Lecture I213E – Class 1

Discrete Signal Processing

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JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY 1990

Course Materials

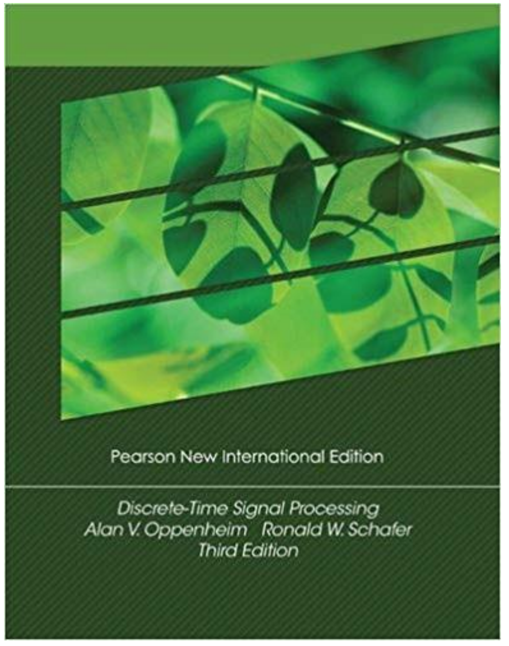
- **Materials**

→ Lecture notes will be uploaded before each lecture
https://jstorage-2018.jaist.ac.jp/s/PGXRrC7iFmN2FWo
Pass: dsp-i213e-2022
(Slide Courtesy of Prof. Nak Young Chong)

- **References**

→ Chi-Tsong Chen:
Linear System Theory and Design, 4th Ed.,
Oxford University Press, 2013.

→ Alan V. Oppenheim and Ronald W. Schafer:
Discrete-Time Signal Processing, 3rd Ed.,
Pearson New International Ed., 2013.



Related Courses & Prerequisite

- **Related Courses**
  - → I212 Analysis for Information Science
  - → I114 Fundamental Mathematics for Information Science

- **Prerequisite**
  - → None

Evaluation

▪ Viewpoint of evaluation

→ Students are able to understand:

- Basic principles in modeling and analysis of linear time-invariant systems
- Applications of mathematical methods and tools to different signal processing problems.

▪ Evaluation method

→ Homework, term project, midterm exam, and final exam

▪ Evaluation criteria

→ Homework/labs (30%), term project (30%) midterm exam (15%), and final exam (25%)

Contact

- ## Lecturer

  → Sakriani Sakti

- ## TA

  Tutorial hours & Term project

  → WANG Lijun (s2010026)

  → TANG Bowen (s2110411)

  Homework

  → PUTRI Fanda Yuliana (s2110425)

- ## Contact Email

  → dsp-i213e-2022@ml.jaist.ac.jp

Schedule

- **December 8th, 2022 – February 9th, 2023**

- **Lecture Course Term 2-2**
  → Tuesday 9:00 — 10:40
  → Thursday 10:50 — 12:30

- **Tutorial Hours**
  → Tuesday 13:30-15:10

Schedule

Dec

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Jan

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	~~5~~	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	~~24~~	25	26	27	28
29	30	31				

Feb

Sun	Mon	Tue	Wed	Thu	Fri	Sat
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26	27	28				

Lecture:
Tuesday 9:00 — 10:40
Thursday 10:50 — 12:30

Tutorial:
Tuesday 13:30 — 15:10

Course review &
term project evaluation
(on tutorial hours)

Midterm & final exam
Thursday 10:50 — 12:30

Syllabus

Class	Date	Lecture Course Tue 9:00 — 10:40 / Thr 10:50 — 12:30	Tutorial Hours Tue 13:30 — 15:10
1	12/08	Introduction to Linear Systems with Applications to Signal Processing	
2	12/13	State Space Description	○
3	12/15	Linear Algebra	
4	12/20	Quantitative Analysis (State Space Solutions) and Qualitative Analysis (Stability)	○
5	12/22	Discrete-time Signals and Systems	
X	01/05		
6	01/10	Discrete-time Fourier Analysis	
7	01/10*	Review of Discrete-time Linear Time-Invariant Signals and Systems (on Tutorial Hours)	
	01/12	Midterm Exam	
8	01/17	Sampling and Reconstruction of Analog Signals	○
9	01/19	z-Transform	
X	01/24		○
10	01/26	Discrete Fourier Transform	
11	01/31	FFT Algorithms	○
12	01/02	Implementation of Digital Filters	
13	02/07	Digital Signal Processors and Design of Digital Filters	
14	02/07*	Review of the Course and Term Project Evaluation (on Tutorial Hours)	
	02/09	Final exam	

Class 1
Introduction to Linear Systems with Applications to Signal Processing

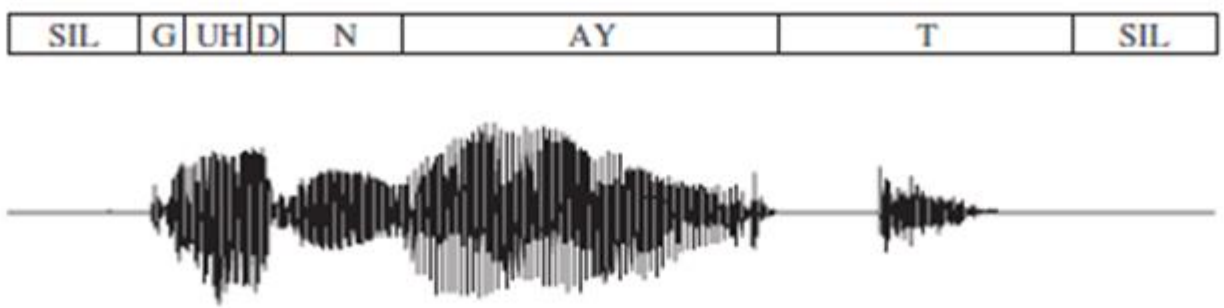
Signal Processing

Discrete-time Signal Processing

- **Signal**

→ ***Carriers of information, both useful and unwanted***

→ The distinction between useful and unwanted information is often subjective as well as objective



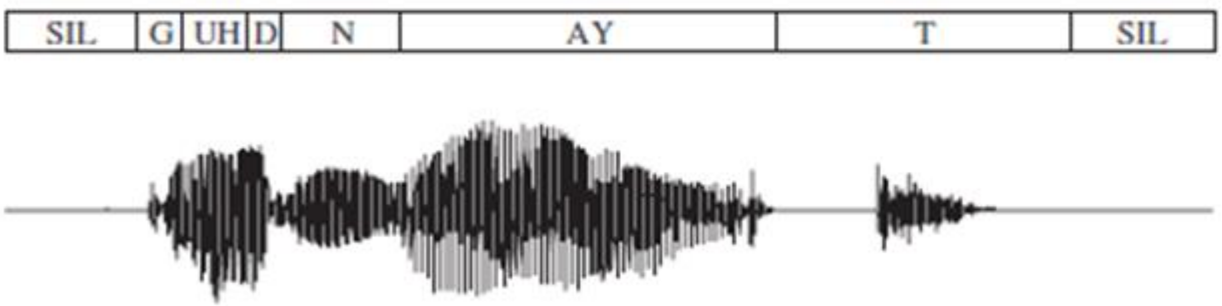
Speech Signal carries various information: linguistic, paralinguistic, speaker information

Discrete-time Signal Processing

▪ Signal Processing

→ An operation designed for extracting, enhancing, storing, and transmitting useful information (from a mix of conflicting information).

→ Signal processing tends to be application dependent



Speech Recognition

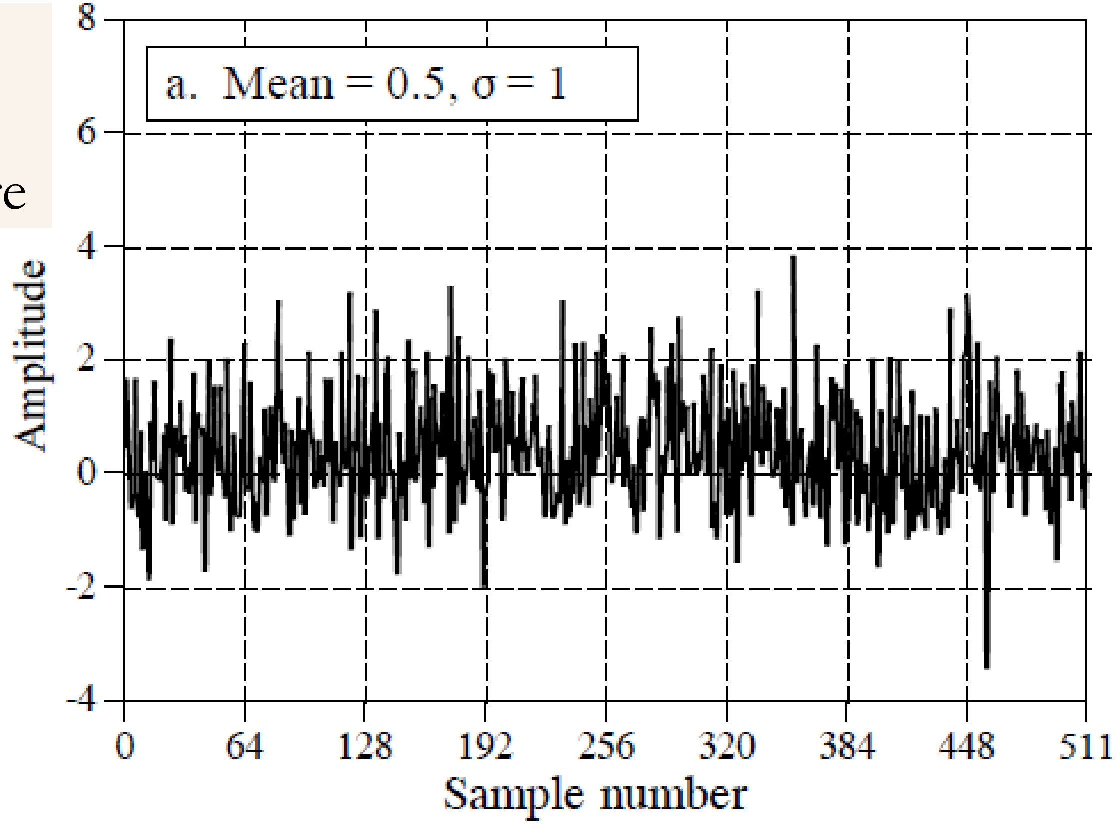
Useful information: linguistic content

Speaker Recognition

Useful information: speaker information

More Examples of Signals

voltage,
light intensity,
sound pressure



Domain: time, frequency, spatial

A Grayscale Image

- **Image: Signals with special characteristics**

  → Measure a parameter over space (distance),
  while most signals are measured over time

Two-dimensional signals:
Represents the intensity of the image at each pixel location

Noisy Signals

- **Noisy Signals:**

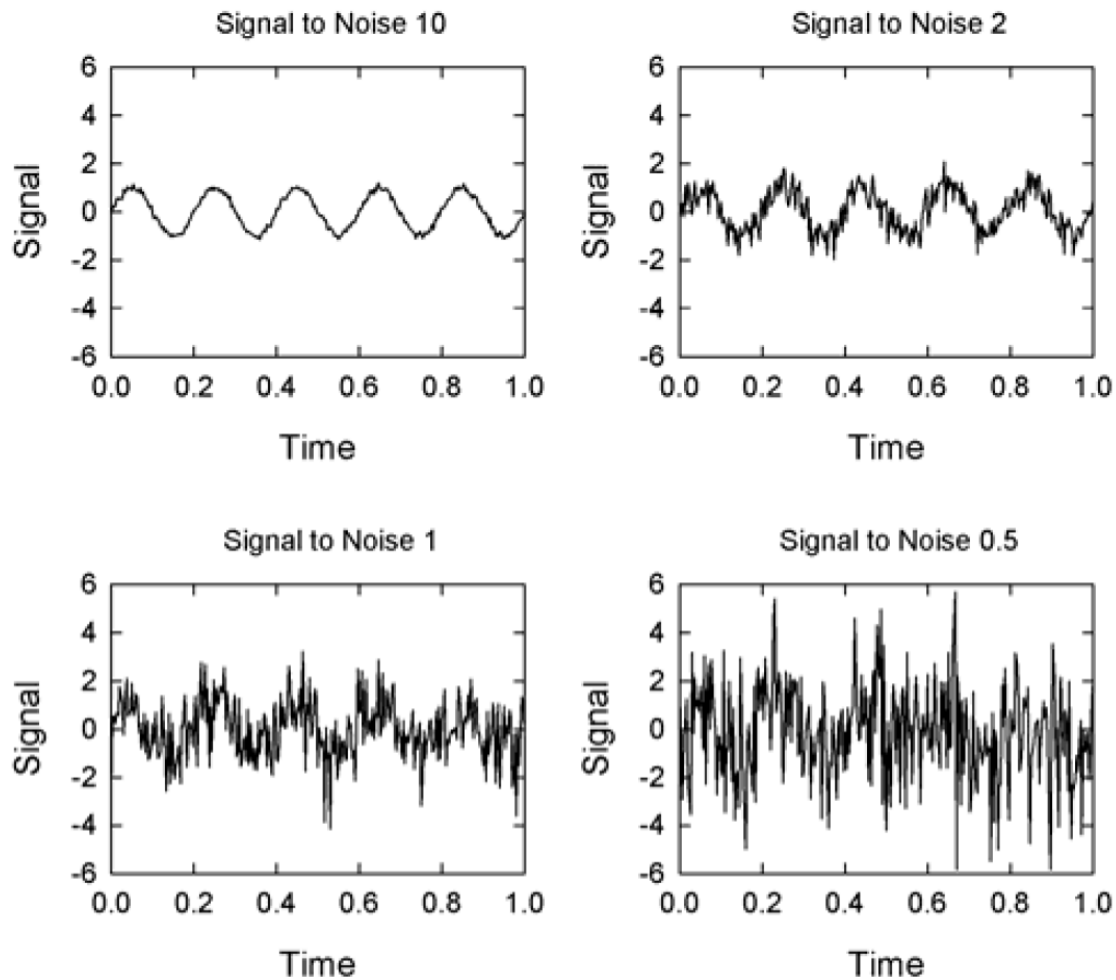
  → Most real-world signals are contaminated by noise

  → Signal-to-noise ratio (SNR or S/N)

$$\boxed{SNR = \frac{P_{Signal}}{P_{Noise}}}$$

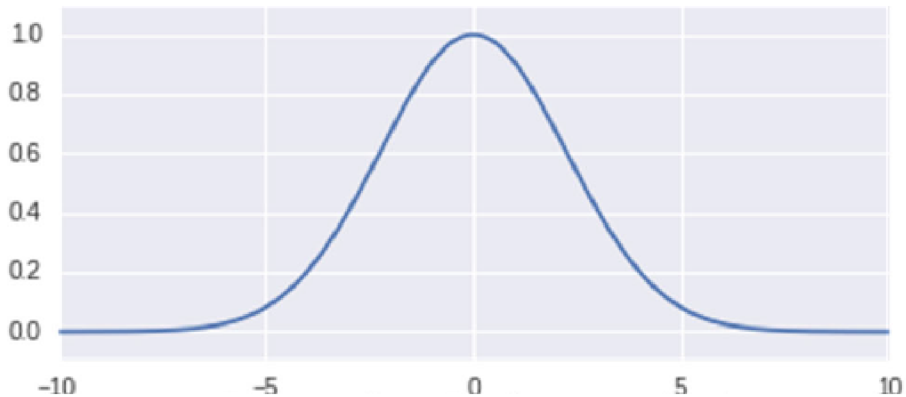
$$SNR\ [dB] = 10\log_{10}\frac{P_{Signal}}{P_{Noise}} = 20\log_{10}\frac{V_{Signal}}{V_{Noise}}$$

where $P = \frac{V^2}{R}$

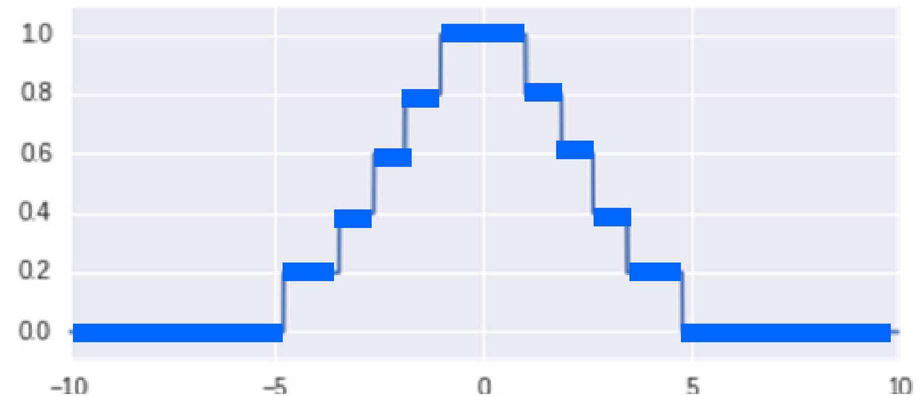


Analog versus Digital Signals

Continuous-time Continuous-valued

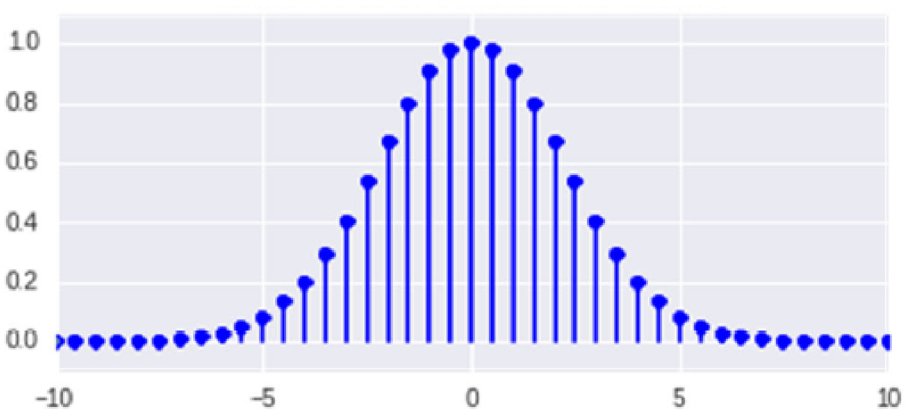


Continuous-time Discrete-valued

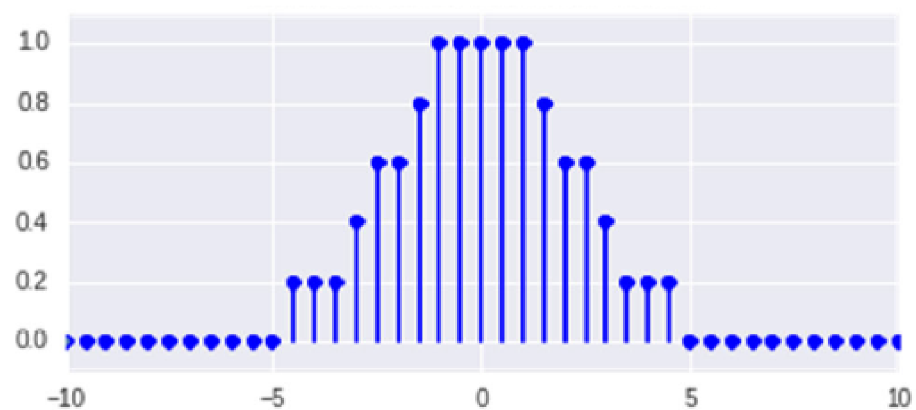


→ **Analog signals**

Discrete-time Continuous-valued



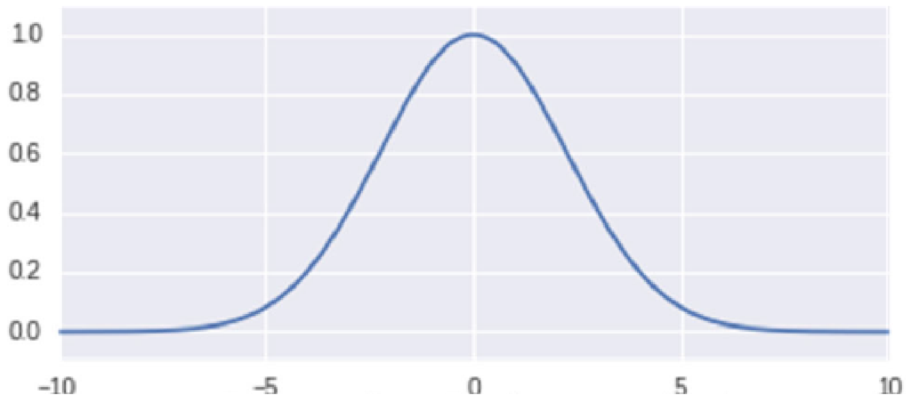
Discrete-time Discrete-valued



→ **Digital signals**

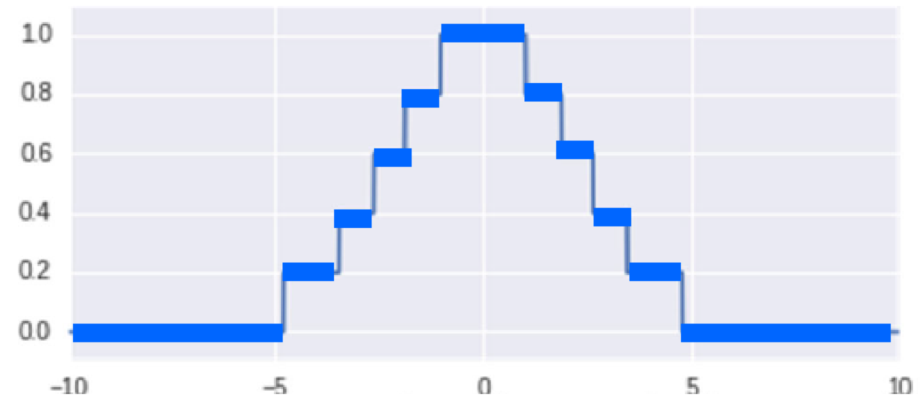
Discrete-time versus Digital Signals

Continuous-time Continuous-valued

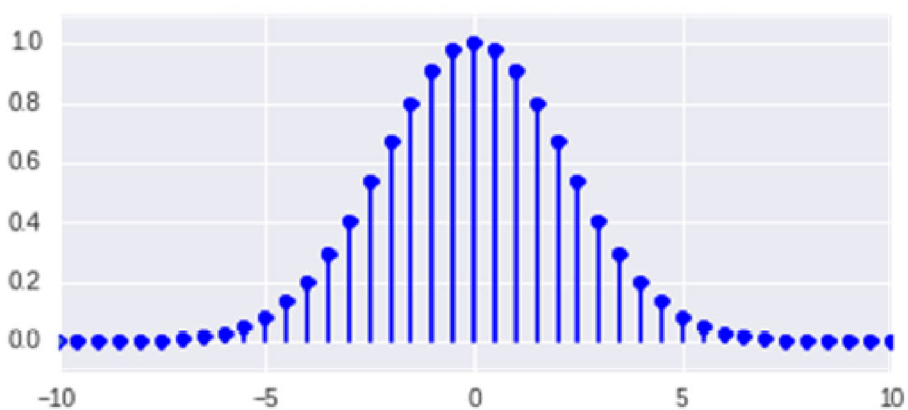


→ Analog signals

Continuous-time Discrete-valued

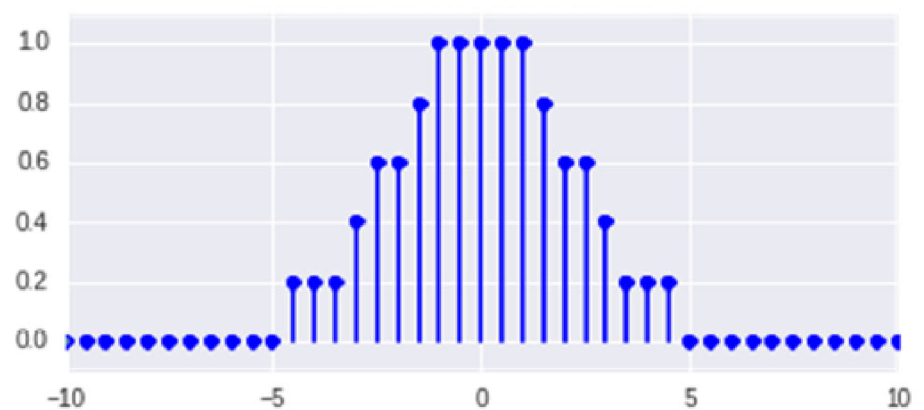


Discrete-time Continuous-valued



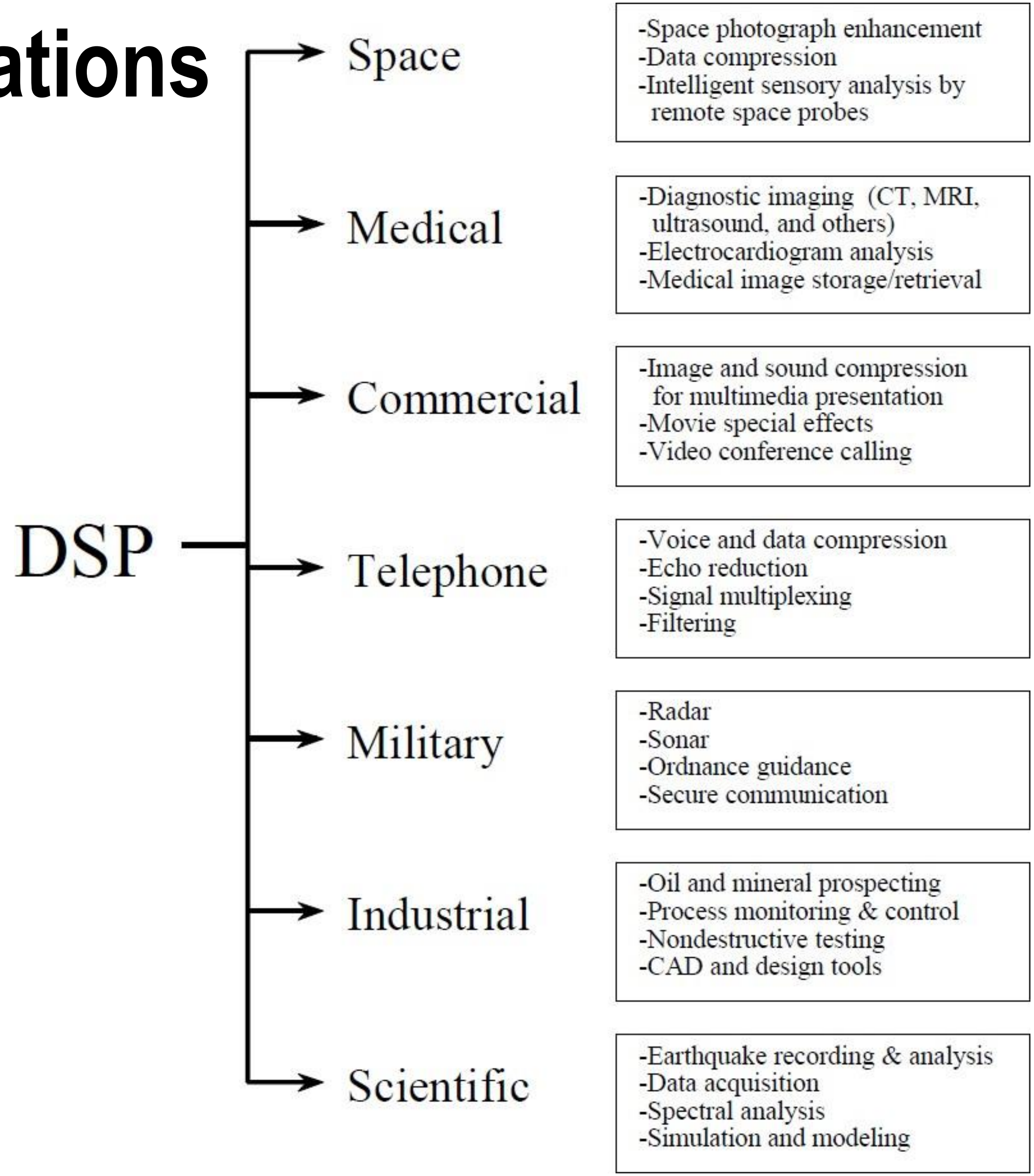
→ Discrete-time signals

Discrete-time Discrete-valued

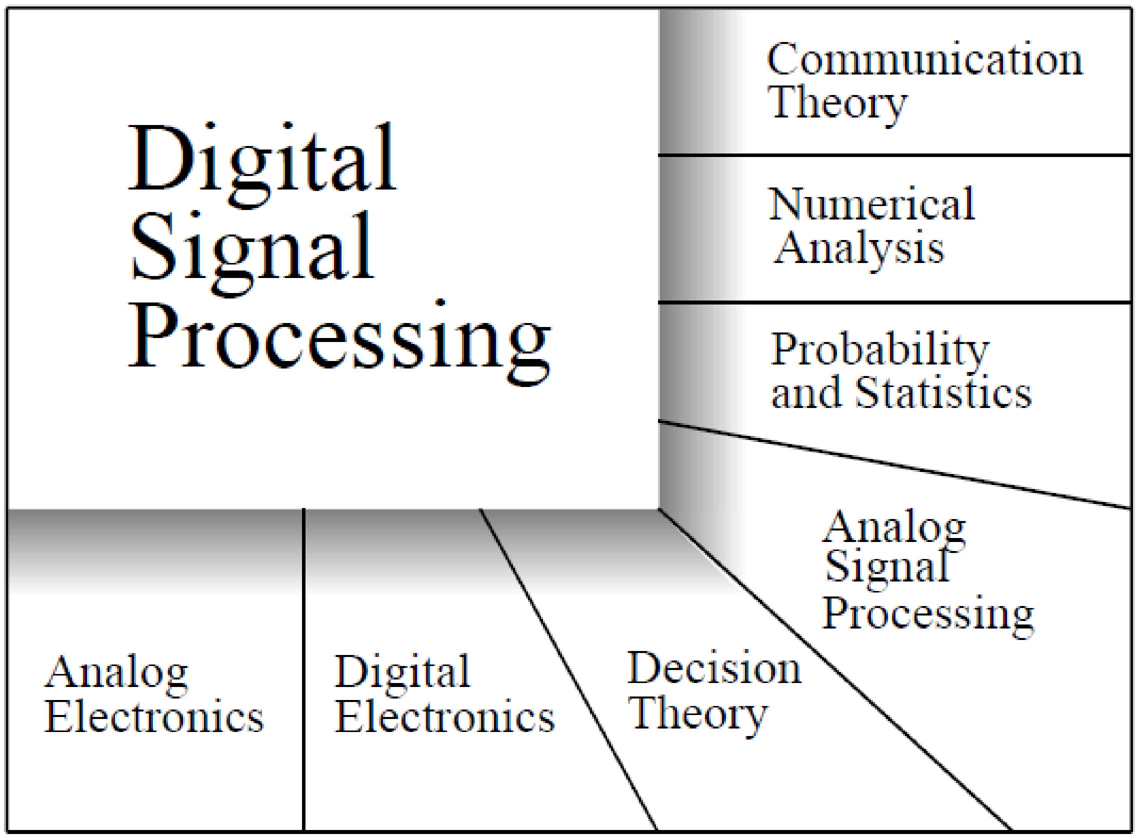


→ Digital signals

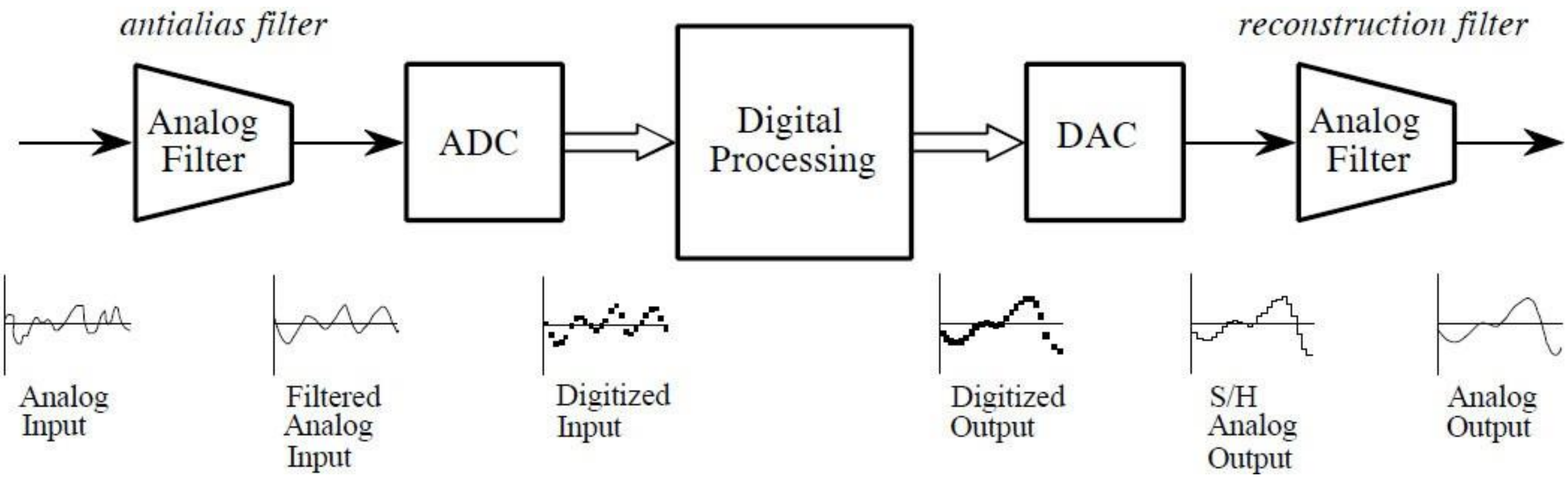
Diverse Applications of DSP



Allied Areas of DSP



A Block Diagram of a DSP System



DSP is the mathematics, algorithms, and techniques used to manipulate signals after they have been converted into a digital form

Advantages & Disadvantages of DSP over ASP

Advantages of DSP

→ The physical size of analog systems is quite large while digital processors are more compact and light in weight (reduces the costs of memories, gates, microprocessors, and so forth)

→ Digital components are less sensitive to environmental changes, noise, and disturbances

→ Digital system is most flexible as software programs & control programs can be easily modified

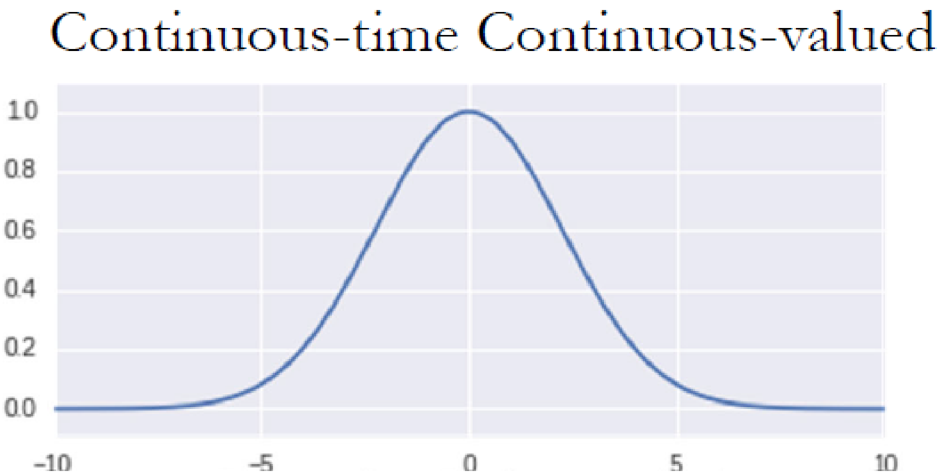
→ Based solely on additions and multiplications, leading to extremely stable processing capability

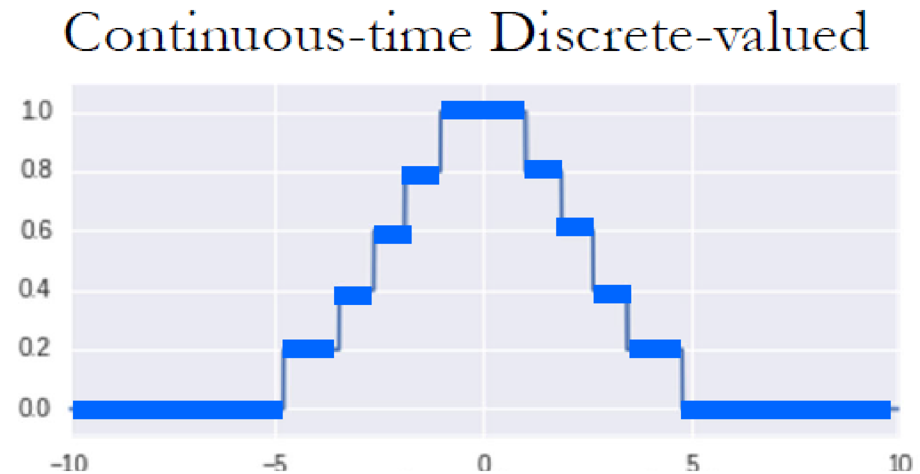
Disadvantages of DSP (Due to conversion from Analog Signals)

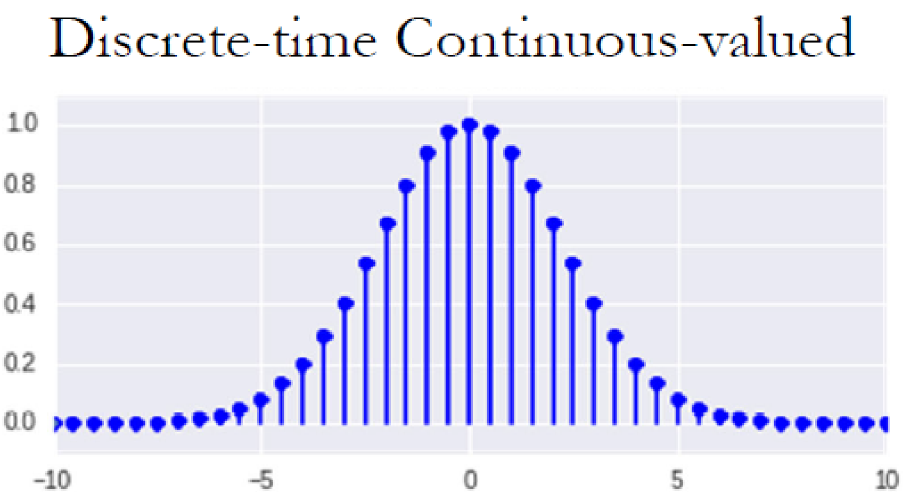
→ Distortion – sampling the signal and quantizing the samples

→ Finite precision effects

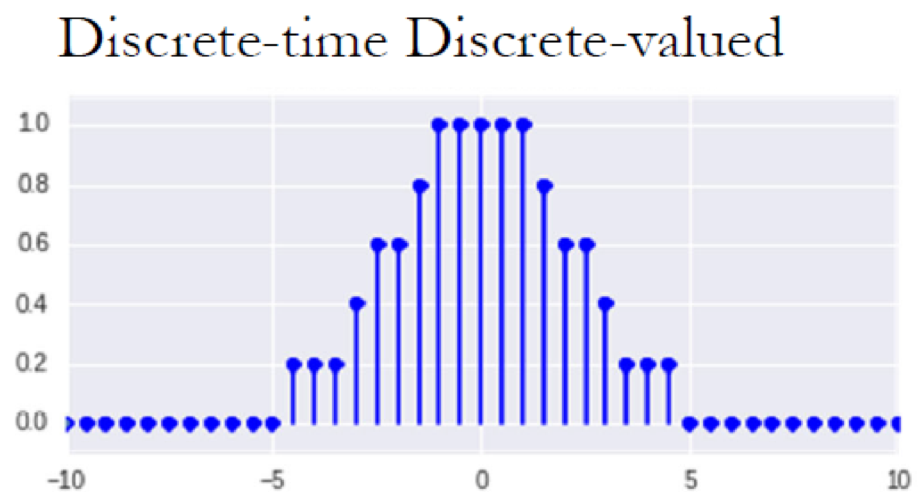
Discrete-time versus Digital Signals





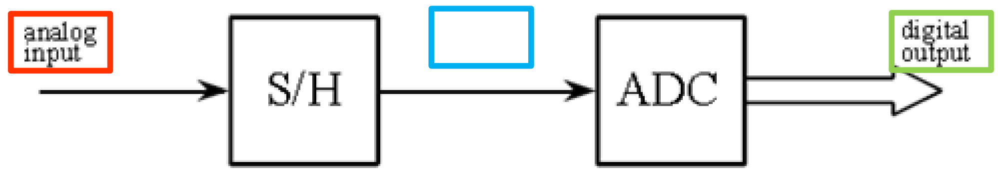


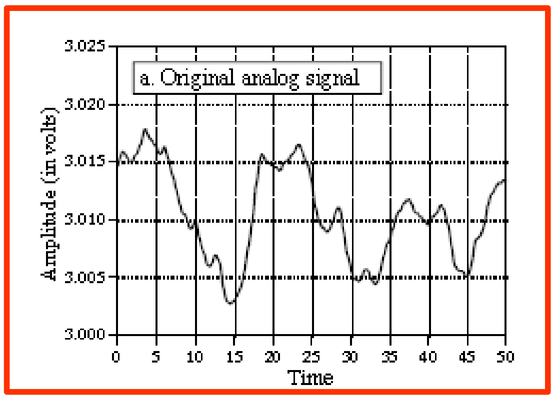
→ **Discrete signals**

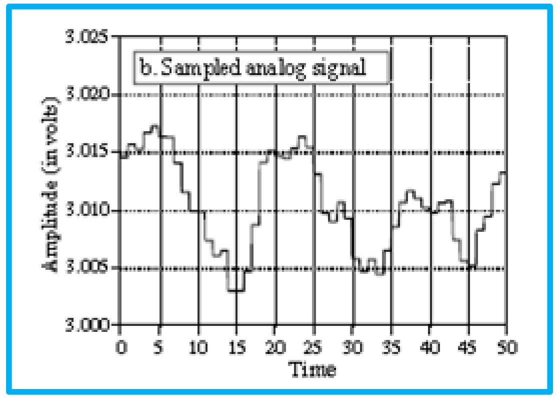


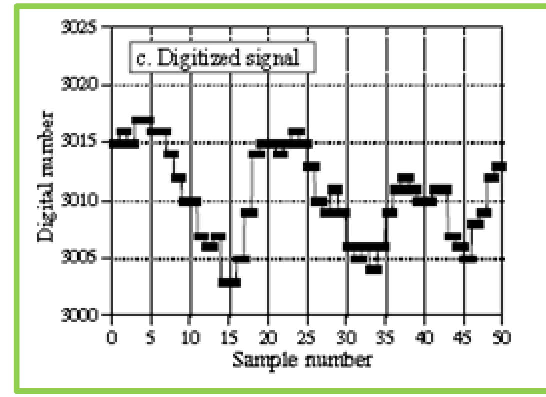
→ **Digital signals**

Difference between Discrete-time & Digital Signal

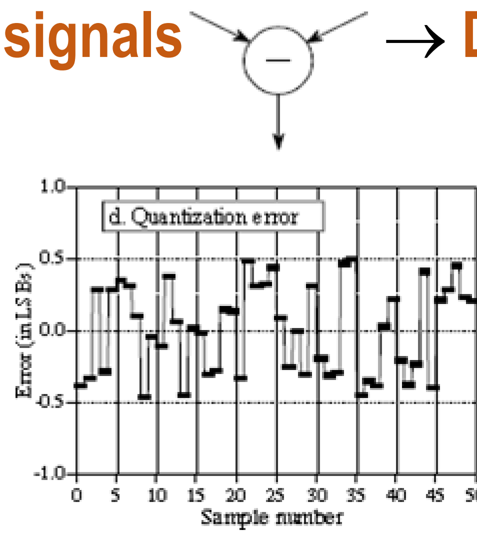




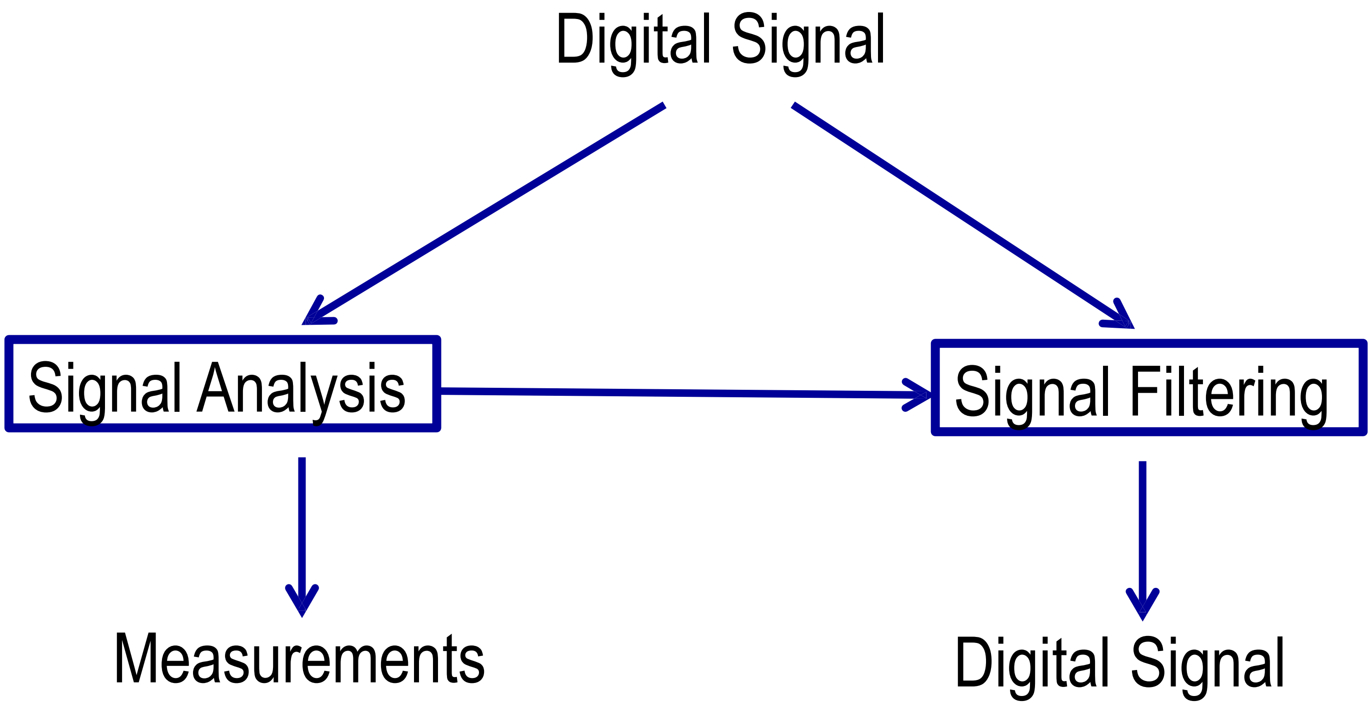




→ **Discrete signals** → **Digital signals**



Two Important Categories of DSP



Signal Analysis

- ## Signal Analysis

Deals with the measurement of signal properties
Generally, a frequency-domain operation

- ## Examples

$\rightarrow$ Spectrum (frequency and/or phase) analysis

Signal Filtering

- **Signal Filtering**

  "Signal in-signal out" situation, generally called *filters*
  Usually (but not always) a time-domain operation

- **Examples**

  $\rightarrow$ Removal of unwanted background noise
  $\rightarrow$ Removal of interference
  $\rightarrow$ Separation of frequency bands
  $\rightarrow$ Shaping of the signal spectrum

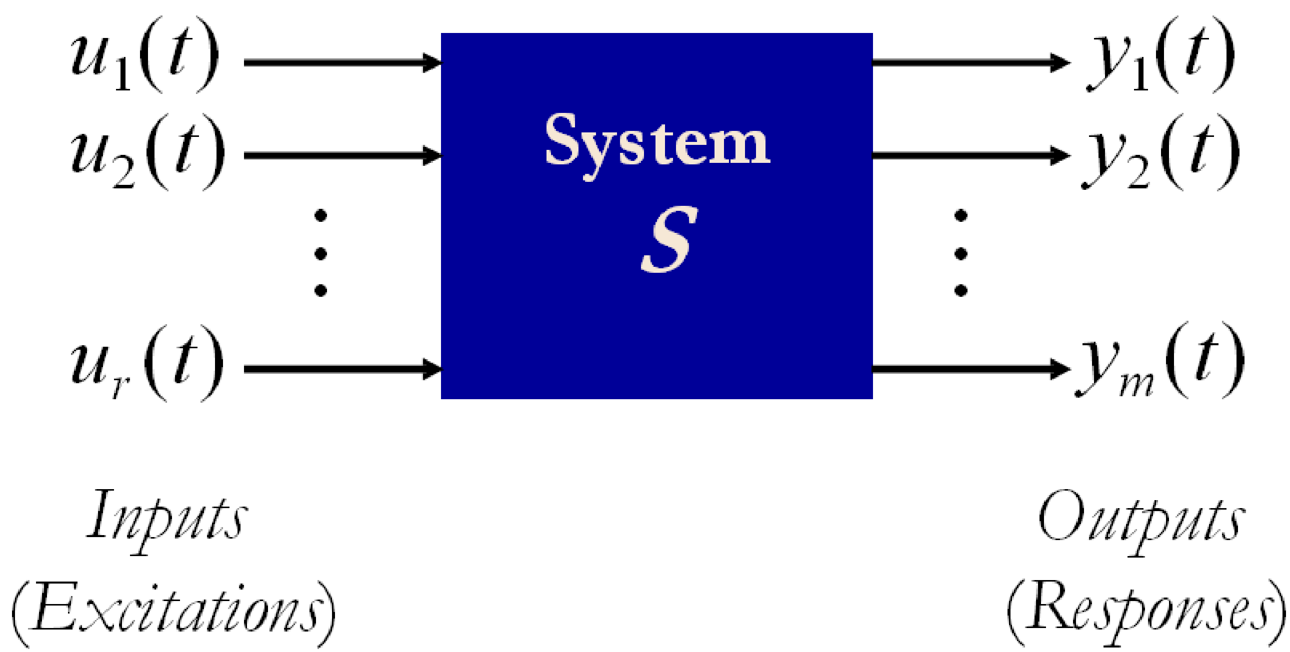
Speech synthesis:

a signal is first analyzed to study its characteristics, which are then used in digital filtering to generate a synthetic voice.

Linear Systems

Physical System

- **Physical System**
  - → An interconnection of physical components that perform a specific function.
  - → These components may be electrical, mechanical, hydraulic, thermal, etc.



Signals in Physical System

▪ Signals in System

→ Associated with every system is a variety of physical quantities such as electrical voltages and currents, mechanical forces and displacements, flow rates, and temperatures

▪ Input/Output Signals

Inputs or excitations

Some of the signals can be directly changed with time in order to effect indirectly desired changes in some other signals of the system that happens to be of particular interest.

Outputs or responses

The system receives inputs and transforms them into outputs!

Linear System

▪ Linear System

→ Systems that satisfy both <u>homogeneity</u> and <u>additivity</u> are considered to be *linear systems*

→ These two rules, taken together, are often referred to as the *principle of superposition*

▪ Linear System Applications

→ Automatic control theory

→ Signal processing

→ Telecommunications

Why Linear Systems Theory?

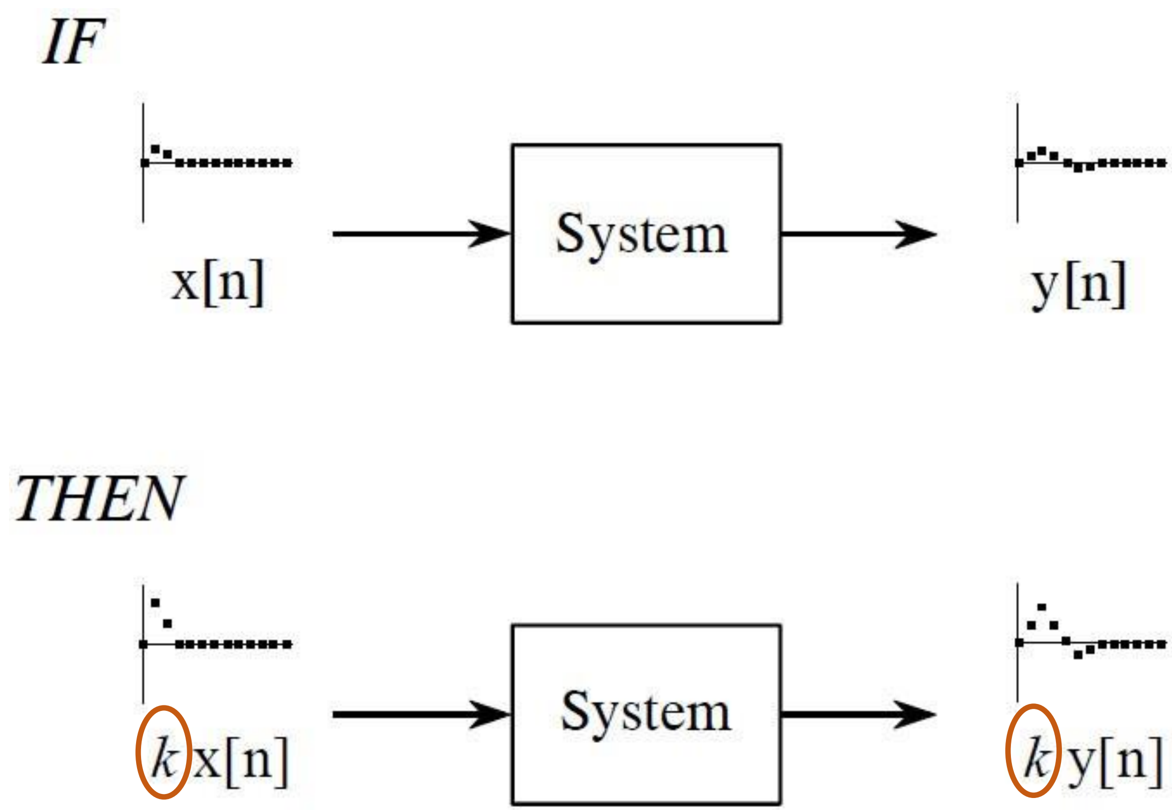
- **Linear System Theory**

  → Characterizing the complete input-output properties of a system by exhaust measurement is usually impossible

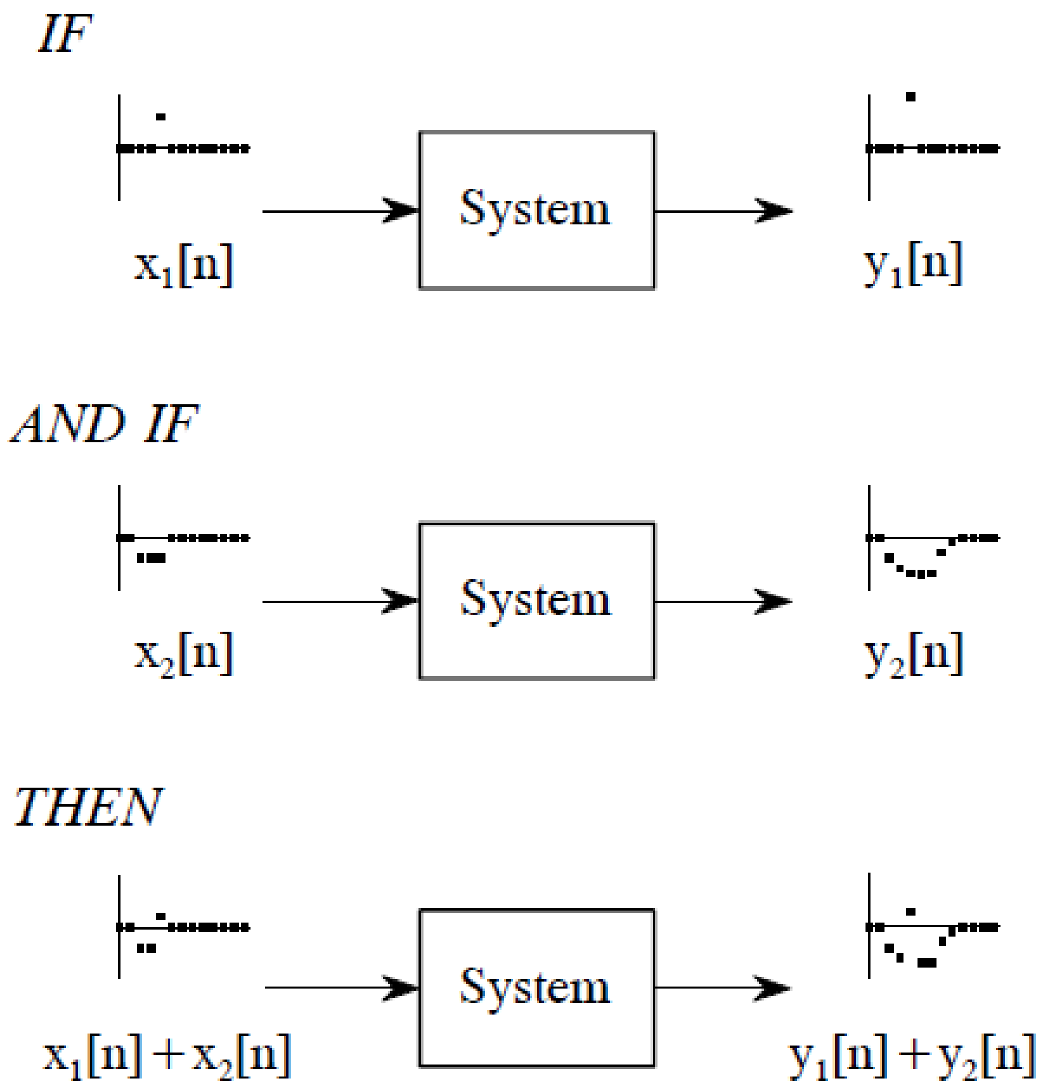
  → When a system qualifies as a linear system, it is possible to use the responses to a small set of inputs to predict the response to any possible input

  → This can save the scientist enormous amounts of work and makes it possible to characterize the system completely

Property of Linear System: Homogeneity

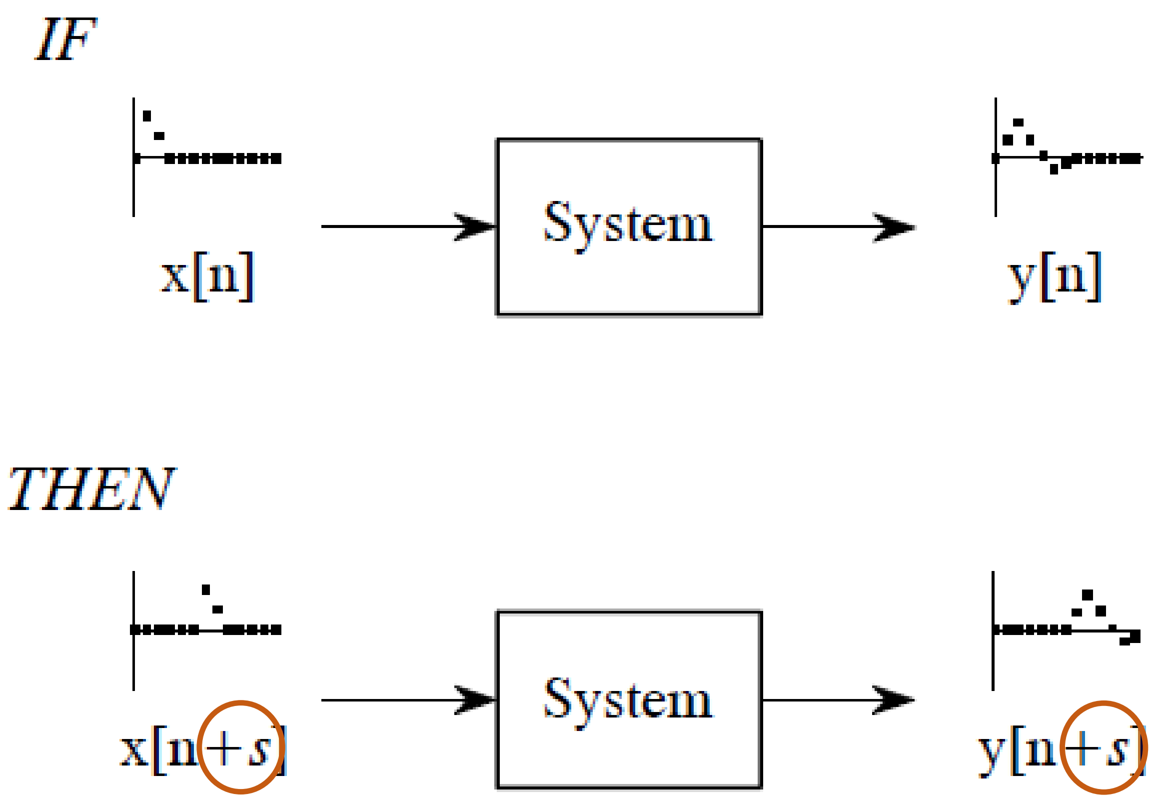


Property of Linear System: Additivity



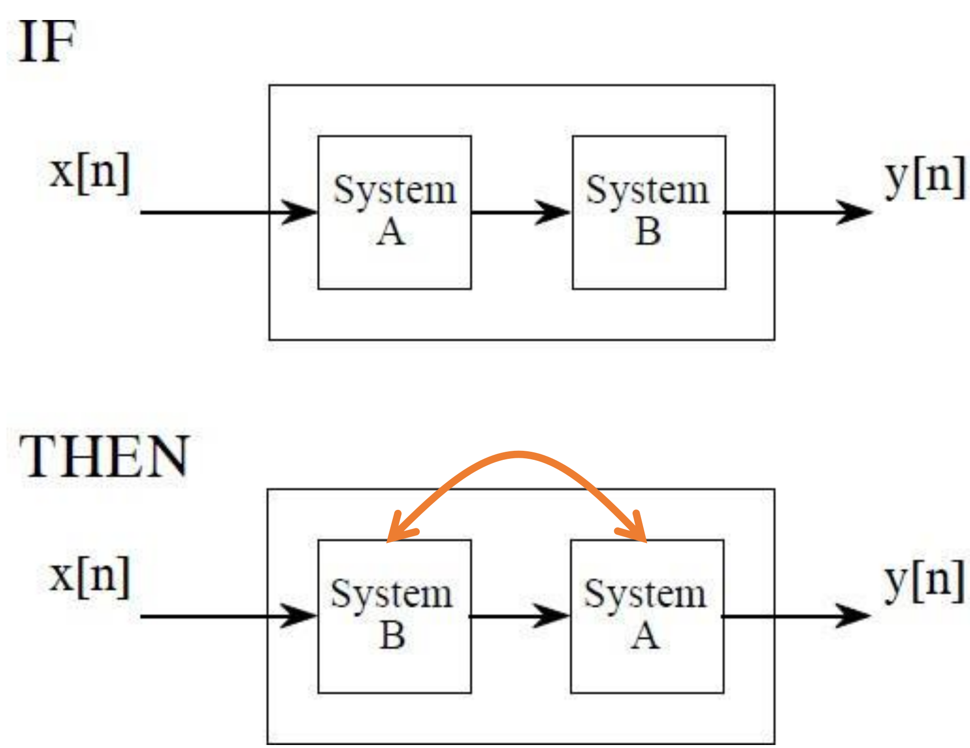
Special Properties:
Shift-Invariant Linear System (SILS)

→ It is not a strict requirement for linearity,
but it is a mandatory property for most DSP

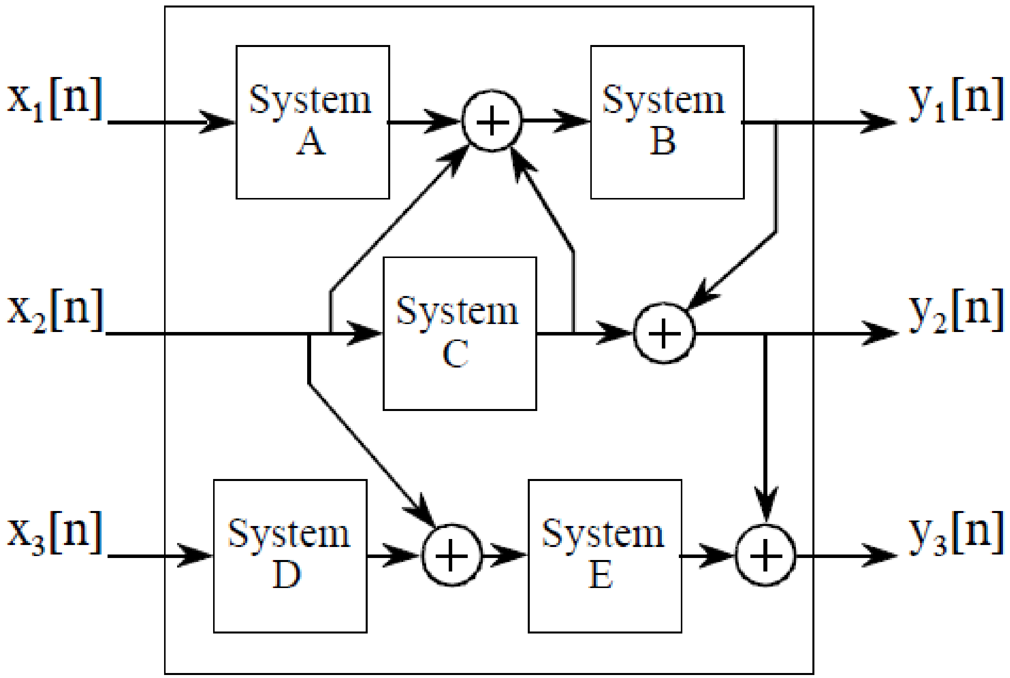


Example: $s = -2$ → shifted right [delayed]; $s = 2$ → shifted left [advanced];

Special Properties: Commutative

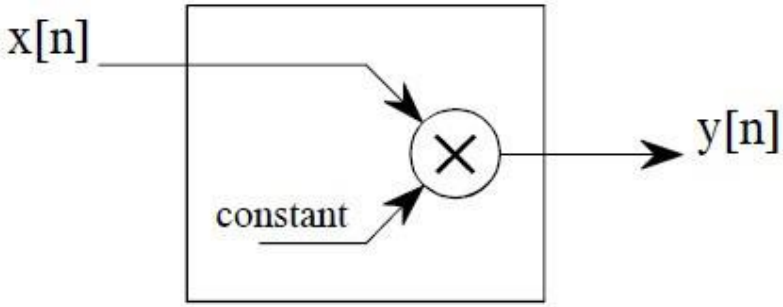


Multiple Inputs and/or Outputs



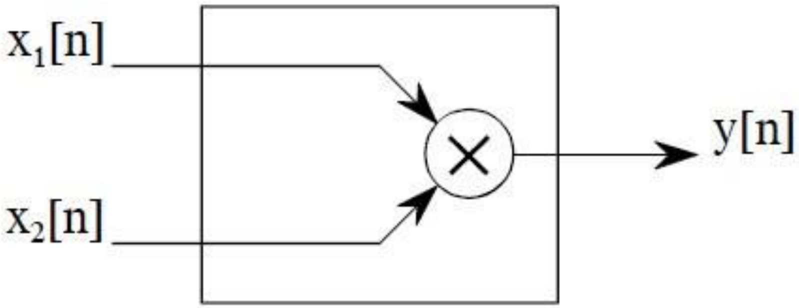
A system with multiple inputs and/or outputs will be linear If it is composed of linear systems and signal addition

Multiplication Signals



Multiplication by a constant

Linear system



Multiplication of two signals

Nonlinear system

The Foundation of DSP: Superposition

- **Superposition Property of Linear Systems**

  $\rightarrow$ The response of a linear system to a sum of signals is the sum of the responses to each individual input signal

- **Objectives of DSP**

  $\rightarrow$ Replace a complicated problem with several easy ones

  $\rightarrow$ **Decomposition & Synthesis:**

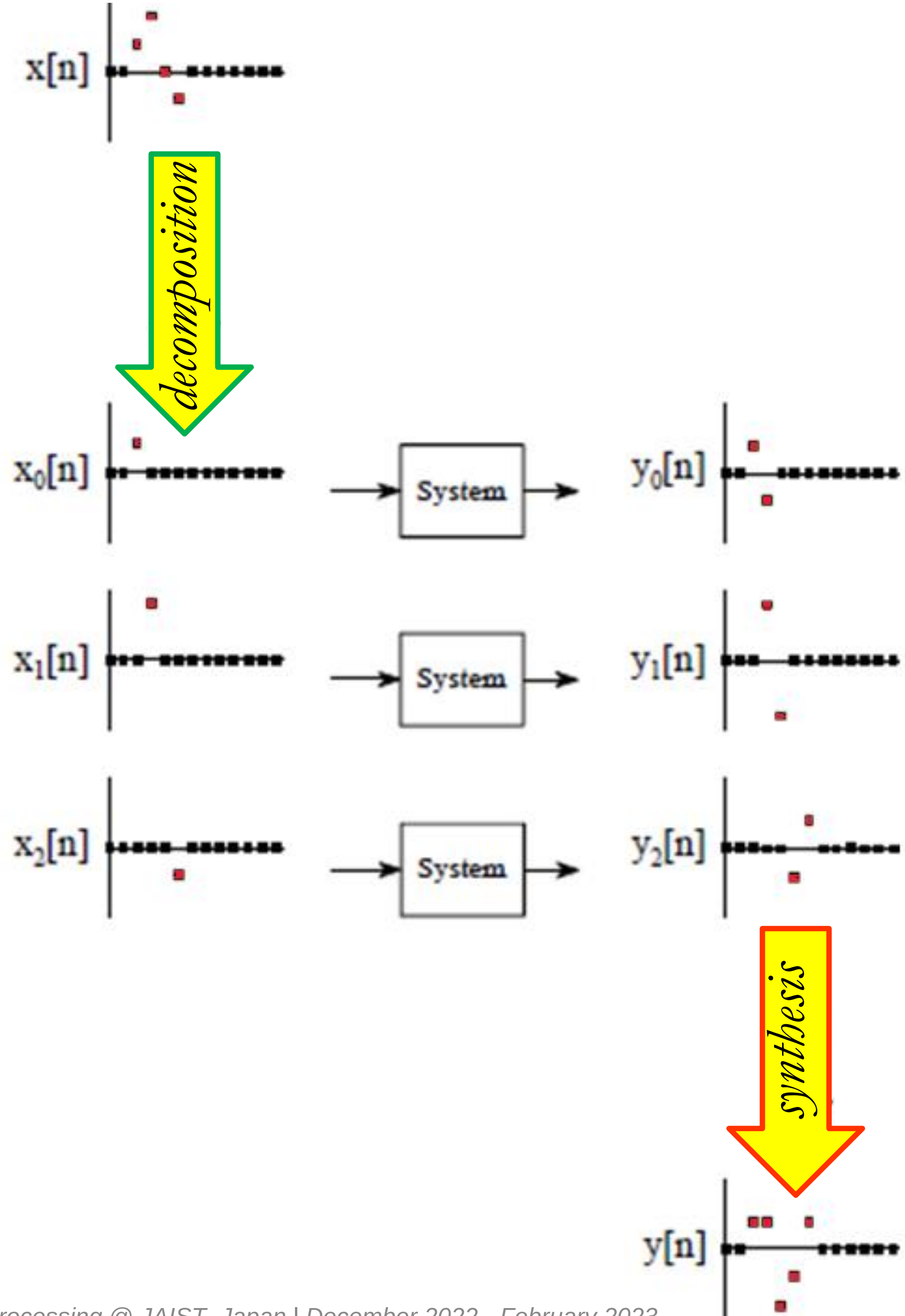
  Decomposition

  A single signal is broken into two or more additive components

  Synthesis

  Combining signals through scaling and addition

Example:



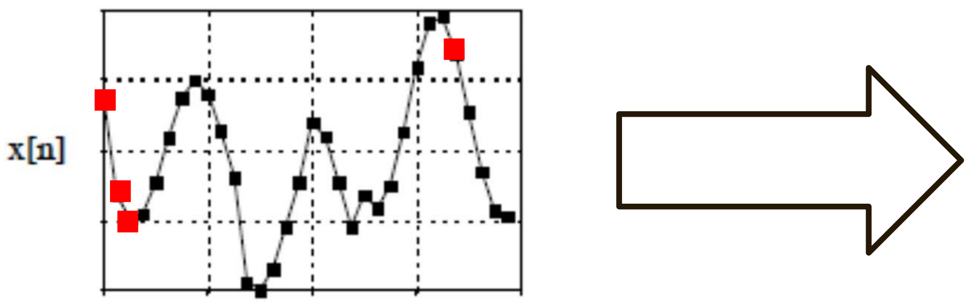
x[n]
decomposition
x0[n]
System
y0[n]
x1[n]
System
y1[n]
x2[n]
System
y2[n]
synthesis
y[n]

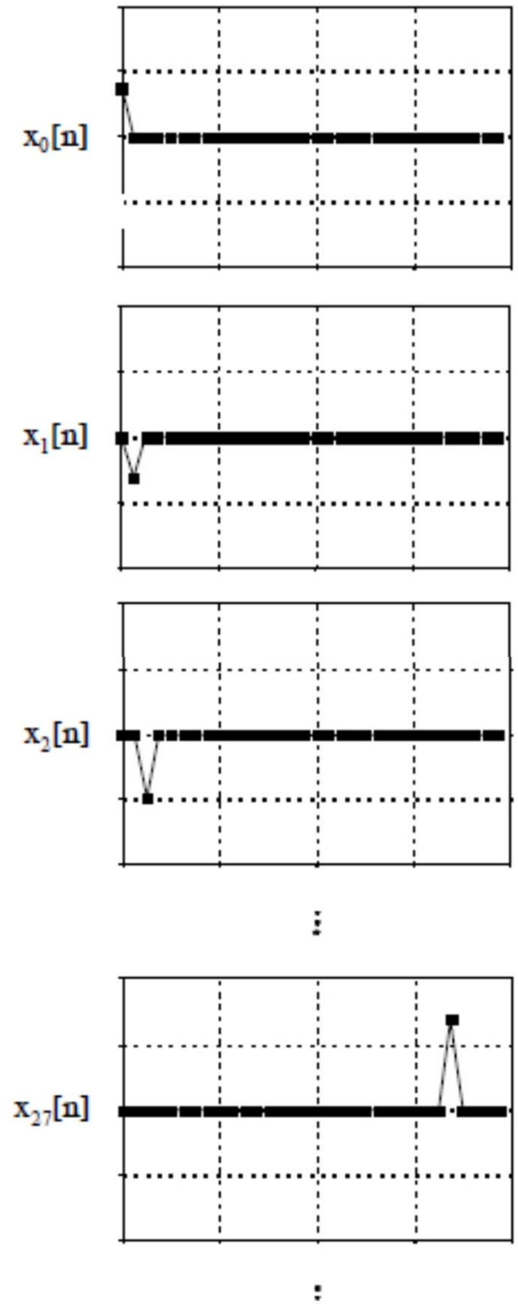
Common Decomposition

- **Various Decomposition**
  - → Impulse Decomposition
  - → Step Decomposition
  - → Even/Odd Decomposition
  - → Interlaced Decomposition
  - → Fourier Decomposition

The main ways:
Impulse decomposition and Fourier decomposition
(Others are only occasionally used)

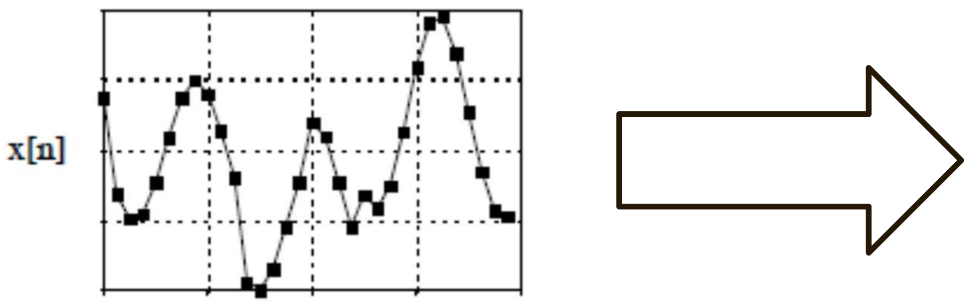
Impulse Decomposition





Impulse: a single non-zero point in a string of zeros

Step Decomposition

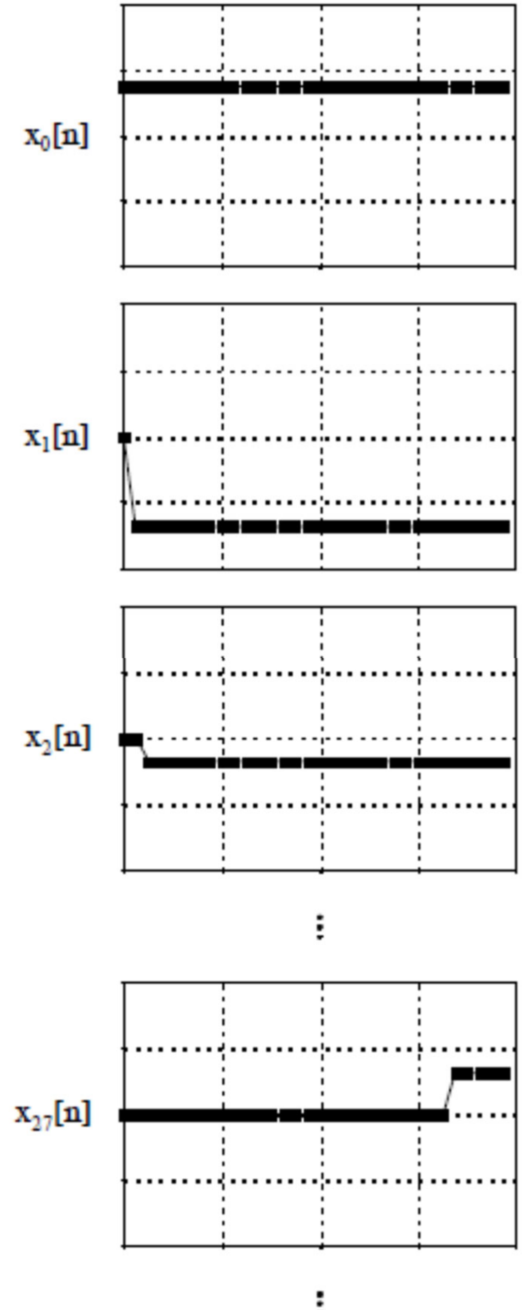


***k*^th component signal** $x_k[n]$

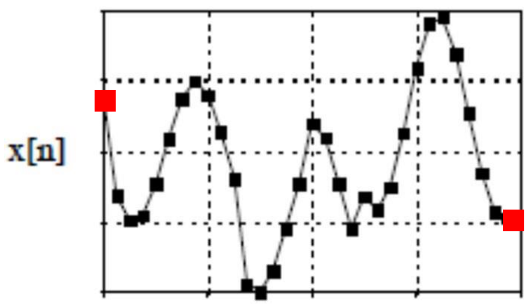
→ *zeros for points 0 through* $k-1$,

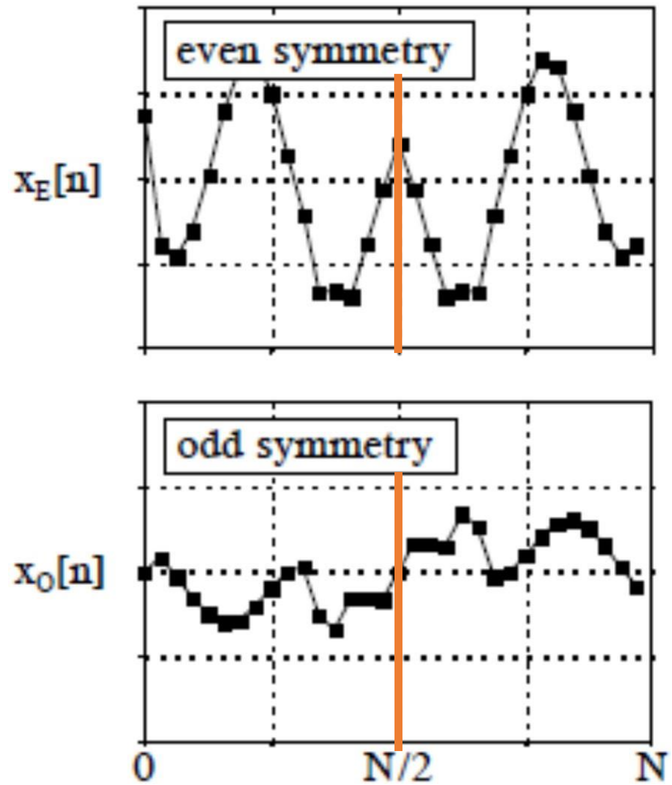
→ *non-zero for remaining points*

with value $x[k]-x[k-1]$



Even/Odd Decomposition

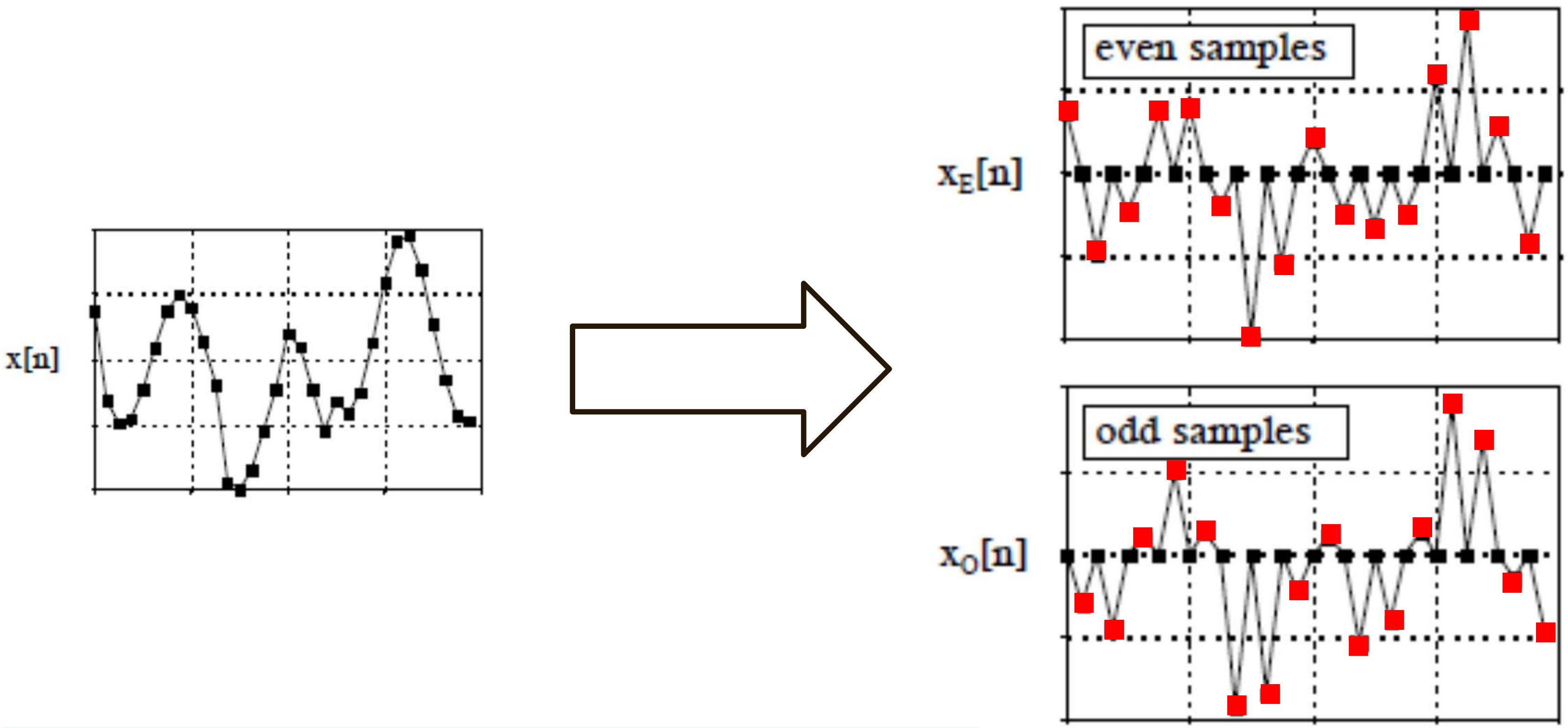




$$x_E[n] = \frac{x[n] + x[N-n]}{2}$$

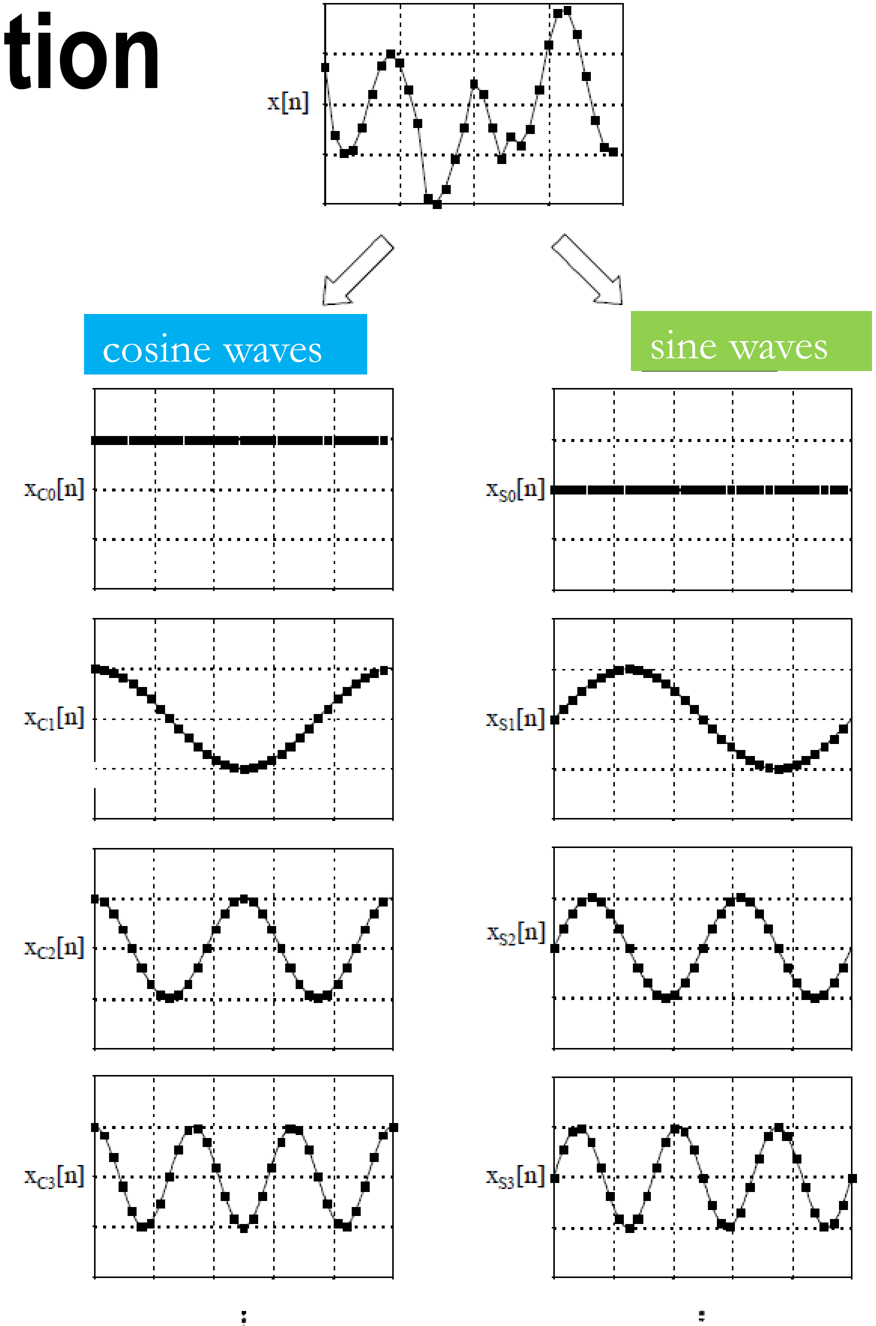
$$x_O[n] = \frac{x[n] - x[N-n]}{2}$$

Interlaced Decomposition



Even samples: odd samples set to zero
Odd samples: even samples set to zero

Fourier Decomposition



Decompose an arbitrary signal
Into sine and cosine waves

Commonly Used Decomposition

- **Various Decomposition**
  - → Impulse Decomposition
  - → Step Decomposition
  - → Even/Odd Decomposition
  - → Interlaced Decomposition
  - → Fourier Decomposition

The main ways:

Impulse decomposition *and* ***Fourier decomposition***

(Others are only occasionally used)

Impulse Decomposition

- **Why impulses are special?**

  → The system's measurement of an impulse can be the key measurement to make

  → The trick is to conceive of the complex stimuli as a combination of impulses

  → We can approximate any complex stimulus as if it were simply the sum of a number of impulses that are scaled copies of one another and shifted in time

Impulse Decomposition

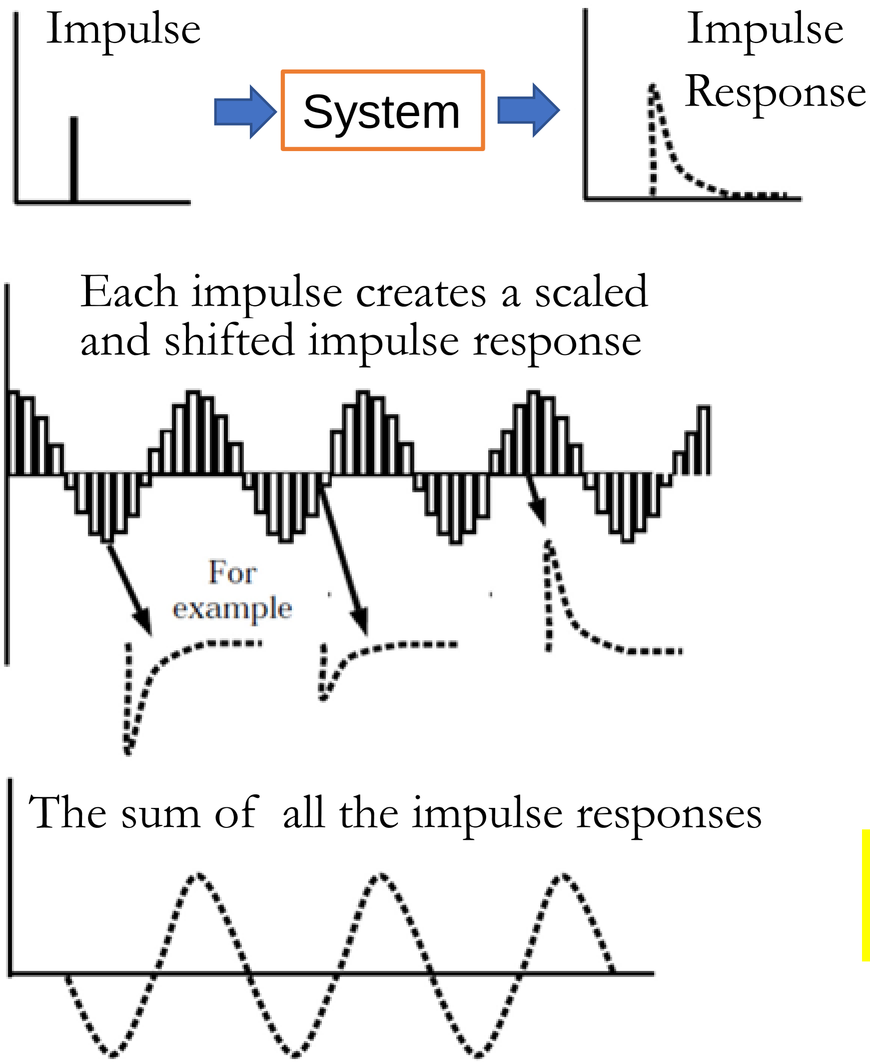
- **For Shift-Invariant Linear Systems (SILS)**

  $\rightarrow$ We can measure the system's response to an impulse
  and we will know how to predict the response to any stimulus
  (combinations of impulses) through the principle of superposition.

  $\rightarrow$ To characterize shift-invariant linear systems,
  we need to measure only one thing:
  the way the system responds to an impulse of a particular
  intensity: the impulse response function of the system

Impulse Decomposition

- **Impulse Response Analysis**



Measure only a single impulse response function

Can predict how the system will respond to any other possible stimulus!

Commonly Used Decomposition

- **Various Decomposition**
  - $\rightarrow$ Impulse Decomposition
  - $\rightarrow$ Step Decomposition
  - $\rightarrow$ Even/Odd Decomposition
  - $\rightarrow$ Interlaced Decomposition
  - $\rightarrow$ Fourier Decomposition

The main ways:

Impulse decomposition *and* ***Fourier decomposition***

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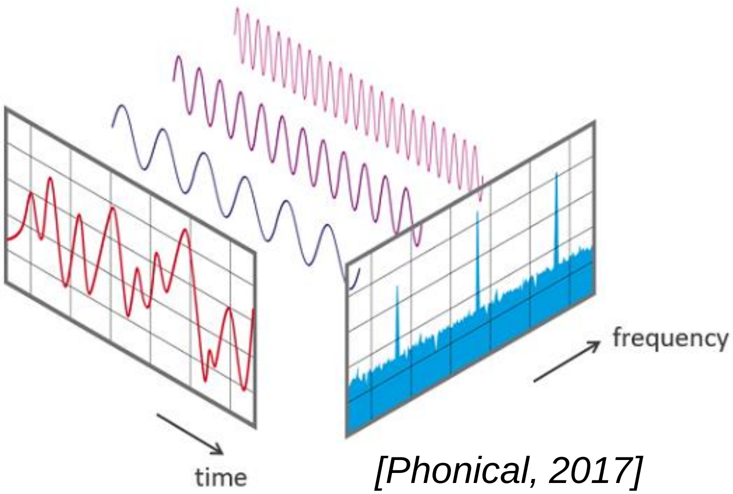
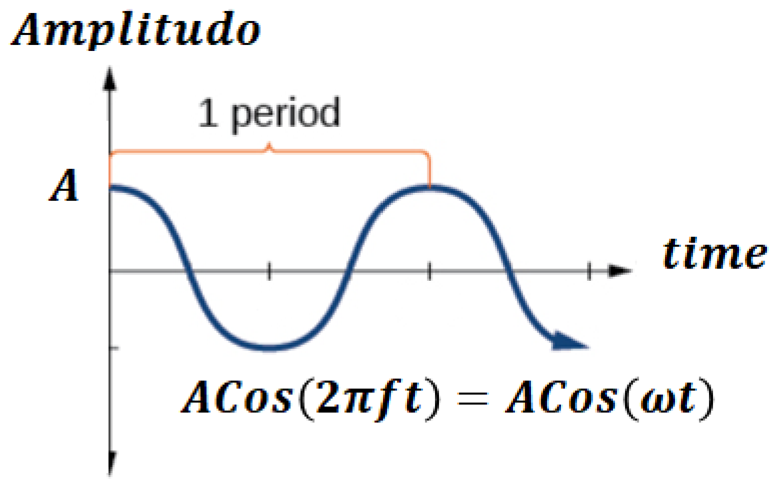
Function Transformation

- **Why Sinusoids are Special?**
  → Most periodic signals are composed of an infinite sum of sinusoids

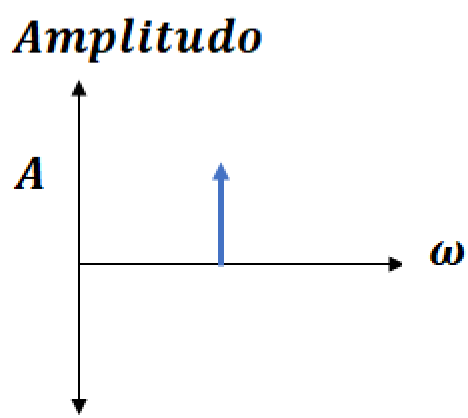


[Phonical, 2017]




```
A = amplitude
T = period of 1 cycle
f = frequency

Example:
1 cycle = 20 sec
f = 1/T = 0.05 Hz
ω = 2π/T = 18 rad/sec
```



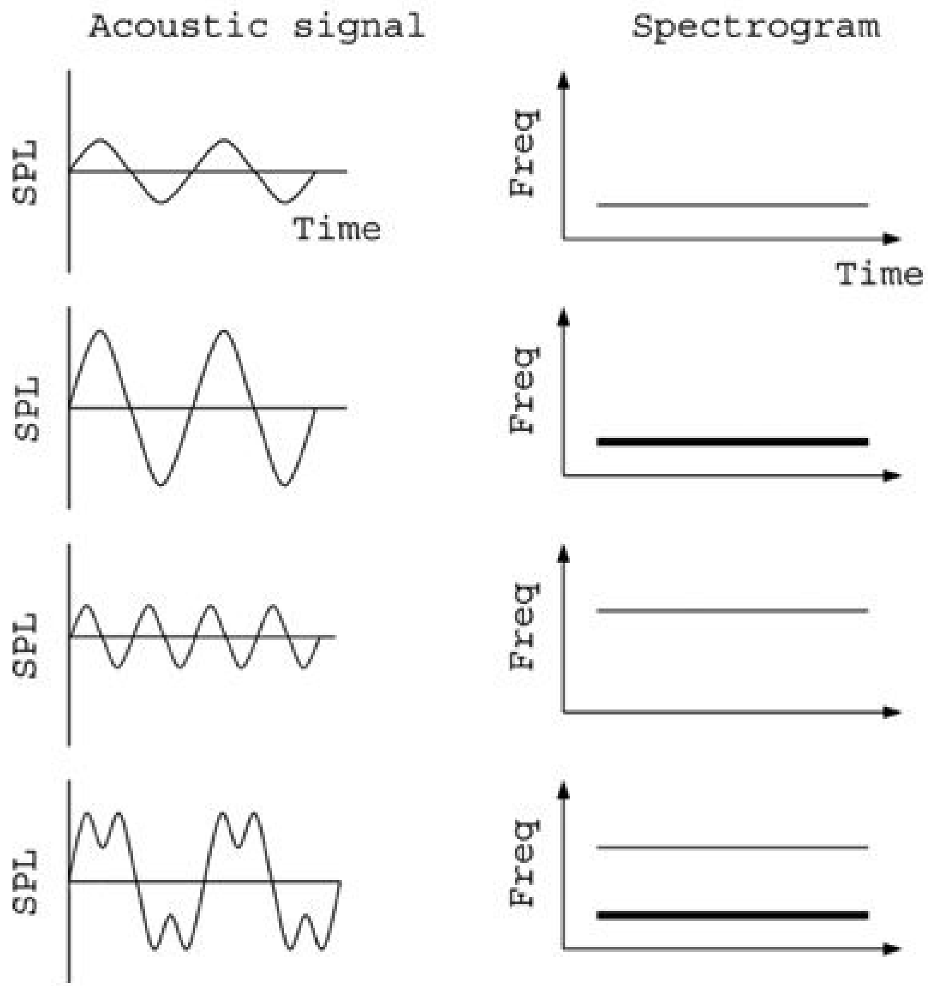
440Hz

1kHz

Fourier Decomposition

- **Why Sinusoids are Special?**

  → Most periodic signals are composed of an infinite sum of sinusoids



Fourier Decomposition

- **Why Sinusoids are Special?**
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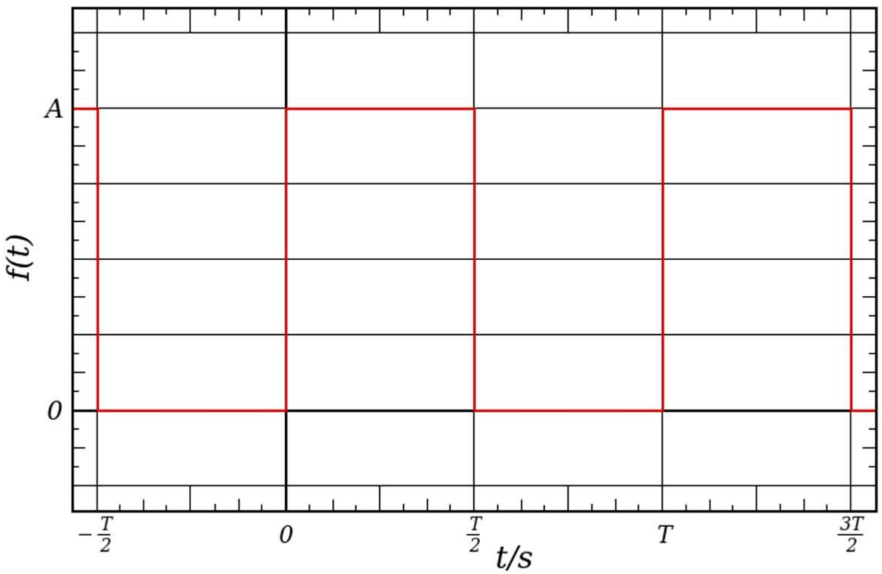
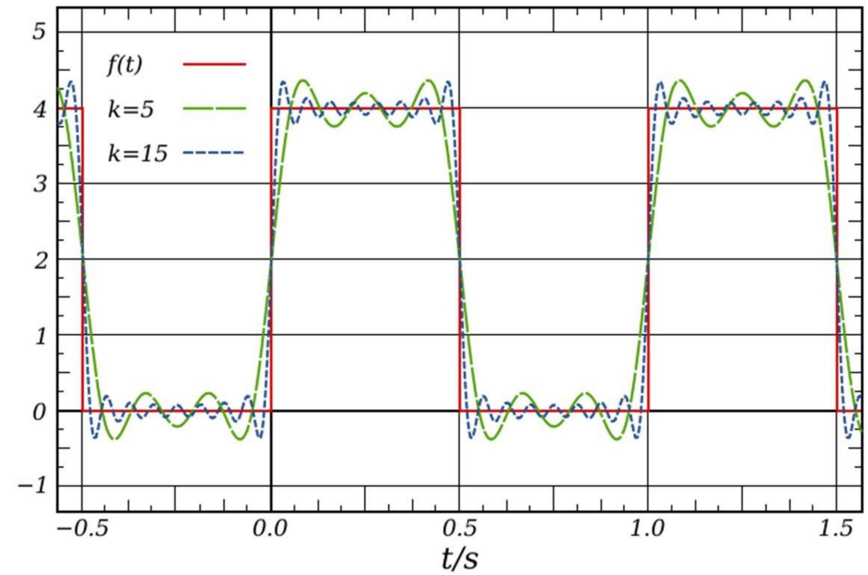




Original squarewave

Fourier series approximation

Fourier Decomposition

Why Sinusoids are Special?

$\rightarrow$ When we use a sinusoidal stimulus as input to a SILS, the system's response is always a (shifted and scaled) copy of the input, at the same frequency as the input.

$\rightarrow$ Sinusoids are the only waveform that has this property: *sinusoidal fidelity*

$$\sin(2\pi ft) \rightarrow \underbrace{A}_{\text{Amount of scaling}}\sin(2\pi ft + \underbrace{\phi}_{\text{Amount of shift}})$$

Measuring the response to a sinusoid for a ***SILS*** *entails measuring only two numbers: the shift and the scale. Quite practical!*

Fourier Decomposition

- **Why Sinusoids are Special?**

$$u(t) = \boxed{e^{j\omega t}}$$

$$y(t) = \int_{\tau=0}^{t} g(t-\tau) e^{j\omega\tau} d\tau$$

integral convolution

$$y(t) = \int_{\tau=0}^{t} g(t-\tau)u(\tau)d\tau$$

$$= \int_{\bar{\tau}=t}^{0} g(\bar{\tau})u(t-\bar{\tau})(-d\bar{\tau}), \quad \bar{\tau} := t-\tau$$

$$\tau = [0\,..\,t]$$
$$\bar{\tau} = [t\,..\,0]$$

$$= \int_{\bar{\tau}=0}^{t} g(\bar{\tau})u(t-\bar{\tau})(d\bar{\tau})$$

$$= \int_{\tau=0}^{t} g(\tau)u(t-\tau)d\tau, \quad \tau \to \bar{\tau}$$

$$= \int_{\tau=0}^{t} g(\tau)e^{j\omega(t-\tau)}d\tau$$

$$= \boxed{e^{j\omega t}} \int_{0}^{t} g(\tau)e^{-j\omega\tau}d\tau$$

Fourier Decomposition

- **Why Sinusoids are Special?**

$$\boxed{e^{at}} \quad \Rightarrow \quad \frac{de^{at}}{dt} = a\boxed{e^{at}}$$

$$\boxed{\sin at} \quad \Rightarrow \quad \frac{d^2 \sin at}{dt^2} = -a^2 \boxed{\sin at}$$

Response of SILSs to Sine Waves

▪ Fourier Series

Express any periodic stimulus as the sum of a series of (shifted and scaled) sinusoids at different frequencies: *Fourier Series* expansion of the stimulus

$$s(t) = A_0 + A_1 \sin(2\pi f_1 t + \phi_1) + A_2 \sin(2\pi f_2 t + \phi_2) + A_3 \sin(2\pi f_3 t + \phi_3) + \cdots$$

▪ Fourier Transform

If you know the stimulus *s*(*t*), you can compute the coefficients by the method called the Fourier Transform (a way of decomposing complex stimuli into their component sinusoids).

Response of SILSs to Sine Waves

- **Frequency Response Analysis**
  - $\rightarrow$ Measure the system's response to sinusoids of all different frequencies
  - $\rightarrow$ Take the input stimulus and use the Fourier Transform to compute the values of the coefficients in the Fourier Series expansion (the stimulus has been broken down as the sum of its component sinusoids)
  - $\rightarrow$ Predict the system's response to the (complex) stimulus simply by adding the responses for all the component sinusoids

Analytical Methods of Linear Systems

Linear Systems Logic

Measure the impulse response

Space/time method

Input Stimulus

Express as sum of scaled and shifted impulses	→	Calculate the response to each impulse	→	Sum the impulse responses to determine the output
Express as sum of scaled and shifted sinusoids	→	Calculate the response to each sinusoid	→	Sum the sinusoidal responses to determine the output

Measure the sinusoidal responses

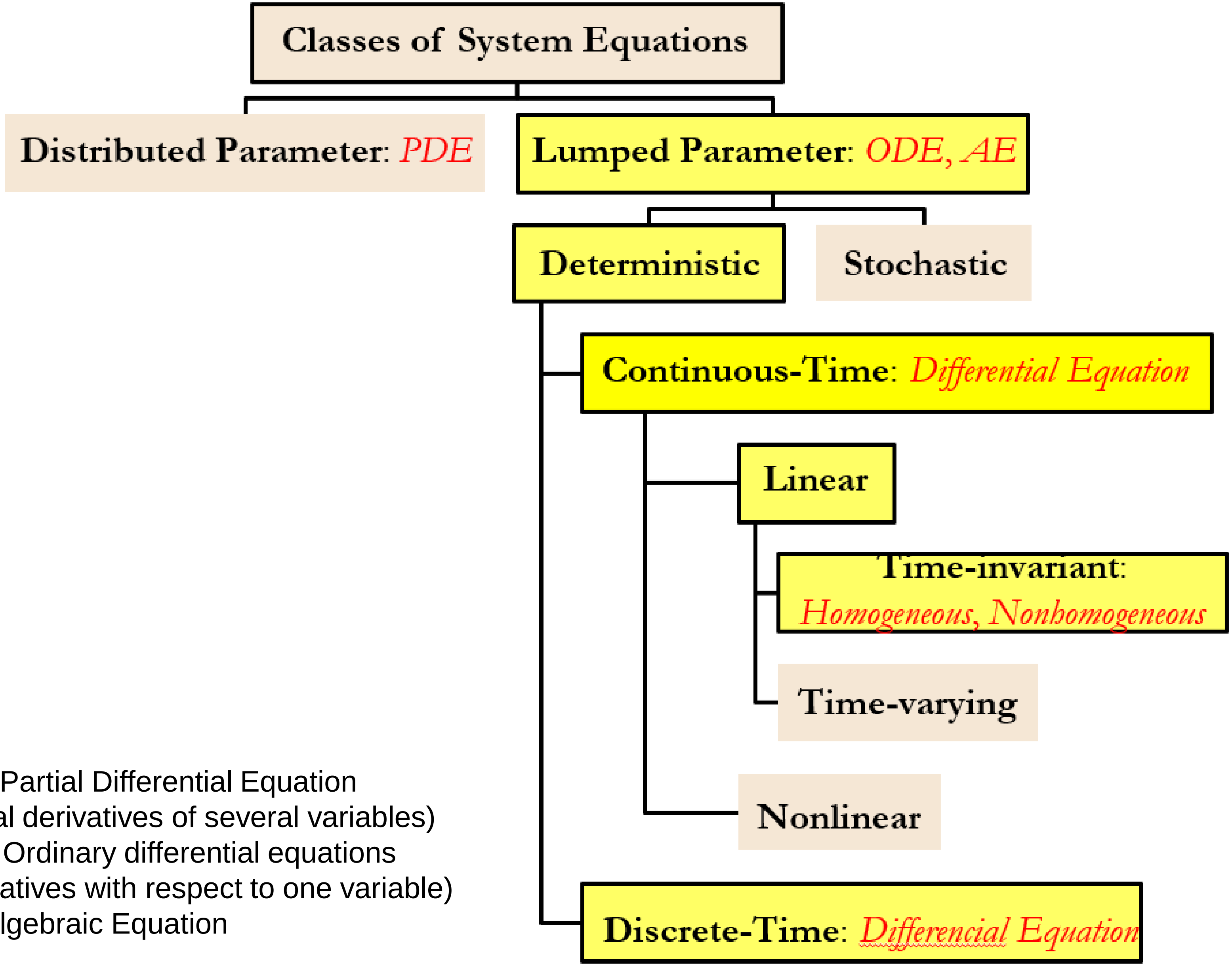
Frequency method

Measure the Response

- **Empirical Method**
  → Apply various signals to a physical system and measure its responses.
  → Trial-and-error
  → May become unworkable if physical systems are complex or too expensive or too dangerous to be experimented on

- **Analytical Method**
  → Modeling
  → Development of mathematical descriptions:
  Kirchhoff's voltage and current laws, Newton's law
  → Analysis
  → Design

Analytical Method: Mathematical Equations



PDE: Partial Differential Equation
(Partial derivatives of several variables)
ODE: Ordinary differential equations
(Derivatives with respect to one variable)
AE: Algebraic Equation

Analytical Method: Analysis & Design

- **Analysis**

  → **<u>Quantitative</u>** – responses of systems excited by certain inputs

  → **<u>Qualitative</u>** – stability, controllability, observability

- **Design**

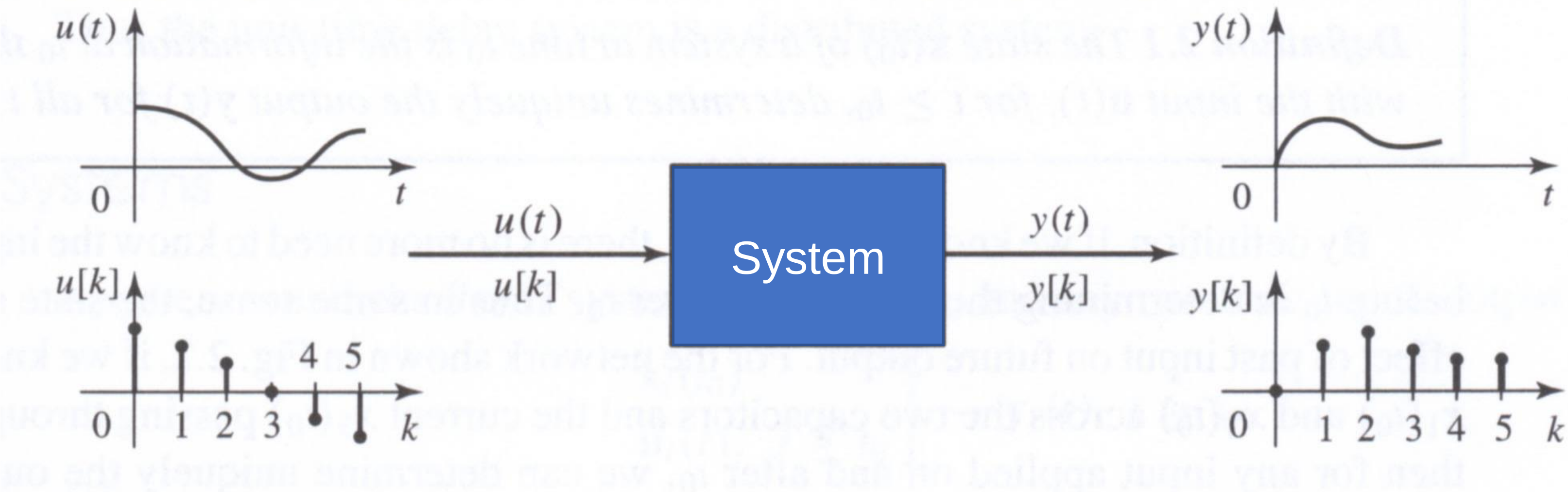
  → If the response of a system is unsatisfactory:

  1. adjust the system parameters
  2. introduce compensators

Selecting a model that is close enough to a physical system and yet simple enough to be studied analytically is the most difficult and important problem in system design.

Mathematical Descriptions of Systems

- **System**



input terminals

output terminals

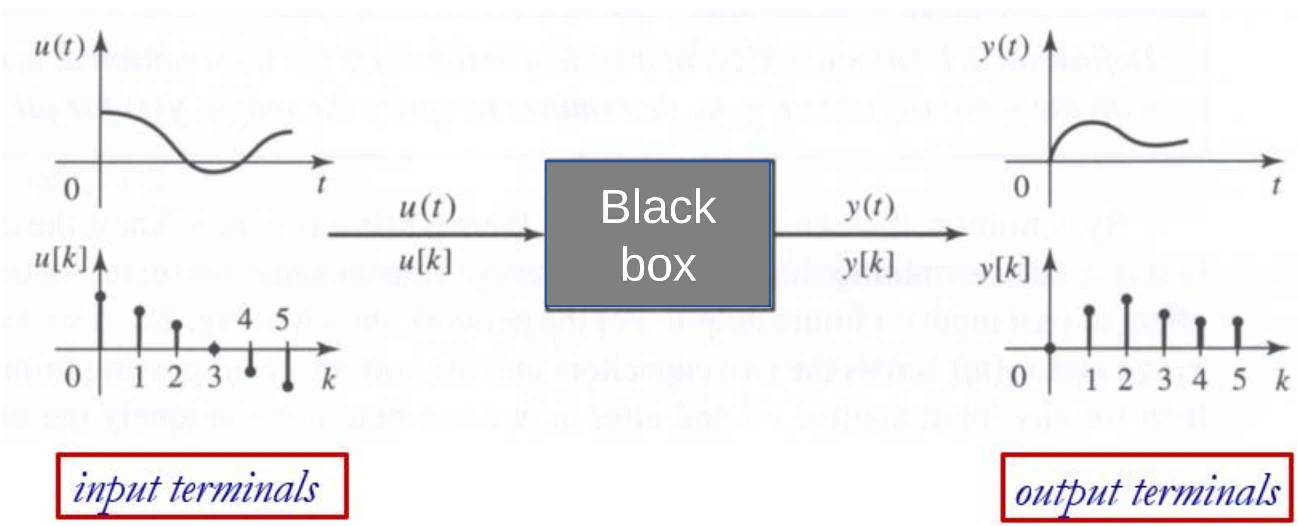
External Description

▪ External Description: Input-Output Description

$\rightarrow$ View the system as a "black box" description:

no information on the internal details of the system

$\rightarrow$ Characterize by the relation of input, output, and system response (impulse response)



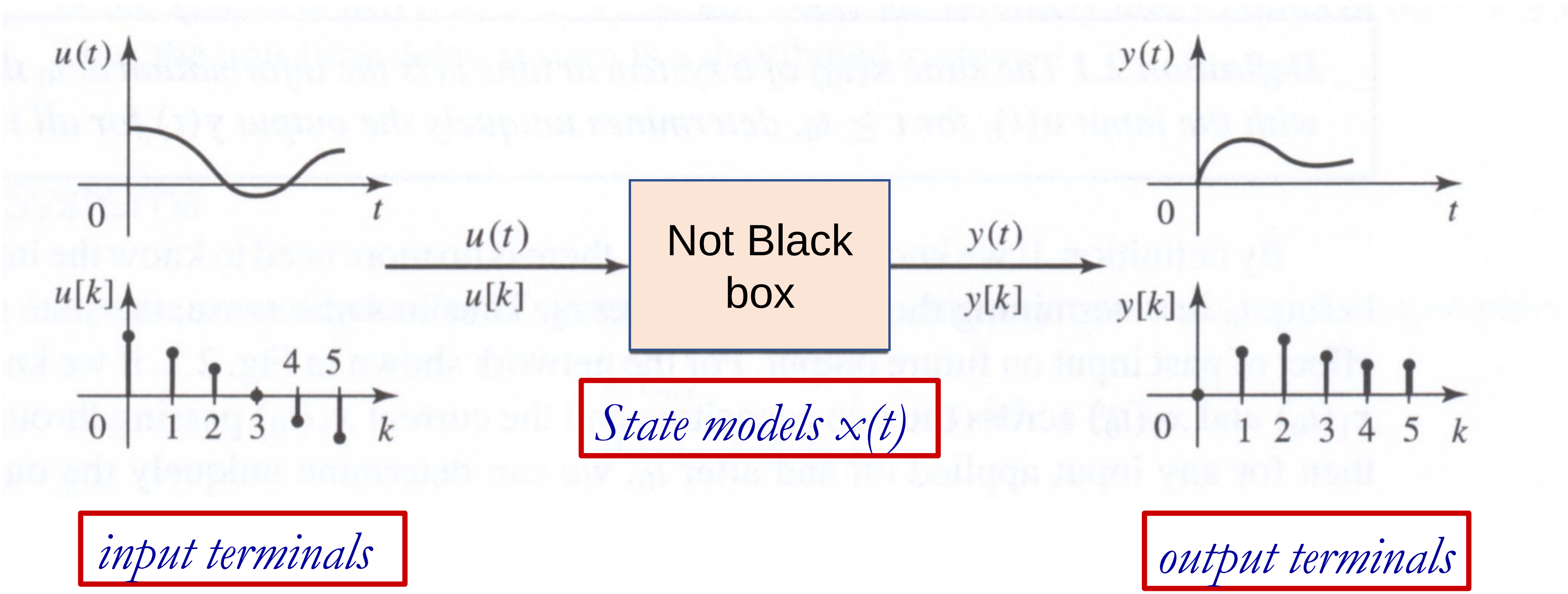
$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

output

input

Mathematical Descriptions of Systems

- ## System



Internal Description

▪ Internal Description: State-Space Description

→ State-space representation:

a mathematical model of a physical system as a set of input, output, and state variables related by first-order differential equations or difference equations

→ If the linear system is lumped (the number of state variables is finite)

state

$$x^{\cdot}(t) = A(t)x(t) + B(t)u(t)$$ 1^{st} order DE

$$y(t) = C(t)x(t) + D(t)u(t)$$ AE

output *input*

→ If a linear system has, in addition, the property of time invariance (SILS)

$$y(t) = \int_0^t G(t-\tau)u(\tau)d\tau$$

$$x^{\cdot}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Mathematical Descriptions of Systems

▪ Type of Systems

$\rightarrow$ SISO (Single Input Single Output) system

$\rightarrow$ MIMO (Multi Input Multi Output) system

$\rightarrow$ SIMO (Single Input Multi Output) system

▪ Type of Inputs/Outputs

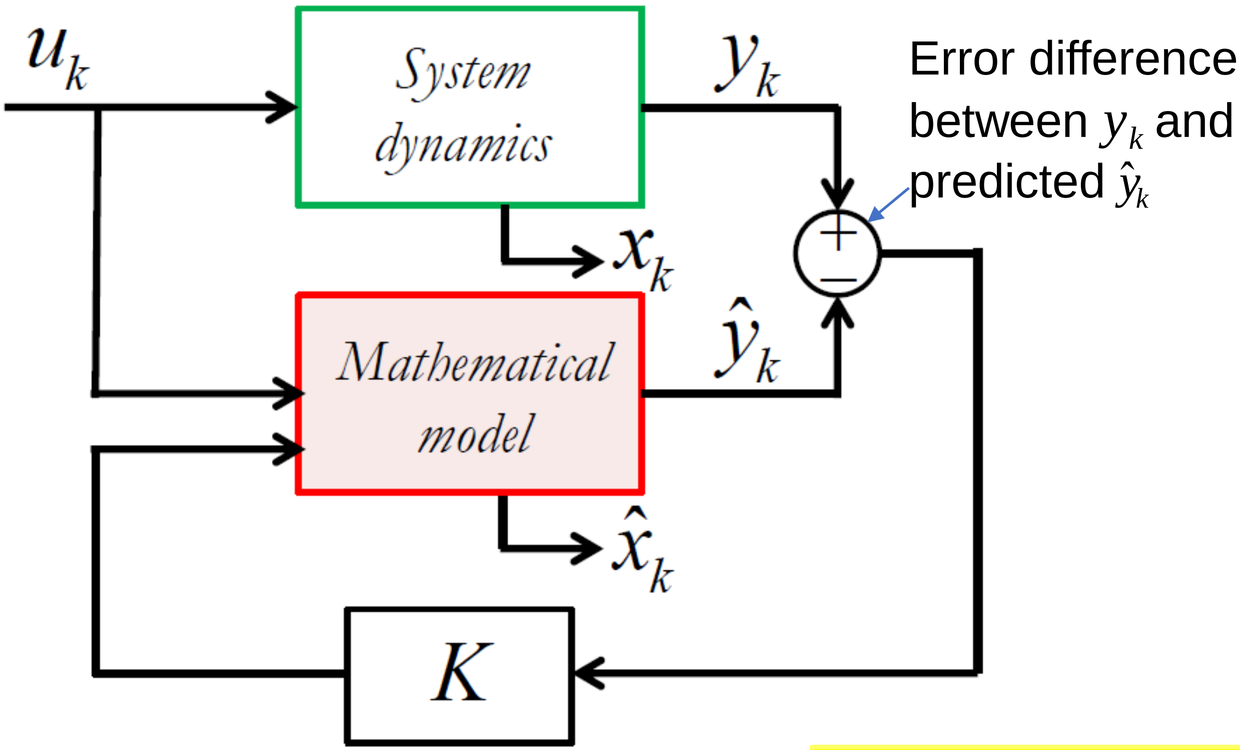
$\rightarrow$ Continuous-time system

$$u(t),\ y(t)$$

$\rightarrow$ Discrete-time system

$$u[k]:=u(kT),\ y[k]:=y(kT)$$

State Observer (Deterministic System)

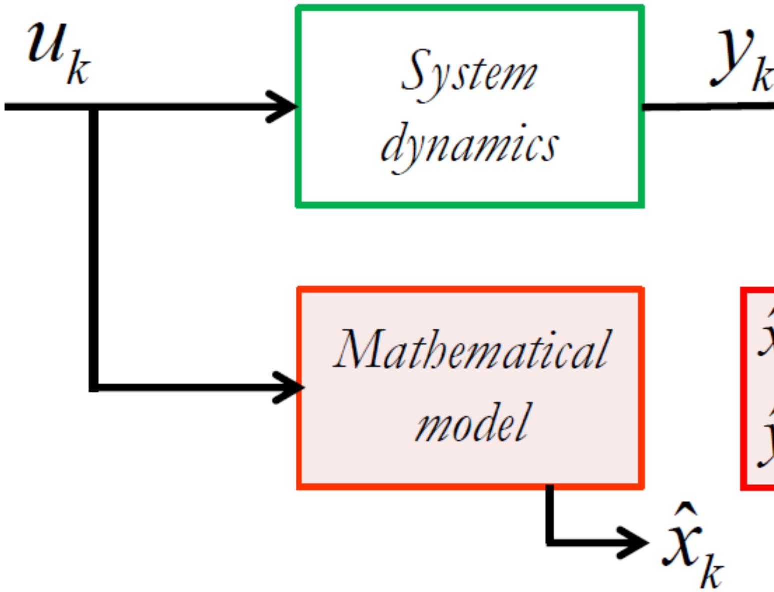


$$x_k = Ax_{k-1} + Bu_k$$
$$y_k = Cx_k$$

$$\hat{x}_k = A\hat{x}_{k-1} + Bu_k$$
$$\hat{y}_k = C\hat{x}_k$$

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + K(y_k - \hat{y}_k) \\ &= A\hat{x}_k + Bu_k + K(y_k - C\hat{x}_k) \end{aligned}$$

State Estimator (Stochastic System)



$$x_k = Ax_{k-1} + Bu_k + w_k, \quad x \in R^n, \quad p(w) \sim N(0,Q)$$

$$y_k = Cx_k + v_k, \quad y \in R^m, \quad p(v) \sim N(0,R)$$

$$\hat{x}_k = A\hat{x}_{k-1} + Bu_k$$

$$\hat{y}_k = C\hat{x}_k$$

$$\hat{x}_k = A\hat{x}_{k-1} + Bu_k + K_k \underbrace{\left[y_k - C(A\hat{x}_{k-1} + Bu_k) \right]}_{y_k - \hat{y}_k = y_k - C\hat{x}_k}$$

Error difference between y_k and predicted $\hat{y}_k$

Properties of Linear Systems

Properties of the System

- **Properties**
  - → Linearity
  - → Time invariance
  - → Causality

Properties of the System: Linearity

$$y[k]=\frac{1}{2}u[k]+\frac{1}{2}u[k-1]$$

$$y_1[k]=\frac{1}{2}u_1[k]+\frac{1}{2}u_1[k-1]; \quad y_2[k]=\frac{1}{2}u_2[k]+\frac{1}{2}u_2[k-1]$$

$$y[k]=G\left\{\alpha_1 u_1[k]+\alpha_2 u_2[k]\right\} \quad \text{input}$$

$$=\frac{1}{2}\left(\alpha_1 u_1[k]+\alpha_2 u_2[k]\right)+\frac{1}{2}\left(\alpha_1 u_1[k-1]+\alpha_2 u_2[k-1]\right)$$

$$=\alpha_1\left(\frac{1}{2}u_1[k]+\frac{1}{2}u_1[k-1]\right)+\alpha_2\left(\frac{1}{2}u_2[k]+\frac{1}{2}u_2[k-1]\right)$$

$$=\alpha_1 y_1[k]+\alpha_2 y_2[k]$$

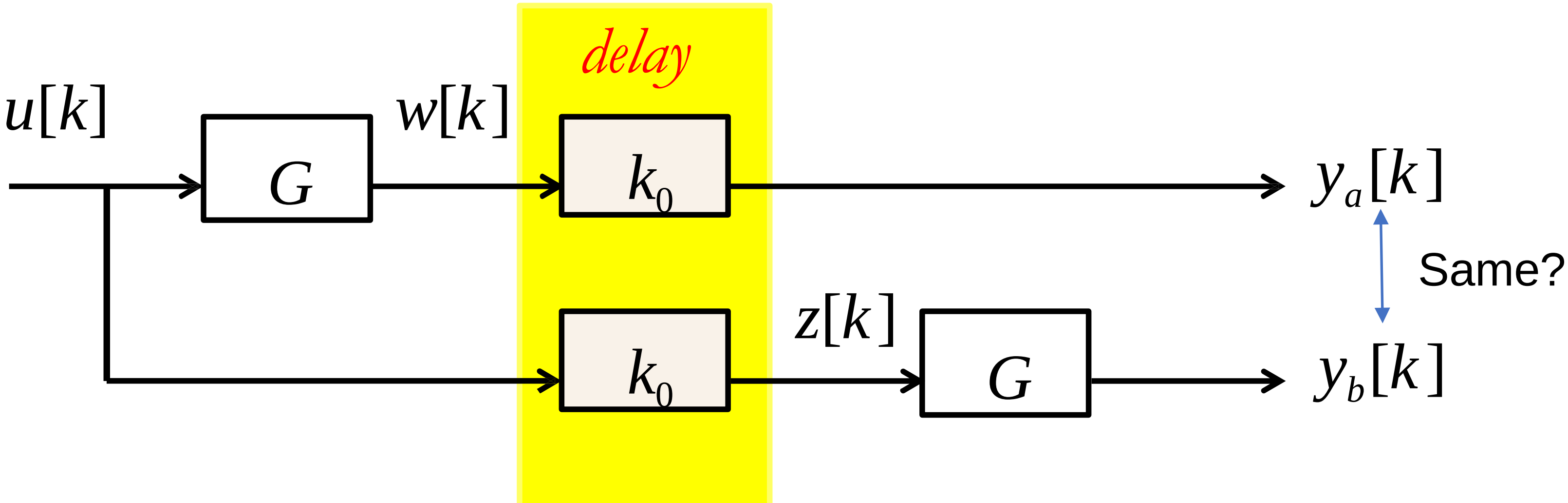
linear

$$y[k]=\cos\left(u[k]\right)$$

$$\cos\left(\alpha_1 u_1[k]+\alpha_2 u_2[k]\right)\neq \alpha_1\cos\left(u_1[k]\right)+\alpha_2\cos\left(u_2[k]\right)$$

nonlinear

Properties of the System: Time Invariance



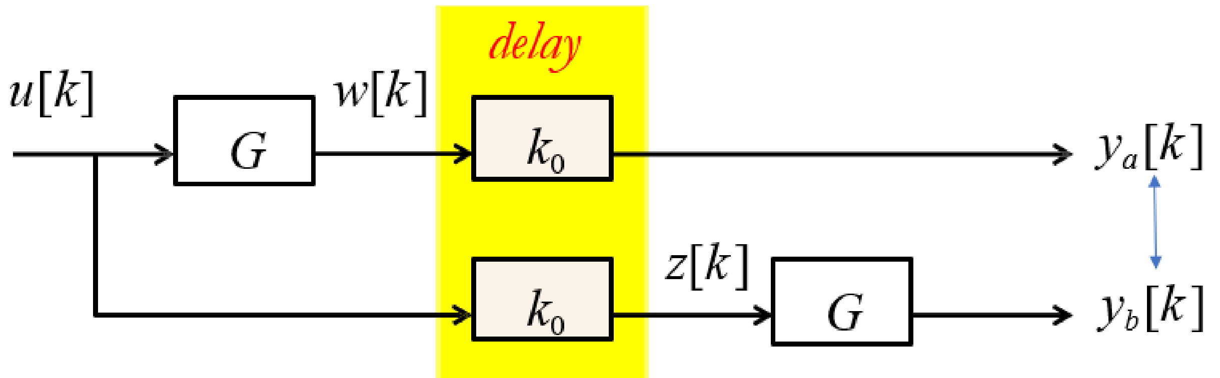
$$w[k] = G\{u[k]\}$$

$$y_a[k] = w[k-k_0] = G\{u[k-k_0]\}$$

$$y_b[k] = G\{z[k]\} = G\{u[k-k_0]\}$$

$$\boxed{y_a[k] = y_b[k] \text{ for any } k_0}$$ *Time invariant*

Properties of the System: Time Invariance



$$y[k] = (-1)^k u[k]$$

$$w[k] = (-1)^k u[k]$$

$$y_a[k] = w[k - k_0] = (-1)^{k-k_0} u[k - k_0]$$

$$z[k] = u[k - k_0]$$

$$y_b[k] = (-1)^k z[k] = (-1)^k u[k - k_0]$$

$$y_a[k] = (-1)^{k-k_0} u[k - k_0] = (-1)^{-k_0} (-1)^k u[k - k_0]$$

$$= \boxed{(-1)^{-k_0}} y_b[k]$$ *not time invariant*

Properties of the System: Causality

- **Lumpedness**

  → A system is called lumpedness system:
  if the system has a finite number of state variables

- **Memoryless**

  → A system is called a memoryless system:
  if its output $y(t_0)$ depends only on the input applied at t_0;
  it is independent of the input applied before or after t_0

  → Most systems have memory
  The output at t_0 depends on $u(t)$ for $t < t_0, t = t_0$, and $t > t_0$

- **Causality**

  → A system is called a causal or nonanticipatory system:
  if its current output depends on past and current inputs
  but not on future input

Every physical system is causal!

Properties of the System: Causality

Current output of a causal system is affected by past input. The input from $-\infty$ to time t has an effect on $y(t)$.

Definition: The state $x(t_0)$ of a system at time t_0 is the information at t_0 that, together with the input $u(t)$, for $t \geq t_0$, determines uniquely the output $y(t)$ for all $t \geq t_0$.

By definition, if we know the state at t_0, there is no more need to know the input $u(t)$ applied before t_0 in determining the output $y(t)$ after t_0. Thus in some sense, the state summarizes the effect of past input on future output.

Homework #1.1 Causality (1 pt.): Due Dec. 15

- **Causality**

  → Determine whether the following linear systems are causal or not and explain why

1) $y(t) = u(t+A)$

2) $y(t) = u(t^{2+B})$

3) $y[k] = 0.5u[k] + 0.5u[k\text{-}C]$; for $n \geq 0$

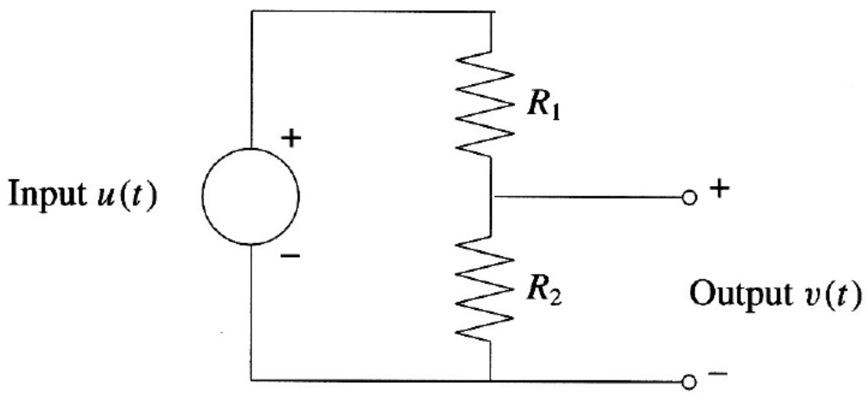
4) $y[k] = 0.25u[k-1] + 0.25u[k+2D]$; for $n \geq 0$

Causal system:
The present output does not depend on future inputs

Use Your ID: sGFEDCBA

Homework #1.2 Memoryless (1 pt.): Due Dec. 15

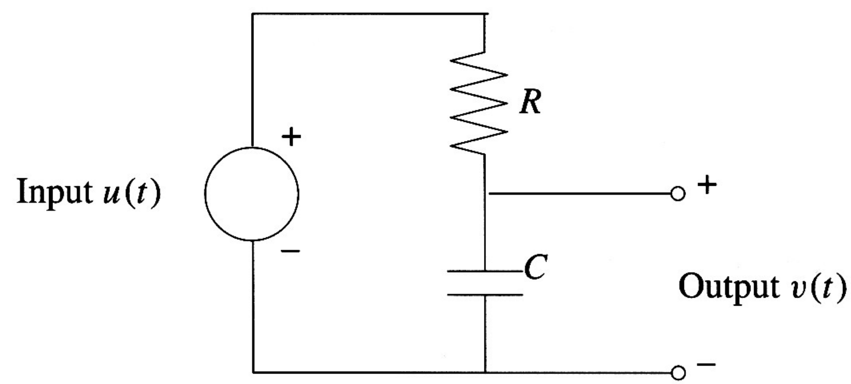
- Memoryless



Output $\qquad$ Input current

$$v(t) = \frac{R_2}{R_1 + R_2} u(t)$$

Memoryless system:
The present output only depend on present inputs

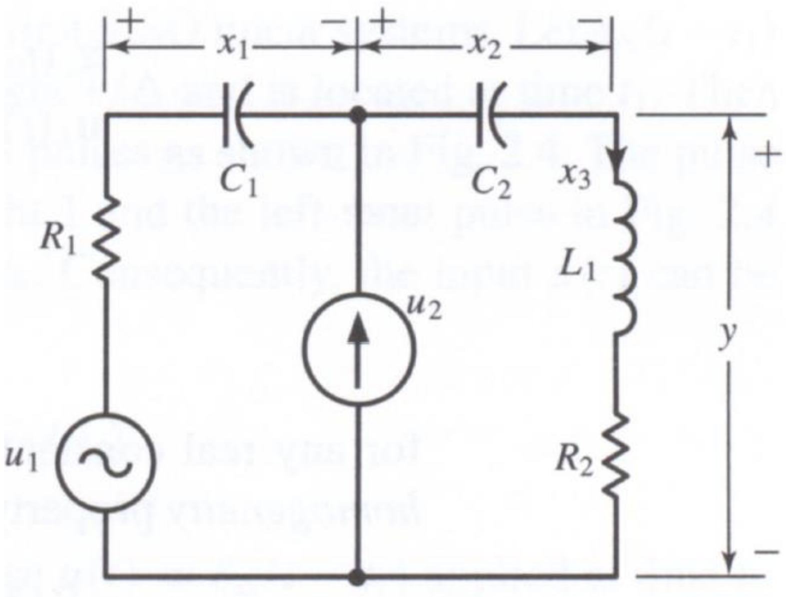


→ When a linear system is memoryless?
→ Determine whether the above linear system is memoryless or not and explain why

Example: Lumped vs Distributed System

▪ Lumped System

Example #1: Network with 3 state variables



$$x(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{bmatrix}$$

the initial state

▪ Distributed System

Example #2: unit time delay system $y(t) = u(t-1)$ *distributed*

To determine $\{y(t),\ \ t \geq t_0\}$, *we need* $\{u(t),\ \ t_0 - 1 \leq t < t_0\}$.

the initial state: infinitely many points

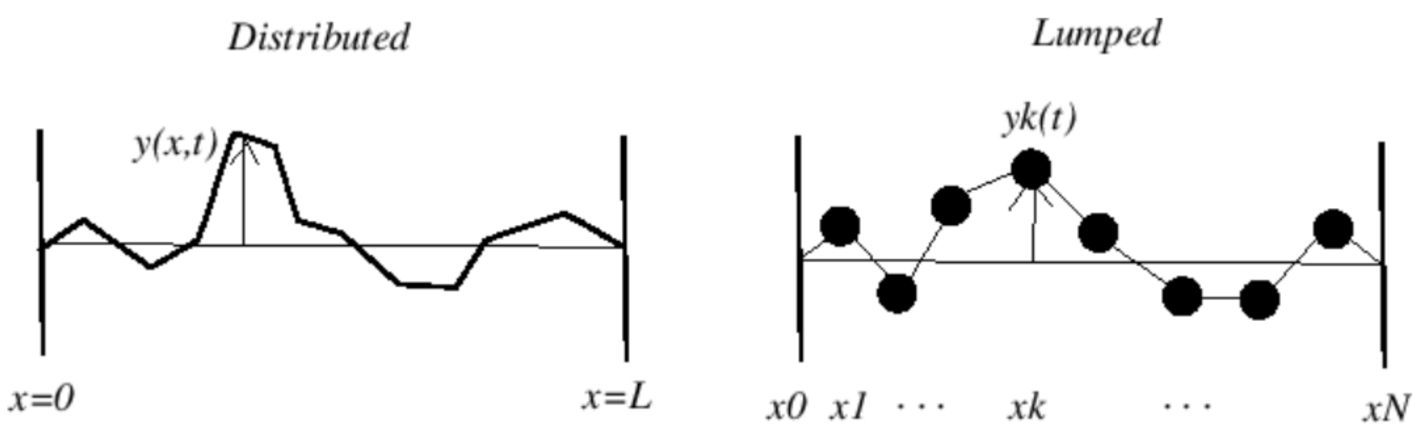
Example: Lumped vs Distributed System

▪ A state input-output pair

$$\left.\begin{array}{ll} \textit{initial state} & x(t_0) \\ \textit{input} & u(t),\quad t \ge t_0 \end{array}\right\} \to y(t),\quad t \ge t_0$$

A system is said to be lumped if its number of state variables is finite or its state is a finite vector.

A system is called a distributed system if its state has infinitely many state variables.



Additivity, Homogeneity, Superposition

A system is called a linear system if for every t_0 and any two state-input-output pairs

$$\left.\begin{array}{c} x_i(t_0) \\ u_i(t), \quad t \geq t_0 \end{array}\right\} \rightarrow y_i(t), \quad t \geq t_0$$

for $i = 1, 2$, we have

$$\left.\begin{array}{c} x_1(t_0) + x_2(t_0) \\ u_1(t) + u_2(t), \quad t \geq t_0 \end{array}\right\} \rightarrow y_1(t) + y_2(t), \quad t \geq t_0 \quad \text{(additivity)}$$

and

$$\left.\begin{array}{c} \alpha x_1(t_0) \\ \alpha u_1(t), \quad t \geq t_0 \end{array}\right\} \rightarrow \alpha y_1(t), \quad t \geq t_0 \qquad \text{(homogeneity)}$$

for any real constant α.

Additivity, Homogeneity, Superposition

These two properties can be combined as

$$\left.\begin{array}{c}\alpha_1 x_1(t_0)+\alpha_2 x_2(t_0)\\ \alpha_1 u_1(t)+\alpha_2 u_2(t),\quad t\geq t_0\end{array}\right\}\rightarrow \alpha_1 y_1(t)+\alpha_2 y_2(t),\quad t\geq t_0$$

for any real constants α_1 and α_2, and is called the superposition property. A system is called a nonlinear system if the superposition property does not hold.

Zero Input Response & Zero State Response

<u>The zero-input response</u>:

if the input $u(t)$ is identically zero for $t \geq t_0$, then the output will be excited exclusively by the initial state $x(t_0)$.

$$\left.\begin{array}{c} x(t_0) \\ u(t) \equiv 0, \quad t \geq t_0 \end{array}\right\} \rightarrow y_{zi}(t), \quad t \geq t_0$$

<u>The zero-state response</u>:

if the initial state $x(t_0)$ is zero, then the output will be excited exclusively by the input.

$$\left.\begin{array}{c} x(t_0) = 0 \\ u(t), \quad t \geq t_0 \end{array}\right\} \rightarrow y_{zs}(t), \quad t \geq t_0$$

Zero Input Response & Zero State Response

The additivity property implies

$$\text{Output due to} \begin{cases} x(t_0) \\ u(t), \quad t \ge t_0 \end{cases} =$$

$$\text{output due to} \begin{cases} x(t_0) \\ u(t) \equiv 0, \quad t \ge t_0 \end{cases}$$

$$+ \ \text{output due to} \begin{cases} x(t_0) = 0 \\ u(t), \quad t \ge t_0 \end{cases}$$

$$\boxed{\text{Response} = \text{zero-input response} + \text{zero-state response}}$$

Zero Input Response & Zero State Response

For nonlinear systems, the complete response can be very different from the sum of the zero-input response and zero-state response. Therefore, we cannot separate the zero-input and zero-state responses in studying nonlinear systems.

If a system is linear, then the additivity and homogeneity properties apply zero-state responses.

$$\{u_i \rightarrow y_i\}$$

$$\{u_1 + u_2 \rightarrow y_1 + y_2\} \text{ and } \{\alpha u_i \rightarrow \alpha y_i\} \text{ for all } \alpha \text{ and all } u_i$$

A similar remark applies to zero-input responses of any linear system.

$$\{x_i(t_0) \rightarrow y_i\}$$

$$\{x_1(t_0) + x_2(t_0) \rightarrow y_1 + y_2\}, \{\alpha x_i(t_0) \rightarrow \alpha y_i\} \text{ for all } \alpha \text{ and all } x_i$$

Thank you

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