Lecture I213E – Class 4

Discrete Signal Processing

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JAIST
JAPAN
ADVANCED INSTITUTE OF
SCIENCE AND TECHNOLOGY
1990

Course Materials

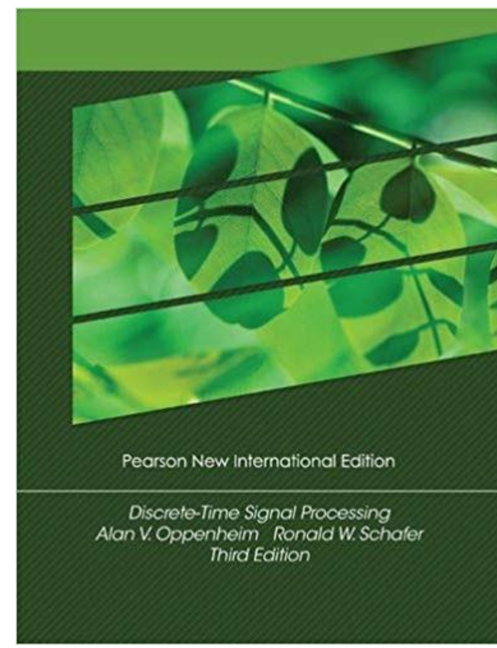
- **Materials**

→ Lecture notes will be uploaded before each lecture
https://jstorage-2018.jaist.ac.jp/s/PGXRrC7iFmN2FWo
Pass: dsp-i213e-2022
(Slide Courtesy of Prof. Nak Young Chong)

- **References**

→ Chi-Tsong Chen:
Linear System Theory and Design, 4th Ed., Oxford University Press, 2013.

→ Alan V. Oppenheim and Ronald W. Schafer:
Discrete-Time Signal Processing, 3rd Ed., Pearson New International Ed., 2013.



Related Courses & Prerequisite

- **Related Courses**
  - → I212 Analysis for Information Science
  - → I114 Fundamental Mathematics for Information Science

- **Prerequisite**
  - → None

Evaluation

- **Viewpoint of evaluation**

  → Students are able to understand:

  - Basic principles in modeling and analysis of linear time-invariant systems
  - Applications of mathematical methods and tools to different signal processing problems.

- **Evaluation method**

  → Homework, term project, midterm exam, and final exam

- **Evaluation criteria**

  → Homework/labs (30%), term project (30%) midterm exam (15%), and final exam (25%)

Contact

- **Lecturer**

  → Sakriani Sakti

- **TA**

  Tutorial hours & Term project
  → WANG Lijun (s2010026)
  → TANG Bowen (s2110411)
  Homework
  → PUTRI Fanda Yuliana (s2110425)

- **Contact Email**

  → dsp-i213e-2022@ml.jaist.ac.jp

Schedule

- **December 8th, 2022 – February 9th, 2023**

- **Lecture Course Term 2-2**
  - → Tuesday 9:00 — 10:40
  - → Thursday 10:50 — 12:30

- **Tutorial Hours**
  - → Tuesday 13:30-15:10

Schedule

Dec

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Jan

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	~~5~~	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	~~24~~	25	26	27	28
29	30	31				

Feb

Sun	Mon	Tue	Wed	Thu	Fri	Sat
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28				

Lecture:
Tuesday 9:00 — 10:40
Thursday 10:50 — 12:30

Tutorial:
Tuesday 13:30 — 15:10

Course review &
term project evaluation
(on tutorial hours)

Midterm & final exam
Thursday 10:50 — 12:30

Syllabus

Class	Date	Lecture Course Tue 9:00 — 10:40 / Thr 10:50 — 12:30	Tutorial Hours Tue 13:30 — 15:10
1	12/08	Introduction to Linear Systems with Applications to Signal Processing	
2	12/13	State Space Description	○
3	12/15	Linear Algebra	
4	12/20	Quantitative Analysis (State Space Solutions) and Qualitative Analysis (Stability)	○
5	12/22	Discrete-time Signals and Systems	
X	01/05		
6	01/10	Discrete-time Fourier Analysis	
7	01/10*	Review of Discrete-time Linear Time-Invariant Signals and Systems (on Tutorial Hours)	
	01/12	Midterm Exam	
8	01/17	Sampling and Reconstruction of Analog Signals	○
9	01/19	z-Transform	
X	01/24		○
10	01/26	Discrete Fourier Transform	
11	01/31	FFT Algorithms	○
12	01/02	Implementation of Digital Filters	
13	02/07	Digital Signal Processors and Design of Digital Filters	
14	02/07*	Review of the Course and Term Project Evaluation (on Tutorial Hours)	
	02/09	Final exam	

Class 4
State-Space Solutions and Stability Analysis

State-Space Solutions & Stability Analysis

- Quantitative analysis:
  Solutions of state-space models
- Qualitative analysis:
  Stability (controllability, observability)

Linear Systems

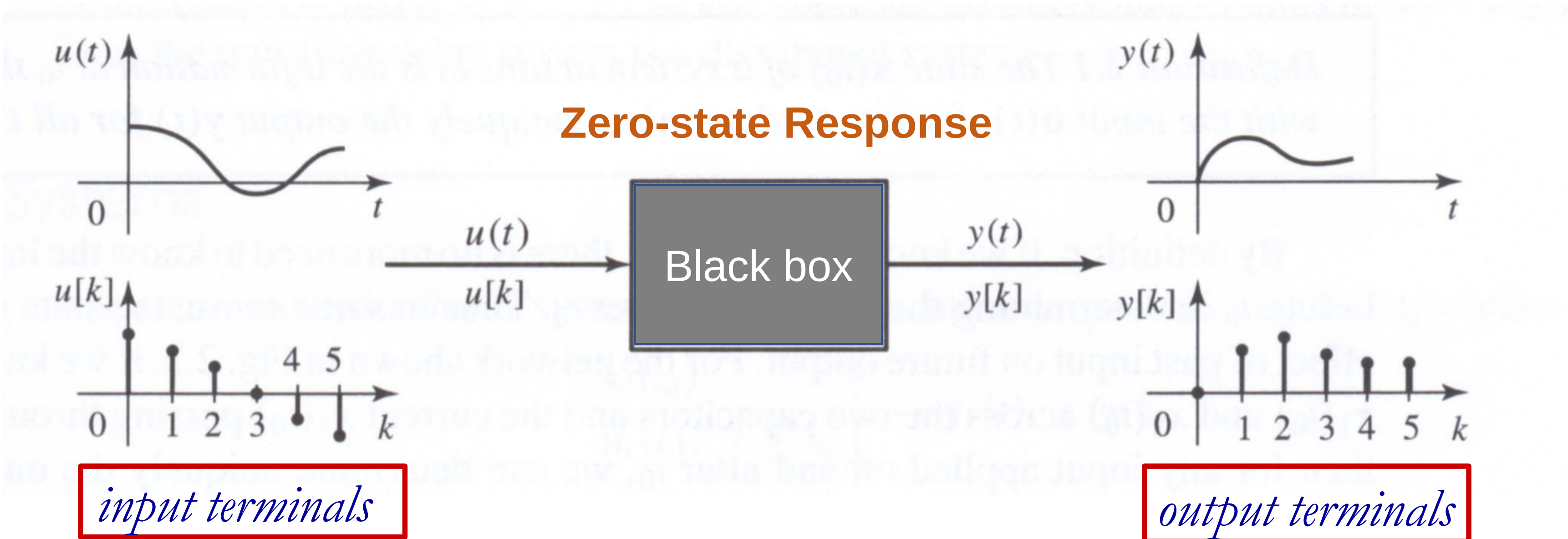
External Description

▪ External Description: Input-Output Description

→ View the system as a "black box" description:

no information on the internal details of the system

→ Characterize by the relation of input, output, and system response (impulse response)



$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

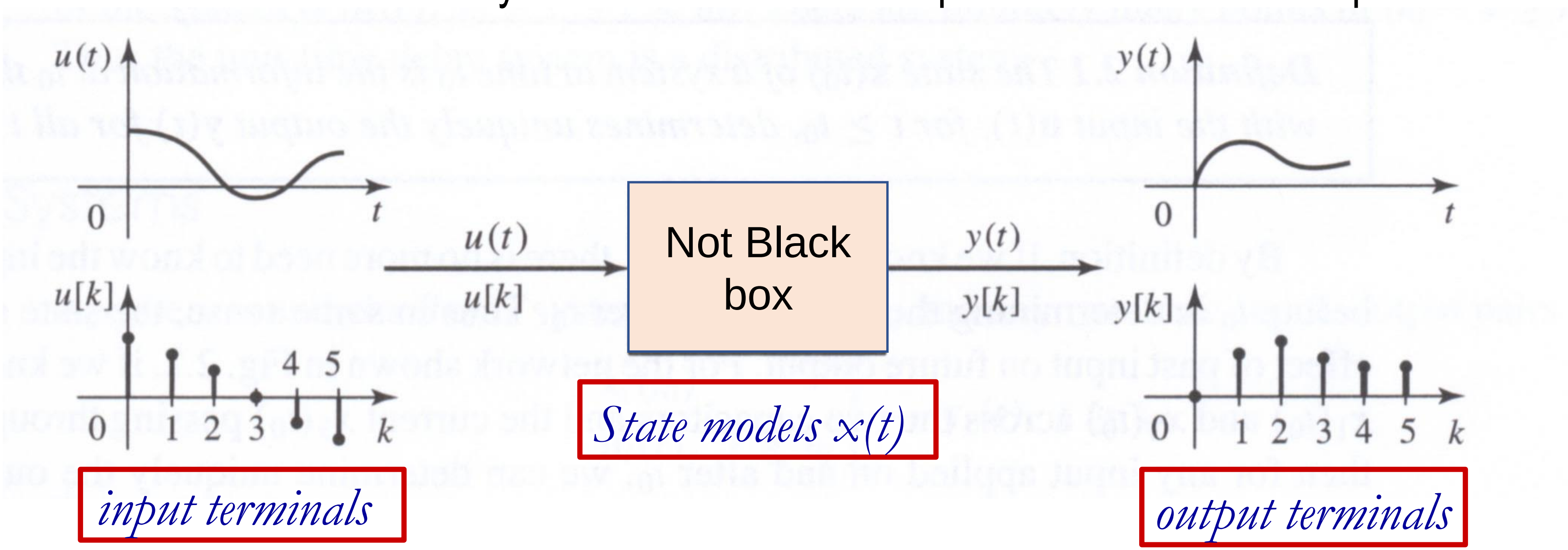
output *input*

Internal Description

▪ Internal Description: State-Space Description

→ State-space representation:

a mathematical model of a physical system as a set of input, output, and state variables related by first-order differential equations or difference equations



state

$$\dot{x}(t) = Ax(t) + Bu(t)$$ 1st order DE

$$y(t) = Cx(t) + Du(t)$$ AE

output

input

Quantitative analysis: Solutions to Input-Output Models

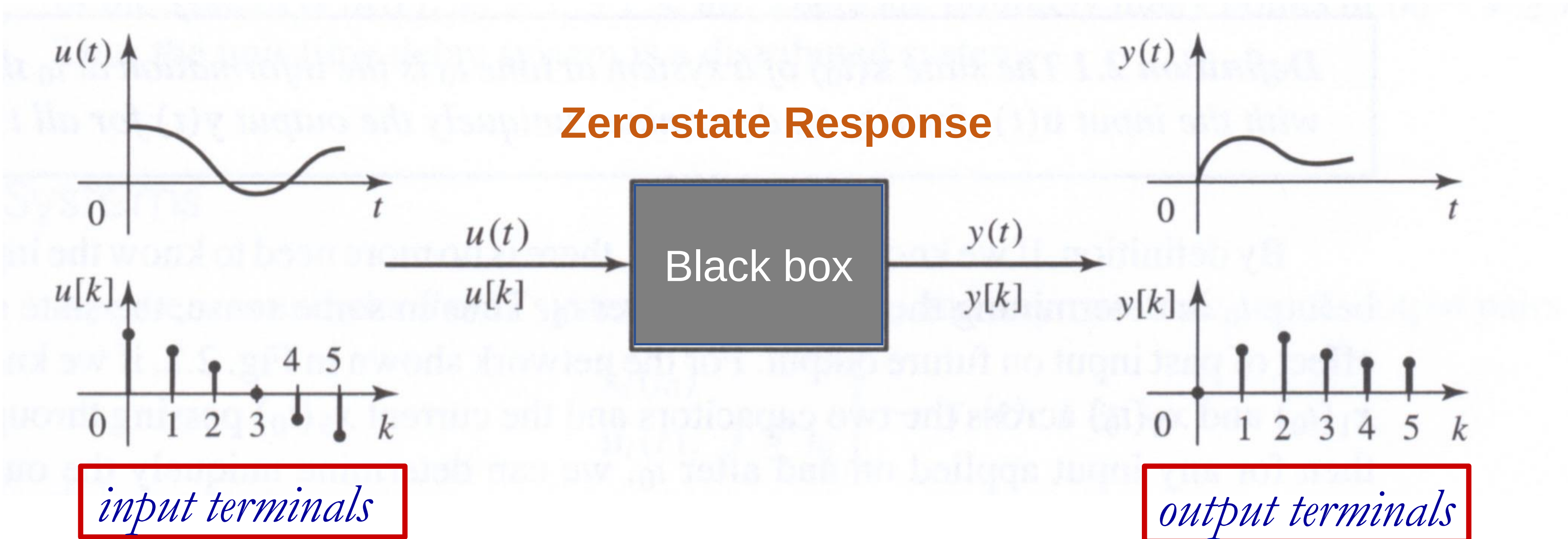
External Description

▪ External Description: Input-Output Description

→ View the system as a "black box" description:

no information on the internal details of the system

→ Characterize by the relation of input, output, and system response (impulse response)



$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

output *input*

How to Find Solution to the Linear System

- **External Description: Input-Output Description**

  $\rightarrow$ **Calculate the Integral: Integration Step Size**

$$y(t) = \int_{\tau=t_0}^{t} g(t,\tau)u(\tau)d\tau$$

: Input-output description

$$\Downarrow$$

$$y(k\Delta) = \sum_{m=k_0}^{k} g(k\Delta, m\Delta)u(m\Delta)\Delta$$

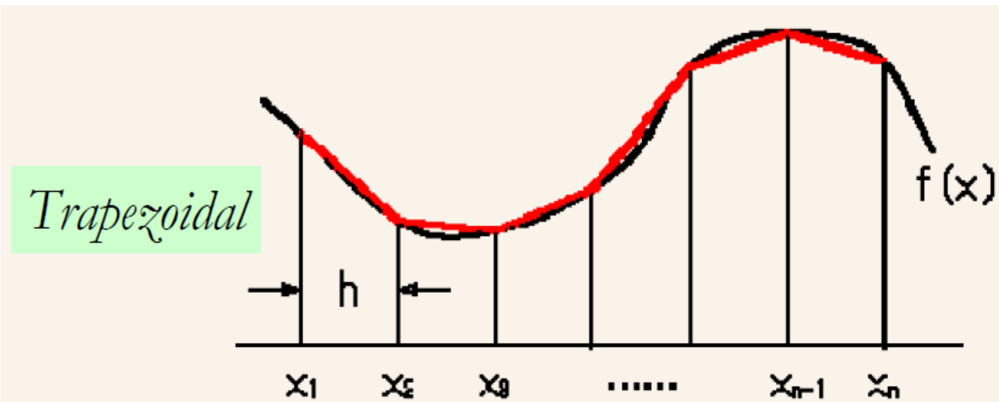
Integration step size

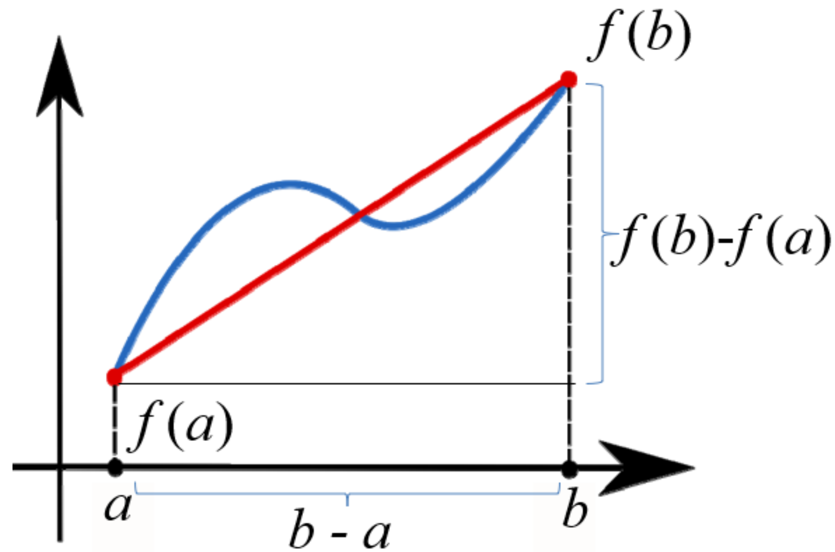
$$\Downarrow$$

easiest, but least accurate

Integration Step Size

- **Example: Integration Method "Trapezoidal"**



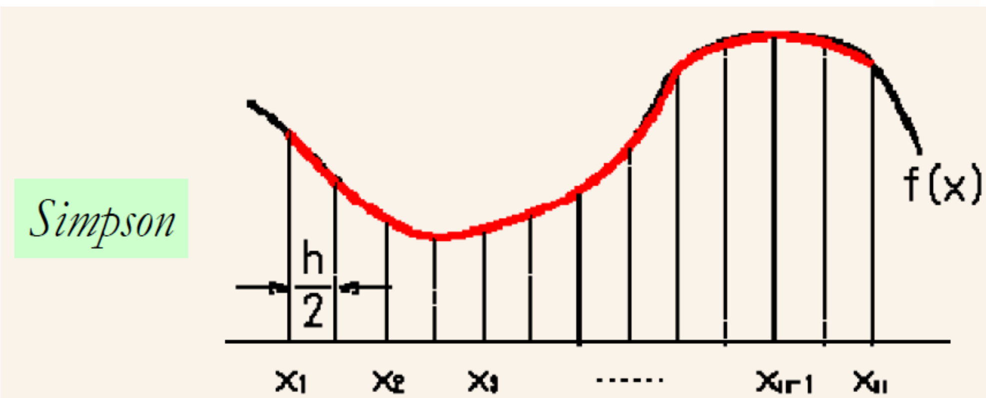


$$(b-a)f(a) + \frac{(b-a)\,(f(b)-f(a))}{2}$$

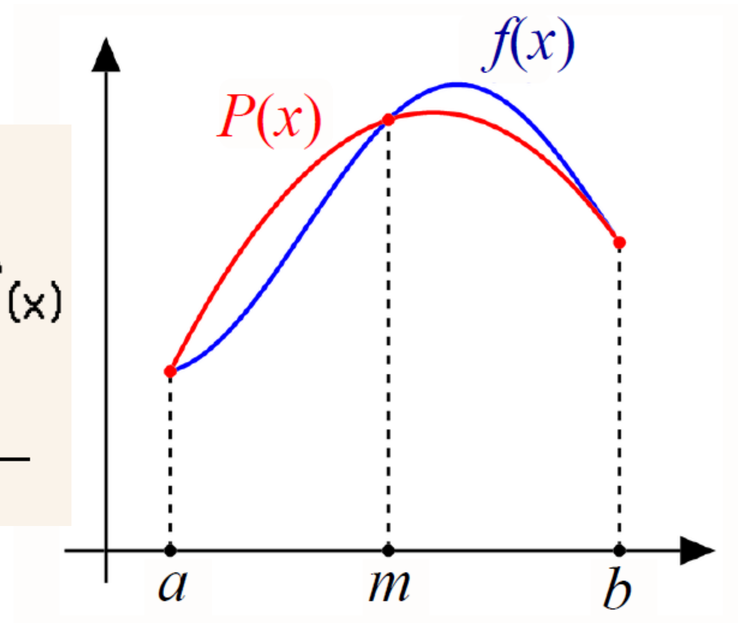
$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$$

Integration Step Size

- **Example: Integration Method "Simpson"**



Simpson's Rule: *parabolas*



$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

How to Find Solution to the Linear System

- **External Description: Input-Output Description**

  $\rightarrow$ **Calculate the Integral:**

  via Laplace Transform & Inverse Laplace Transform

$$y(t) = \int_{\tau=t_0}^{t} g(t,\tau)u(\tau)d\tau \quad : \textit{Input-output description}$$

For the LTI case,

$$\hat{y}(s) = \hat{g}(s)\hat{u}(s)$$

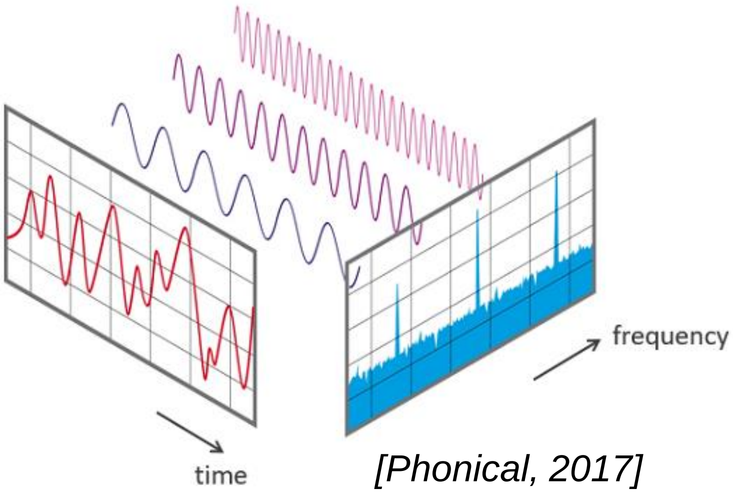
If the system is lumped, $\hat{g}(s), \hat{u}(s) \rightarrow$ *rational functions*

$$y(t) = \mathrm{L}^{-1}\{\hat{g}(s)\hat{u}(s)\}$$

Laplace Transform

Function Transformation

- **Analyzing signals from a different perspective:**



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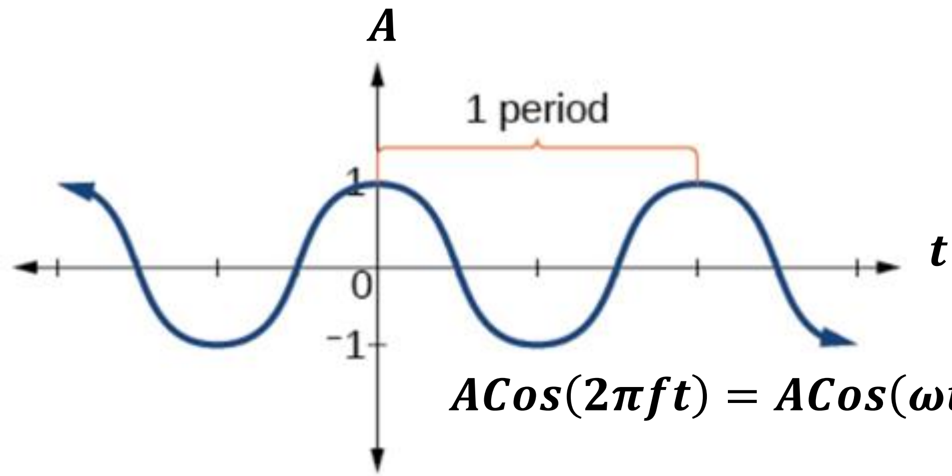

$$ACos(2\pi f t) = ACos(\omega t)$$

```
A = amplitude
T = period of 1 cycle
f = frequency

Example:
1 cycle = 20 sec
f = 1/T = 0.05 Hz
ω = 2π/T = 18 rad/sec
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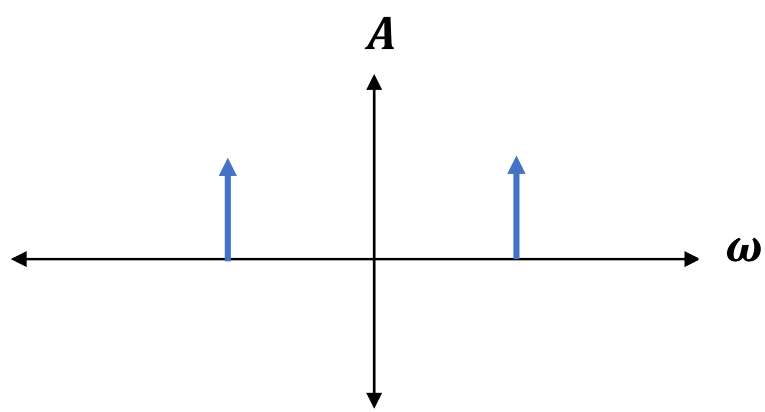
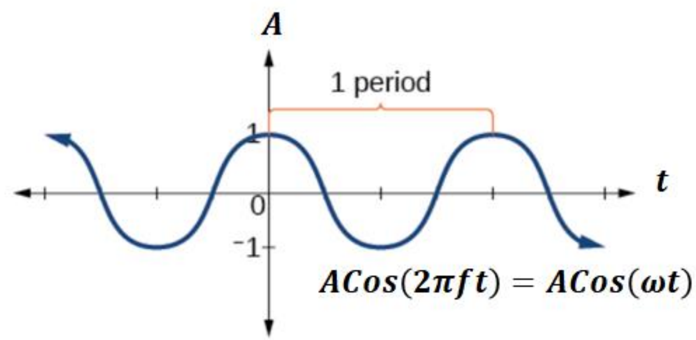


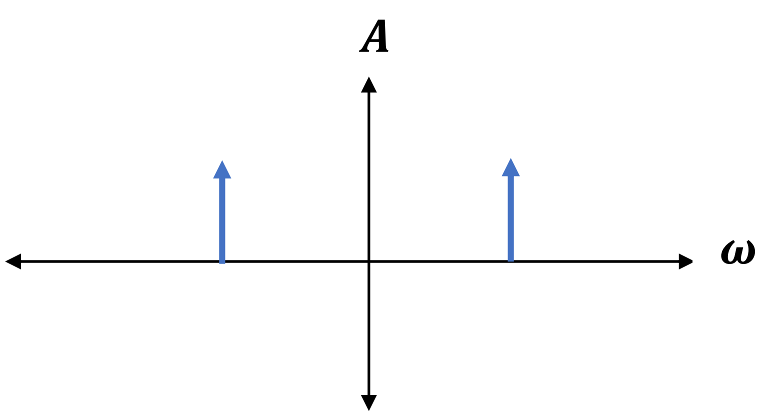
Function Transformation

▪ Fourier versus Laplace Transform



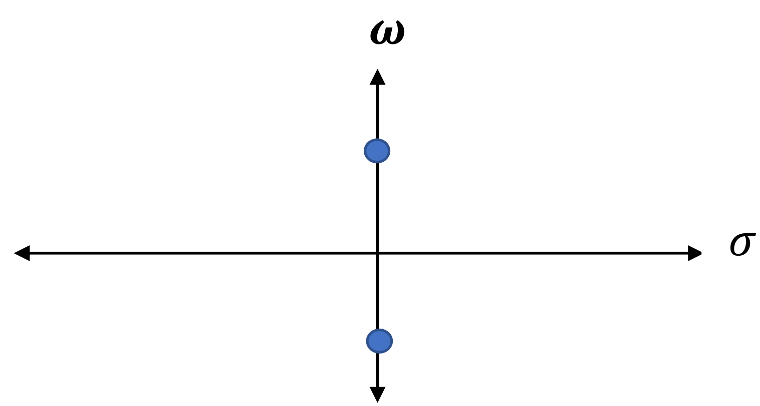
Fourier transform:
Map to the frequency domain

$$Y(\omega) = \mathcal{F}\big(y(t)\big) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$



Laplace transform:
Map to the S-domain (freq+exponential)

$$Y(s) = \mathcal{L}\big(y(t)\big) = \int_{0}^{\infty} y(t)\, e^{-st}\, dt$$

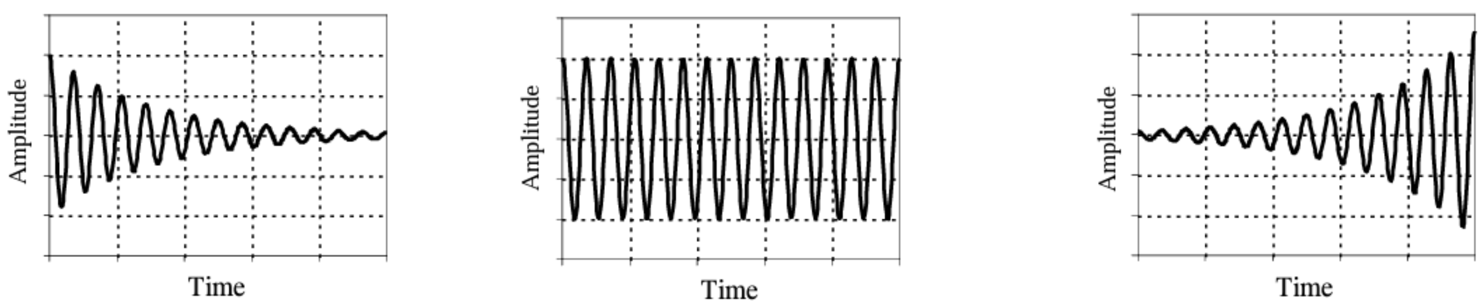


Function Transformation

- **Fourier versus Laplace Transform**

Fourier $$\boxed{Y(\omega)} = \int_{-\infty}^{\infty} \boxed{y(t)} e^{-j\omega t} dt$$

Which frequencies or sinusoids are present in a function?
A slice of the Lapalce transform.

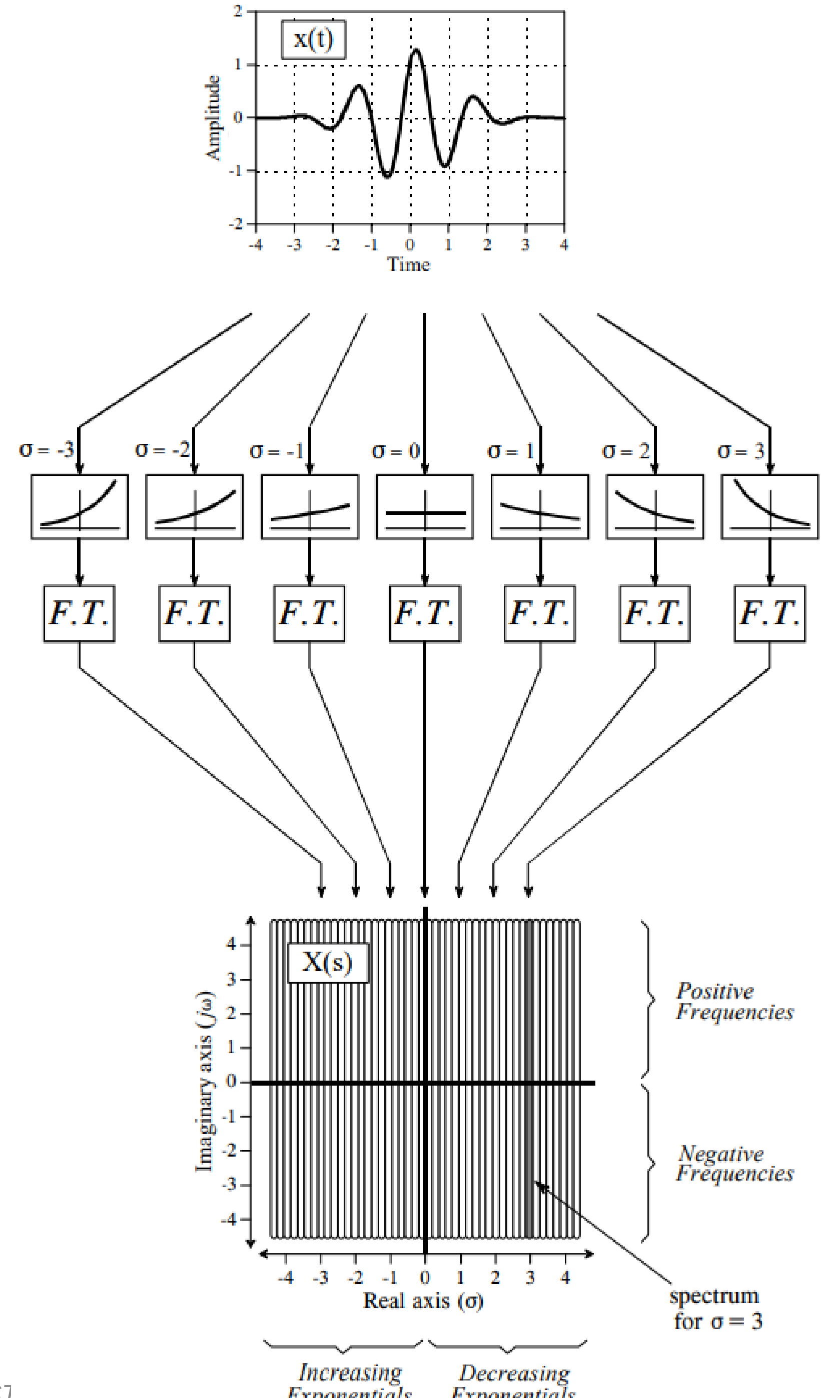


Laplace $$\hat{y}(s) = \int_{0}^{\infty} y(t) e^{-st} dt, \quad s = \sigma + j\omega$$

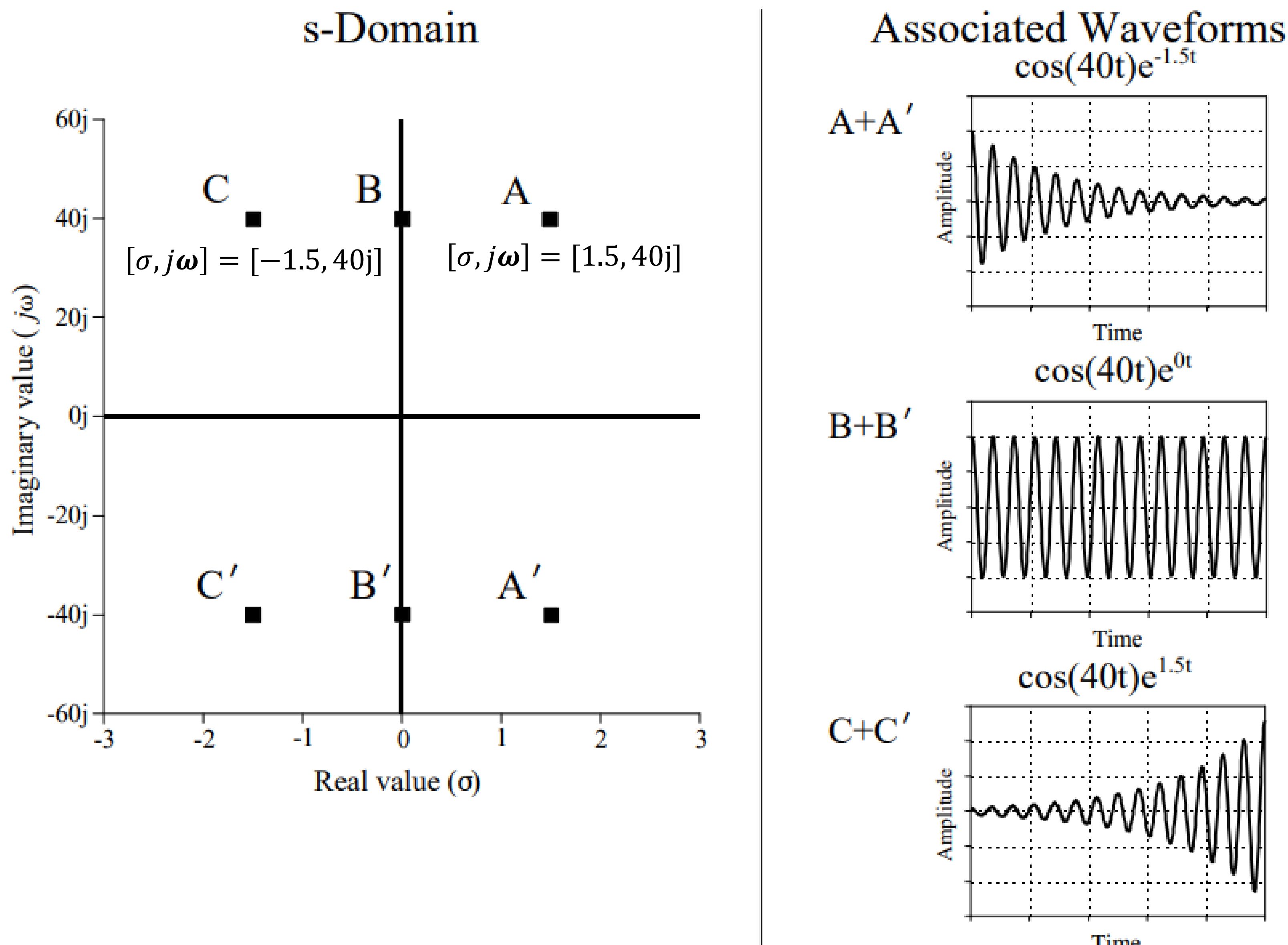
$$= \int_{0}^{\infty} \boxed{y(t) e^{-\sigma t}} e^{-j\omega t} dt$$

Which sinusoids and exponentials are present in a function?

Laplace Transform



Laplace Transform

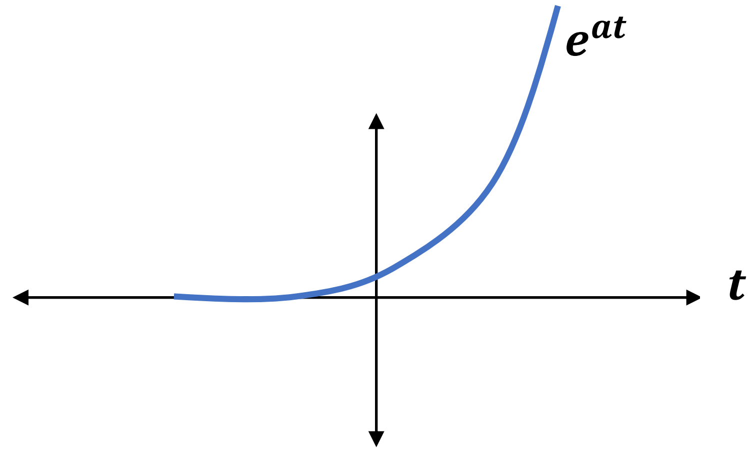


Laplace Transform

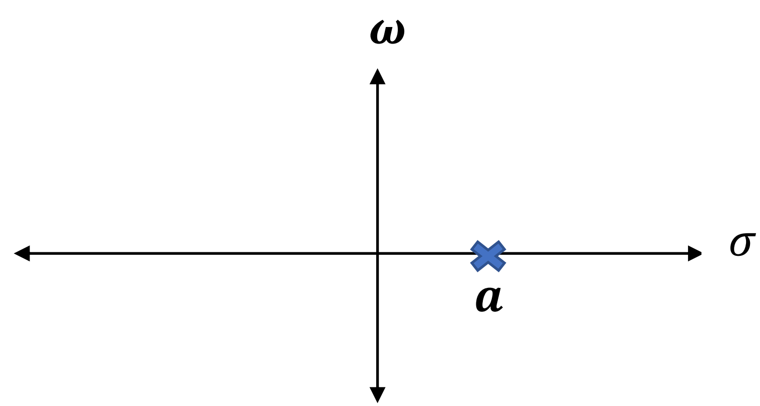
- **Example:**

if $f(t) = e^{at}$ find $L\{f(t)\}$.

$$
\begin{aligned}
L\{e^{at}\} &= \int_0^{\infty} e^{-st} \cdot e^{at} \\
&= \lim_{B \to \infty} \int_0^{B} e^{-t(s-a)} dt \\
&= \lim_{B \to \infty} \left[\frac{e^{-t(s-a)}}{-(s-a)} \right]_0^B \\
&= \lim_{B \to \infty} \left(\frac{e^{-B(s-a)} - 1}{-(s-a)} \right) \\
&= \frac{1}{s-a} \qquad\qquad s > 0
\end{aligned}
$$



Pole-Zero Diagram



How about $f(t) = e^{-at}$?

How to Find Solution to the Linear System

- **Some Laplace Transform Pairs**

$f(t), t > 0$	$\hat{f}(s)$
$\delta(t)$ *unit impulse*	1
$\delta(t-T)$ *delayed impulse*	e^{-Ts}
e^{-at}	$\dfrac{1}{s+a}$
$\dfrac{1}{(n-1)!} t^{n-1} e^{-at}$ $n=1,2,3,\ldots$	$\dfrac{1}{(s+a)^n}$
$\dfrac{1}{b-a}(e^{-at} - e^{-bt})$	$\dfrac{1}{(s+a)(s+b)}$

How to Find Solution to the Linear System

- **Some Laplace Transform Pairs**

$f(t), t > 0$	$\hat{f}(s)$
$\frac{1}{a-b}(ae^{-at} - be^{-bt})$	$\frac{s}{(s+a)(s+b)}$
$\frac{1}{b-a}[(z_1 - a)e^{-at} - (z_1 - b)e^{-bt}]$	$\frac{s+z_1}{(s+a)(s+b)}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \phi)$	$\frac{s \sin\phi + \omega\cos\phi}{s^2+\omega^2}$
$\frac{1}{\omega}e^{-at} \sin \omega t$	$\frac{1}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

How to Find Solution to the Linear System

Some Laplace Transform Pairs

$f(t), t > 0$	$\hat{f}(s)$
$1(t)$ *unit step*	$\dfrac{1}{s}$
$1(t-T)$ *delayed step*	$\dfrac{1}{s}e^{-Ts}$
$1(t)-1(t-T)$ *rectangular pulse*	$\dfrac{1}{s}(1-e^{-Ts})$
$\dfrac{1}{a}(1-e^{-at})$	$\dfrac{1}{s(s+a)}$
$\dfrac{1}{\omega^2}(1-\cos\omega t)$	$\dfrac{1}{s(s^2+\omega^2)}$
t *unit ramp*	$\dfrac{1}{s^2}$
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$ $n=1,2,3,\ldots$

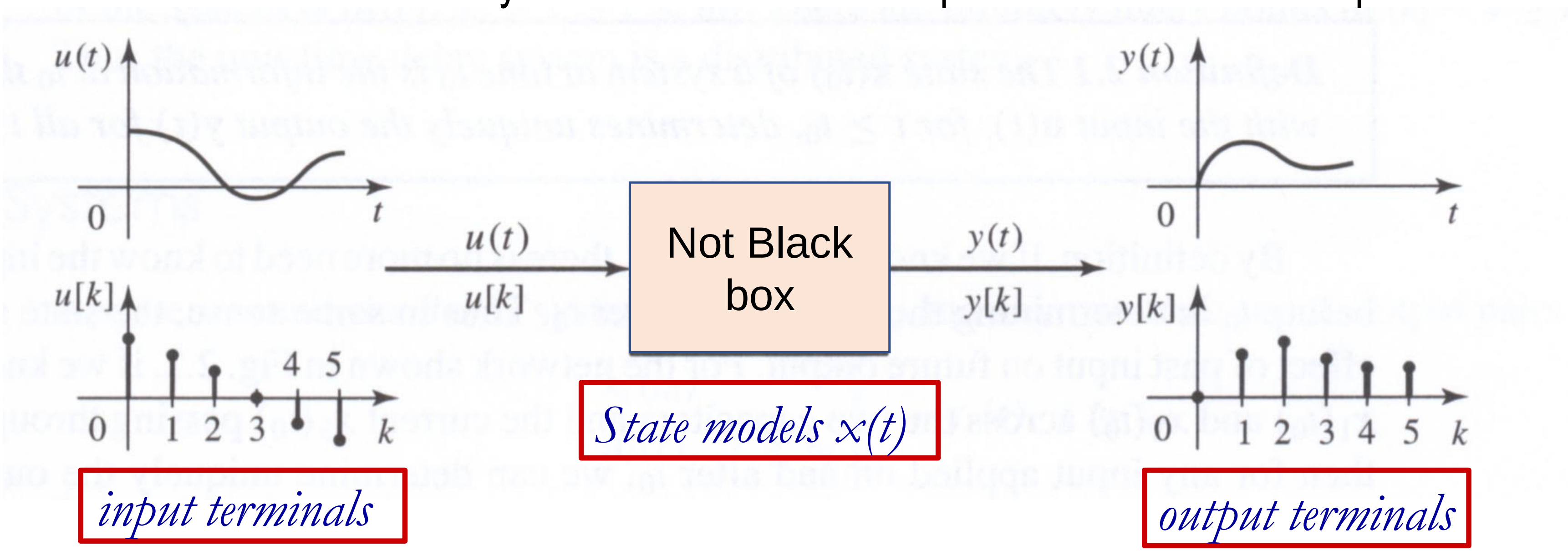
Quantitative analysis: Solutions of state-space models

Internal Description

▪ Internal Description: State-Space Description

→ State-space representation:

a mathematical model of a physical system as a set of input, output, and state variables related by first-order differential equations or difference equations



$$\dot{x}(t) = Ax(t) + Bu(t)$$ 1st order DE

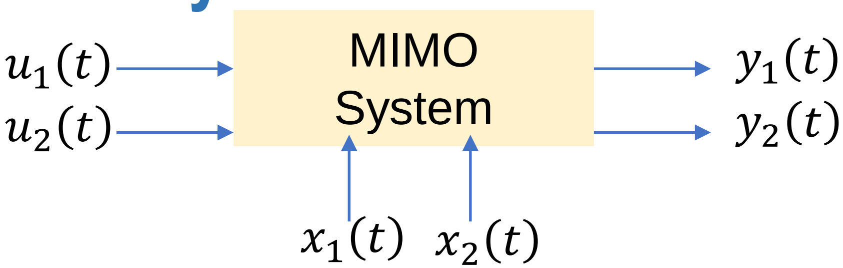
state

$$y(t) = Cx(t) + Du(t)$$ AE

output *input*

State-Space Model

▪ MIMO System



→ **Output Equation (Define the output of system based on input and current state)**:

$$y_1(t) = c_{11}x_1(t) + c_{11}x_2(t) + d_{11}u_1(t) + d_{12}u_2(t)$$

$$y_2(t) = c_{21}x_1(t) + c_{22}x_2(t) + d_{21}u_1(t) + d_{22}u_2(t)$$

$$\begin{bmatrix} y_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_1(t) \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_1(t) \end{bmatrix}$$

$$y(t) = Cx(t) + Du(t)$$

→ **State Equation (Define how the system's state change)**:

$$\frac{d(x_1(t))}{dt} = \dot{x_1}(t) = a_{11}x_1(t) + a_{11}x_2(t) + b_{11}u_1(t) + b_{12}u_2(t)$$

$$\frac{d(x_2(t))}{dt} = \dot{x_2}(t) = a_{21}x_1(t) + a_{22}x_2(t) + b_{21}u_1(t) + b_{22}u_2(t)$$

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_1(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_1(t) \end{bmatrix}$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

State-Space Solution

1. *Solutions* of *state equations*
2. How to transform *transfer functions* into *state equations*. $\rightarrow$ $\boxed{\textit{the realization problem}}$

Consider the linear time-invariant (LTI) state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where A, B, C, and D are, respectively, $n \times n$, $n \times p$, $q \times n$, and $q \times p$ constant matrices. The problem is to find the solution excited by the initial state $x(0)$ and the input $u(t)$.

State-Space Solution

State Equation

→ Linear system (Normal / dynamic system): $\dot{x}(t) = Ax(t) + Bu(t)$

→ Linear Homogeneous (zero-input) system: $\dot{x}(t) = Ax(t)$

State Transition Matrix in Linear System

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Bu(t)$$

$$\underbrace{e^{-At}\dot{x}(t) - e^{-At}Ax(t)}_{\text{Partial derivatives}} = e^{-At}Bu(t)$$

$$\frac{d(e^{-At}x(t))}{dt} = e^{-At}Bu(t)$$

the integral of a derivative is the function itself

$$\int_{\tau=0}^{t} \frac{d(e^{-A\tau}x(\tau))}{d\tau} d\tau = \int_{\tau=0}^{t} e^{-A\tau}Bu(\tau)\, d\tau$$

$$e^{-A\tau}x(\tau)|_{\tau=0}^{t} = \int_{\tau=0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$e^{-At}x(t) - e^{0}x(0) = \int_{\tau=0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$x(t) - e^{At}x(0) = \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$\boxed{\begin{aligned} x(t) &= e^{At}x(0) + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= Ce^{At}x(0) + C\int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{aligned}}$$

State Transition Matrix in Linear Homogeneous System

→ The homogeneous response has exponential form

$$\boxed{x(t) = \varphi(t)x(0) = e^{At}x(0)}$$

State-Space Solution via Laplace Transform

State Equation

→ Linear system (Normal / dynamic system): $\dot{x}(t) = Ax(t) + Bu(t)$

→ Linear Homogeneous (zero-input) system: $\dot{x}(t) = Ax(t)$

State Transition Matrix in Linear Homogeneous System

$$x(t) = \varphi(t)x(0)$$

→ Take Laplace Transform of $\dot{x}(t) = Ax(t)$

$$s\hat{x}(s) - \hat{x}(0) = A\hat{x}(s)$$

$$Is\hat{x}(s) - \hat{x}(0) = A\hat{x}(s)$$

$$\hat{x}(s)[sI - A] = \hat{x}(0)$$

$$\hat{x}(s) = [sI - A]^{-1}\hat{x}(0)$$

→ Inverse Laplace Transform

$$\boxed{x(t) = \mathcal{L}^{-1}[sI - A]^{-1}x(0)}$$

Only depend on initial state

State Transition Matrix in Linear System

$$\hat{x}(s) = (sI - A)^{-1}[x(0) + B\hat{u}(s)]$$

$$\hat{y}(s) = C(sI - A)^{-1}[x(0) + B\hat{u}(s)] + D\hat{u}(s)$$

$$\boxed{\begin{aligned} x(t) &= \mathcal{L}^{-1}[sI - A]^{-1}x(0) + \mathcal{L}^{-1}\{[sI - A]^{-1}B\hat{u}(s)\} \\ y(t) &= C\mathcal{L}^{-1}[sI - A]^{-1}x(0) + C\mathcal{L}^{-1}\{[sI - A]^{-1}B\hat{u}(s)\} + \mathcal{L}^{-1}\{D\hat{u}(s)\} \end{aligned}}$$

State-Space Solution via Laplace Transform

- **Transfer Function**

$$\hat{y}(s) = C(sI - A)^{-1}[x(0) + B\hat{u}(s)] + D\hat{u}(s)$$

$$\hat{G}(s) = \frac{y(s)}{\hat{u}(s)} = \frac{C(sI - A)^{-1}x(0) + C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s)}{\hat{u}(s)}$$

$$= \frac{1}{\hat{u}(s)}C(sI - A)^{-1}x(0) + C(sI - A)^{-1}B + D$$

- **Transfer Function when Zero-state Response**

$$\hat{G}(s) = \frac{y(s)}{\hat{u}(s)} = \frac{C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s)}{\hat{u}(s)}$$

$$= C(sI - A)^{-1}B + D$$

Solving
State Transition Matrix

State Transition Matrix

- **State Equation**

  → Classical Equation $$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- **Key: Find solution for State Transition Matrix** e^{At}

→Solution:

1. Using Theorem 5
2. Using diagonalization $A = Q\hat{A}Q^{-1}$. Then e^{At} = $Qe^{\hat{A}t}Q^{-1}$
3. Using infinite power series

Example Solution 1

- **Find Solution where** $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
- **Solution 1:** Using Theorem 5

Characteristic Polynomial

$$\det(A - \lambda I) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 1 & -2-\lambda \end{bmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$$

Repeated eigenvalues $\lambda_1 = \lambda_2 = -1$

Define polynomial $h(\lambda) = \beta_0 + \beta_1 \lambda + \cdots + \beta_{n-1}\lambda^{n-1}$

$$h(\lambda) = \beta_0 + \beta_1 \lambda$$

Solving equation

$$f(A) = e^{At} \rightarrow f(\lambda) = e^{\lambda t}$$

$$f(\lambda_i) = h(\lambda_i) \rightarrow f(\lambda_1 = -1) = e^{-t} = \beta_0 - \beta_1$$

$$f'(\lambda_1 = -1) = -te^{-t} = -\beta_1$$

$$\beta_1 = te^{-t} \qquad \beta_0 = e^{-t} + te^{-t}$$

$$f(A) = h(\boldsymbol{A}) = \beta_0 I + \beta_1 \boldsymbol{A}$$

$$e^{At} = \begin{bmatrix} e^{-t} + te^{-t} & 0 \\ 0 & e^{-t} + te^{-t} \end{bmatrix} + \begin{bmatrix} 0 & -te^{-t} \\ te^{-t} & -2te^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Example Solution 2

- **Find Solution where** $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
- **Solution 2:** Using diagonalization $A = Q\hat{A}Q^{-1}$. Then $e^{At} = Qe^{\hat{A}t}Q^{-1}$

Using Jordan form of A: Let $A = Q\hat{A}Q^{-1}$; then $e^{At} = Qe^{\hat{A}t}Q^{-1}$, where $\hat{A}$ is in Jordan form and $e^{\hat{A}t}$ can be readily be obtained by using

$$\hat{A} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

$$e^{\hat{A}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! & t^3 e^{\lambda_1 t}/3! \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! \\ 0 & 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & e^{\lambda_1 t} \end{bmatrix}.$$

Example Solution 2

- **Find Solution where** $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
- **Solution 2:** Using diagonalization $A = Q\hat{A}Q^{-1}$. Then $e^{At} = Qe^{\hat{A}t}Q^{-1}$

$$e^{\hat{A}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{bmatrix}$$

Repeated eigenvalues $\lambda_1 = \lambda_2 = -1$

$$(A - \lambda I)q_1 = \left(\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) q_1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} q_1 = 0 \longrightarrow q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda I)q_2 = \left(\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) q_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} q_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow q_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e^{At} = Qe^{\hat{A}t}Q^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Example Solution 3

- **Find Solution where** $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
- **Solution 3:** Using infinite power series

$$e^{At} = I + At + \frac{t^2}{2!}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k$$

$$\begin{aligned}
e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} t + \frac{t^2}{2!} \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & -2t \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} + \cdots \\
&= \begin{bmatrix} 1 - 0t - \frac{t^2}{2} - \cdots & 0 - t + \frac{2t^2}{2} + \cdots \\ 0 + t - \frac{2t^2}{2} + \cdots & 1 - 2t + \frac{3t^2}{2} + \cdots \end{bmatrix} \\
&= \begin{bmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{bmatrix}
\end{aligned}$$

$$\boxed{\begin{aligned}
e^{-t} &= 1 - t + \frac{t^2}{2!} + \cdots \\
-e^{-t} &= 0 - 1 + \frac{2t}{2!} + \cdots \\
-te^{-t} &= 0 - t + \frac{2t^2}{2!} + \cdots \\
te^{-t} &= 0 + t - \frac{2t^2}{2!} + \cdots
\end{aligned}}$$

State Transition Matrix via Laplace Transform

- **State Equation**

→ Classical Equation $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

→ Via Laplace Transform $x(t) = \mathcal{L}^{-1}[sI - A]^{-1}x(0) + \mathcal{L}^{-1}\{ [sI - A]^{-1}B\hat{u}(s)\}$

- **Key: Find solution for State Transition Matrix**

$$e^{At} = \mathcal{L}^{-1}[sI - A]^{-1}$$

→Solution:

1. Taking the inverse of $(sI - A)$
2. Using Theorem 5
3. Using diagonalization $(sI - A)^{-1} = Q\left(sI - \hat{A}\right)^{-1}Q^{-1}$
4. Using infinite power series
5. Using the Leverrier algorithm

Example Solution 1

- **Find Solution where** $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
- **Solution 1:** Taking the inverse of $(sI - A)$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2 + 2s + 1} \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}$$

$$= \frac{1}{(s+1)^2} \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

$$e^{At} = L^{-1}[sI - A]^{-1} = \begin{bmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Example Solution 2

- **Find Solution where** $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
- **Solution 2:** Using Theorem 5

Characteristic Polynomial $\Delta\lambda = \prod_{i=1}^{m} (\lambda - \lambda_i)^{n_i} = (\lambda + 1)^2$

Repeated eigenvalues $\lambda_1 = \lambda_2 = -1$

Define polynomial $h(\lambda) = \beta_0 + \beta_1 \lambda + \cdots + \beta_{n-1} \lambda^{n-1}$

$$h(\lambda) = \beta_0 + \beta_1 \lambda$$

Solving equation

$$f(A) = (sI - A)^{-1} \rightarrow f(\lambda) = (s - \lambda)^{-1}$$

$$f(\lambda_i) = h(\lambda_i) \rightarrow f(-1) = (s+1)^{-1} = \beta_0 - \beta_1$$

$$f'(-1) = (s+1)^{-2} = \beta_1$$

$$h(\lambda) = \beta_0 + \beta_1 \lambda$$

$$h(\lambda) = [(s+1)^{-1} + (s+1)^{-2}] + (s+1)^{-2} \lambda$$

$$(sI - A)^{-1} = h(A) = [(s+1)^{-1} + (s+1)^{-2}]\boldsymbol{I} + (s+1)^{-2} A$$

$$= \begin{bmatrix} \frac{s+2}{(s+1)^2} & 0 \\ 0 & \frac{s+2}{(s+1)^2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{-2}{(s+1)^2} \end{bmatrix} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Example Solution 5

- **Find Solution where** $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
- **Solution 5:** Using the Leverrier algorithm

$$(sI - A)^{-1} := \frac{1}{\Delta(s)}\left[R_0 s^{n-1} + R_1 s^{n-2} + \cdots + R_{n-2}s + R_{n-1}\right]$$

$$\Delta(s) := \det(sI - A) := s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

$$\alpha_1 = -\frac{tr(AR_0)}{1}, \quad R_0 = I$$

$$\alpha_2 = -\frac{tr(AR_1)}{2}, \quad R_1 = AR_0 + \alpha_1 I = A + \alpha_1 I$$

$$\alpha_3 = -\frac{tr(AR_2)}{3}, \quad R_2 = AR_1 + \alpha_2 I = A^2 + \alpha_1 A + \alpha_2 I$$

$$\vdots$$

$$\alpha_{n-1} = -\frac{tr(AR_{n-2})}{n-1}, \quad R_{n-1} = AR_{n-2} + \alpha_{n-1} I = A^{n-1} + \alpha_1 A^{n-2} + \cdots + \alpha_{n-2} A + \alpha_{n-1} I$$

$$\alpha_n = -\frac{tr(AR_{n-1})}{n}, \quad 0 = AR_{n-1} + \alpha_n I$$

Example

- **Find Solution where** $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
- **Solution 5:** Using the Leverrier algorithm

$$(sI - A)^{-1} = \frac{1}{\Delta(s)}[R_0 s^{n-1} + R_1 s^{n-2} + \cdots + R_{n-2}s + R_{n-1}]$$

$$\Delta(s) = det(sI - A) = s^2 + 2s + 1$$

$$R_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \alpha_1 = -\frac{tr(AR_0)}{1} = -\frac{tr(A)}{1} = 2$$

$$R_1 = A + \alpha_1 I = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{\Delta(s)}[R_0 s + R_1]$$

$$= \frac{1}{s^2 + 2s + 1}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

$$= \frac{1}{(s+1)^2}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-t} + te^{-t} & -te^{-t} \\ te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Homework #4.1 State Transition Matrix (1 pt.): Due Dec. 26

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Calculate state transition matrix:

$$\varphi(s) = \ [sI - A]^{-1}$$

$$\varphi(t) = \ \mathcal{L}^{-1}[sI - A]^{-1}$$

Choose 2 methods that you like.

Discretization

Discretization

- **Continuous-time state equation**

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

- **Discrete-time state equation**

$$\dot{x}(t) = \lim_{T \to 0} \frac{x(t+T) - x(t)}{T} \longrightarrow \dot{x}(t)T = x(t+T) - x(t)$$

$$x(t+T) = x(t) + \dot{x}(t)T$$

$$= x(t) + Ax(t)T + Bu(t)T$$

$$\boxed{x(t+T) = x(t) + Ax(t)T + Bu(t)T}$$

Discretization

If we compute $x(t)$ and $y(t)$ only at $t = kT$ for $k = 0, 1, \ldots,$ then

$$\boxed{\begin{aligned} x\big((k+1)T\big) &= (I + TA)x(kT) + TBu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \end{aligned}}$$

This is a discrete-time state-space equation.

1. *Can easily be computed on a digital computer*
2. *The easiest to carry out*
3. *Yields the least accurate results for the same* T

State-Space Solution

If an input $u(t)$ is generated by a digital computer followed by a DA converter, then $u(t)$ will be piecewise constant.

$$u(t) = u(kT) =: u[k] \qquad \text{for } \; kT \le t < (k+1)T$$

$k = 0, 1, 2, \ldots.$

This input changes values only at discrete-time instants.

$$x[k] := x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau$$

$$x[k+1] := x\big((k+1)T\big) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

State-Space Solution

$$x[k+1] = e^{AT}\left[e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau\right]$$

$$+\int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}Bu(\tau)d\tau$$

$\alpha := kT + T - \tau,$ *: Introducing the new variable*

$$x[k+1] = e^{AT}x[k] + \left(\int_0^{T} e^{A\alpha}d\alpha\right)Bu[k]$$

Discrete-time State-Space Equation

Thus, if an input changes value only at discrete-time instants kT and if we compute only the responses at $t = kT$, then

$$\boxed{\begin{aligned} x[k+1] &= A_d x[k] + B_d u[k] \\ y[k] &= C_d x[k] + D_d u[k] \\ \\ A_d = e^{AT},\ \ B_d &= \left(\int_0^T e^{A\tau} d\tau \right) B,\ \ C_d = C,\ \ D_d = D \end{aligned}}$$

This is a discrete-time state-space equation.

Note that there is no approximation involved in this derivation.

Discrete-time State-Space Equation

Consider the discrete-time state-space equation

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

A general form of solutions:

$$x[1] = Ax[0] + Bu[0]$$
$$x[2] = Ax[1] + Bu[1] = A^2 x[0] + ABu[0] + Bu[1]$$

for $k > 0$,

$$\boxed{\begin{aligned} x[k] &= A^k x[0] + \sum_{m=0}^{k-1} A^{k-1-m} Bu[m] \\ y[k] &= CA^k x[0] + \sum_{m=0}^{k-1} CA^{k-1-m} Bu[m] + Du[k] \end{aligned}}$$

Equivalent State Equation

Equivalent State Equation

Consider the n-dimensional state equation

$$\boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}}$$

This state equation can be considered to be associated with the orthonormal basis. <u>We study the effect on the equation by choosing a different basis.</u>

Definition: Let P be an $n \times n$ real nonsingular matrix and let $\bar{x} = Px$. Then the state equation

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}u(t) \\ y(t) &= \bar{C}\bar{x}(t) + \bar{D}u(t) \end{aligned} \qquad \boxed{\begin{aligned} \bar{A} &= PAP^{-1} & \bar{B} &= PB \\ \bar{C} &= CP^{-1} & \bar{D} &= D \end{aligned}}$$

is said to be (algebraically) equivalent to the above state equation and $\bar{x} = Px$ is called an equivalence transformation.

Determinant

Square matrices of order n

$$AB \neq BA$$

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

the determinant of the product is equal to the product of the determinants.

the commutative law of multiplication applies.

Characteristic Polynomial

$$\bar{\Delta}(\lambda) = \det(\lambda I - \bar{A}) = \det(\lambda P P^{-1} - P A P^{-1}) = \det\left[P(\lambda I - A)P^{-1}\right]$$
$$= \det(P)\det(\lambda I - A)\det(P^{-1}) = \det(\lambda I - A) = \Delta(\lambda)$$

$$\hat{\bar{G}}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = CP^{-1}\left[P(sI - A)P^{-1}\right]^{-1} PB + D$$
$$= CP^{-1}P(sI - A)^{-1}P^{-1}PB + D = C(sI - A)^{-1}B + D = \hat{G}(s)$$

Thus equivalent state equations have the same characteristic polynomial and, consequently, the same set of eigenvalues and the same transfer matrix.

Zero-State Equivalent

Two state equations are said to be *zero-state equivalent* if they have the same transfer matrix or

$$D + C(sI - A)^{-1}B = \overline{D} + \overline{C}(sI - \overline{A})^{-1}\overline{B}$$

$$s^{-1}\sum_{k=0}^{\infty}(s^{-1}A)^k = s^{-1}I + s^{-2}A + s^{-3}A^2 + \cdots$$

$$= s^{-1}(I - s^{-1}A)^{-1} = [s(I - s^{-1}A)]^{-1} = (sI - A)^{-1}$$

$$D + CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \cdots =$$

$$\overline{D} + \overline{C}\overline{B}s^{-1} + \overline{C}\overline{A}\overline{B}s^{-2} + \overline{C}\overline{A}^2\overline{B}s^{-3} + \cdots$$

Zero-State Equivalent

Theorem 10

Two linear time-invariant state equations $\{A, B, C, D\}$ and $\{\overline{A}, \overline{B}, \overline{C}, \overline{D}\}$ are zero-state equivalent or have the same transfer matrix if and only if

$$D = \overline{D} \quad \text{and}$$

$$CA^m B = \overline{C}\,\overline{A}^m \overline{B} \qquad m = 0,\ 1,\ 2,\ \ldots$$

In order for two state equations to be equivalent, they must have the same dimension. This is not the case for zero-state equivalence.

Example:

$\dot{x}(t) = Ax(t) + Bu(t)$ $\longrightarrow$ $\dot{x}(t) = \begin{bmatrix} -0.1 & 2 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 10 \\ 0.1 \end{bmatrix} u(t)$

$y(t) = Cx(t) + Du(t)$ $\longrightarrow$ $y(t) = \begin{bmatrix} 0.2 & -1 \end{bmatrix} x(t)$

Let us introduce new state variables:

$$\bar{x}_1(t) = \frac{20}{100} x_1(t) = 0.2\, x_1(t) \longrightarrow \bar{x}_2(t) = \frac{20}{0.1} x_2(t) = 200\, x_2(t)$$

$$\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ 0 & 200 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \longrightarrow P = \begin{bmatrix} 0.2 & 0 \\ 0 & 200 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 5 & 0 \\ 0 & 0.005 \end{bmatrix}$$

$$\bar{x} = Px$$

Equivalent state equation:

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) = PAP^{-1}\bar{x}(t) + PBu(t)$$

$$\dot{\bar{x}}(t) = \begin{bmatrix} 0.2 & 0 \\ 0 & 200 \end{bmatrix} \begin{bmatrix} -0.1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0.005 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0.2 & 0 \\ 0 & 200 \end{bmatrix} \begin{bmatrix} 10 \\ 0.1 \end{bmatrix} u(t)$$

$$\dot{\bar{x}}(t) = \begin{bmatrix} -0.1 & 0.002 \\ 0 & -1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 2 \\ 20 \end{bmatrix} u(t)$$

$$y(t) = \bar{C}\bar{x}(t) = CP^{-1}\bar{x}(t) = \begin{bmatrix} 0.2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0.005 \end{bmatrix} \bar{x}(t) = \begin{bmatrix} 1 & -0.005 \end{bmatrix} x(t)$$

Block Diagram

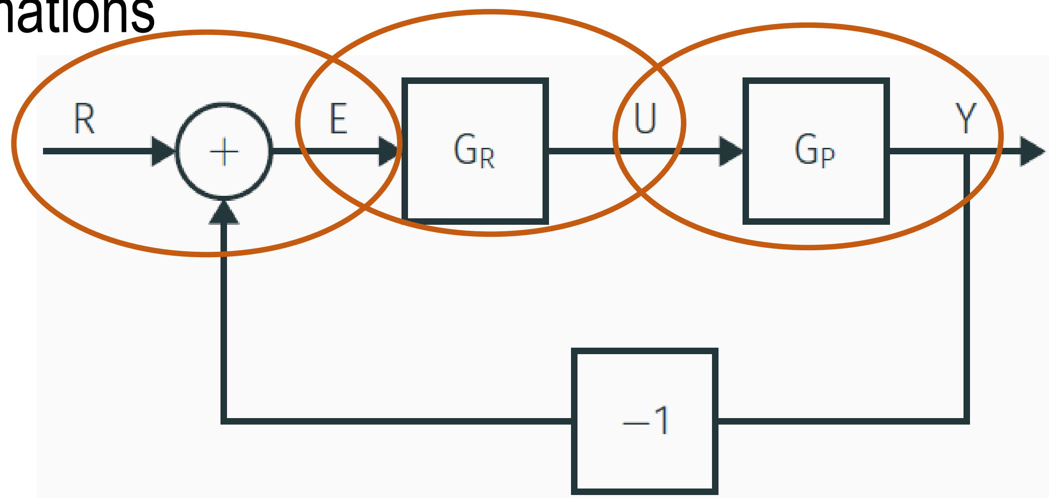
Block Diagram

- **Block Diagram consists of**

  $\longrightarrow$ Blocks - Transfer functions

  $\longrightarrow$ Arrows - Signals

  $\longrightarrow$ Summations



$$Y = G_P U, \quad U = G_R E, \quad E = R - Y$$

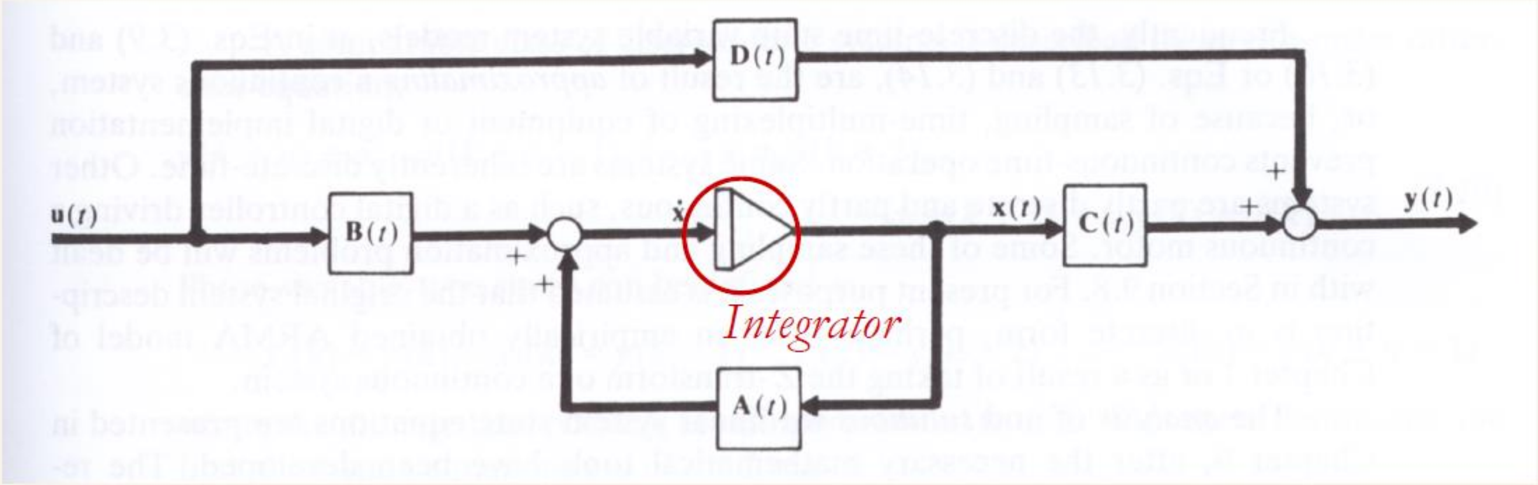
$$\boxed{Y = \frac{G_P G_R}{1 + G_P G_R} R}$$

Continuous-time Linear System

Continuous-time Linear System: Block Diagram Form

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

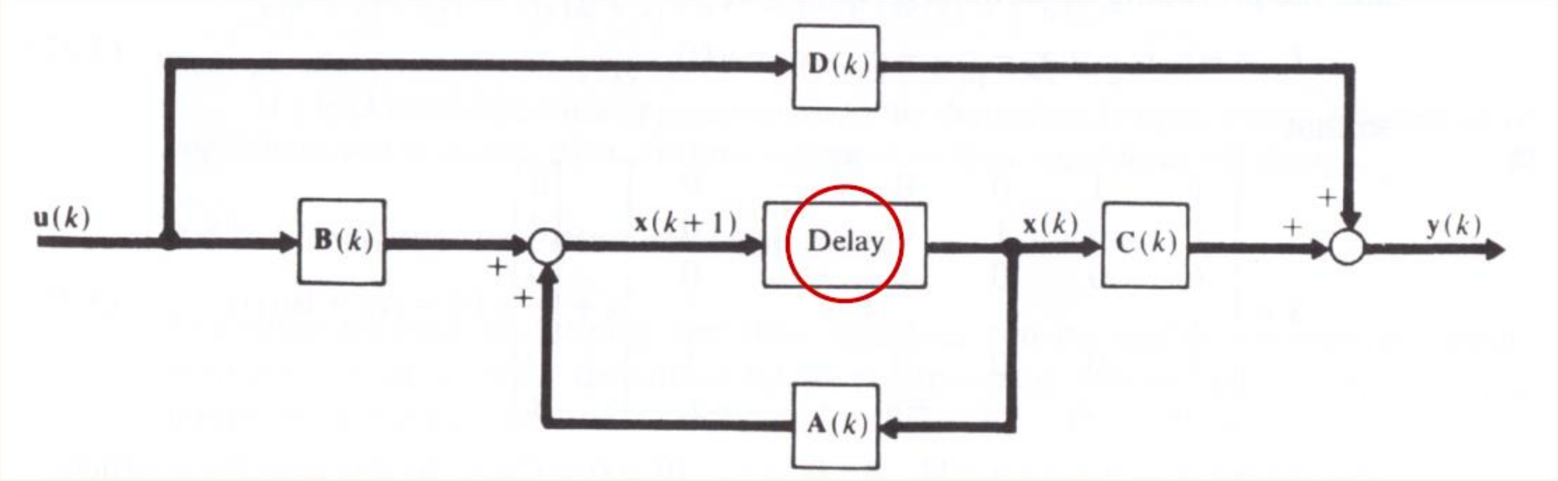
$$y(t) = C(t)x(t) + D(t)u(t)$$



Discrete-time Linear System

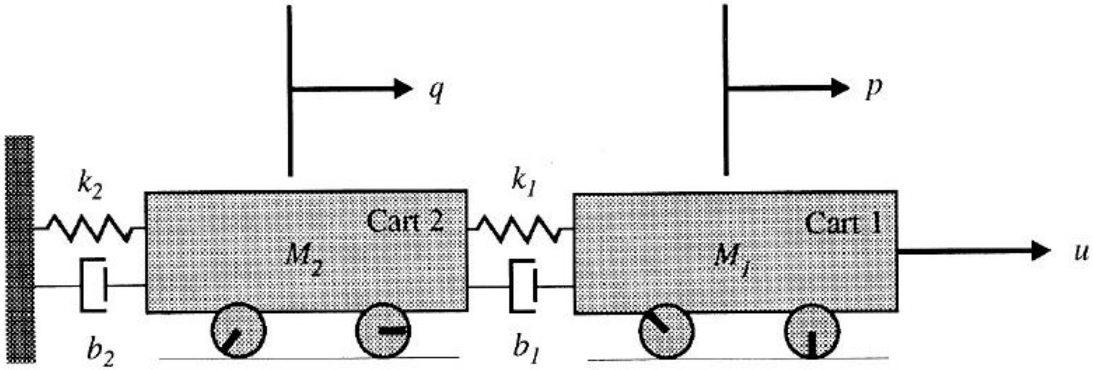
Discrete-time Linear System: Block Diagonal Form

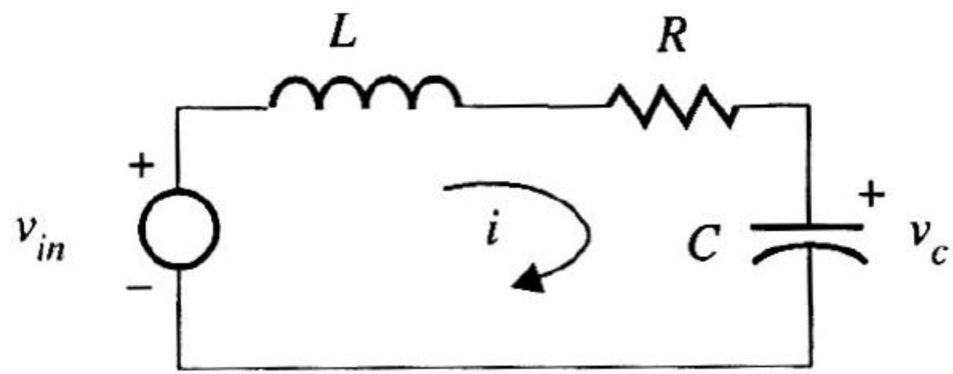
$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$



Modeling Mechanical & Electrical Systems

Mechanical-Electrical Equivalency

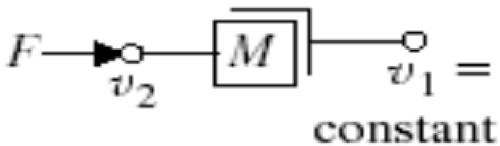




$$\left\{\begin{array}{l}\text{voltage}\\ \text{velocity}\\ \text{pressure}\end{array}\right\} \text{ and } \left\{\begin{array}{l}\text{current}\\ \text{force}\\ \text{volume flow rate}\end{array}\right\} \text{ in the } \left\{\begin{array}{l}\text{electrical}\\ \text{mechanical}\\ \text{fluidic}\end{array}\right\} \text{ domains.}$$

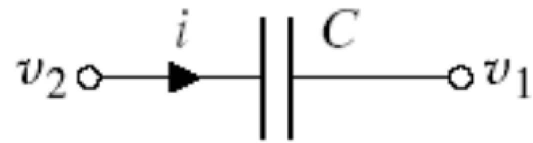
Mechanical-Electrical Equivalency

- **Translational Mass – Electrical Capacitance**



$$F = M\frac{dv_{12}}{dt}$$

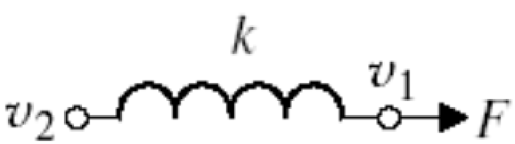
$$\frac{dv_{12}}{dt} = \frac{F}{M}$$



$$i = C\frac{dv_{12}}{dt}$$

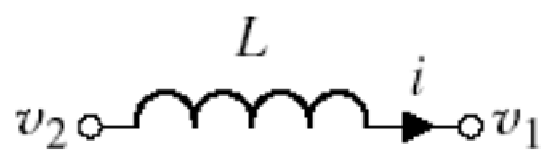
$$\frac{dv_{12}}{dt} = \frac{i}{C}$$

- **Translational Spring – Electrical Inductance**



$$v_{12} = \frac{1}{k}\frac{dF}{dt}$$

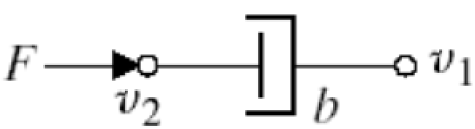
$$\frac{dF}{dt} = kv_{12}$$



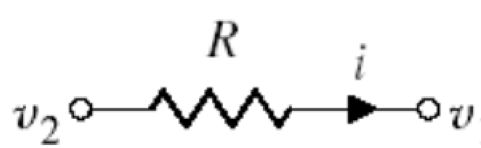
$$v_{12} = L\frac{di}{dt}$$

$$\frac{di}{dt} = \frac{1}{L}v_{12}$$

- **Translational Damper – Electrical Resistance**



$$F = bv_{12}$$



$$i = \frac{v_{12}}{R}$$

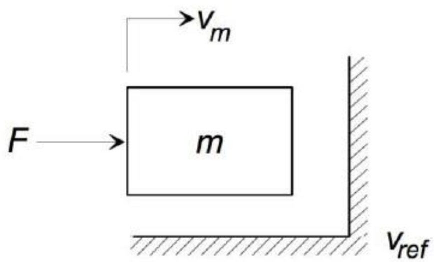
State-Space Model

▪ State Variables

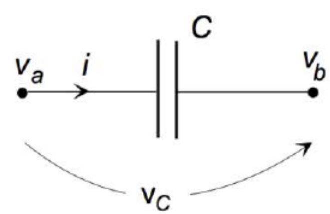
# of state variables = # of <u>independent</u> energy storage elements

▪ Energy Storage Elements

→ **Mass and Capacitor**: Stored energy as a function of Across-variable

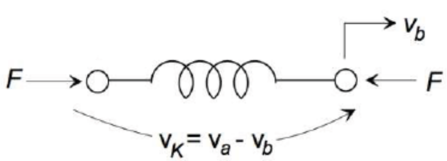


$$\begin{aligned} F &= m\frac{dv}{dt} \\ E &= \int_{-\infty}^{t} vF\,\mathrm{d}t = \int_{0}^{t} mv\,\mathrm{d}v \\ &= \frac{1}{2}mv^2 \end{aligned}$$

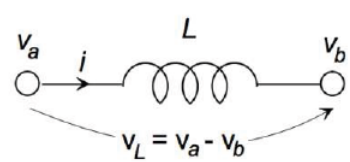


$$\begin{aligned} i &= C\frac{dv}{dt} \\ E &= \int_{-\infty}^{t} vi\,\mathrm{d}t = \int_{0}^{t} Cv\,\mathrm{d}v \\ &= \frac{1}{2}Cv^2 \end{aligned}$$

→ **Spring and Inductor**: Stored energy as a function of Through-variable



$$\begin{aligned} v &= \frac{1}{K}\frac{dF}{dt} \\ E &= \int_{-\infty}^{t} vF\,\mathrm{d}t = \frac{1}{K}\int_{0}^{t} F\,\mathrm{d}F \\ &= \frac{1}{2K}F^2 \end{aligned}$$



$$\begin{aligned} v &= L\frac{di}{dt} \\ E &= \int_{-\infty}^{t} vi\,\mathrm{d}t = \int_{0}^{t} Li\,\mathrm{d}i \\ &= \frac{1}{2}Li^2 \end{aligned}$$

→ **Damper and Resistor:** do not store energy, they dissipate energy

State-Space Equation for Electrical Systems

▪ State Variables for Electrical Systems

→ Assign all capacitor voltages and all inductor currents as **state variables**

→ If the capacitor voltage is assigned as x_i then its current is $C\dot{x}_i$

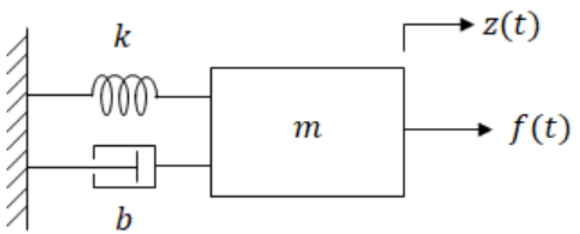
→ If the inductor current is assigned as x_j then its voltage is $L\dot{x}_j$

→ Use Kirchoff's current or voltage law to express the current or voltage in terms of state variables

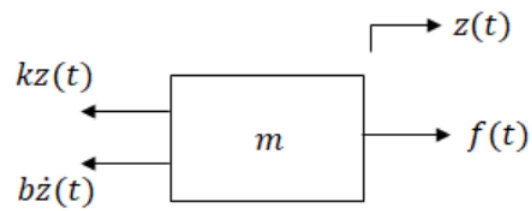
▪ State Variables for Mechanical Systems

→ When there is a mass, its position & velocity are chosen as **state variables**

→ Use Newton's 2nd Law to express mass and acceleration in terms of state variables



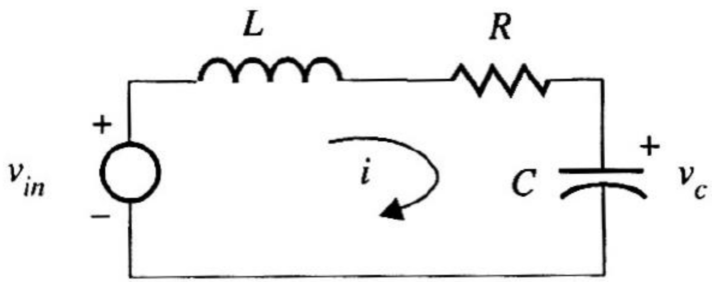
(input)	$f(t)$	=	external force applied on mass	
(output)	$z(t)$	=	position	unknown
	m	=	mass	
	k	=	spring constant	
	b	=	damping coefficient	



Newton's 2nd law:

$$\boxed{f(t) - b\frac{dz}{dt} - kz(t) = m\frac{d^2z}{dt^2}}$$

Example: Electrical System



→ **Kirchoff's voltage law:**

$$v_{in} = v_L + v_R + v_C = L\frac{di}{dt} + Ri + v_C$$

→ **Energy storage elements**

$$x_1 = v_C \longrightarrow i = C\dot{x}_1 \longrightarrow \dot{x}_1 = \frac{i}{C} = \frac{1}{C}x_2$$

$$x_2 = i \longrightarrow v_L = L\dot{x}_2 \longrightarrow \dot{x}_2 = \frac{di}{dt} = \frac{1}{L}(v_{in} - v_C - Ri) = \frac{1}{L}v_{in} - \frac{1}{L}x_1 - \frac{R}{L}x_2$$

→ **State equation**

$$\dot{x} = Ax + Bu \longrightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_C \\ i \end{bmatrix} \quad A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \quad u = v_{in}$$

$$\dot{x} = \begin{bmatrix} \frac{dv_C}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v_C \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_{in}$$

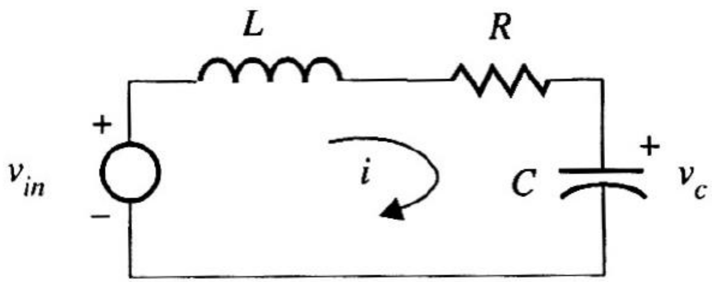
→ **Output equation for v_C**

$$y = v_C = Cx + Du = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i \end{bmatrix} + [0]v_{in}$$

→ **Output equation for *i***

$$y = i = Cx + Du = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_C \\ i \end{bmatrix} + [0]v_{in}$$

Example: Electrical System



→ **Kirchoff's voltage law:**

$$v_{in} = L\frac{di}{dt} + Ri + v_C$$

→ **Transfer Function when output is v_C**

$$\frac{\hat{v}_C(s)}{\hat{v}_{in}(s)} = \hat{G}(s) = C(sI - A)^{-1}B + D \longrightarrow (sI - A) = \begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{R}{L} \end{bmatrix}$$

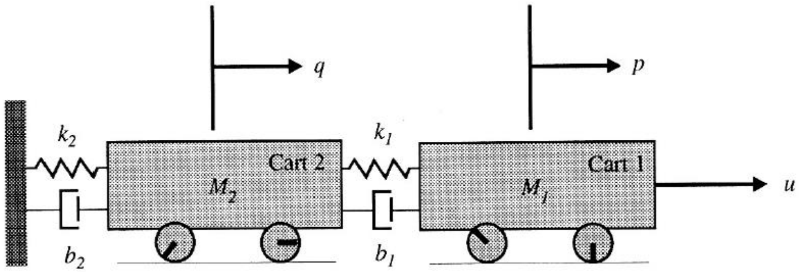
$$(sI - A)^{-1} = \frac{1}{s\left(s+\frac{R}{L}\right)+\frac{1\,1}{CL}} \begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix} = \frac{1}{s^2+\left(\frac{R}{L}\right)s+\frac{1}{LC}} \begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$

$$\hat{G}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\frac{1}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}} \begin{bmatrix} s + \frac{R}{L} & \frac{1}{C} \\ -\frac{1}{L} & s \end{bmatrix} \right) \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} + [0] = \frac{1}{LC} \frac{1}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}}$$

→ **Transfer Function when output is i**

$$\hat{G}(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\frac{1}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}} \begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix} \right) \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} + [0] = \frac{1}{L} \frac{s}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}}$$

Example: Mechanical System



→ **Newton's second law:**

Sum of forced equals mass multiplied by acceleration

$$M_1\ddot{p} = u - k_1(p-q) - b_1(\dot{p}-\dot{q})$$
$$M_2\ddot{q} = k_1(p-q) + b_1(\dot{p}-\dot{q}) - k_2 q - b_2\dot{q}$$

→ **Energy storage elements**

$$x_1 = p \longrightarrow \dot{x_1} = \dot{p} \qquad x_3 = \dot{p} \longrightarrow \dot{x_3} = \ddot{p} = -\frac{k_1}{M_1}x_1 + \frac{k_1}{M_1}x_2 - \frac{b_1}{M_1}x_3 + \frac{b_1}{M_1}x_4 + \frac{1}{M_1}u$$

$$x_2 = q \longrightarrow \dot{x_2} = \dot{q} \qquad x_4 = \dot{q} \longrightarrow \dot{x_4} = \ddot{q} = \frac{k_1}{M_2}x_1 - \frac{(k_1+k_2)}{M_2}x_2 + \frac{b_1}{M_2}x_3 - \frac{(b_1+b_2)}{M_2}x_4$$

→ **State equation**

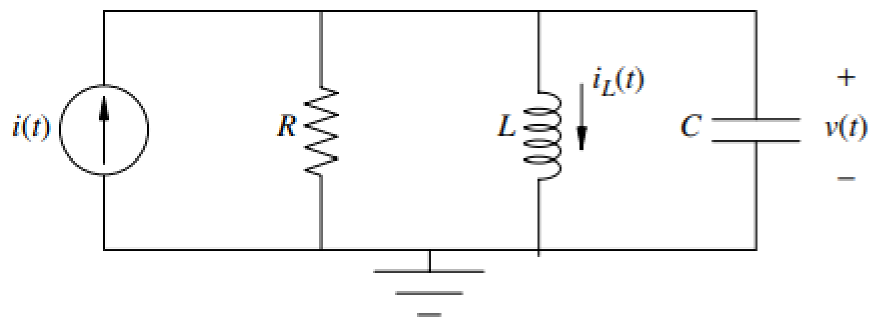
$$\dot{x} = Ax + Bu \longrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p \\ q \\ \dot{p} \\ \dot{q} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \ddot{p} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & -\frac{b_1}{M_1} & \frac{b_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{(k_1+k_2)}{M_2} & \frac{b_1}{M_2} & -\frac{(b_1+b_2)}{M_2} \end{bmatrix} \begin{bmatrix} p \\ q \\ \dot{p} \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix} u$$

→ **Output equation for *p***

$$y = p = Cx + Du = [1 \quad 0 \quad 0 \quad 0]x$$

Homework #4.2 Solution for Electrical System (1 pt.): Due Dec. 26



→ Write the state-space equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

→ Define A, B, C, D

Qualitative analysis: Stability

Stability of Linear Systems

Stability: a basic requirement

If a system is not stable, the system may *burn out*, *disintegrate*, or *saturate* when a signal, no matter how small, is applied.

Why Stability?

1. In many cases, we wish to be able to determine whether a given system will operate properly (in only a very rough sense) without obtaining a detailed solution of the system equation.
2. Even more important, since we could obtain solutions for only a few typical inputs, we might wish to know if there is *any input which could cause improper behavior.*

Stability of Linear Systems

▪ Total Response of Linear System

Total response = zero-input response + zero-state response

▪ Input-Output Stability: Zero-state Response

$\rightarrow$ Response due to the input only (initial state condition is zero)

$\rightarrow$ **BIBO (bounded-input bounded-output) stability**

Whether the output $y(t)$ stays bounded (not infinity) for a given bounded input $u(t)$

▪ Internal Stability: Zero-input Response

$\rightarrow$ Response due to initial state condition only (there is no other input)

$\rightarrow$ **Marginal and asymptotic stabilities**

The $x(t)$ stays bounded or go to zero

Input-Output Stability
(Zero-state Response)

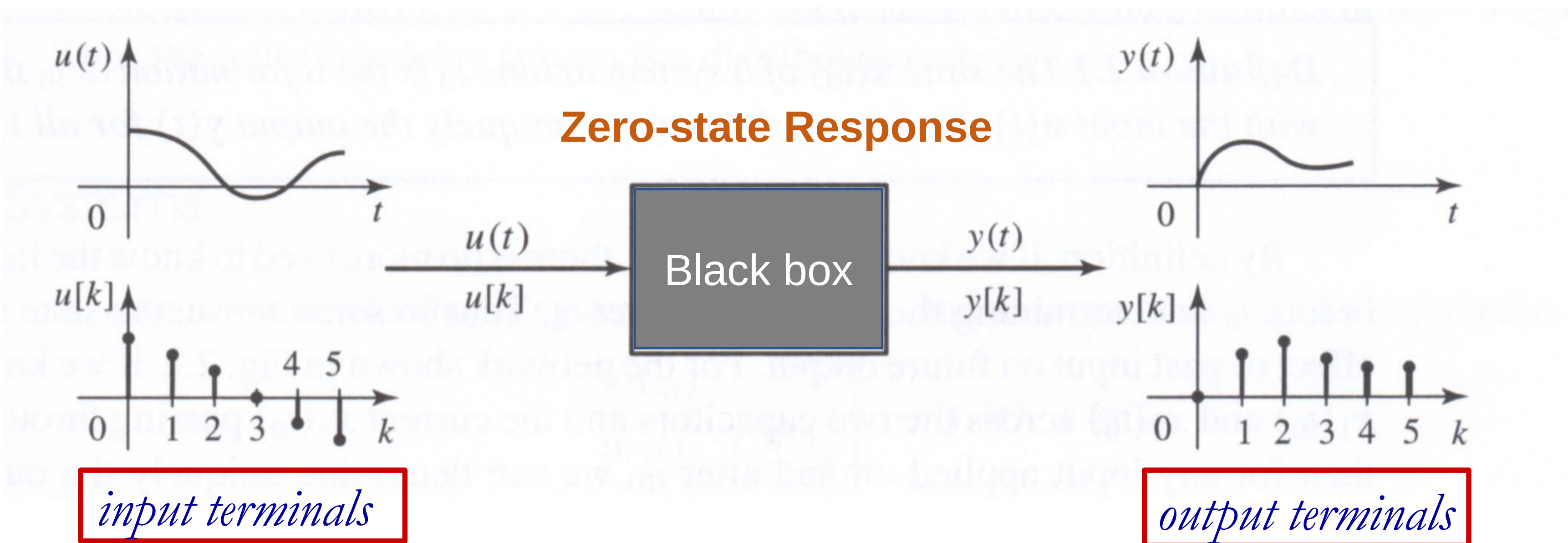
External Description

External Description: Input-Output Description

→ View the system as a "black box" description:

no information on the internal details of the system

→ Characterize by the relation of input, output, and system response (impulse response)



$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

output *input*

Input-Output Stability

▪ Bounded-input-bounded-output (BIBO) stability

$\rightarrow$ A system is said to be **BIBO stable**
if every bounded input excites a bounded output

▪ Input Stability

An input $u(t)$ is **said to be bounded** if $u(t)$ does not grow to positive or negative infinity, or equivalently, $\exists$ a constant u_m s.t.

$$|u(t)| \leqslant u_m < \infty, \quad \forall t \geqslant 0.$$

▪ Output Stability

$$\forall \|u(t)\| < \epsilon \quad \forall t \geq 0, \text{ and } x_0 = 0 \Rightarrow \|y(t)\| < \delta \quad \forall t \geq 0.$$

Impulse Response & Transfer Function Stability

▪ Impulse Response Stability

For a continuous causal linear time invariant system, the condition for BIBO stability is that the impulse response be absolutely integrable, i.e.,

$$\int_0^{\infty} |g(t)|dt$$

exists.

▪ Transfer Function Stability

An LTI system which has a proper rational transfer function

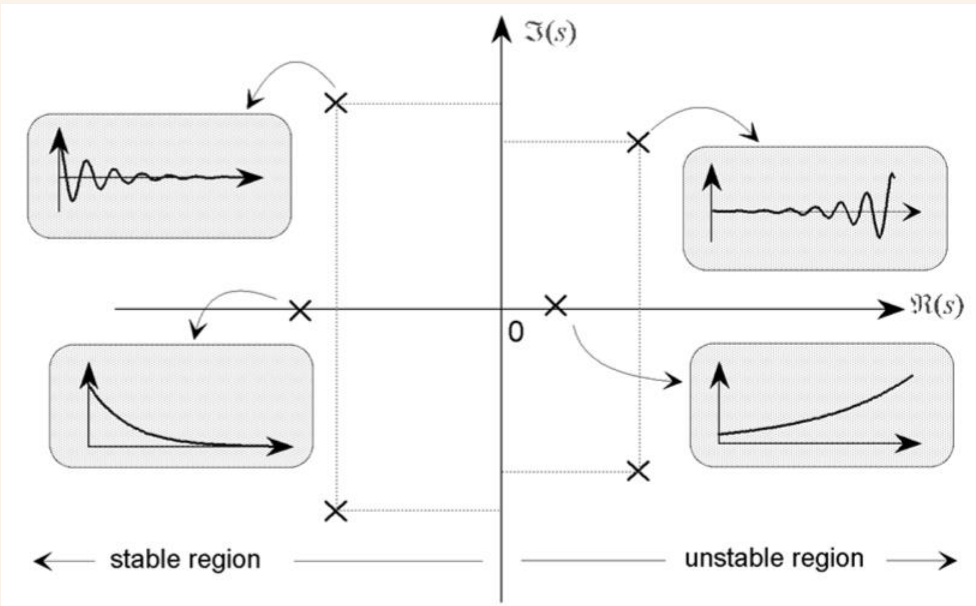
$$G(s) = \frac{n(s)}{d(s)}$$

is BIBO stable if and only if all the poles of $G(s)$ have negative real parts (in other words, all the poles of $G(s)$ are in the open left half complex plane (OLHP)).

Transfer Function Stability

Theorem 17

A SISO system with proper rational transfer function $\hat{g}(s)$ is BIBO stable if and only if every pole of $\hat{g}(s)$ has a negative real part or, equivalently, lies inside the left-half s-plane.



Transfer Function – Eigenvalue

The BIBO stability of state equations:

transfer matrix

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}\right\} \Rightarrow \hat{G}(s) = C(sI - A)^{-1}B + D$$

The zero-state response of this equation is BIBO *stable iff every pole of* $\hat{G}(s)$ *has a negative real part.*

Transfer Function – Eigenvalue

The relationship between the poles of $\hat{G}(s)$ and the eigenvalues of A:

$$\hat{G}(s) = \frac{1}{\det(sI - A)} C[Adj(sI - A)]B + D$$

Every pole of $\hat{G}(s)$ is an eigenvalue of A. Thus if every eigenvalue of A has a negative real part, then every pole has a negative real part and the state equation is BIBO stable.

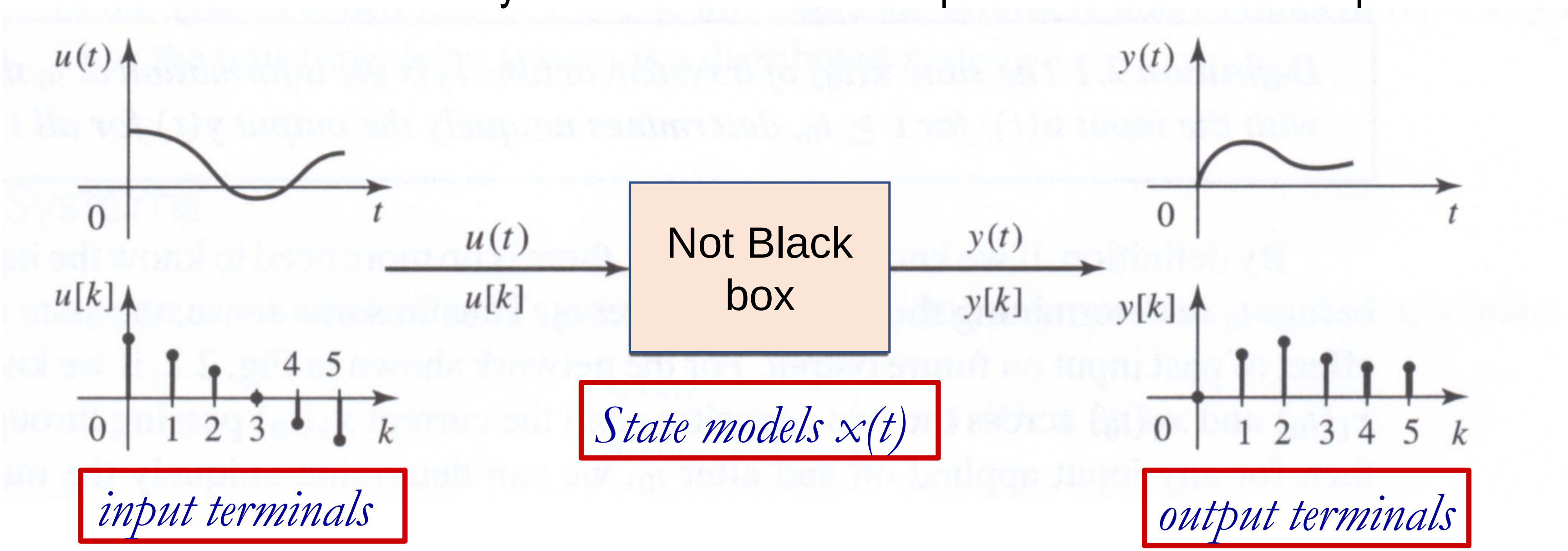
Internal Stability
(Zero-input Response)

Internal Description

▪ Internal Description: State-Space Description

→ State-space representation:

a mathematical model of a physical system as a set of input, output, and state variables related by first-order differential equations or difference equations



state

$$\dot{x}(t) = Ax(t) + Bu(t)$$ 1st order DE

$$y(t) = Cx(t) + Du(t)$$ AE

output   input

Internal Stability

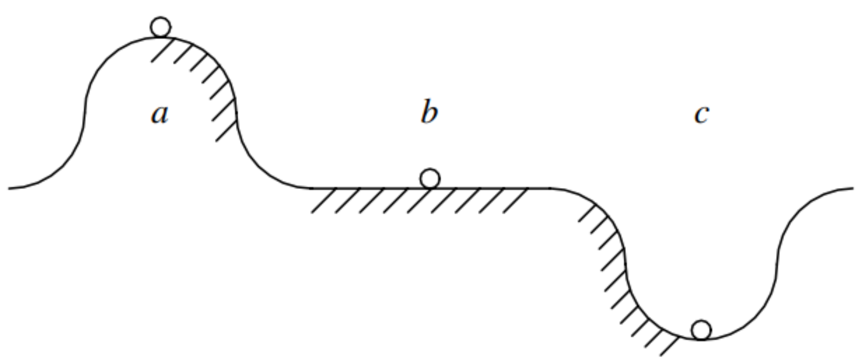
▪ Equilibrium States

Depicted below are equilibrium states a, b and c:

– **a is unstable:** even a tiny movement will move the state away from equilibrium

– **b is stable:** a small movement will move the state a small distance

– **c is asymptotically stable:** a small movement will move the state, but it will eventually return to the original point.



Internal Stability & Eigenvalues

$$\dot{x}(t) = Ax(t) \quad \rightarrow \quad x(t) = e^{At}x_0$$

(AS) - A system S is said to be **asymptotically stable** if its state zero-input response **converges** to the origin for **any** initial condition

(MS) - A system S is said to be **marginally stable** if its state zero-input response remains **bounded** for **any** initial condition

(U) - A system S is said to be **unstable** if its state zero-input response **diverges** for **some** initial condition

Internal Stability & Eigenvalues

$$\dot{x}(t) = Ax(t) \quad \rightarrow \quad x(t) = e^{At}x_0$$

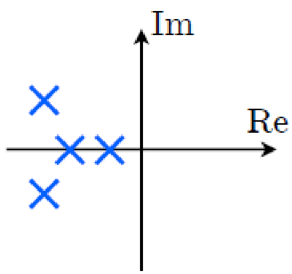
A LTI system is **asymptotically stable**
if and only if
all the eigenvalues have **strictly negative real part**

A LTI system is **marginally stable**
if and only if
all the eigenvalues have **non positive** real part
and those which have **zero real part** have **scalar Jordan blocks**

A LTI system is **unstable**
if and only if
there exists at least one eigenvalue with **positive real part** or a **Jordan block** corresponding to an eigenvalue with **zero real part** of dimension **greater than** 1

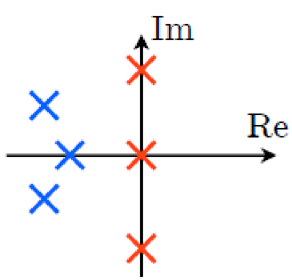
Internal Stability & Eigenvalues

$$\dot{x}(t) = Ax(t) \quad \rightarrow \quad x(t) = e^{At}x_0$$



asymptotic stability

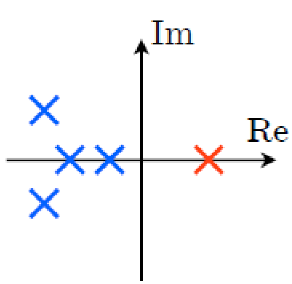
all eigenvalues in the open left half-plane



distinct eigenvalues case

marginal stability

some eigenvalues may be on the Im axis



instability

at least one eigenvalue with positive real part

Internal Stability & Eigenvalues

- **More details on Marginally Stable**

Theorem 5.4 (Internal Stability): $\dot{x}(t) = Ax(t)$ is Marginally (or Lyapunov) stable if and only if all the eigenvalues of A have zero or negative real parts, and

- ❑ those with zero real part are associated with a Jordan block of order 1 or
- ❑ those with zero real part are simple (distinct) roots.
- ❑ Note that 2 superimposed zero roots can be either 2 district roots or 2 repeated (hence not distinct) roots.

Internal Stability & Eigenvalues

- **Example**

Marginally stable

$$\dot{x} = Ax = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x \longrightarrow \Delta\lambda = \det(\lambda I - A) = 0$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix}$$

$\Delta\lambda = \lambda^2(\lambda + 1) \longrightarrow$ Eigenvalues 0,0, -1

$\Psi(\lambda) = \lambda(\lambda + 1) \longrightarrow$

$$\Psi(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The matrix A has eigenvalues $\lambda_0 = 0$ with multiplicity 2, and $\lambda_1 = -1$ with multiplicity 1.
- The eigenvalue $\lambda_0 = 0$ is a simple root of the minimal polynomial, hence the system is marginally stable.
- Or, the eigenvalue $\lambda_0 = 0$ is associated to Jordan blocks of order 1, so the equation is marginally stable.

Internal Stability & Eigenvalues

- **Example**

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x \longrightarrow \Delta\lambda = \det(\lambda I - A) = 0$$

Not Marginally stable

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix}$$

$\Delta\lambda = \lambda^2(\lambda + 1) \longrightarrow$ Eigenvalues 0,0, -1

$$\Psi(\lambda) = \lambda(\lambda + 1) \longrightarrow \Psi(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Psi(\lambda) = \lambda^2(\lambda + 1) \longrightarrow \Psi(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

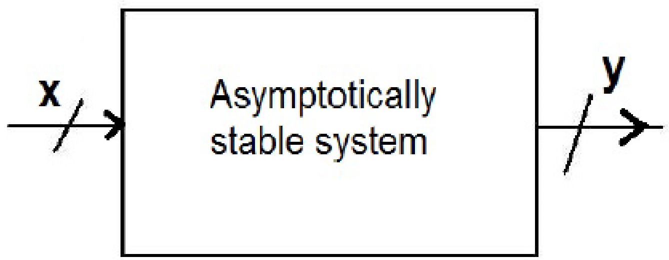
- The matrix A has the same eigenvalues and same multiplicities of the previous example.
- Now, however, the eigenvalue $\lambda_0 = 0$ is not a simple root of the minimal polynomial.
- Or, the repeated eigenvalue $\lambda_0 = 0$ is associated with a Jordan block of order 2, so the equation is unstable.

Homework #4.3 Stability (1 pt.): Due Dec. 26

$$\dot{x} = Ax = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x$$

marginally stable? Asymptotically stable?

LTI Stability



Let the input x equal zero. If every initial condition set results $y(t)$ converging to zero, then the system is **asymptotically stable**.
For the LTI systems BIBO stability and asymptotic stability are equivalent.

Checking for Stability

Consider a general transfer function:

$$H(s) = \frac{q(s)}{p(s)}$$

where q and p are polynomials, and $\deg(q) \leq \deg(p)$.

We need tools for checking stability: whether or not all roots of $p(s) = 0$ lie in OLHP.

For simple polynomials, can just factor them "by inspection" and find roots.

Now, this is hard to do for high-degree polynomials — it's computationally intensive, especially symbolically.

But: often we *don't need to know* precise pole locations, just need to know that they are strictly stable.

Routh-Hurwitz Condition

Let $G(s) = \frac{p(s)}{q(s)}$, where

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots a_1 s + a_0$$
$$= a_n (s - r_1)(s - r_2) \dots (s - r_n)$$

where r_i are the roots of $q(s) = 0$.

By multiplying out, $q(s) = 0$ can be written as

$$q(s) = a_n s^n - a_n (r_1 + r_2 + \dots + r_n) s^{n-1}$$
$$+ a_n (r_1 r_2 + r_2 r_3 + \dots) s^{n-2}$$
$$- a_n (r_1 r_2 r_3 + r_1 r_2 r_4 + \dots) s^{n-3}$$
$$+ \dots + (-1)^n a_n (r_1 r_2 r_3 \dots r_n) = 0$$

If all r_i are real and in left half plane, what is sign of coeffs of s^k?
the same!

Routh-Hurwitz Condition

Consider $G(s) = \frac{p(s)}{q(s)}$. Poles are solutions to $q(s) = 0$; i.e.,

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

Construct a table of the form

$$\begin{array}{l|cccc}
\text{Row } n & a_n & a_{n-2} & a_{n-4} & \ldots \\
\text{Row } n-1 & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
\text{Row } n-2 & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
\text{Row } n-3 & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ldots \\
\text{Row } 0 & h_{n-1} & & &
\end{array}$$

where

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \qquad c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

Routh-Hurwitz Condition

Now consider the table that we have just constructed

Row n	a_n	a_{n-2}	a_{n-4}	$\dots$
Row $n-1$	a_{n-1}	a_{n-3}	a_{n-5}	$\dots$
Row $n-2$	b_{n-1}	b_{n-3}	b_{n-5}	$\dots$
Row $n-3$	c_{n-1}	c_{n-3}	c_{n-5}	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$
Row 0	h_{n-1}			

Loosely speaking:

- Number of roots in the right half plane is equal to the number of sign changes in the first column of the table
- Stability iff no sign changes in the first column

Now let's move towards a more sophisticated statement

Routh-Hurwitz Condition

1. Consider $q(s)$ with $a_n > 0$

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots a_1 s + a_0 = 0$$

2. Construct a table of the form

$$\begin{array}{l|cccc}
\text{Row } n & a_n & a_{n-2} & a_{n-4} & \ldots \\
\text{Row } n-1 & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
\text{Row } n-2 & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
\text{Row } n-3 & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ldots \\
\text{Row } 0 & h_{n-1} & & &
\end{array}$$

   Procedure provided on the following slides

3. Count the sign changes in the first column

4. That is the number of roots in the right half plane

Stability (poles in LHP) iff all terms in first col. have same sign

Some Special Case

Case 1

Zero as 1st element in a row and no other term in the row; or

Zero as 1st element in a row and a nonzero term in the row

Example 1
$$\begin{array}{cc} * & * \\ * & \\ 0 & \end{array}$$

Example 2
$$\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 7 & 0 & 2 & \end{array}$$

Example 3
$$\begin{array}{ccc} * & * & * \\ * & * & \\ 0 & 2 & \end{array}$$

Solution: Replace 0 with a positive number ε

Some Special Case

Case 2

Zero as 1st element in a row and all others are zero

$$\text{Example} \quad \begin{array}{cccccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \end{array}$$

Solution: Use the above row to form the auxiliary polynomial.

Case 3

Array ends before reaching the s^0 row

$$\text{Example} \quad \begin{array}{cc} s^5 & * \\ s^4 & * \\ s^3 & \\ s^2 & \\ s^1 & \\ s^0 & \end{array}$$

Solution: If the procedure terminates before reaching the s^0 row, say at s^k row then there is a root at $s = 0$ with multiplicity k.

Example

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

$$\begin{array}{rlll}
 & s^4 & 1 & 3 & 5 \\
\text{no ch.}\downarrow & & & & \\
 & s^3 & 2 & 4 & \\
\text{no ch.}\downarrow & & & & \\
 & s^2 & \frac{2\times3-4\times1}{2} = 1 & \frac{2x5-0}{2} = 5 & \\
\text{ch.}\downarrow & & & & \\
 & s^1 & \frac{1\times4-2\times5}{1} = -6 & & \\
\text{ch.}\downarrow & & & & \\
 & s^0 & 5 & &
\end{array}$$

Conclusion: Two roots have positive real parts.

Homework #4.4 Routh-Hurwitz Condition (1 pt.): Due Dec. 26

Use Routh-Hurwitz's criterion to determine the number of roots in the right-half *s*-plane for the following equations:

$$f(s) = s^3 + 11s^2 + 38.36s + 41.8$$

Discrete-time Input-Output Stability

Discrete-time Input-Output Stability

Consider a discrete-time SISO system described by

$$y[k] = \sum_{m=0}^{k} g[k-m]u[m] = \sum_{m=0}^{k} g[m]u[k-m]$$

linear,
time-invariant,
causal,
initially relaxed

where $g[k]$ is the impulse response sequence or the output sequence excited by an impulse sequence applied at $k = 0$.

An input sequence $u[k]$ is said to be *bounded* if $u[k]$ does not grow to positive or negative infinity or there exists a constant u_m such that

$$|u[k]| \leq u_m < \infty \text{ for } k = 0,1,2,\ldots$$

Discrete-time Input-Output Stability

A system is said to be *BIBO stable* if every bounded-input sequence excites a bounded-output sequence.

A discrete-time SISO system is BIBO stable *iff* $g[k]$ is absolutely summable in $[0,\infty)$

$$\sum_{k=0}^{\infty} \left| g[k] \right| \leq M < \infty$$

for some constant M.

Absolutely summable means $\sum_{k=-\infty}^{\infty} \left| g[k] \right|$ *exists and is finite.*

Discrete-time Input-Output Stability

A discrete-time SISO system with proper rational transfer function $\hat{g}(z)$ is BIBO stable *iff* every pole of $\hat{g}(z)$ has a magnitude less than 1 or, equivalently, lies inside the unit circle on the z-plane.

If $\hat{g}(z)$ has pole p_i with multiplicity m_i, then its partial fraction expansion contains the factors

$$\frac{1}{z-p_i}, \frac{1}{(z-p_i)^2}, \cdots, \frac{1}{(z-p_i)^{m_i}}$$

Thus the inverse z-transform of $\hat{g}(z)$ contains the factors

$$p_i^k, \;\; kp_i^k, \;\; \cdots, \;\; k^{m_i-1}p_i^k$$

Z-Transform Pairs

$f(k), k = 0,1,2,\ldots$	$\hat{f}(z)$
1 *at* k, 0 *elsewhere* *Kronecker delta sequence*	z^{-k}
e^{-akT}	$\dfrac{z}{z-e^{-aT}}$
kTe^{-akT}	$\dfrac{Te^{-aT}z}{(z-e^{-aT})^2}$
1 *unit step sequence*	$\dfrac{1}{(z-1)}$
kT *unit ramp sequence*	$\dfrac{Tz}{(z-1)^2}$

Z-Transform Pairs

$f(k), k = 0,1,2,\ldots$	$\hat{f}(z)$
$\sin \omega kT$	$\dfrac{z \sin \omega T}{z^2 - 2z\cos\omega T + 1}$
$\cos \omega kT$	$\dfrac{z(z - \cos\omega T)}{z^2 - 2z\cos\omega T + 1}$
$e^{-akT} \sin \omega kT$	$\dfrac{z e^{-aT} \sin \omega T}{z^2 - 2z e^{-aT}\cos\omega T + e^{-2aT}}$
$e^{-akT} \cos \omega kT$	$\dfrac{z(z - e^{-aT}\cos\omega T)}{z^2 - 2z e^{-aT}\cos\omega T + e^{-2aT}}$
$\dfrac{1}{a-b}(a^{k-1} - b^{k-1}), k > 0$ 0 *for* $k=0$	$\dfrac{1}{(z-a)(z-b)}$
$\dfrac{1}{a-b}(a^k - b^k)$	$\dfrac{z}{(z-a)(z-b)}$
$1 - a^k$	$\dfrac{z(1-a)}{(z-1)(z-a)}$

Z-Transform Pairs

Partial fractions

$$G(z) = \frac{N(z)}{(z-p_i)^{m_i}}$$

$$N(z) = (z-p_i)^{m_i-1} q_1 + r_1(z)$$

$$r_1(z) = (z-p_i)^{m_i-2} q_2 + r_2(z)$$

$$r_2(z) = (z-p_i)^{m_i-3} q_3 + r_3(z)$$

$$\vdots$$

$$N(z) \longleftarrow \frac{2z^2+z-3}{(z+2)^3}$$

$$2z^2+z-3 = (z+2)^2 \cdot 2 + (-7z-11)$$

$$-7z-11 = (z+2)\cdot(-7)+3$$

$$2z^2+z-3 = (z+2)^2 \cdot 2 + (z+2)\cdot(-7)+3$$

$$\frac{2z^2+z-3}{(z+2)^3} = \frac{2}{z+2} - \frac{7}{(z+2)^2} + \frac{3}{(z+2)^3}$$

Relationship between S-plane & Z-plane

Consider the continuous signal

$$f(t) = e^{-at}, \quad t > 0,$$

$$\hat{f}(s) = \mathrm{L}[e^{-at}] = \frac{1}{s+a} \quad \rightarrow s = -a$$

$$\hat{f}(z) = Z[e^{-akT}] = \frac{z}{z - e^{-aT}} \quad \rightarrow z = e^{-aT}$$

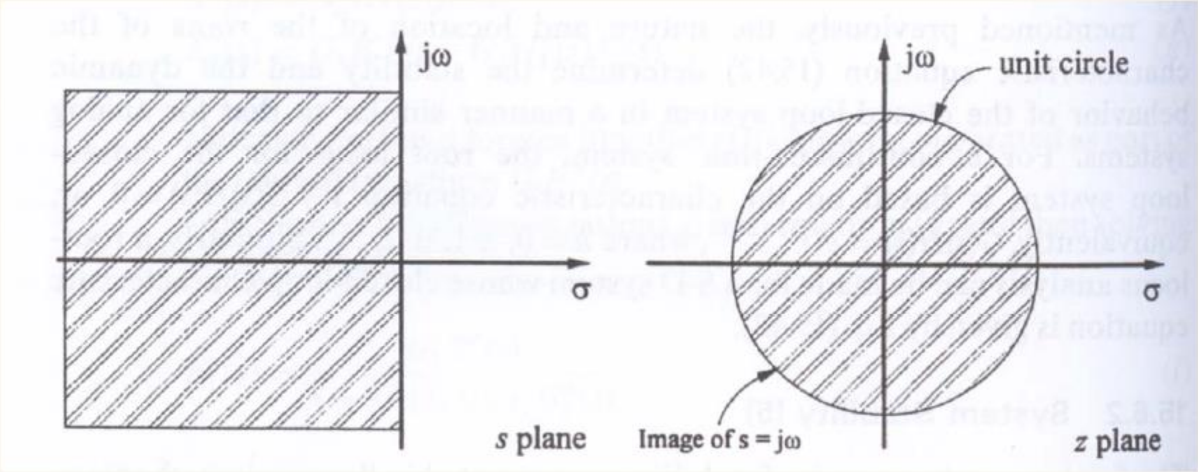
This means that a pole at $s = -a$ *in the* s*-plane corresponds to a pole at* $z = e^{-aT}$ *in the discrete domain.*

The equivalent characteristics in the z*-plane are related to those in the* s*-plane by the expression* $z = e^{sT}$ *where* T *is the sample period.*

Relationship between S-plane & Z-plane

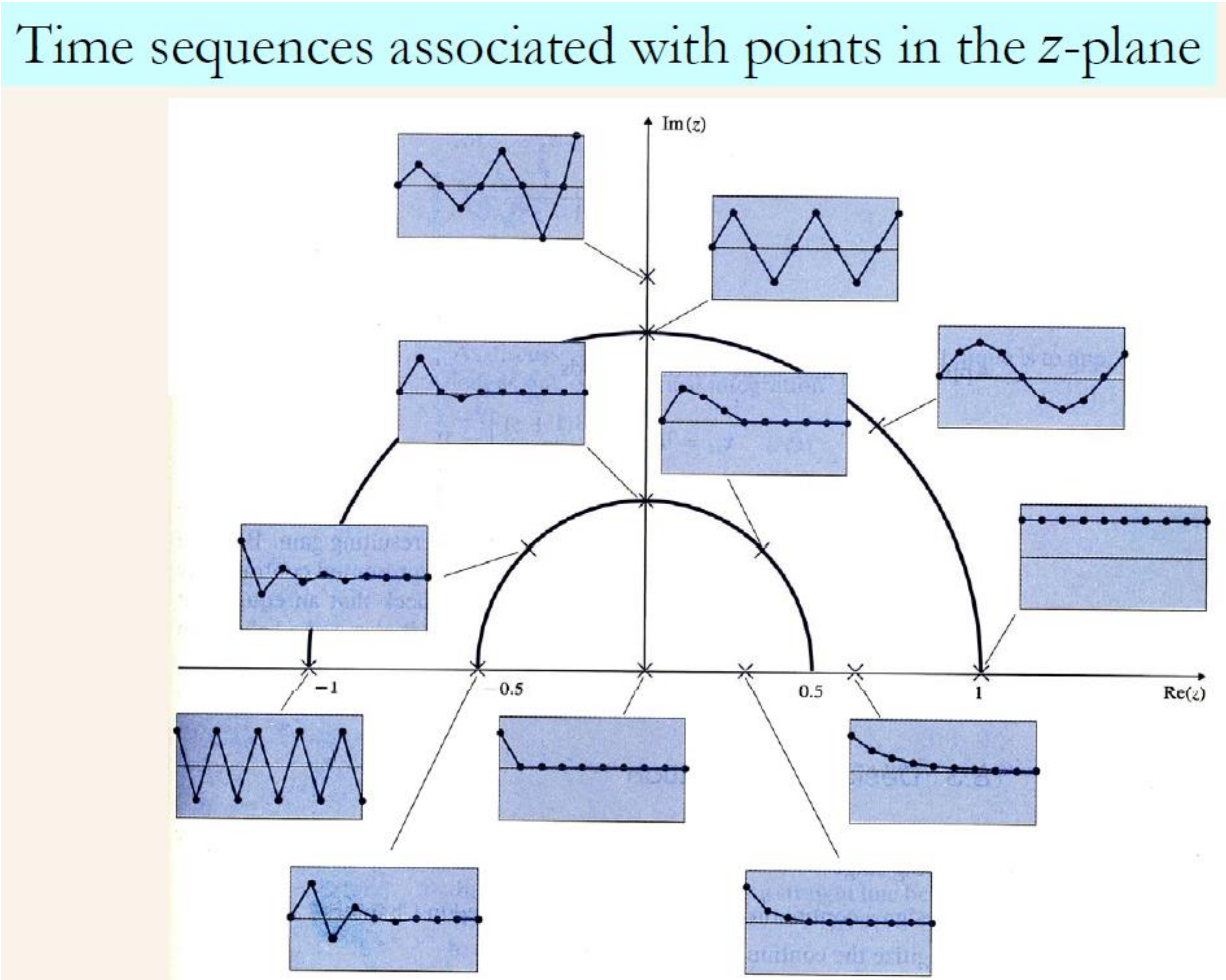
The mapping of the s plane into the z plane by means of

$$z = e^{Ts}.$$



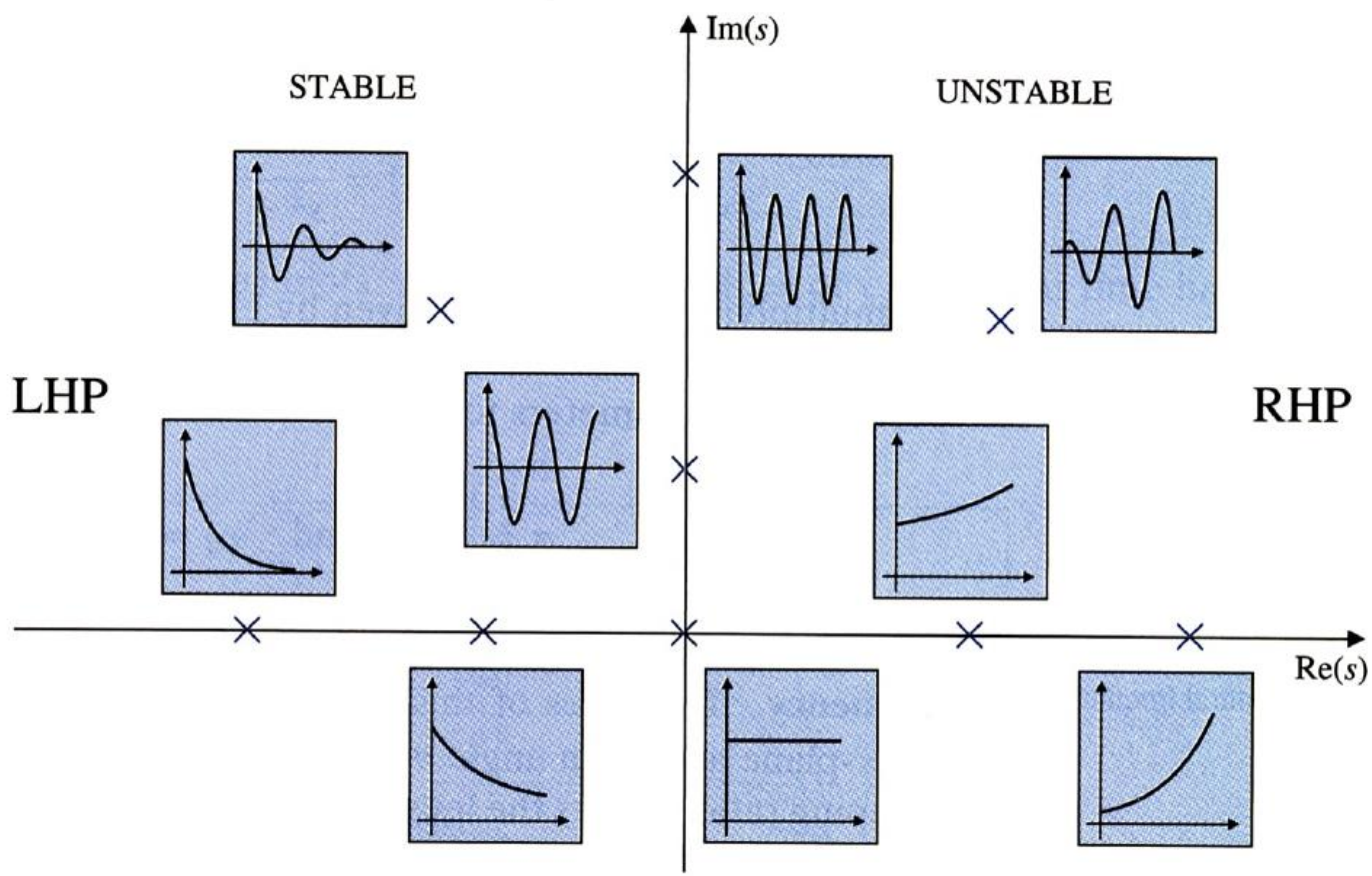
The strips in the left-half s plane $(\sigma < 0)$ map into the region inside the unit circle in the z-plane, and the strips in the right-half s plane $(\sigma > 0)$ map into the region outside the unit circle in the z-plane.

Relationship between S-plane & Z-plane



Relationship between S-plane & Z-plane

Time functions associated with points in the s-plane



Impulse Response & Transfer Function

A MIMO discrete-time system with impulse response sequence matrix $G[k]=[g_{ij}[k]]$ is BIBO stable *iff* every $g_{ij}[k]$ is absolutely summable.

A MIMO discrete-time system with discrete proper rational transfer matrix $\hat{G}[z]=[\hat{g}_{ij}(z)]$ is BIBO stable *iff* every pole of every $\hat{g}_{ij}(z)$ has a magnitude less than 1.

Impulse Response & Transfer Function

The BIBO stability of discrete-time state equations:

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

The zero-state response of the equation is BIBO stable *iff* every pole of

$$\hat{G}(z) = C(zI - A)^{-1}B + D$$

has a magnitude less than 1.

$$\hat{G}(z) = \frac{1}{\det(zI - A)} C[Adj(zI - A)]B + D$$

Every pole of $\hat{G}(z)$ is an eigenvalue of A. Thus *if every eigenvalue of A has a magnitude less than* 1, then the equation is BIBO stable.

Discrete-time
Internal Stability
(Zero-input Response)

Discrete-time Internal Stability

Consider the discrete-time state space equation

$$x[k+1] = Ax[k] + Bu[k] \quad \textit{zero}$$

$$\rightarrow x[k] = A^{k}x[0] + \sum_{m=0}^{k-1} A^{k-1-m}Bu[m]$$

(*stable in the sense of Lyapunov*)

The equation is *marginally stable* if every finite initial state x_0 excites a bounded response. It is *asymptotically stable* if every finite initial state excites a bounded response, which, in addition, approaches 0 as $k \to \infty$.

Discrete-time Internal Stability

The equation $x[k+1] = Ax[k]$ is *marginally stable iff* all eigenvalues of A have magnitude less than or equal to 1 and those equal to 1 are simple roots of the minimal polynomial of A.

The equation $x[k+1] = Ax[k]$ is *asymptotically stable iff* all eigenvalues of A have magnitude less than 1.

Jury-Marden Stability Criterion

$$\hat{G}(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{\sum_{i=0}^{N} b_i z^{N-i}}$$

- $N(z)$ and $D(z)$ have no common factors.
- $b_0 > 0$

Row	*Coefficients*							
1	b_0	b_1	b_2	b_3	$\cdots$	b_{N-2}	b_{N-1}	b_N
2	b_N	b_{N-1}	b_{N-2}	b_{N-3}	$\cdots$	b_2	b_1	b_0
3	c_0	c_1	c_2	$\cdots$	c_{N-3}	c_{N-2}	c_{N-1}	
4	c_{N-1}	c_{N-2}	c_{N-3}	$\cdots$	c_2	c_1	c_0	
5	d_0	d_1	d_2	$\cdots$	d_{N-3}	d_{N-2}		
6	d_{N-2}	d_{N-3}	d_{N-4}	$\cdots$	d_1	d_0		
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$2N-3$	r_0	r_1	r_2					

$$c_i = \begin{vmatrix} b_i & b_N \\ b_{N-i} & b_0 \end{vmatrix}, \quad 0, \quad 1, \quad \ldots, \quad N-1 \qquad d_i = \begin{vmatrix} c_i & c_{N-1} \\ c_{N-1-i} & c_0 \end{vmatrix}, \quad 0, \quad 1, \quad \ldots, \quad N-2$$

Jury-Marden Stability Criterion

$D(z)$ has roots inside the unit circle of the z-plane, *iff*

1. $D(1) > 0$
2. $(-1)^N D(-1) > 0$
3. $b_0 > |b_N|$

$$|c_0| > |c_{N-1}|$$

$$|d_0| > |d_{N-2}|$$

$$\vdots \qquad \vdots$$

$$|r_0| > |r_2|$$

Example:

$$\hat{G}(z) = \frac{z^4}{4z^4 + 3z^3 + 2z^2 + z + 1}$$

$$D(z) = 4z^4 + 3z^3 + 2z^2 + z + 1$$ *denominator polynomial*

1. $D(1) > 0$
2. $(-1)^N D(-1) > 0$
3. $b_0 > |b_N|$
   $|c_0| > |c_{N-1}|$
   $|d_0| > |d_{N-2}|$
   $\vdots \quad \vdots$
   $|r_0| > |r_2|$

1. $D(1) = 11 > 0$
2. $(-1)^4 D(-1) = 3 > 0$
3.

1	$\boxed{4}$	3	2	1	1	$b_0 > \lvert b_4 \rvert$
2	$\boxed{1}$	1	2	3	4	
3	$\boxed{15}$	11	6	1		$\lvert c_0 \rvert > \lvert c_3 \rvert$
4	$\boxed{1}$	6	11	15		
5	$\boxed{224}$	159	$\boxed{79}$			$\lvert d_0 \rvert > \lvert d_2 \rvert$

This filter is stable!

Example:

$$\hat{G}(z) = \frac{z^2 + 2z + 1}{z^4 + 6z^3 + 3z^2 + 4z + 5}$$

$$D(z) = z^4 + 6z^3 + 3z^2 + 4z + 5$$

denominator polynomial

1. $D(1) > 0$
2. $(-1)^N D(-1) > 0$
3. $b_0 > |b_N|$

   $|c_0| > |c_{N-1}|$

   $|d_0| > |d_{N-2}|$

   $\vdots \qquad \vdots$

   $|r_0| > |r_2|$

1. $D(1) = 19 > 0$
2. $(-1)^4 D(-1) = -1 < 0$

This filter is unstable!

Thank you

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