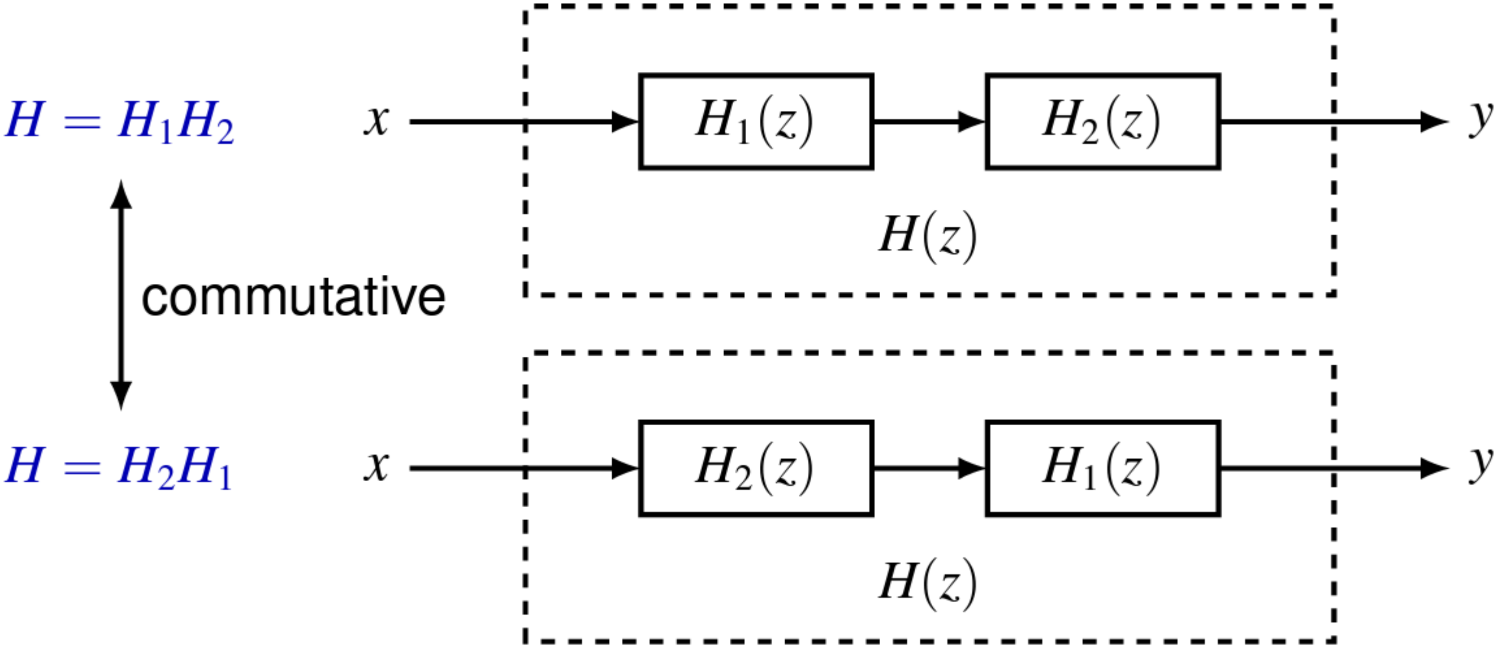




Block Diagram Representation

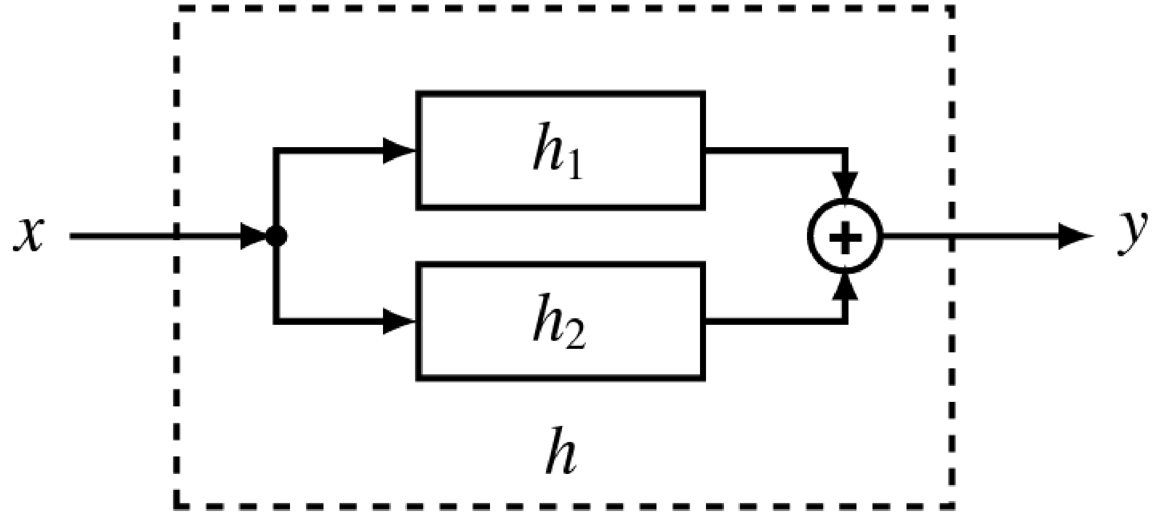
Series Connection

$$Y = (XH_1)H_2 = X(H_1H_2) = (XH_2)H_1$$

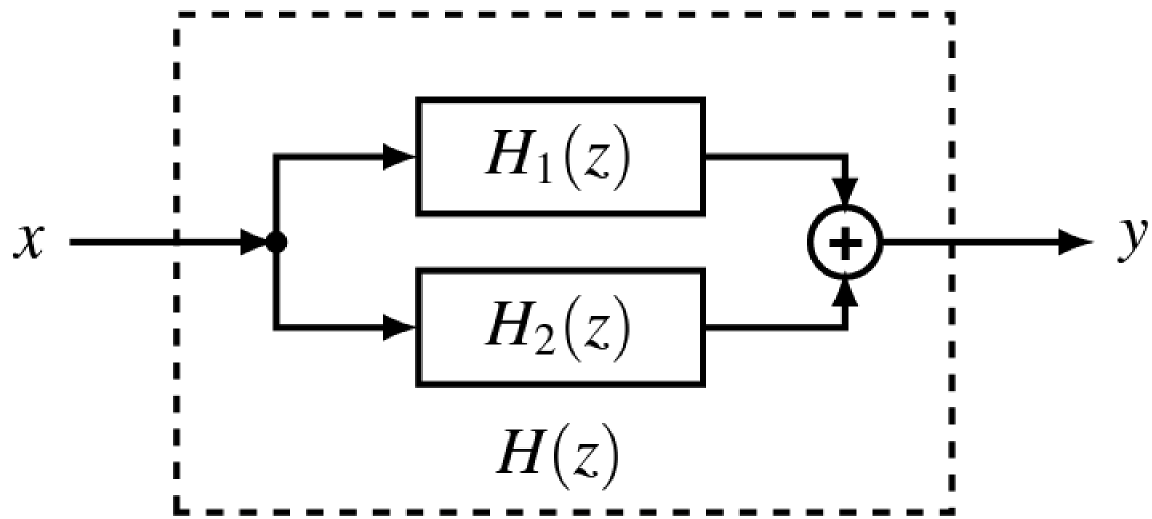


Parallel Connection

$$h = h_1 + h_2$$



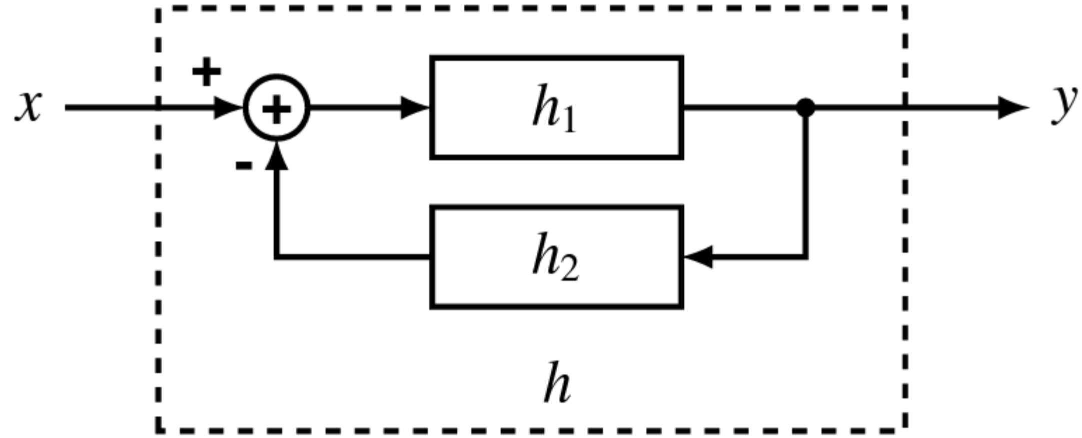
$$H = H_1 + H_2$$



Feedback Connection

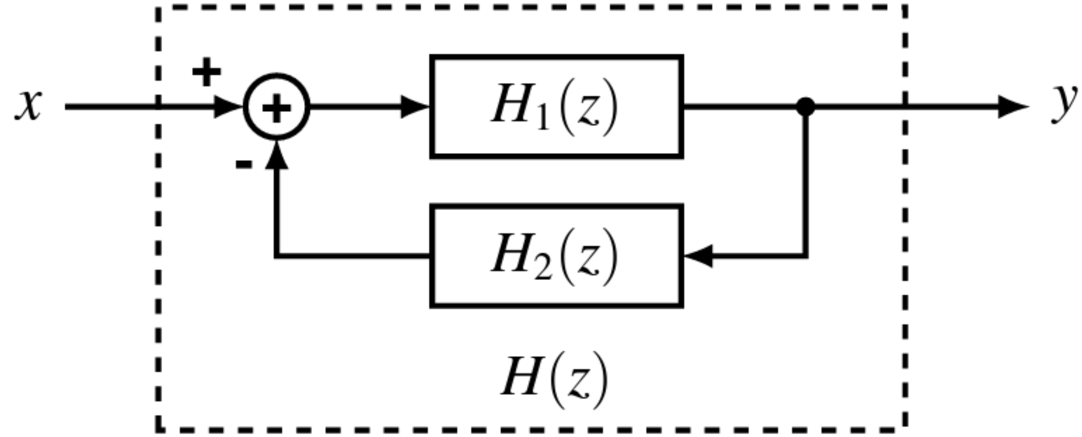
$$y = h_1 * (x - h_2 * y)$$

$$h = ?$$



$$Y = H_1 X - H_1 H_2 Y$$

$$H = \frac{Y}{X} = \frac{H_1}{1 + H_1 H_2}$$



Example

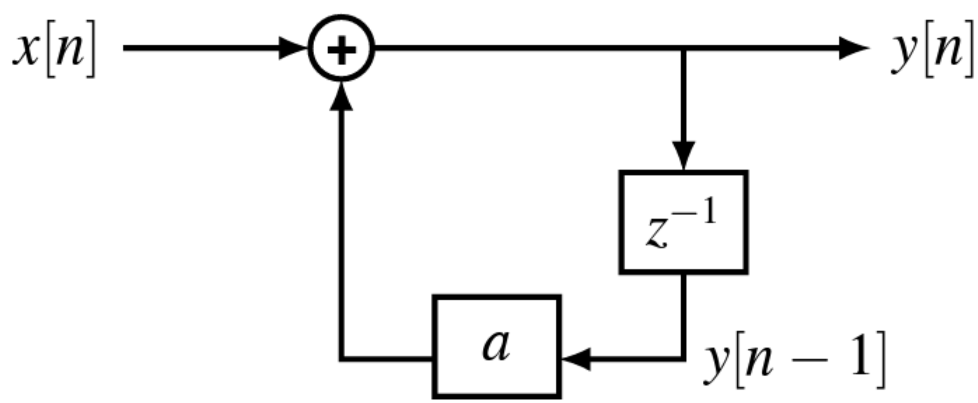
Causal LTI systems with system function

$$H(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

Equivalent description by difference equation

$$y[n] - ay[n-1] = x[n]$$

with initial rest condition.



Example

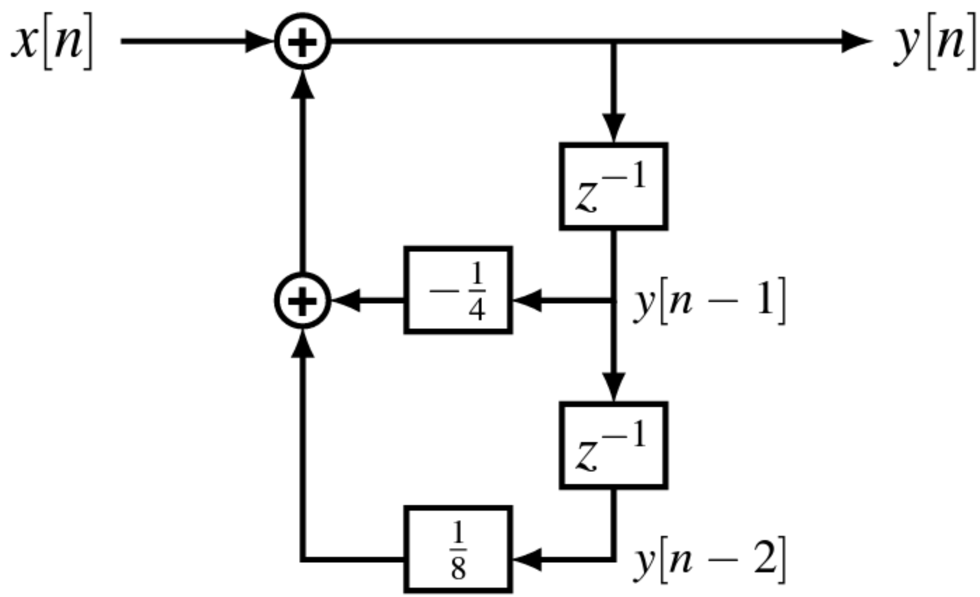
Causal LTI systems with system function

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

Equivalent description by different equation

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$

Direct form

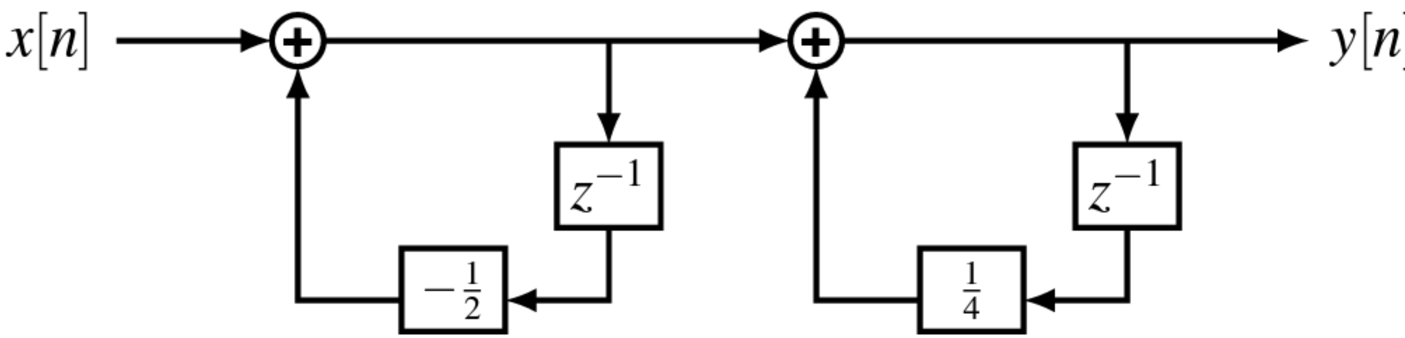


Example

Causal LTI systems with system function

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} \cdot \frac{1}{1 - \frac{1}{4}z^{-1}}$$

Cascade form



Unilateral Z-Transform

Z-Transform: Bilateral & Unilateral

- **Two-sided / Bilateral**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- **One-sided / Unilateral**

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Unilateral Z-Transform

Definition:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Remarks:

- The unilateral z-transform ignores $x[-1], x[-2], \ldots$ and, hence, is typically only used for sequences that are zero for $n < 0$ (sometimes called causal sequences).
- If $x[n] = 0$ for all $n < 0$ then the unilateral and bilateral transforms are identical.

Homework #9.3 Unilateral Z-Transform (2 pt.): Due Jan. 26

Determine the *one-sided* z-transforms of the following signals.

(1) $x_1(k) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

(2) $x_2(k) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

(3) $x_3(k) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$

(4) $x_4(k) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

Thank you