Homework 3

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Name : TRAN, Thanh Cong
Student ID: 2210421

3.1)

$$M = \begin{bmatrix} -1 & 1 & 1 \\ 0 & A & -B \\ 0 & 1 & 0 \end{bmatrix}$$

- $$M - \lambda I = \begin{bmatrix} -1-\lambda & 1 & 1 \\ 0 & A-\lambda & -B \\ 0 & 1 & -\lambda \end{bmatrix}$$

- $$\begin{aligned} \det(M - \lambda I) &= (-1-\lambda)\det\begin{bmatrix} A-\lambda & -B \\ 1 & -\lambda \end{bmatrix} - \det\begin{bmatrix} 0 & -B \\ 0 & -\lambda \end{bmatrix} + \det\begin{bmatrix} 0 & A-\lambda \\ 0 & 1 \end{bmatrix} \\ &= (-1-\lambda)(-\lambda(A-\lambda)+B) - 0 + 0 \\ &= (-1-\lambda)(-\lambda A + \lambda^2 + B) \\ &= \lambda A - \lambda^2 - B + \lambda^2 A - \lambda^3 - \lambda B \\ &= -\lambda^2(\lambda+1) + \lambda A(\lambda+1) - B(\lambda+1) \\ &= (\lambda+1)(-\lambda^2 + \lambda A - B) \end{aligned}$$

- $$\det(M - \lambda I) = 0 \Rightarrow \left[\begin{aligned} &\lambda = -1 \\ &\lambda = \frac{A - \sqrt{A^2-4B}}{2} \\ &\lambda = \frac{A + \sqrt{A^2-4B}}{2} \end{aligned}\right.$$

- $\lambda_1 = -1$

$$(M - \lambda_1 I)v_1 = 0 \Leftrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & A+1 & -B \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{cases} v_{12} + v_{13} = 0 \\ (A+1)v_{12} - Bv_{13} = 0 \\ v_{12} + v_{13} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \forall v_{11} \\ v_{12} = -v_{13} \\ v_{13} = v_{13}(A+1)/B \end{cases}$$. The system of LEs has solution only when $\frac{A+1}{B} = -1$

If $\frac{A+1}{B} = -1$, λ_1 has 2 eigenvectors $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

- $\lambda_2 = \frac{A - \sqrt{A^2-4B}}{2}$

$$(M - \lambda_2 I)v_2 = 0 \Leftrightarrow \begin{bmatrix} -1 - \frac{A-\sqrt{A^2-4B}}{2} & 1 & 1 \\ 0 & A - \frac{A-\sqrt{A^2-4B}}{2} & -B \\ 0 & 1 & \frac{-A+\sqrt{A^2-4B}}{2} \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{cases} v_{21}\left(-1 - \frac{A-\sqrt{A^2-4B}}{2}\right) + v_{22} + v_{23} = 0 \\ v_{22}\left(A - \frac{A-\sqrt{A^2-4B}}{2}\right) - Bv_{23} = 0 \\ v_{22} + v_{23}\left(\frac{-A+\sqrt{A^2-4B}}{2}\right) = 0 \end{cases}$$

$$\Leftrightarrow \Big(\quad v_{21} = (v_{22} + v_{23}) / \left(1 + \frac{A-\sqrt{A^2-4B}}{2}\right)$$

$$\Leftrightarrow \begin{cases} v_{21} = (v_{22} + v_{23}) \Big/ \left(1 + \dfrac{A - \sqrt{A^2-4B}}{2}\right) \\ v_{22} = v_{23} B \Big/ \left(A - \dfrac{A-\sqrt{A^2-4B}}{2}\right) \\ v_{22} = v_{23} \left(\dfrac{A - \sqrt{A^2-4B}}{2}\right) \end{cases}$$

This system of LEs has solution only when

$$B \Big/ \left(A - \frac{A - \sqrt{A^2-4B}}{2}\right) = \frac{A - \sqrt{A^2-4B}}{2}$$

$$\Leftrightarrow B = \left(\frac{A + \sqrt{A^2-4B}}{2}\right)\left(\frac{A - \sqrt{A^2-4B}}{2}\right)$$

$$\Leftrightarrow B = \frac{A^2 - A^2 + 4B}{4}$$

$$\Leftrightarrow B = B \text{ satisfied } \forall A, B \Leftrightarrow \begin{cases} v_{21} = v_{23} \\ v_{22} = v_{23}\left(\dfrac{A - \sqrt{A^2-4B}}{2}\right) \\ \forall v_{23} \in R \setminus \{0\} \end{cases}$$

λ_2 has 1 eigenvector $v_2 = \begin{bmatrix} 1 \\ \dfrac{A - \sqrt{A^2-4B}}{2} \\ 1 \end{bmatrix}$

- $\lambda_3 = \dfrac{A + \sqrt{A^2-4B}}{2}$

$$(M - \lambda_3 I) v_3 = 0 \Leftrightarrow \begin{cases} v_{31}\left(-1 - \dfrac{A + \sqrt{A^2-4B}}{2}\right) + v_{32} + v_{33} = 0 \\ v_{32}\left(A - \dfrac{A + \sqrt{A^2-4B}}{2}\right) - B v_{33} = 0 \\ v_{32} + v_{33}\left(\dfrac{-A - \sqrt{A^2-4B}}{2}\right) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} v_{31} = (v_{32} + v_{33}) \Big/ \left(1 + \dfrac{A + \sqrt{A^2-4B}}{2}\right) \\ v_{32} = v_{33} B \Big/ \left(\dfrac{A - \sqrt{A^2-4B}}{2}\right) \\ v_{32} = v_{33}\left(\dfrac{A + \sqrt{A^2-4B}}{2}\right) \end{cases}$$

This system of LEs has solution only when:

$$B \Big/ \left(\frac{A - \sqrt{A^2-4B}}{2}\right) = \frac{A + \sqrt{A^2-4B}}{2}$$

$$\Leftrightarrow B = \frac{A^2 - A^2 + 4B}{4}$$

$$\Leftrightarrow B = B \text{ satisfied } \forall A, B \Leftrightarrow \begin{cases} v_{31} = v_{33} \\ v_{32} = v_{33}\left(\dfrac{A + \sqrt{A^2-4B}}{2}\right) \\ \forall v_3 \in R \setminus \{0\} \end{cases}$$

λ_3 has 1 eigenvector $v_3 = \begin{bmatrix} 1 \\ \dfrac{A + \sqrt{A^2-4B}}{2} \\ 1 \end{bmatrix}$

3.2)

$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$; compute $f(A)$ given $f(\lambda) = \lambda^{100}$

- $\Delta(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \end{bmatrix} = \lambda(\lambda+2) + 1 = \lambda^2 + 2\lambda + 1 = \psi(\lambda)$

$A = \begin{bmatrix} \dots \\ -1 & -2 \end{bmatrix}$; compute $f(A)$ given $f(\lambda) = \lambda$

- $\Delta(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix} = \lambda(\lambda+2) + 1 = \lambda^2 + 2\lambda + 1 = \psi(\lambda)$

- Let $h(\lambda) = \beta_0 + \beta_1 \lambda$, using the Cayley - Hamilton Theorem 5

$$\begin{cases} f(-1) = h(-1) \\ f'(-1) = h'(-1) \end{cases} \Leftrightarrow \begin{cases} (-1)^{100} = \beta_0 - \beta_1 \\ 100 \cdot (-1)^{(100-1)} = \beta_1 \end{cases} \Leftrightarrow \begin{cases} \beta_0 = (-1)^{(100-1)} (100-1) \\ \beta_1 = (-1)^{(100-1)} \cdot 100 \end{cases}$$

- $f(A) = h(A) \Leftrightarrow f(A) = (-1)^{(100-1)}(100-1) I + (-1)^{(100-1)} \cdot 100\, A$

$$= \begin{bmatrix} (-1)^{(100-1)}(100-1) & (-1)^{(100-1)} \cdot 100 \\ (-1)^{100} \cdot 100 & (-1)^{100} \; (100+1) \end{bmatrix}$$

- Using the Cayley - Hamilton Theorem 4 we have

$$\Delta(A) = A^2 + 2A + I = 0 \Leftrightarrow A^2 = -2A - I$$

$$\Leftrightarrow A^3 = -2A^2 - A = -2(-2A - I) - A = 3A + 2I$$

$$\vdots$$

continue to A^{100}

3.3)

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$; $AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

- $\det(AA^T - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 = 0 \Leftrightarrow \lambda = 2$ with multiplicity 2. $S = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$

$(AA^T - \lambda I) v = 0 \Leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} v = 0$ satisfied $\forall v \Leftrightarrow E_{AA^T}(\lambda) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- $\det(A^T A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix}$

$$= (1-\lambda) \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} + \det \begin{bmatrix} 0 & 1 & 0 \\ 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda) \left[(1-\lambda)^3 - (1-\lambda) \right] + 1 - (1-\lambda)^2$$

$$= (1-\lambda)(1 - 3\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda) + 1 - 1 + 2\lambda - \lambda^2$$

$$= (1-\lambda)(-\lambda^3 + 3\lambda^2 - 2\lambda) - \lambda^2 + 2\lambda$$

$$= -\lambda^3 + 3\lambda^2 - 2\lambda + \lambda^4 - 3\lambda^3 + 2\lambda^2 - \lambda^2 + 2\lambda$$

$$= \lambda^4 - 4\lambda^3 + 4\lambda^2$$

$$= \lambda^2(\lambda^2 - 4\lambda + 4)$$

$$= \lambda^2(\lambda - 2)^2$$

$\det(A^T A - \lambda I) = \lambda^2(\lambda-2)^2 = 0 \Leftrightarrow \begin{cases} \lambda_1 = 2 \text{ with multiplicity } 2 \\ \lambda_2 = 0 \end{cases}$

- $(A^T A - \lambda_1 I) v = 0 \Leftrightarrow \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0 \Leftrightarrow \begin{cases} -v_1 + v_3 = 0 \\ -v_2 + v_4 = 0 \\ v_1 - v_3 = 0 \\ v_2 - v_4 = 0 \end{cases} \Leftrightarrow \begin{cases} v_1 = v_3 \\ v_2 = v_4 \\ \forall v_3, v_4 \end{cases} \Leftrightarrow E_{A^TA}(\lambda_1) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \dots \end{bmatrix} \begin{bmatrix} v_1 \\ \dots \end{bmatrix}$ $\begin{cases} v_1 + v_3 = 0 \\ \dots \end{cases}$ $\begin{cases} v_1 = -v_3 \\ \dots \end{cases}$ $\left\{ \begin{bmatrix} -1 \\ \dots \end{bmatrix}, \begin{bmatrix} 0 \\ \dots \end{bmatrix} \right\}$

* $(A^T A - \lambda_2 I)v = 0 \Leftrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0 \Leftrightarrow \begin{cases} v_1 + v_3 = 0 \\ v_2 + v_4 = 0 \\ v_1 + v_3 = 0 \\ v_2 + v_4 = 0 \end{cases} \Leftrightarrow \begin{cases} v_1 = -v_3 \\ v_2 = -v_4 \\ \forall v_3, v_4 \end{cases} \Rightarrow E_{A^T A}(\lambda_2) = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$Q = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \quad Q_n = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

* SVD of A:

$$A = R\,S\,Q_n^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$