Lecture I213E – Class 6

Discrete Signal Processing

Sakriani Sakti

JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY 1990

Course Materials

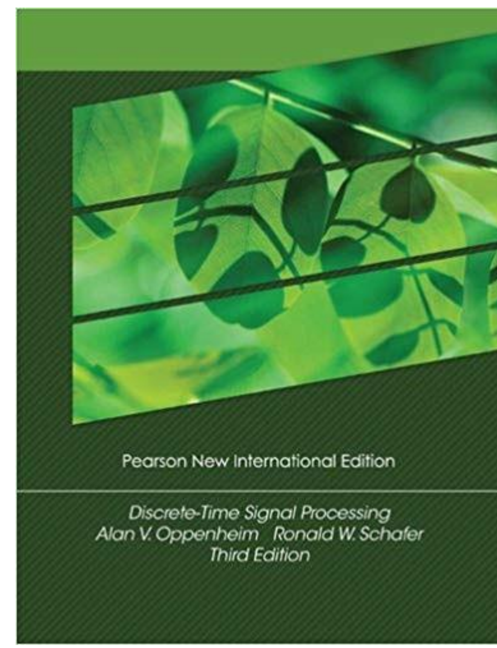
- **Materials**

→ Lecture notes will be uploaded before each lecture
https://jstorage-2018.jaist.ac.jp/s/PGXRrC7iFmN2FWo
Pass: dsp-i213e-2022
(Slide Courtesy of Prof. Nak Young Chong)

- **References**

→ Chi-Tsong Chen:
Linear System Theory and Design, 4th Ed., Oxford University Press, 2013.

→ Alan V. Oppenheim and Ronald W. Schafer:
Discrete-Time Signal Processing, 3rd Ed., Pearson New International Ed., 2013.



Related Courses & Prerequisite

- **Related Courses**
  - → I212 Analysis for Information Science
  - → I114 Fundamental Mathematics for Information Science

- **Prerequisite**
  - → None

Evaluation

Viewpoint of evaluation

→ Students are able to understand:

- Basic principles in modeling and analysis of linear time-invariant systems
- Applications of mathematical methods and tools to different signal processing problems.

Evaluation method

→ Homework, term project, midterm exam, and final exam

Evaluation criteria

→ Homework/labs (30%), term project (30%)
midterm exam (15%), and final exam (25%)

Contact

- ## Lecturer

  → Sakriani Sakti

- ## TA

  Tutorial hours & Term project

  → WANG Lijun (s2010026)

  → TANG Bowen (s2110411)

  Homework

  → PUTRI Fanda Yuliana (s2110425)

- ## Contact Email

  → dsp-i213e-2022@ml.jaist.ac.jp

Schedule

- **December 8th, 2022 – February 9th, 2023**

- **Lecture Course Term 2-2**

  → Tuesday 9:00 — 10:40

  → Thursday 10:50 — 12:30

- **Tutorial Hours**

  → Tuesday 13:30-15:10

Schedule

Dec

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Jan

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	~~5~~	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	~~24~~	25	26	27	28
29	30	31				

Feb

Sun	Mon	Tue	Wed	Thu	Fri	Sat
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28				

Lecture:
Tuesday 9:00 — 10:40
Thursday 10:50 — 12:30

Tutorial:
Tuesday 13:30 — 15:10

Course review &
term project evaluation
(on tutorial hours)

Midterm & final exam
Thursday 10:50 — 12:30

Syllabus

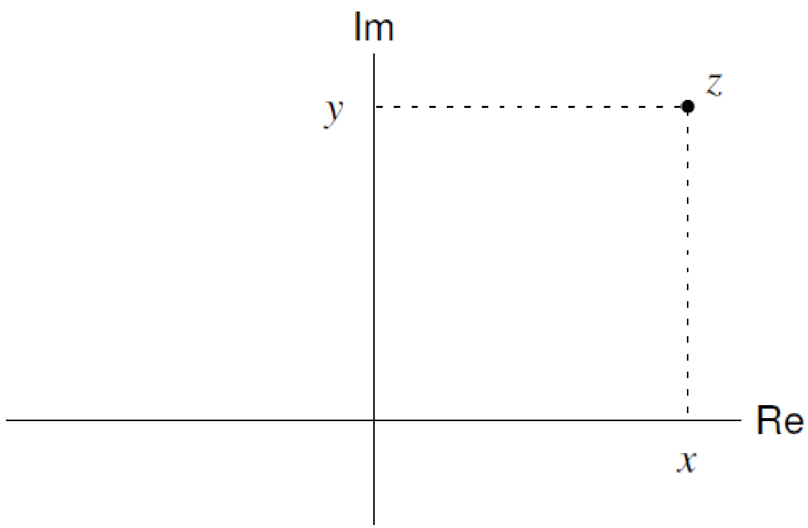
Class	Date	Lecture Course Tue 9:00 — 10:40 / Thr 10:50 — 12:30	Tutorial Hours Tue 13:30 — 15:10
1	12/08	Introduction to Linear Systems with Applications to Signal Processing	
2	12/13	State Space Description	○
3	12/15	Linear Algebra	
4	12/20	Quantitative Analysis (State Space Solutions) and Qualitative Analysis (Stability)	○
5	12/22	Discrete-time Signals and Systems	
X	01/05		
6	01/10	Discrete-time Fourier Analysis	
7	01/10*	Review of Discrete-time Linear Time-Invariant Signals and Systems (on Tutorial Hours)	
	01/12	Midterm Exam	
8	01/17	Sampling and Reconstruction of Analog Signals	○
9	01/19	z-Transform	
X	01/24		○
10	01/26	Discrete Fourier Transform	
11	01/31	FFT Algorithms	○
12	01/02	Implementation of Digital Filters	
13	02/07	Digital Signal Processors and Design of Digital Filters	
14	02/07*	Review of the Course and Term Project Evaluation (on Tutorial Hours)	
	02/09	Final exam	

Class 6
Discrete-time
Fourier Analysis

Review of Complex Analysis

Complex Numbers

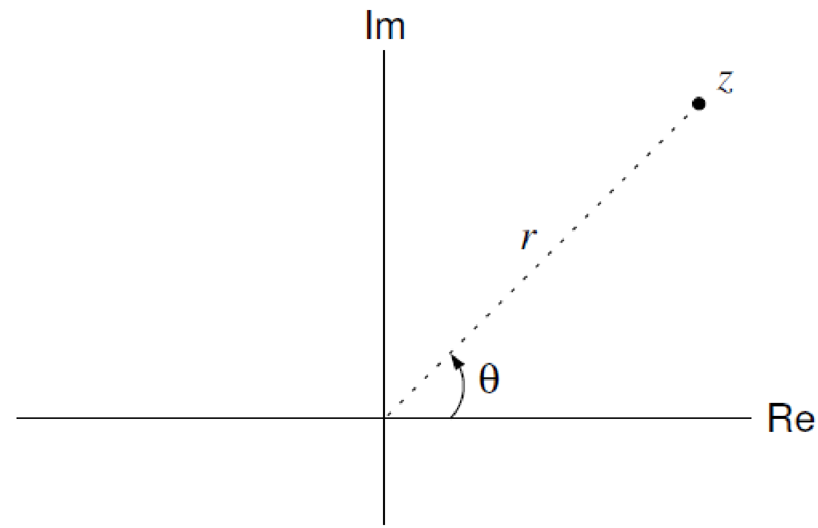
- **Cartesian (Rectangular) Form**



$$\boxed{z = x + jy}$$

where $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$

- **Polar Form**



$$\boxed{z = r(\cos\theta + j\sin\theta) = re^{j\theta}}$$

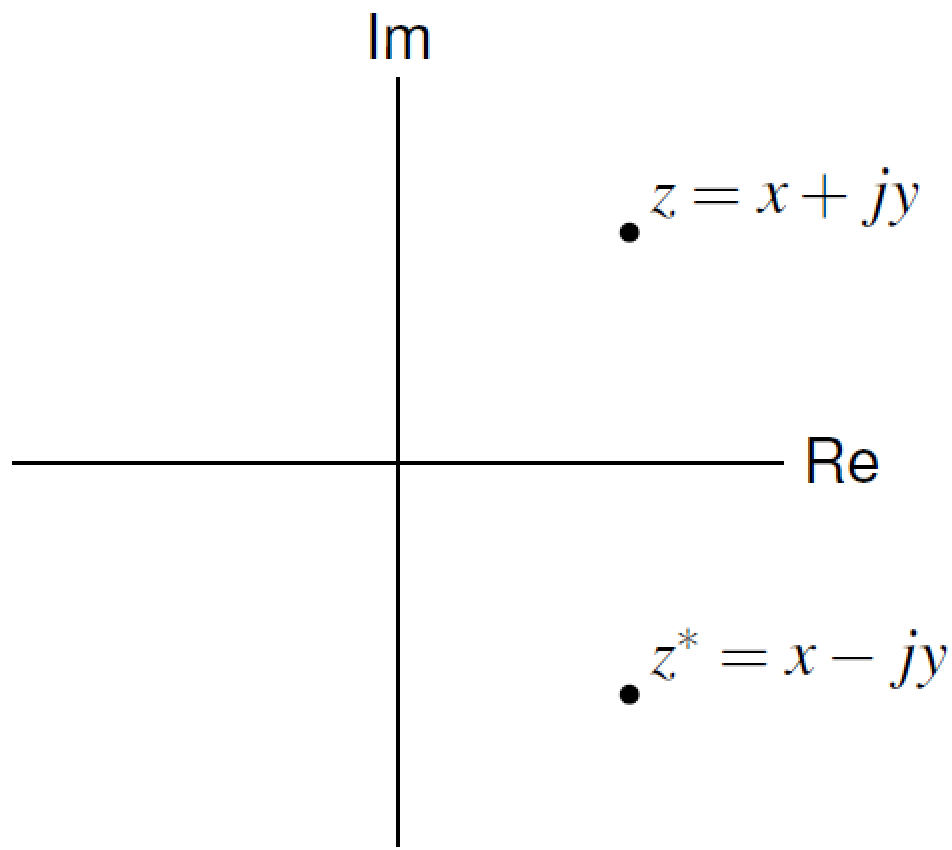
where $r = |z|$ and $\theta = \arg z$

Conjugate

- **Conjugate Complex Numbers**

  $\rightarrow$ The conjugation operation reflects a point in the complex plane about the real axis

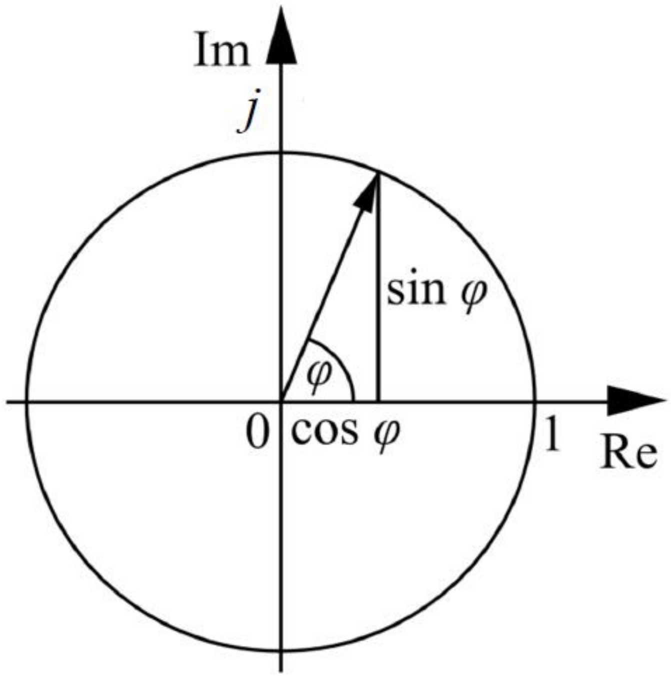
  $$\boxed{z^* = x - jy}$$



Complex Exponential

- **Euler's Relation**

$$\boxed{e^{j\varphi} = \cos\varphi + j\sin\varphi}$$



$$e^{j\varphi} = 1 + j\varphi + \frac{(j\varphi)^2}{2!} + \frac{(j\varphi)^3}{3!}\cdots = \sum_{k=0}^{\infty}\frac{(j\varphi)^k}{k!}$$

$$= \boxed{1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots} + j\left(\boxed{\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots}\right)$$

$$= \cos\varphi + j\sin\varphi$$

Even function Odd function

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\varphi^{2k+1}$$

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\varphi^{2k}$$

Complex Exponential

- ## Euler's Relation

$$\boxed{e^{j\omega t} = \cos \omega t + j \sin \omega t}$$

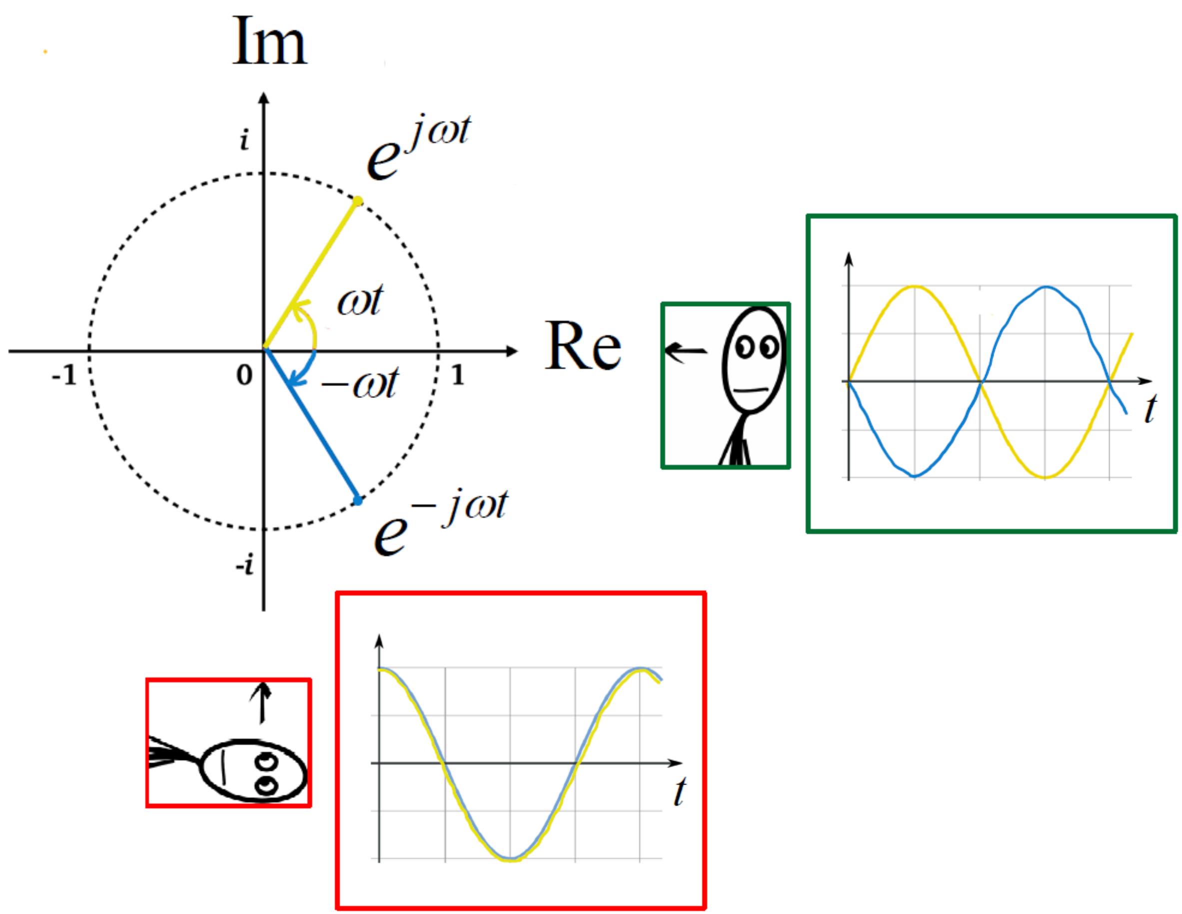
1. Convenient representations of sinusoidal signals
2. A mixture of two orthogonal signals
   → the in-phase component $\cos \omega t$
   → the phase-quadrature component $\sin \omega t$

90 degrees out of phase,
A relative phase shift of $\pm \pi / 2$

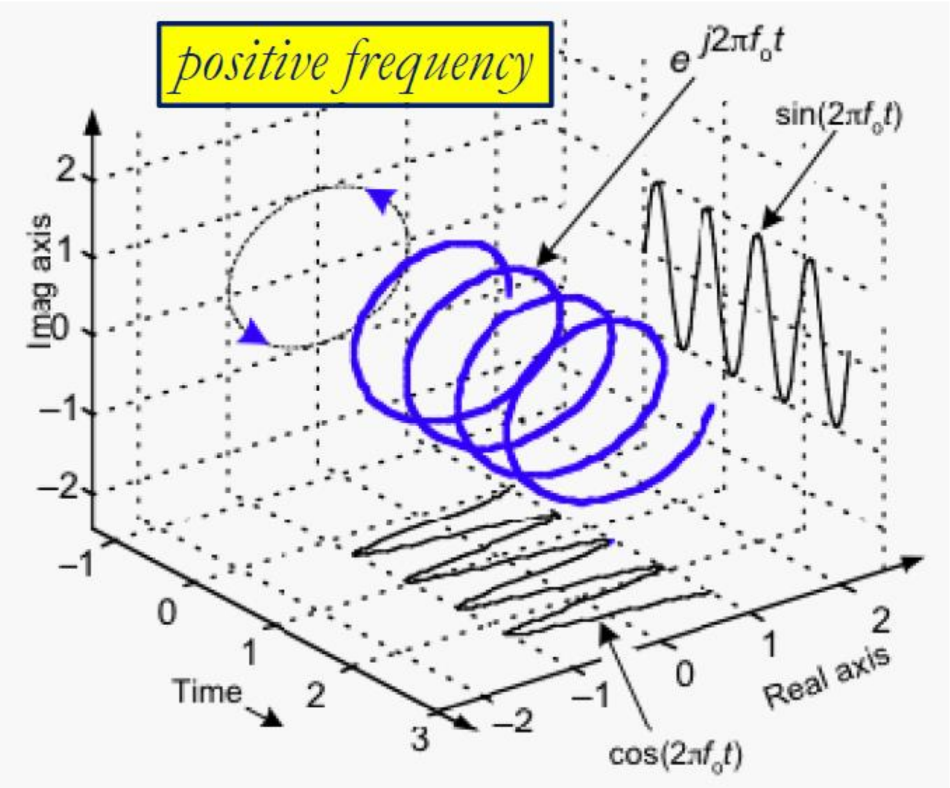
Complex Exponential

- **Euler's Relation**

$$\boxed{e^{j\omega t} = \cos \omega t + j \sin \omega t}$$



Complex Exponential



Fourier Analysis

Fourier Analysis

- **Fourier**

→ Named after **Jean Baptiste Joseph Fourier (1768-1830)**

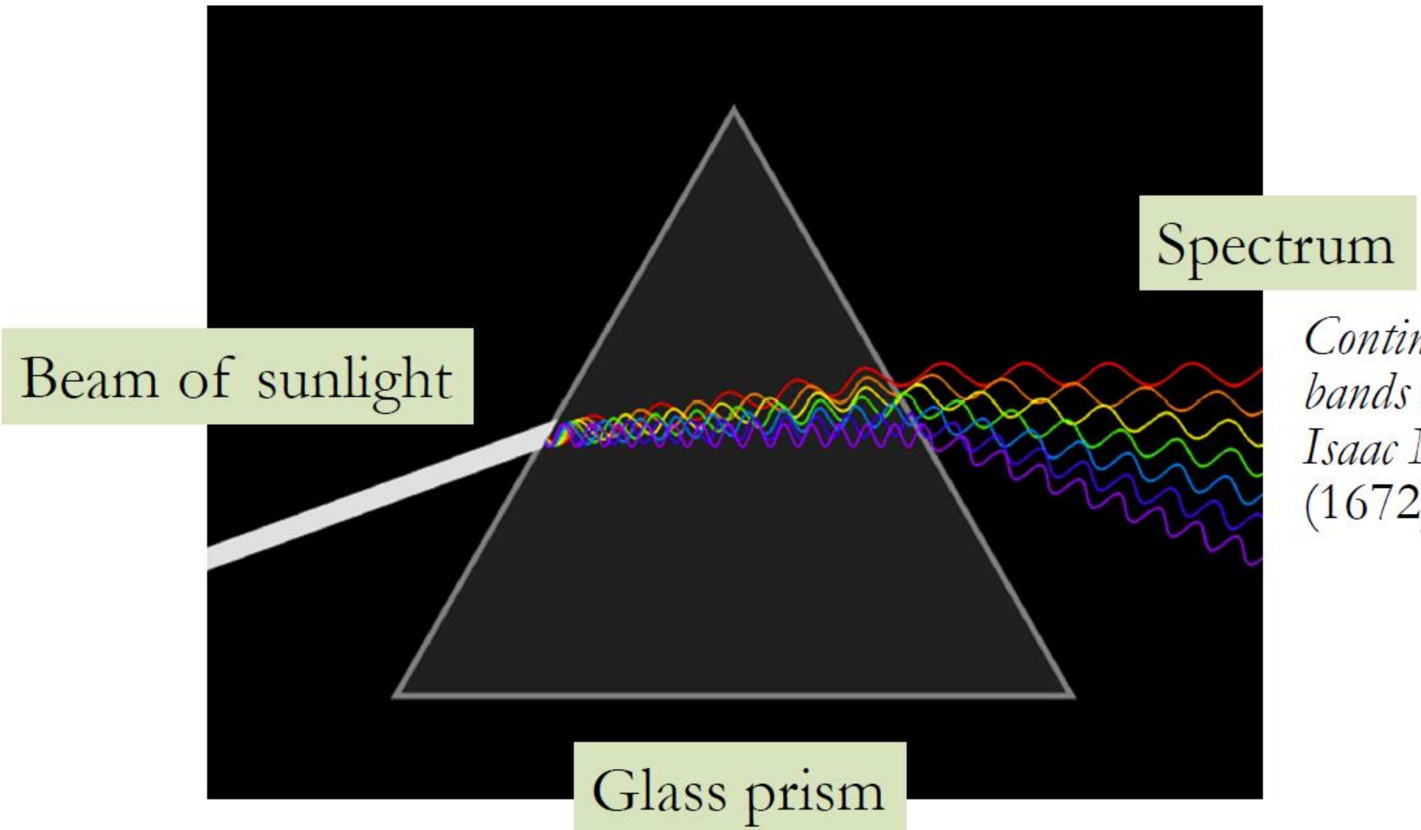
→ French mathematician and physicist

- **Fourier Analysis**

→ The analysis of a complex waveform expressed as a series of sinusoidal functions, the frequencies of which form a harmonic series

Analysis of the White Light

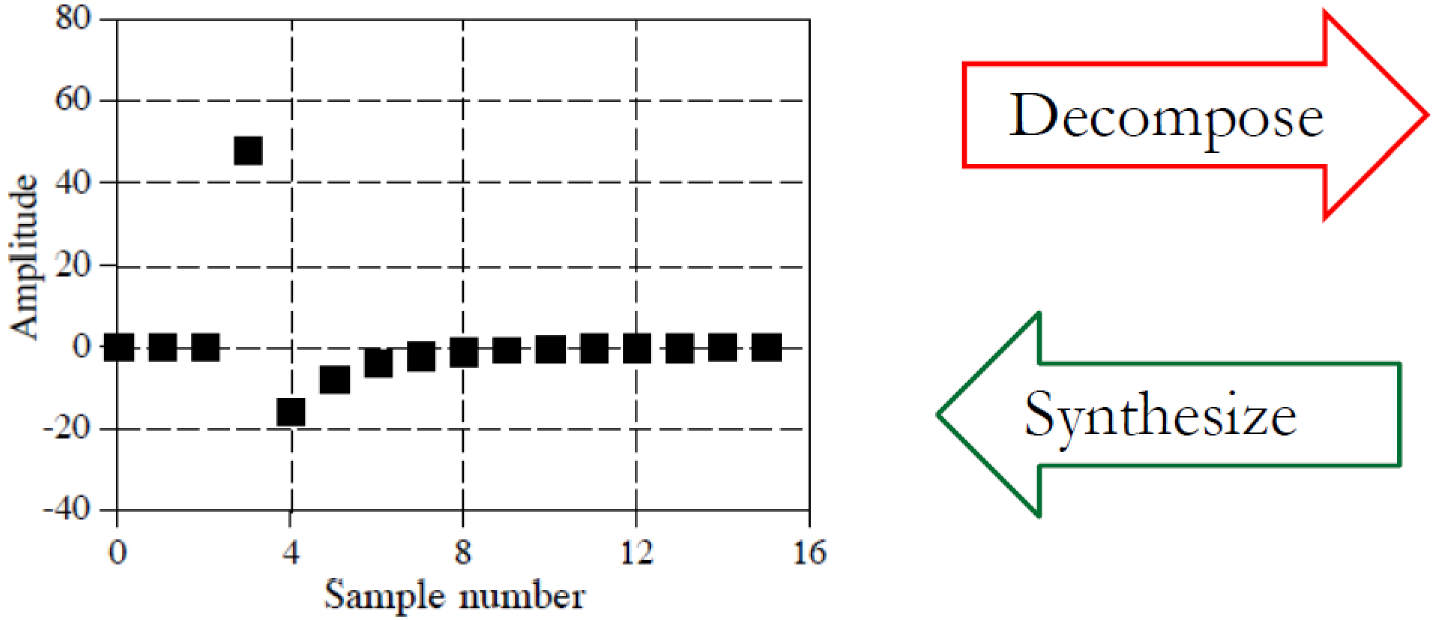
Analysis of the white light (sunlight) using glass prism



Continuous bands of colors, Isaac Newton (1672)

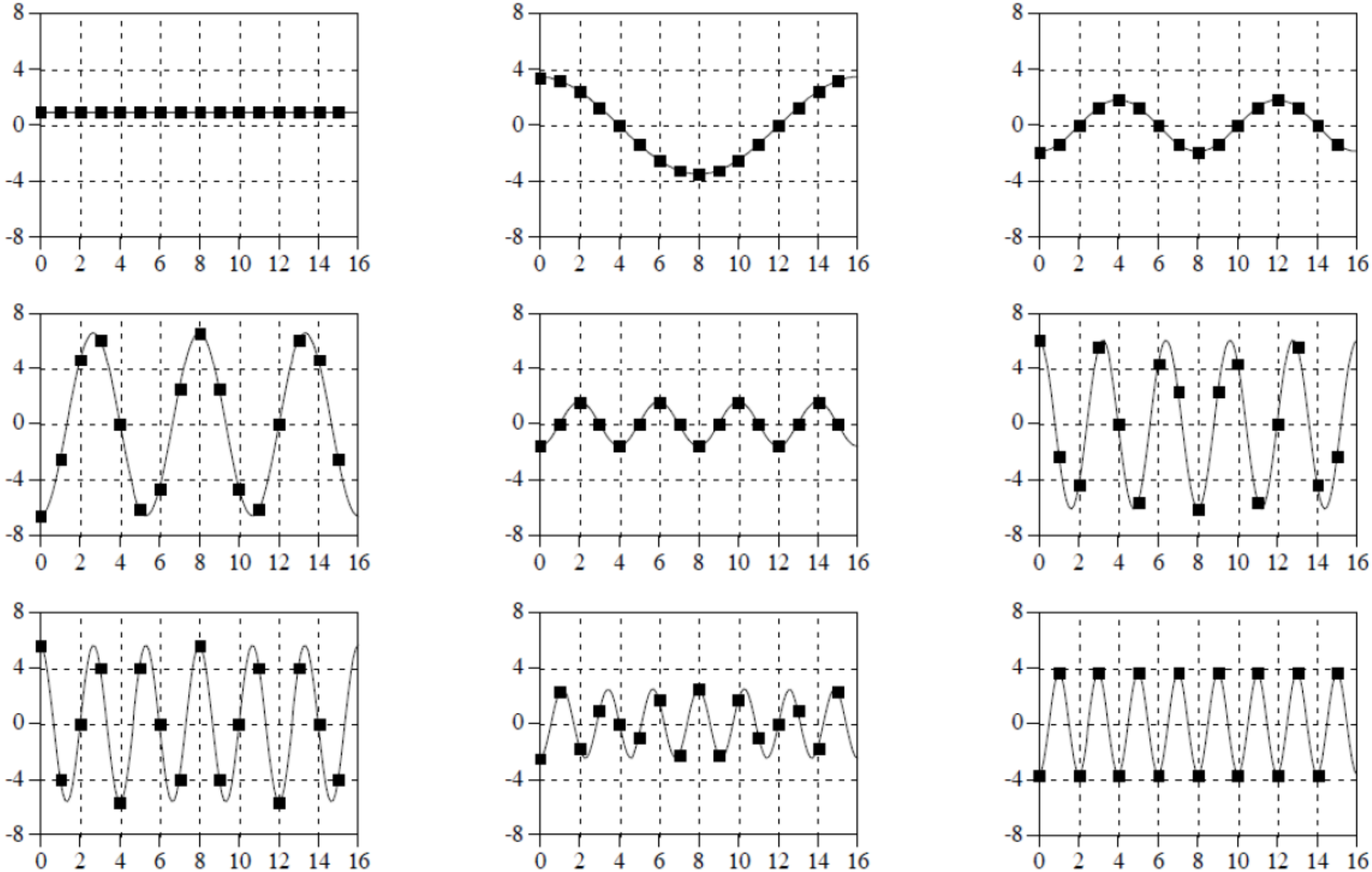
Fourier Decomposition

Example of Fourier decomposition: a 16 point signal



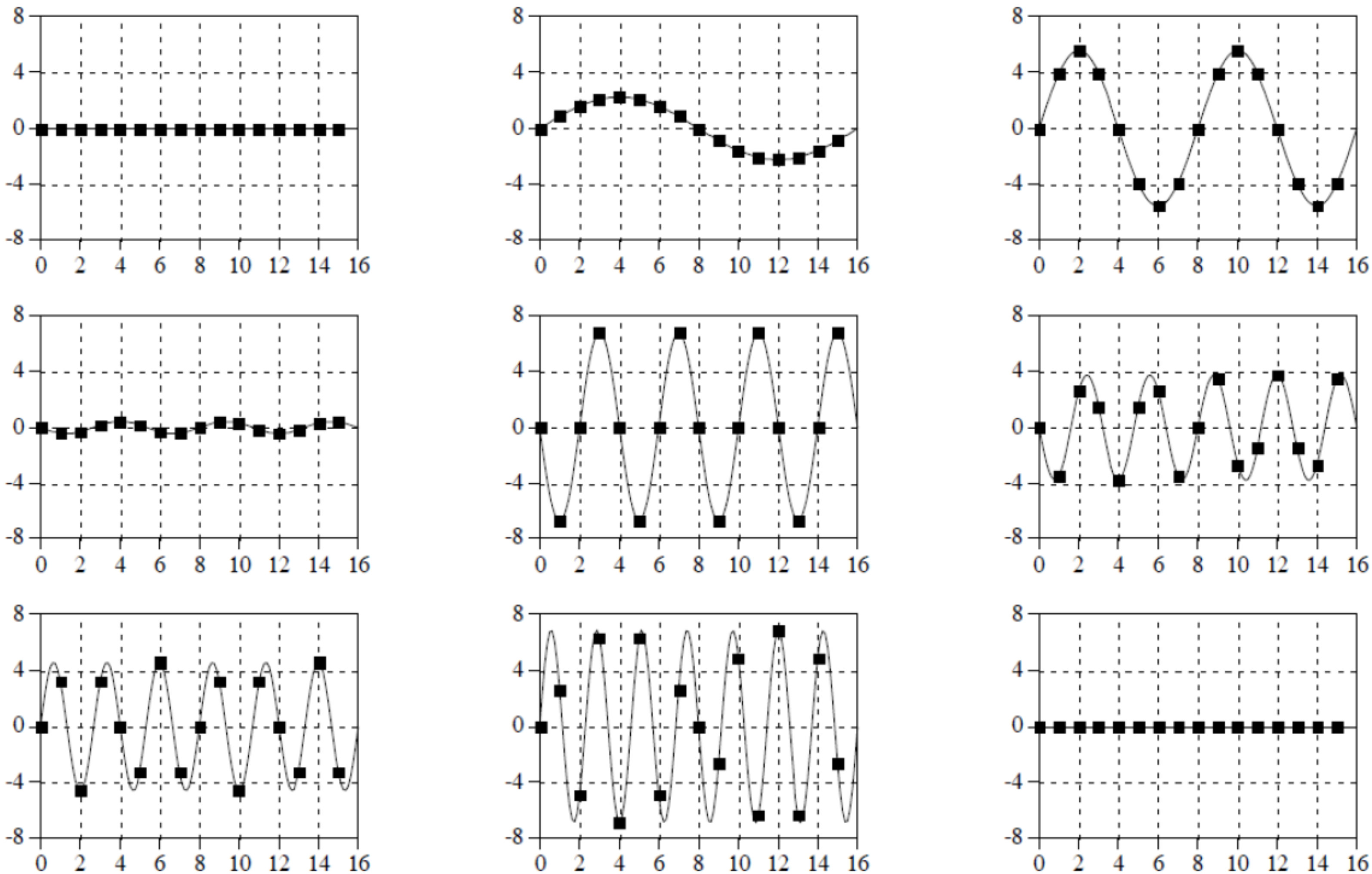
Fourier Decomposition

9 cosine waves, each with a different frequency and amplitude



Fourier Decomposition

9 sine waves, each with a different frequency and amplitude



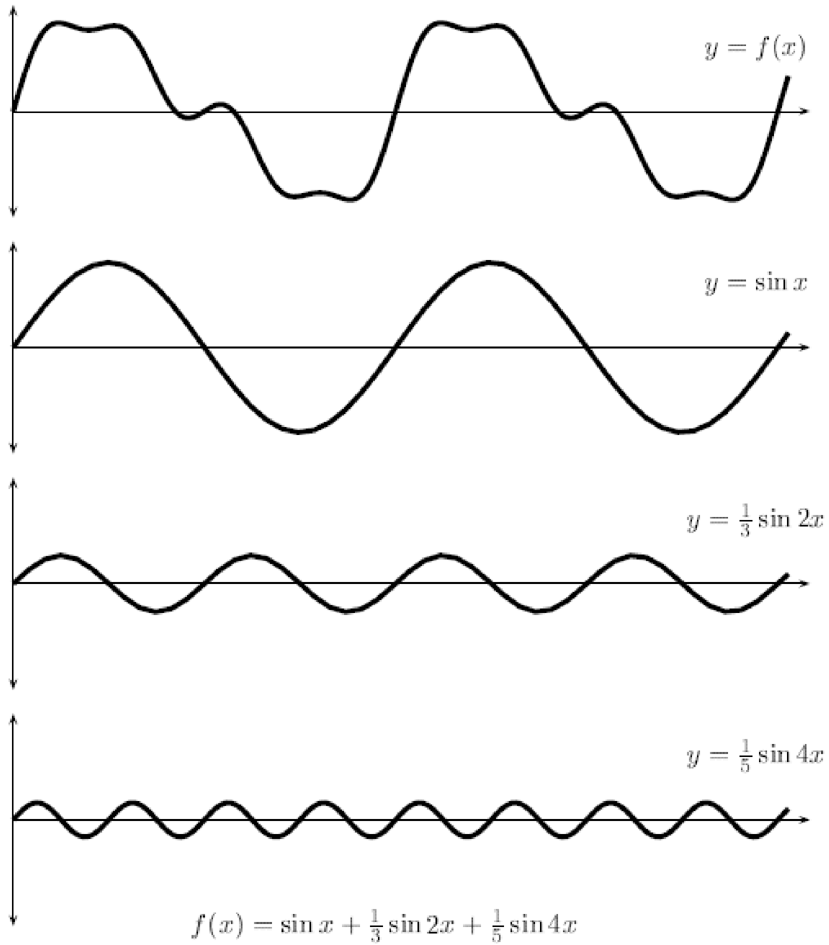
Categories of Fourier Transform

Type of Transform	Example Signal
Fourier Transform *signals that are continuous and aperiodic*	
Fourier Series *signals that are continuous and periodic*	
Discrete Time Fourier Transform *signals that are discrete and aperiodic*	
Discrete Fourier Transform *signals that are discrete and periodic*	

Fourier Series

Fourier Series Representation

- ## Approximation of Periodic Signal



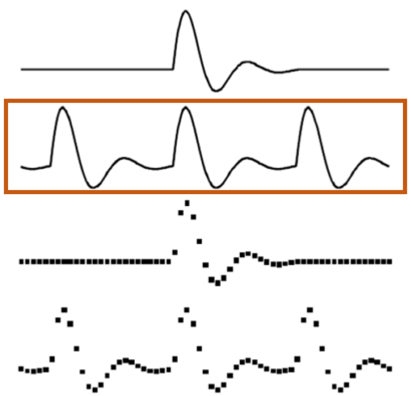
Continuous-time Fourier Series (CTFS)

▪ Fourier Series Representation

→ (Almost) any periodic function $x(t)$ with fundamental frequency ω_0 can be described as a sum of sinusoids

<u>Periodic Signal</u>

$$x(t) = x(t+nT), \quad n \in Z \quad (integers)$$



<u>Fourier Series</u>: Infinite sum of sines and cosines

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\Omega_0 t}, \quad \Omega_0 = 2\pi F_0 \;\; fundamental\ frequency$$

<u>Fourier Series Coefficients</u>

$$C_k = \frac{1}{T}\int_{<T>} x(t) e^{-jk\Omega_0 t} dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt$$

Continuous-time Fourier Series (CTFS)

Fourier Series Representation

$$\cos\theta = \tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin\theta = \tfrac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\Omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\Omega_0 t)$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \left(\frac{e^{jk\Omega_0 t} + e^{-jk\Omega_0 t}}{2}\right) + \sum_{k=1}^{\infty} b_k \left(\frac{e^{jk\Omega_0 t} - e^{-jk\Omega_0 t}}{2j}\right)$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \left(\frac{e^{jk\Omega_0 t} + e^{-jk\Omega_0 t}}{2}\right) + \sum_{k=1}^{\infty} b_k \left(\frac{-je^{jk\Omega_0 t} + je^{-jk\Omega_0 t}}{2}\right)$$

$$= a_0 + \sum_{k=1}^{\infty} \left(\frac{a_k - jb_k}{2}\right) e^{jk\Omega_0 t} + \sum_{k=1}^{\infty} \left(\frac{a_k + jb_k}{2}\right) e^{-jk\Omega_0 t}$$

$$= \underbrace{a_0}_{C_0} + \sum_{k=1}^{\infty} \underbrace{\left(\frac{a_k - jb_k}{2}\right)}_{C_k \ \textit{for positive } k} e^{jk\Omega_0 t} + \sum_{k=-\infty}^{-1} \underbrace{\left(\frac{a_{-k} + jb_{-k}}{2}\right)}_{C_k \ \textit{for negative } k} e^{jk\Omega_0 t}, \quad k := -k$$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\Omega_0 t}}$$

Continuous-time Fourier Series (CTFS)

- **Deriving Fourier Coefficient**

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\Omega_0 t}$$

$$x(t)e^{-jn\Omega_0 t} = \sum_{k=-\infty}^{\infty} C_k e^{jk\Omega_0 t} e^{-jn\Omega_0 t}$$

$$\int_{-T/2}^{T/2} x(t)e^{-jn\Omega_0 t} dt = \sum_{k=-\infty}^{\infty} C_k \underbrace{\int_{-T/2}^{T/2} e^{jk\Omega_0 t} e^{-jn\Omega_0 t} dt}_{\begin{cases} 0, & k \neq n \\ T, & k = n \end{cases}}$$

$$= C_k T$$

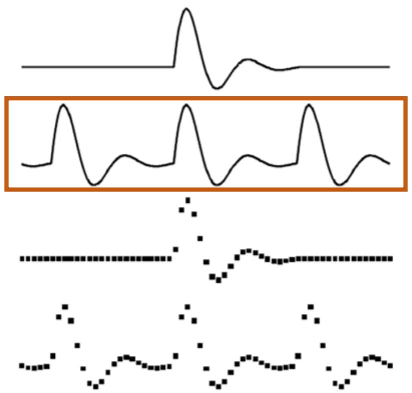
$$\boxed{C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_0 t} dt}$$

Discrete-time Fourier Series (DTFS)

Continuous-time Fourier Series (CTFS)

<u>Fourier Series</u>

$$x(t)=\sum_{k=-\infty}^{\infty} C_k e^{jk\Omega_0 t}, \quad \Omega_0 = 2\pi F_0 \ \textit{fundamental frequency}$$



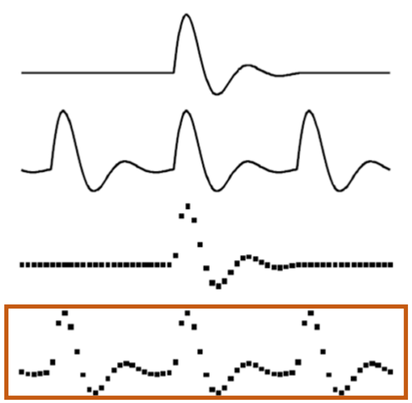
<u>Fourier Series Coefficient</u>

$$C_k = \frac{1}{T}\int_{<T>} x(t)e^{-jk\Omega_0 t}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_0 t}\,dt$$

Discrete-time Fourier Series (DTFS)

<u>Fourier Series</u>

$$x[n]=\sum_{k=<N>} C_k e^{jkn\frac{2\pi}{N}}$$



<u>Fourier Series Coefficients</u>

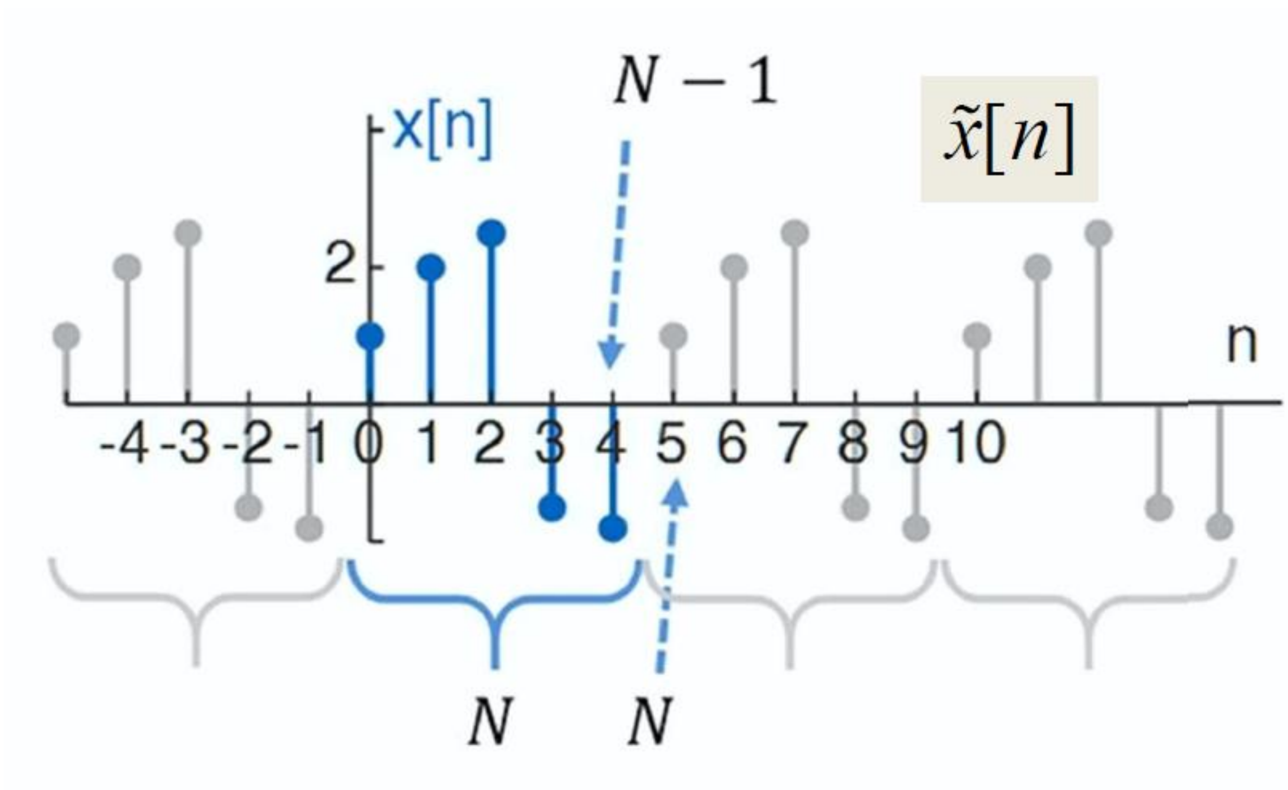
$$C_k = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi}{T}kt}\,dt \xrightarrow[T=NT_s]{t=nT_s} \frac{1}{NT_s}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{NT_s}knT_s}$$

$$C_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkn\frac{2\pi}{N}}$$

Discrete-time Fourier Series (DTFS)

- **Aperiodic and Periodic Signals**

$$\boxed{x[n] \rightarrow \tilde{x}[n]}$$



Fourier Transform

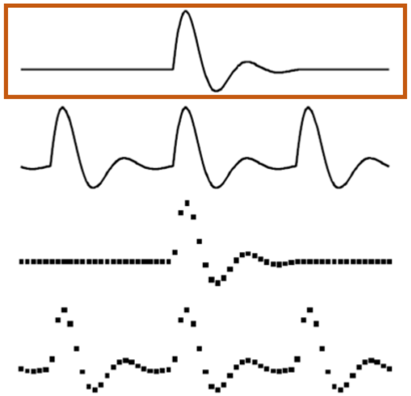
Continuous-time Fourier Transform (CTFT)

▪ Fourier Transform

Aperiodic Signal

$$x(t) = x(t + nT), \quad n \in Z \quad (integers)$$

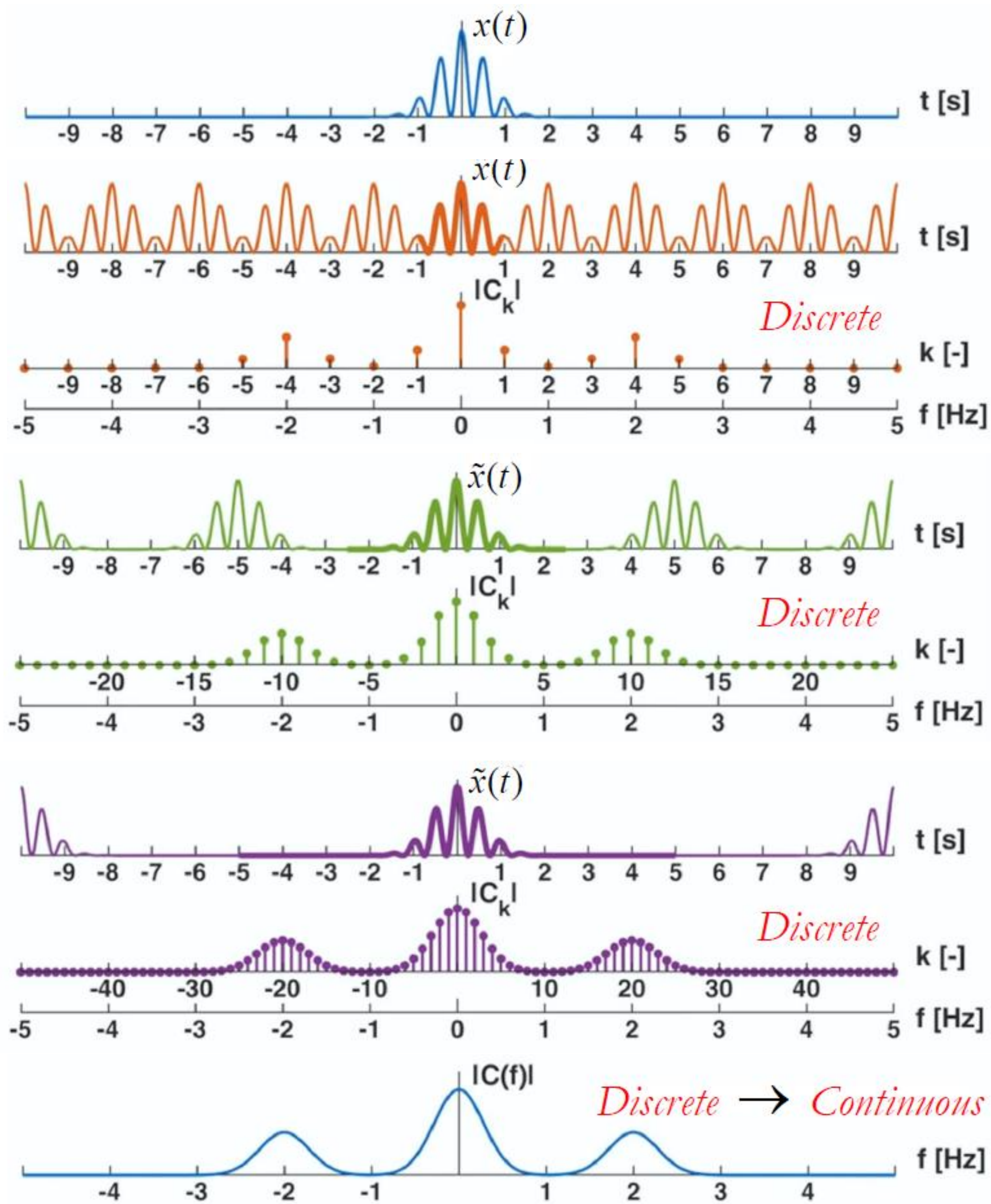
$$T \to \infty$$



Fourier Transform

$$\boxed{X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt}$$

Aperiodic & Periodic Signals



aperiodic

$T = 2s$

$F_0 = 0.5Hz$

$T = 5s$

$F_0 = 0.2Hz$

$T = 10s$

$F_0 = 0.1Hz$

$T \to \infty$

$\Delta F \to 0$

Continuous-time Fourier Transform (CTFT)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\Omega_0 t}, \qquad C_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\Omega_0 t} dt \quad \textit{discrete spectra}$$

$$= \sum_{k=-\infty}^{\infty} \left[\frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(\tau) e^{-jk\Omega_0 \tau} d\tau \right] e^{jk\Omega_0 t}, \qquad \Omega_0 = \frac{2\pi}{T} \rightarrow \frac{1}{T} = \frac{\Omega_0}{2\pi}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} \tilde{x}(\tau) e^{-jk\Omega_0 \tau} d\tau \right] \Omega_0 e^{jk\Omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} \tilde{x}(\tau) e^{-jk\Omega_0 \tau} d\tau \right] e^{jk\Omega_0 t} \Delta\Omega, \qquad \Delta\Omega = (k+1)\Omega - k\Omega = \Omega_0$$

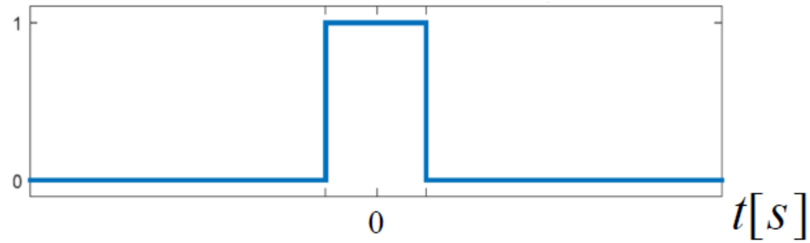
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \tilde{x}(\tau) e^{-j\Omega\tau} d\tau \right] e^{j\Omega t} d\Omega, \quad T \rightarrow \infty$$

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} x(\tau) e^{-j\Omega\tau} d\tau \right]}_{X(j\Omega) \ \textit{continuous spectra}} e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega}$$

Continuous-time Fourier Transform (CTFT)

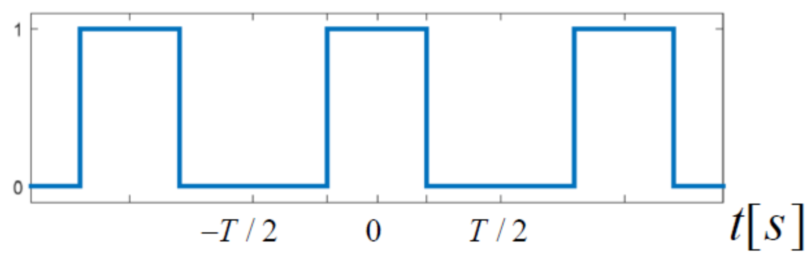
▪ Relationship between FS and CTFT

aperiodic pulse function $x(t)$



$$X(j\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt=\int_{-T/2}^{T/2}x(t)e^{-j\Omega t}dt$$

periodic pulse function $\tilde{x}(t)$



$$C_k=\frac{1}{T}\int_{<T>}\tilde{x}(t)e^{-jk\Omega_0 t}dt=\frac{1}{T}\int_{-T/2}^{T/2}x(t)e^{-jk\Omega_0 t}dt$$

Discrete-time Fourier Transform (DTFT)

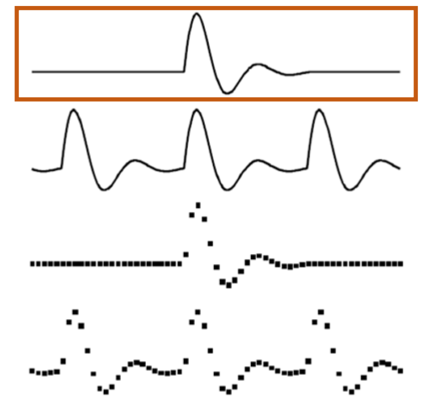
- **Continuous-time Fourier Transform (CTFT)**

<u>Aperiodic Signal</u>

$$x(t) = x(t + nT), \quad n \in Z \quad (\textit{integers})$$
$$T \to \infty$$

<u>Fourier Transform</u>

$$\boxed{X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt}$$



- **Discrete-time Fourier Transform (DTFT)**

<u>Aperiodic Signal</u>

$$t = nT_s, \quad n \in Z$$
$$x(t) \xrightarrow{t = nT_s} x(nT_s) = x[n]$$

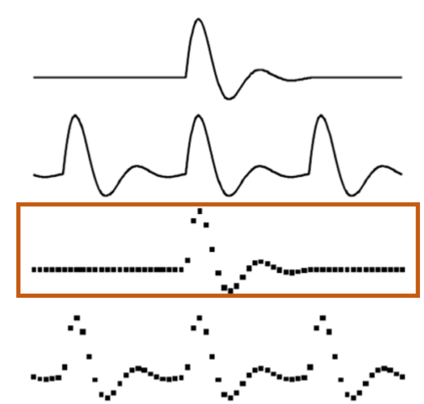
$$\Omega T_s = 2\pi \frac{F}{F_s} = \omega$$

normalized frequency

<u>Fourier Transform</u>

$$\int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \xrightarrow{t = nT_s} \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n T_s}$$

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}}$$

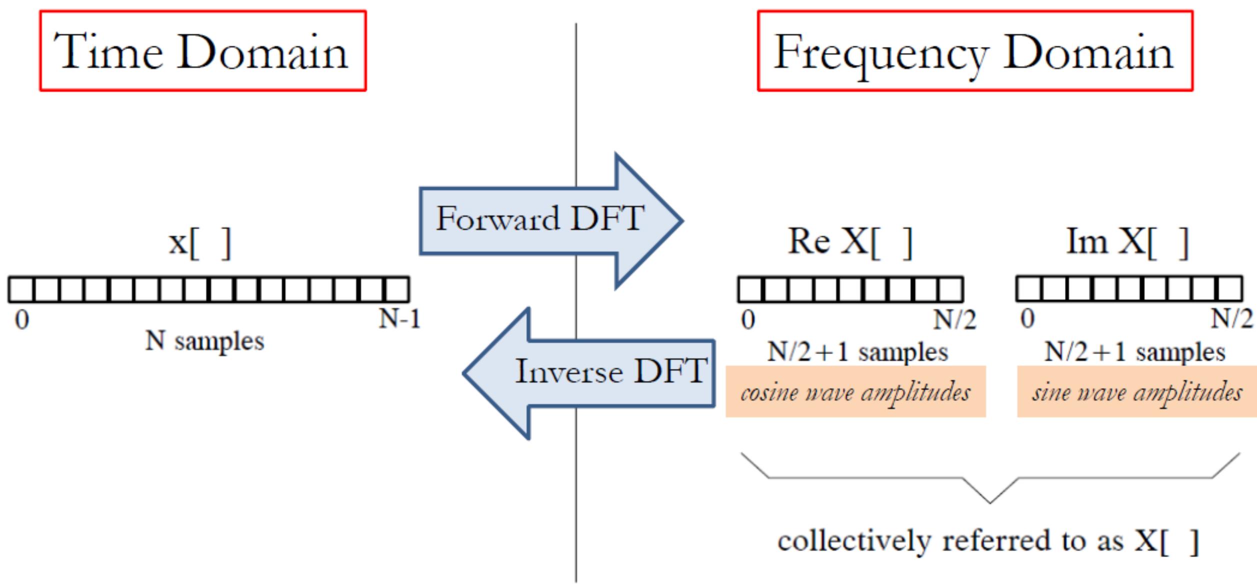


Discrete Fourier Transform (DFT)

Discrete Fourier Transform (DFT)

Discrete Fourier Transform

- $\rightarrow$ Changes an N point input signal into two $\frac{N}{2} - 1$ point output signals
- $\rightarrow$ The input signal contains the signal being decomposed, while the two output signals contain the amplitudes of the component sine and cosine waves

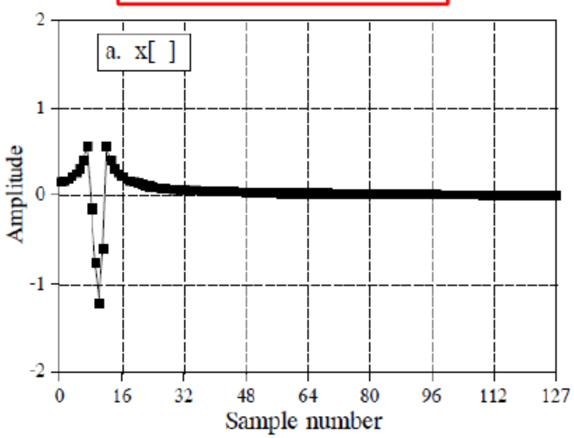


- decomposition, analysis, *the* forward DFT, *the* DFT
- synthesis, *the* inverse DFT

Independent Variables (Frequency Domain)

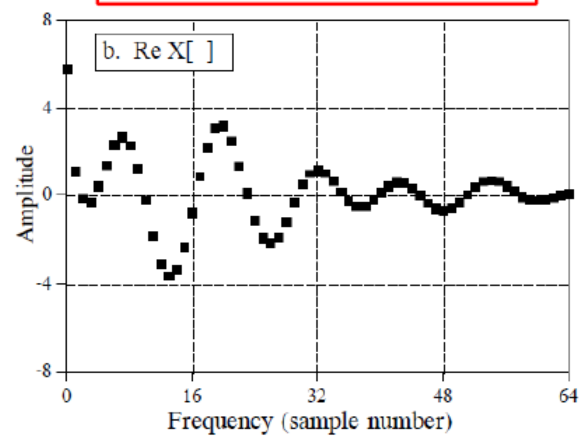
- **Horizontal Axis of the Frequency Domain**

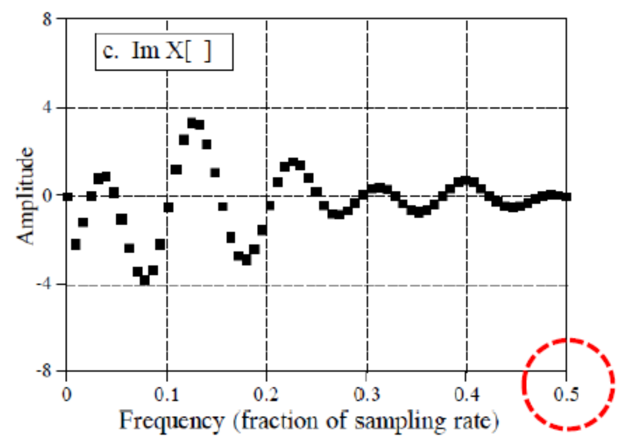
Time Domain



1. Samples
   $$c[n] = \cos(2\pi kn / N)$$
2. fraction of sampling rate
   $$c[n] = \cos(2\pi fn)$$
3. natural frequency [rad]
   $$c[n] = \cos(\omega n)$$
4. analog frequency

Frequency Domain

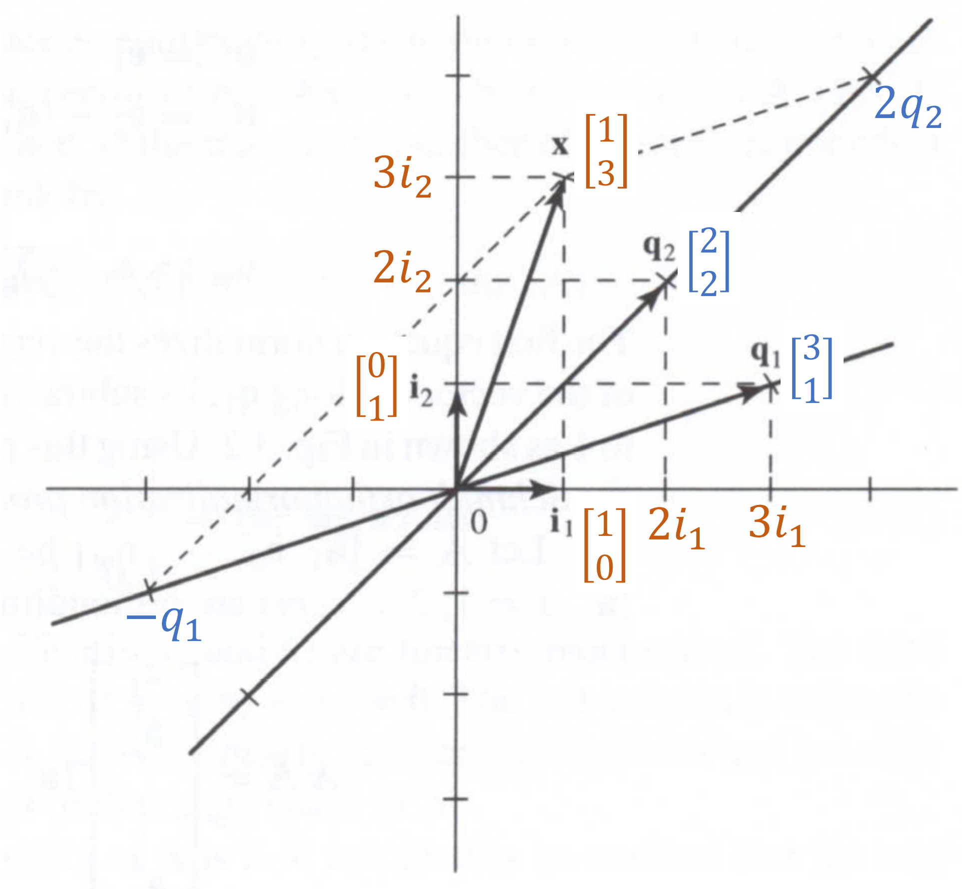




Basis Function

- **Review:**
  → Different representation of vector x

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



Representation of x
with respect to the basis $\{i_1, i_2\}$

$$x = Q \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Representation of x
with respect to the basis $\{q_1, q_2\}$

$$x = Q \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find representation of x with basis $\{q_2, i_2\}$

DFT Basis Functions

Basis Functions:

→ A set of sine and cosine waves with unity amplitude

$$c_k[i] = \cos(2\pi ki / N) \qquad \leftarrow \mathrm{Re}\, X[k]$$

$$s_k[i] = \sin(2\pi ki / N) \qquad \leftarrow \mathrm{Im}\, X[k]$$

→ The output of the DFT is a set of numbers that represent amplitudes

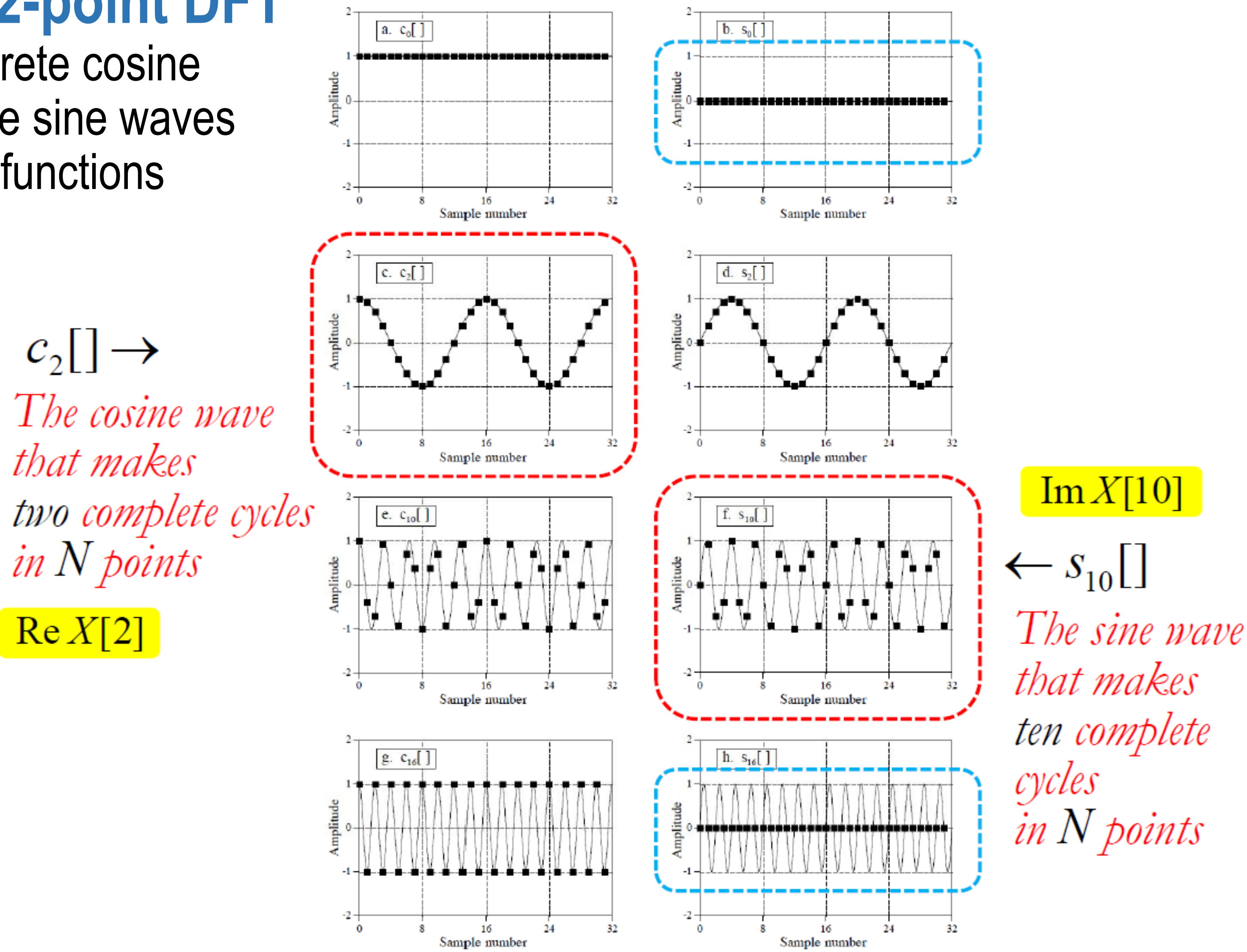
Assign each amplitude (the frequency domain) to the proper sine or cosine waves (the basis functions)

↓

A set of scaled sine and cosine waves that can be added to form the time domain signal.

DFT Basis Functions

- **Example: 32-point DFT**
  → Has 17 discrete cosine & 17 discrete sine waves for its basis functions

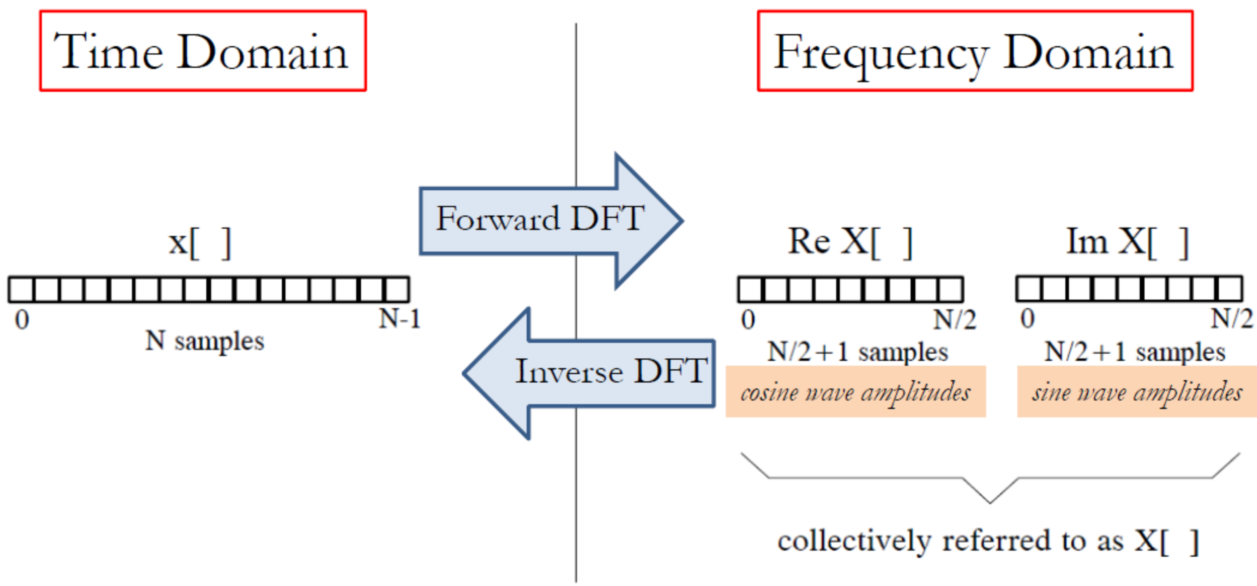


Discrete Fourier Transform (DFT)

Discrete Fourier Transform

→ Changes an N point input signal into two $\frac{N}{2}+1$ point output signals

→ The input signal contains the signal being decomposed, while the two output signals contain the amplitudes of the component sine and cosine waves



- decomposition, analysis, *the* forward DFT, *the* DFT
- synthesis, *the* inverse DFT

Analysis

- **Analysis: Calculating the DFT**

→ Three different ways:

1. **Simultaneous Equation**
   - ✓ Given N values from the time domain, calculate the N values of the frequency domain
   - ✓ Basic algebra: to solve for N unknowns, we need N linearly independent equations
   - ✓ Too inefficient to be of practical use
2. **Correlations**
   - ✓ Correlating the input signal with each basis function
   - ✓ When a particular sine wave is present, it produces the value. Otherwise, it will be zero.

$$ReX[k] = \sum_{i=0}^{N-1} x[i] \cos(2\pi k i / N)$$

$$ImX[k] = -\sum_{i=0}^{N-1} x[i] \sin(2\pi k i / N)$$

3. **Fast Fourier Transform (FFT)**
   - ✓ An ingenious algorithm that decomposes a DFT with N points, into N DFTs each with a single point.

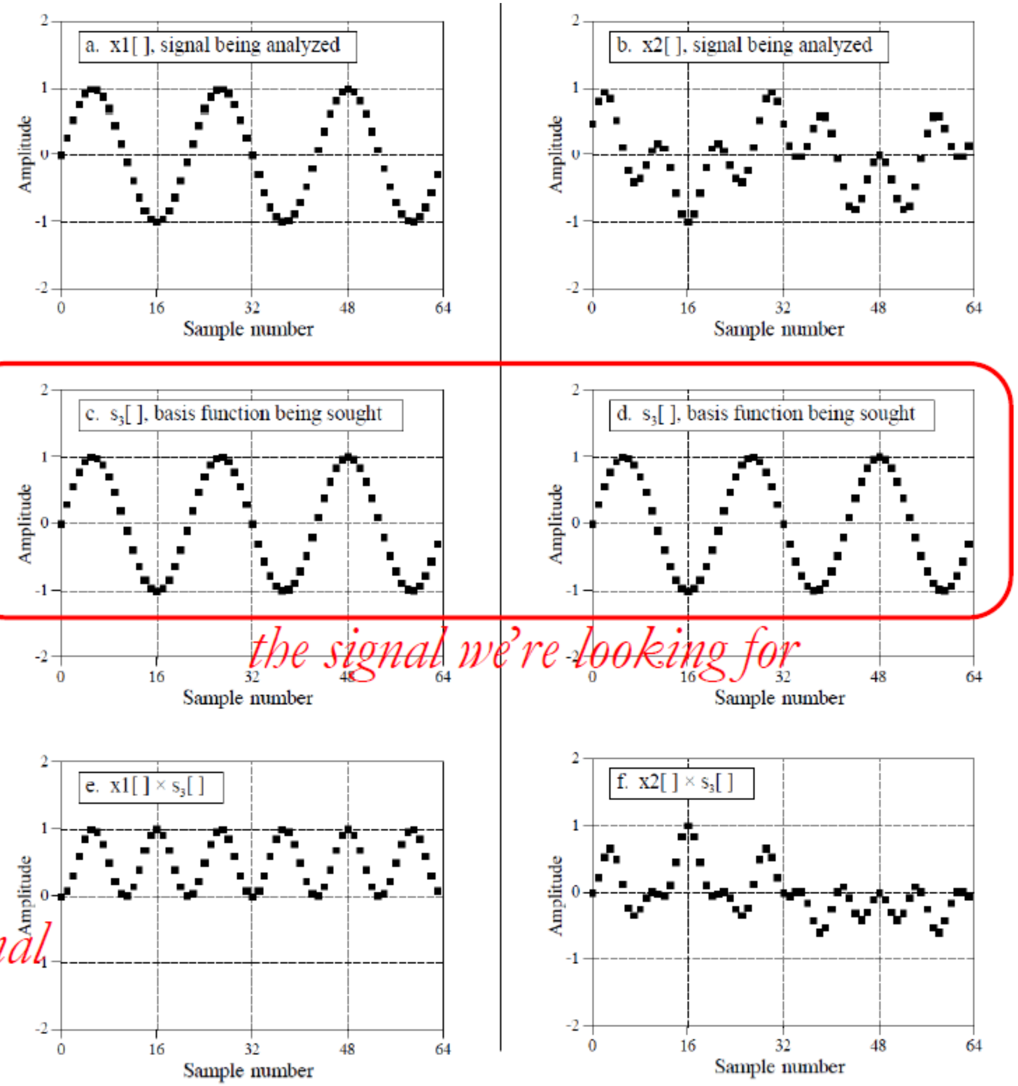
Analysis

Analysis

$$\operatorname{Re} X[k] = \sum_{i=0}^{N-1} x[i]\cos(2\pi ki / N)$$

$$\operatorname{Im} X[k] = -\sum_{i=0}^{N-1} x[i]\sin(2\pi ki / N)$$

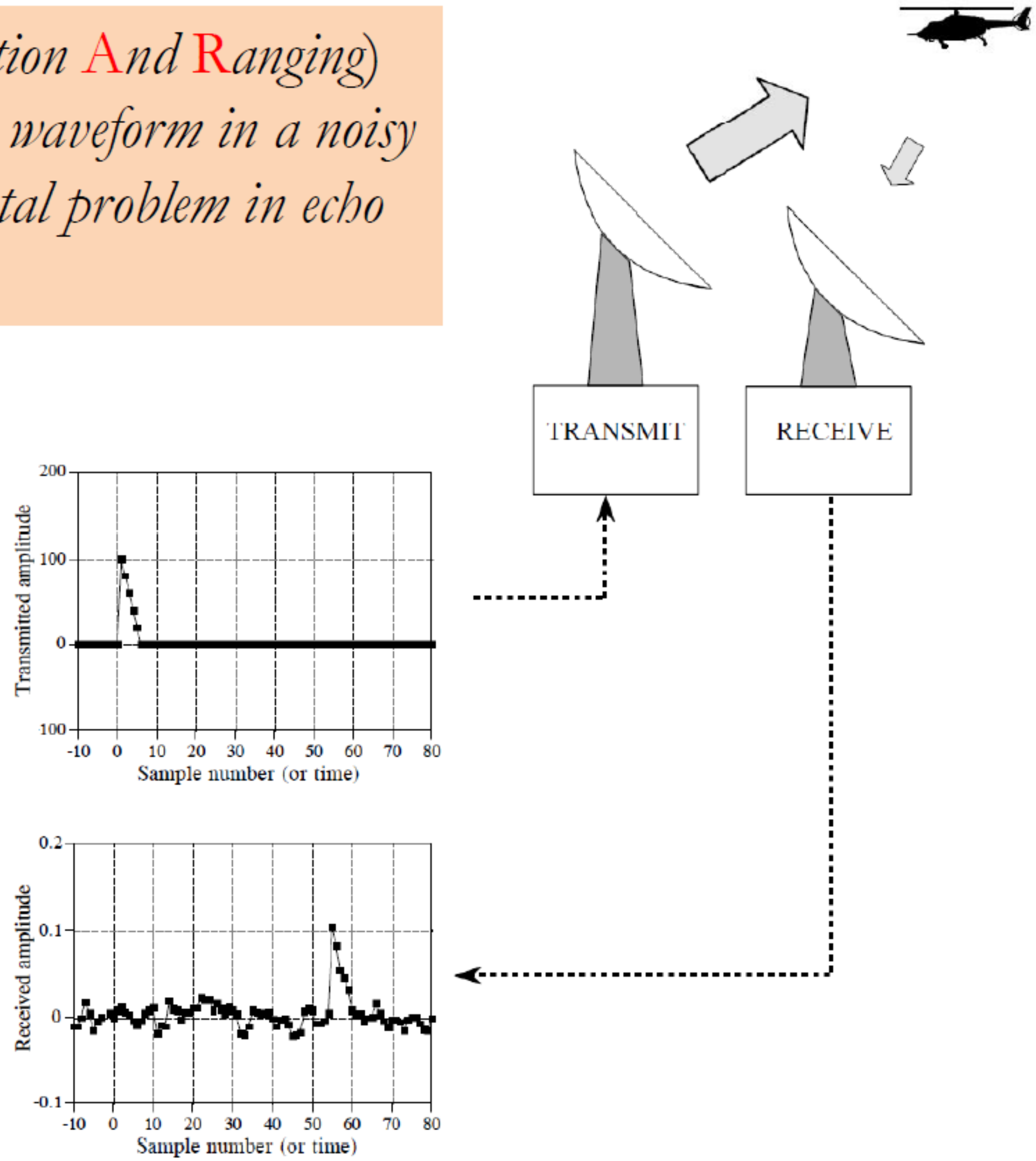
Correlating the input signal with each basis function!



Detect a known waveform contained in another signal.

Analysis

Radar (RAdio Detection And Ranging)
Detection of a known waveform in a noisy signal is the fundamental problem in echo location: correlation



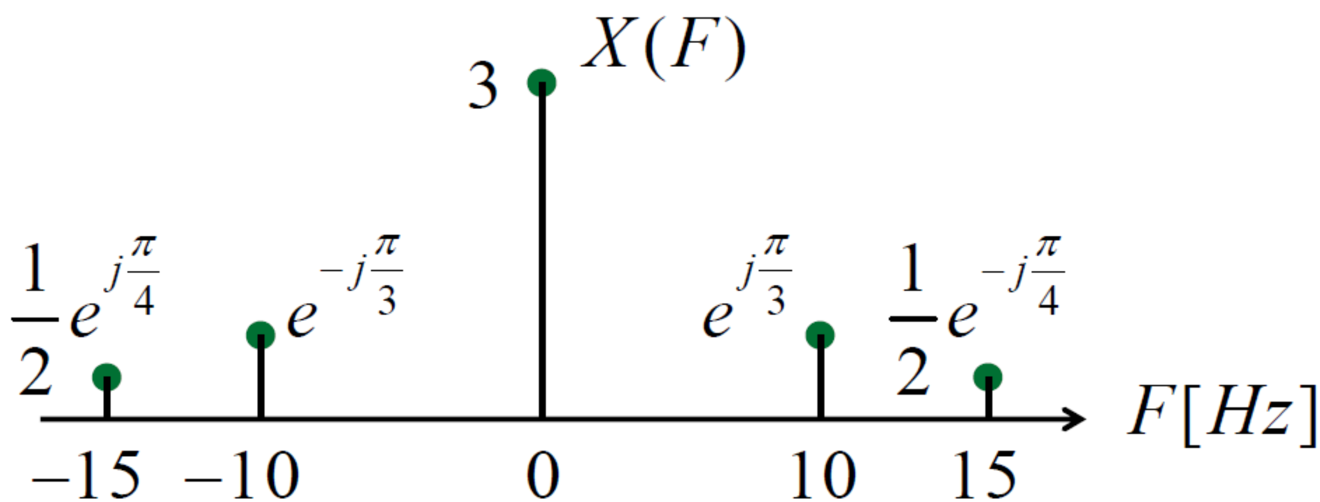
Example

- **Frequency Spectrum of a Signal**

$$x(t) = 3 + 2\cos\left(20\pi t + \frac{\pi}{3}\right) + \cos\left(30\pi t - \frac{\pi}{4}\right)$$

$$= 3 + \frac{2}{2}e^{j\frac{\pi}{3}}e^{j2\pi 10t} + \frac{2}{2}e^{-j\frac{\pi}{3}}e^{-j2\pi 10t}$$

$$+ \frac{1}{2}e^{-j\frac{\pi}{4}}e^{j2\pi 15t} + \frac{1}{2}e^{j\frac{\pi}{4}}e^{-j2\pi 15t}$$



Synthesis

- **Synthesis: Calculating the Inverse DFT (IDFT)**
  - $\rightarrow$ Any N point signal, $x[i]$, can be created by adding $\frac{n}{2}+1$ cosine & $\frac{n}{2}+1$ sine waves.
  - $\rightarrow$ The amplitudes of the cosine & sine waves are held in the arrays $\mathrm{Re}\bar{X}[k]$ & $\mathrm{Im}\bar{X}[k]$

$$\boxed{x[i]=\sum_{k=0}^{N/2}\mathrm{Re}\,\bar{X}[k]\cos(2\pi ki/N)+\sum_{k=0}^{N/2}\mathrm{Im}\,\bar{X}[k]\sin(2\pi ki/N)}$$

$$\mathrm{Re}\,\bar{X}[k]=\frac{\mathrm{Re}\,X[k]}{N/2}$$

$$\mathrm{Im}\,\bar{X}[k]=-\frac{\mathrm{Im}\,X[k]}{N/2}$$

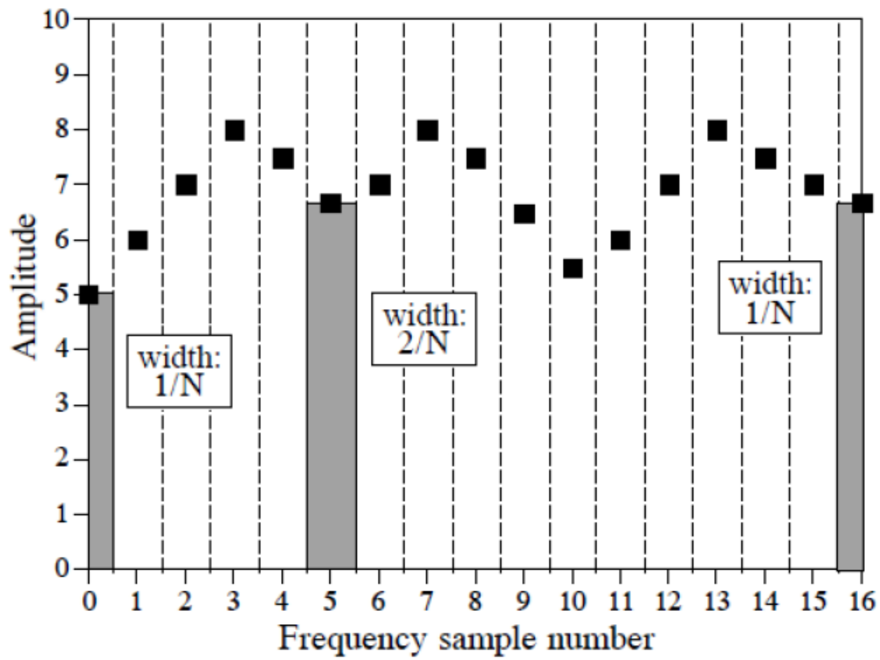
$$\mathrm{Re}\,\bar{X}[0]=\frac{\mathrm{Re}\,X[0]}{N}$$

$$\mathrm{Im}\,\bar{X}[N/2]=-\frac{\mathrm{Im}\,X[N/2]}{N}$$

Synthesis

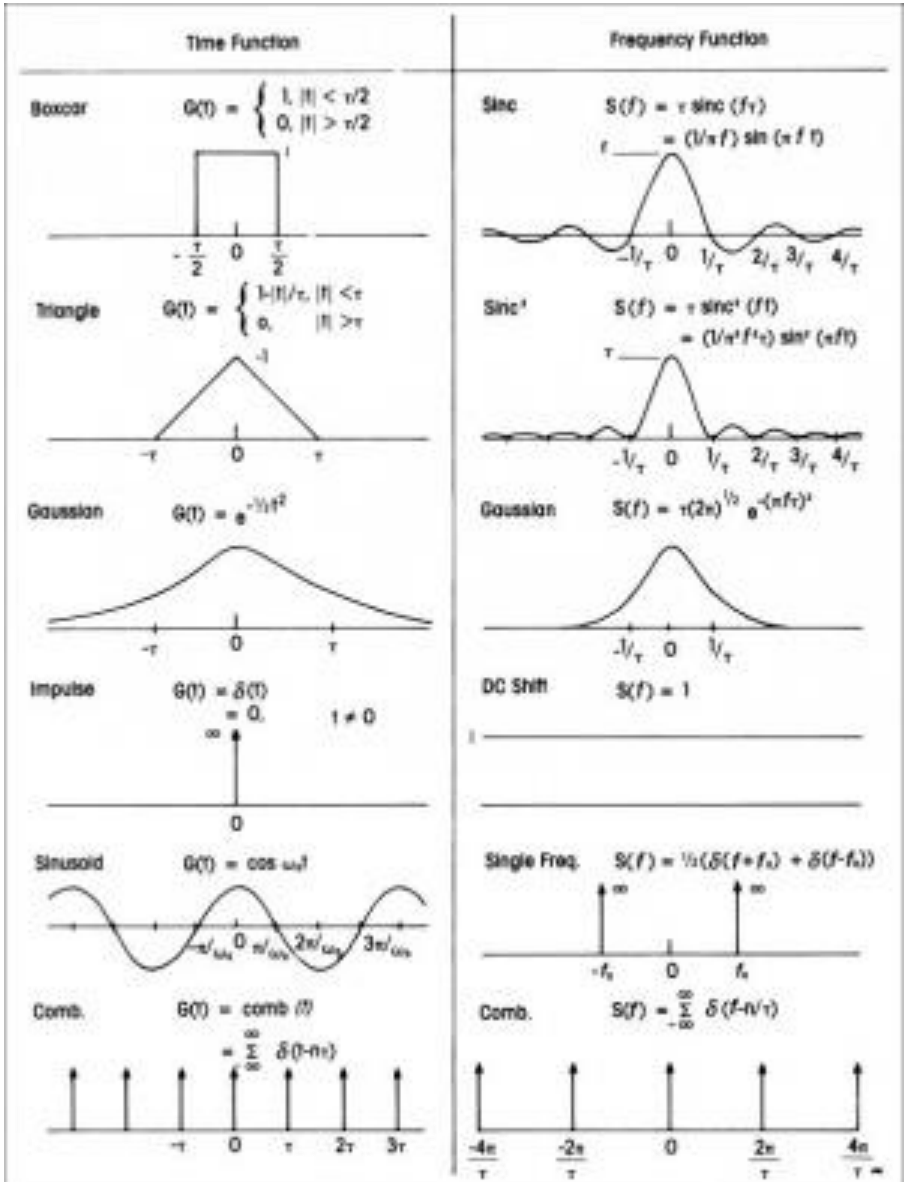
The frequency domain is defined as a spectral density:

How much signal (amplitude) is present per unit of bandwidth?



Each sample in the frequency domain can be thought of being contained in the frequency band of width $2/N$.

Fourier Transform Pairs



Discrete-time Fourier Transform

Discrete-time Fourier Transform (DTFT)

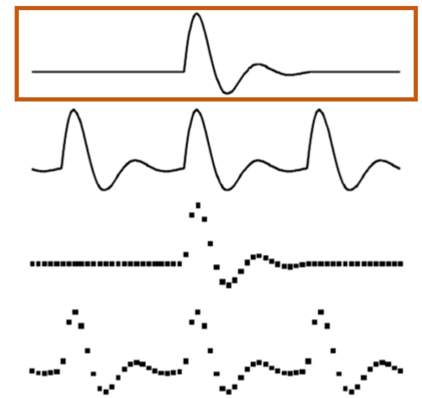
- **Continuous-time Fourier Transform (CTFT)**

<u>Aperiodic Signal</u>

$$x(t) = x(t + nT), \quad n \in Z \quad (integers)$$
$$T \to \infty$$

<u>Fourier Transform</u>

$$\boxed{X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt}$$



- **Discrete-time Fourier Transform (DTFT)**

<u>Aperiodic Signal</u>

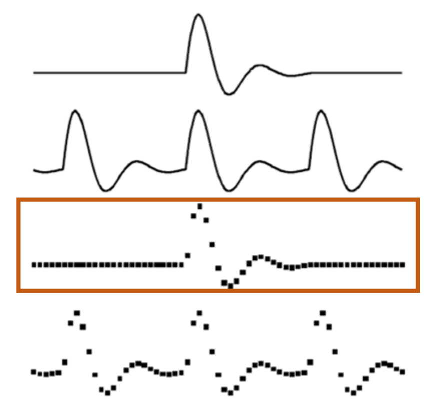
$$t = nT_s, \quad n \in Z$$
$$x(t) \xrightarrow{t=nT_s} x(nT_s) = x[n]$$

$$\Omega T_s = 2\pi \frac{F}{F_s} = \omega$$
normalized frequency

<u>Fourier Transform</u>

$$\int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \xrightarrow{t=nT_s} \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n T_s}$$

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}}$$



Discrete-time Fourier Transform (DTFT)

If $x(k)$ *is absolutely summable, that is,* $\sum_{k=-\infty}^{\infty} |x(k)| < \infty$, *then its* DTFT *is given by*

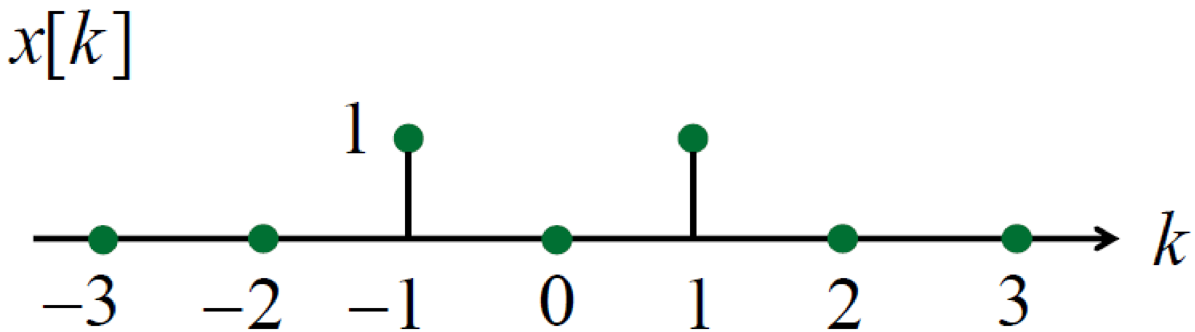
$$X(e^{j\omega}) \equiv F[x(k)] = \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k}$$

The inverse discrete-time Fourier transform (IDTFT) *of* $X(e^{j\omega})$ *is given by*

$$x(k) \equiv F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega k} d\omega$$

Example

- **DTFT**



$$x[k] = \delta[k-1] + \delta[k+1]$$

$$\begin{aligned} X(e^{j\omega}) &= 1 \cdot e^{-j(-1)\omega} + 1 \cdot e^{-j(1)\omega} \\ &= e^{j\omega} + e^{-j\omega} \\ &= 2\cos\omega \end{aligned}$$

Example

- **Inverse DTFT (IDTFT)**

$$X(e^{j\omega}) = 2\cos\omega$$

Product-to-sum identities

$$\boxed{\begin{aligned} 2\cos\theta\cos\varphi &= \cos(\theta-\varphi)+\cos(\theta+\varphi) \\ 2\cos\theta\sin\varphi &= \sin(\theta+\varphi)-\sin(\theta-\varphi) \end{aligned}}$$

$$x[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos\omega e^{j\omega k}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[2\cos\omega\cos\omega k + 2\cos\omega\, j\sin\omega k\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{\left[\cos(1-k)\omega + \cos(1+k)\omega\right] + j\left[\sin(1+k)\omega - \sin(1-k)\omega\right]\right\} d\omega$$

$$= \begin{cases} \dfrac{1}{2\pi}\cdot 2\pi = 1, & k = \ \ 1 \\[2ex] \dfrac{1}{2\pi}\cdot 2\pi = 1, & k = -1 \end{cases}$$

Example

DTFT: Finite & Infinite Duration

infinite duration

Determine the discrete-time Fourier transform of step function $x(k) = (0.5)^k u(k)$.

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(k)e^{-j\omega k} = \sum_{0}^{\infty}(0.5)^k e^{-j\omega k}$$

$$= \sum_{0}^{\infty}(0.5e^{-j\omega})^k = \frac{1}{1-0.5e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-0.5}$$

finite duration

Determine the discrete-time Fourier transform of the following finite-duration sequence: $x(k) = \{1,\underset{\uparrow}{2},3,4,5\}$

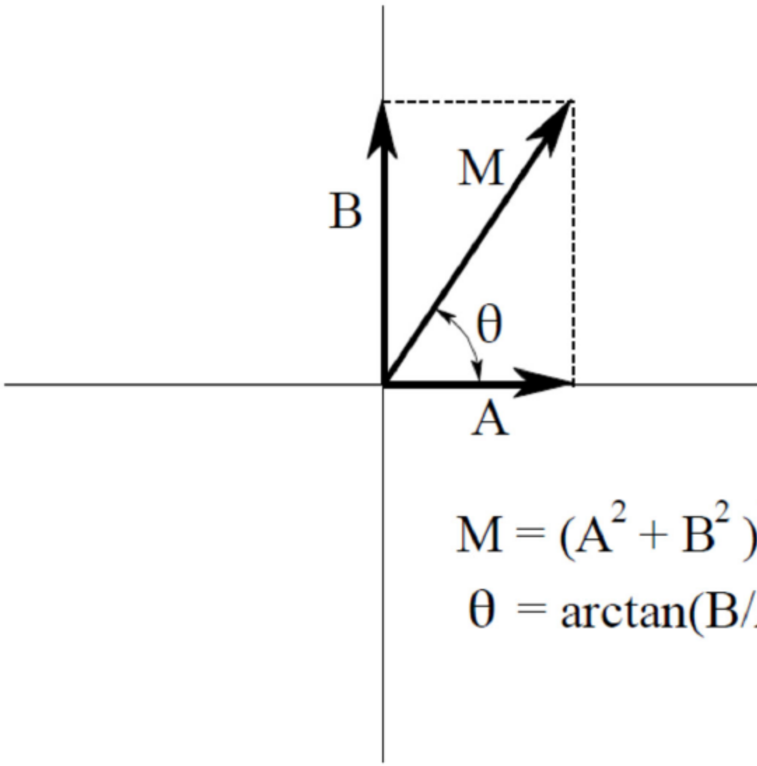
$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(k)e^{-j\omega k} = e^{j\omega} + 2 + 3e^{-j\omega} + 4e^{-j2\omega} + 5e^{-j3\omega}$$

Some Common DTFT Pairs

$x[k]$	$X(e^{j\omega}), -\pi \le \omega \le \pi$
$\delta[k]$	1
1	$2\pi\delta(\omega)$
$u[k]$	$\dfrac{1}{1-e^{-j\omega}} + \pi\delta(\omega)$
$\alpha^k u[k]$	$\dfrac{1}{1-\alpha e^{-j\omega}}$
$e^{j\omega_0 k}$	$2\pi\delta(\omega-\omega_0)$
$\cos[\omega_0 k]$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin[\omega_0 k]$	$j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]$
$\alpha^{\lvert k \rvert}u[k]$	$\dfrac{1-\alpha^2}{1-2\alpha\cos(\omega)+\alpha^2}$

Rectangular & Polar Form

- **Rectangular-to-Polar Conversion**



$$M = (A^2 + B^2)^{1/2}$$
$$\theta = \arctan(B/A)$$

$$MagX[k] = (\mathrm{Re}\,X[k]^2 + \mathrm{Im}\,X[k]^2)^{1/2}$$

$$PhaseX[k] = \arctan\left(\frac{\mathrm{Im}\,X[k]}{\mathrm{Re}\,X[k]}\right)$$

$$\mathrm{Re}\,X[k] = MagX[k]\cos(PhaseX[k])$$

$$\mathrm{Im}\,X[k] = MagX[k]\sin(PhaseX[k])$$

Rectangular & Polar Form

Rectangular Form

Rectangular coordinates: rarely used in signal processing

$$X(\omega) = X_R(\omega) + j\ X_I(\omega)$$

where $X_R(\omega), X_I(\omega) \in \mathbb{R}$.

Polar Form

Polar coordinates: more intuitive way to represent frequency content

$$X(\omega) = |X(\omega)|\ e^{j\Theta(\omega)}$$

where $|X(\omega)|, \Theta(\omega) = \angle X(\omega) \in \mathbb{R}$.

- $|X(\omega)|$: determines the relative presence of a sinusoid $e^{j\omega n}$ in $x(n)$
- $\Theta(\omega) = \angle X(\omega)$: determines how the sinusoids line up relative to one another to form $x(n)$

Polar Form: Cosine & Sine Wave

- ## Cosine Waves

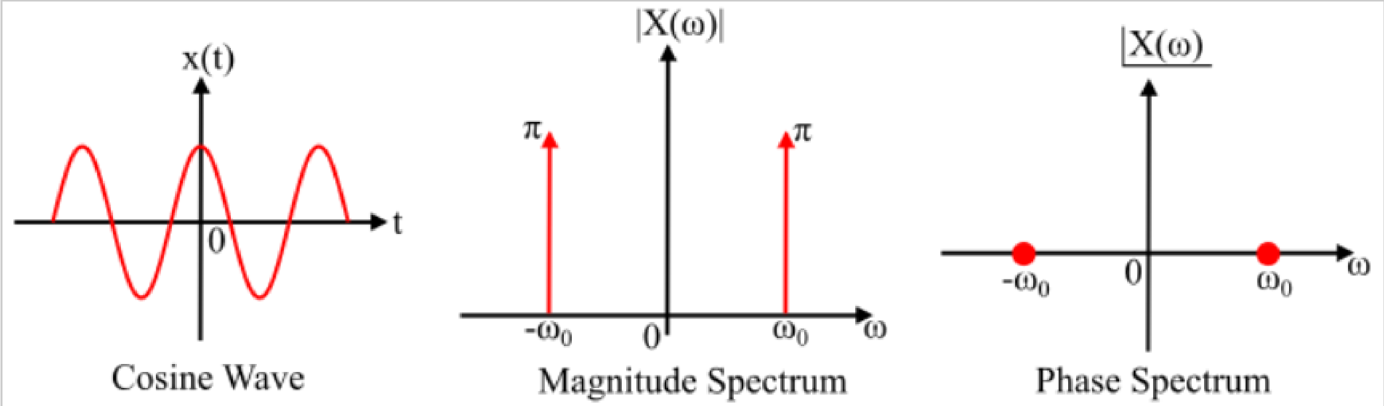


Figure-2

- ## Sine Waves

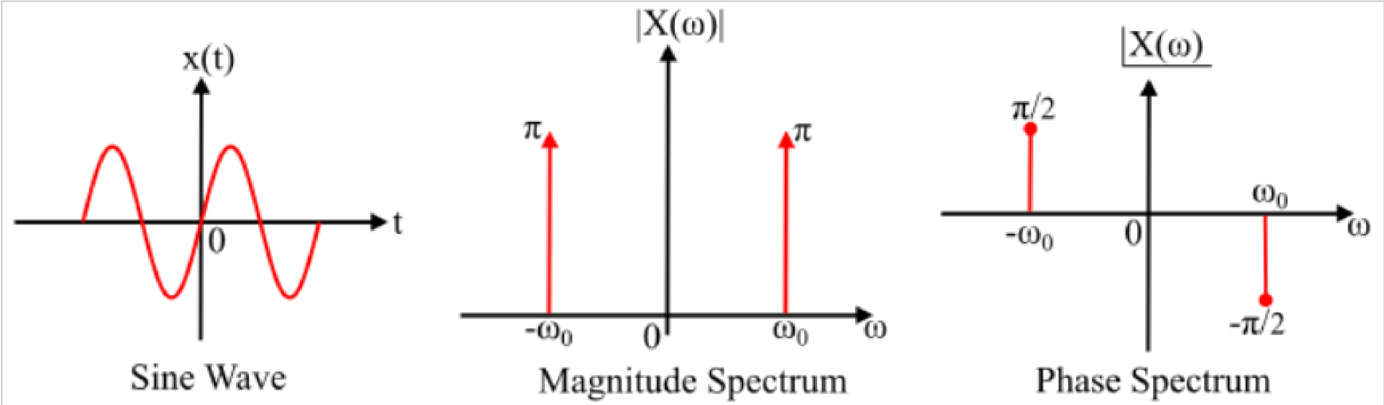
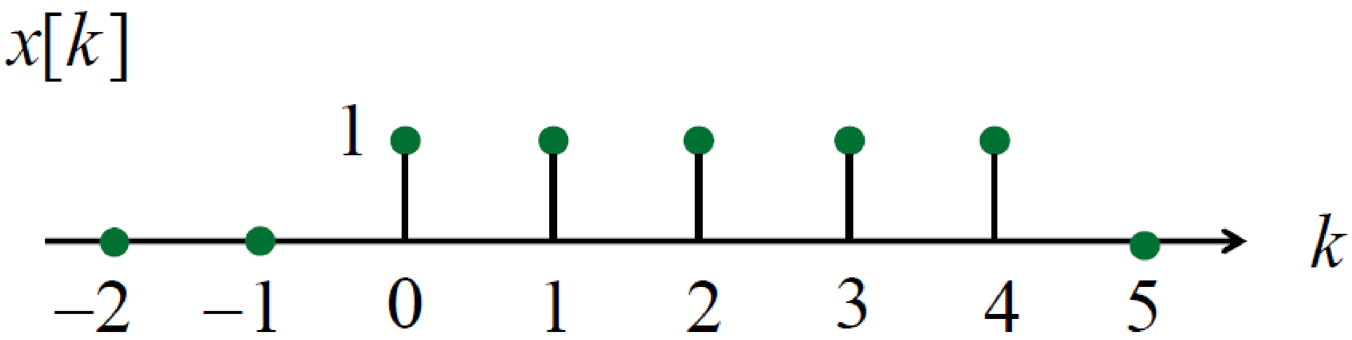


Figure-1

Example (1)

- **Rectangular Form**



$$X(e^{j\omega}) = \sum_{k=0}^{4} x[k] e^{-jwk}$$

$$= 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega}$$

Example (1)

- **Polar Form** $\boxed{\left|X(e^{j\omega})\right|, \quad \measuredangle X(e^{j\omega})}$

$$X(e^{j\omega}) = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega}$$

$$= \sum_{k=0}^{4} e^{-j\omega k}$$

$$\sum_{k=0}^{N-1} \alpha^k = \frac{1-\alpha^N}{1-\alpha}, \quad \forall \alpha$$

$$= \frac{1-e^{-j\omega(5)}}{1-e^{-j\omega}}$$

$$\boxed{\alpha = e^{-j\omega}, \quad N = 5}$$

$$= \frac{e^{-j5\omega/2}\left[e^{j5\omega/2} - e^{-j5\omega/2}\right]}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

$$= e^{-j2\omega} \boxed{\frac{\sin 5\omega/2}{\sin \omega/2}}$$

$$\boxed{\left|X(e^{j\omega})\right|}$$

$$\measuredangle X(e^{j\omega}) = -2\omega$$

Example (2)

$$x[k] = a^{|k|}, \quad |a| < 1$$

$$x[k] = a^{k}, \quad k \geq 0$$

$$= a^{-k}, \quad k < 0$$

$$\boxed{\sum_{k=0}^{\infty} \alpha^k \to \frac{1}{1-\alpha}, \quad |\alpha| < 1}$$

$$X_1(e^{j\omega}) = \sum_{k=0}^{\infty} x[k] e^{-j\omega k} = \sum_{k=0}^{\infty} a^k e^{-j\omega k} = \sum_{k=0}^{\infty} [ae^{-j\omega}]^k$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

$$X_2(e^{j\omega}) = \sum_{-\infty}^{-1} a^{-k} e^{-j\omega k} = \sum_{1}^{\infty} a^k e^{j\omega k} = \sum_{1}^{\infty} [ae^{j\omega}]^k$$

$$= \sum_{0}^{\infty} [ae^{j\omega}]^k - 1 = \frac{1}{1 - ae^{j\omega}} - 1 = \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

Example (2)

$$X(e^{j\omega}) = X_1(e^{j\omega}) + X_2(e^{j\omega})$$

$$= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$= \boxed{\frac{1 - a^2}{1 - 2a\cos\omega + a^2}}$$

$$\boxed{\left| X(e^{j\omega}) \right|}$$

$$\measuredangle X(e^{j\omega}) = 0$$

$$\cos\omega = \frac{e^{j\omega} + e^{-j\omega}}{2}$$

$$\sin\omega = \frac{e^{j\omega} - e^{-j\omega}}{2j}$$

Example (3): CTFT

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\Omega t}dt$$

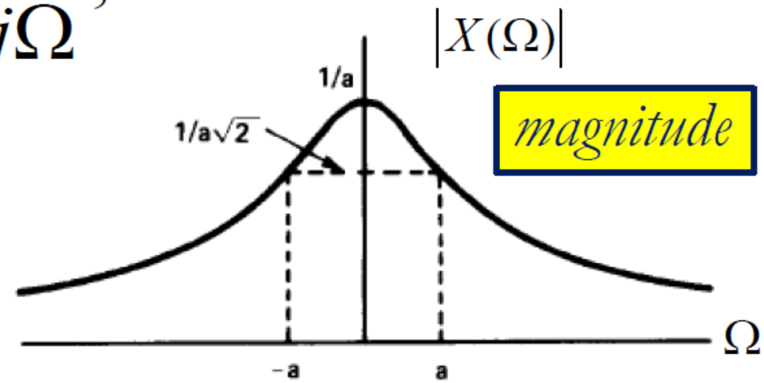
$$= \int_{0}^{\infty} e^{-at}e^{-j\Omega t}dt$$

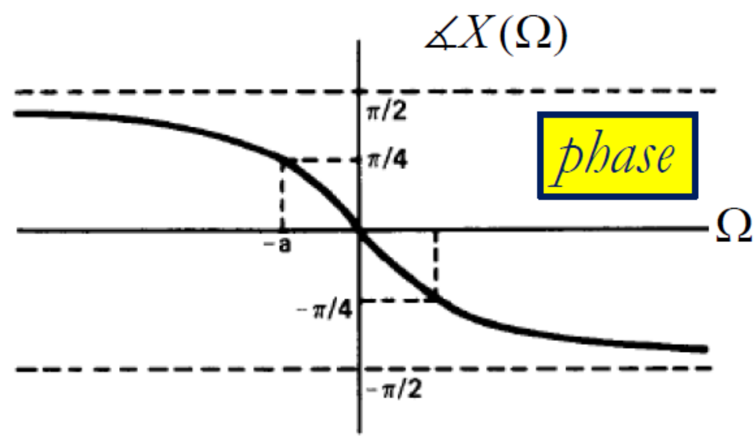
$$= \int_{0}^{\infty} e^{-(a+j\Omega)t}dt$$

$$= \frac{-1}{a+j\Omega}e^{-(a+j\Omega)t}\Big|_0^{\infty} = \frac{-1}{a+j\Omega}(0-1) = \frac{1}{a+j\Omega}; \quad a > 0$$

Example (3): CTFT

$$e^{-at}u(t) \leftrightarrow \frac{1}{a + j\Omega}; \quad a > 0$$



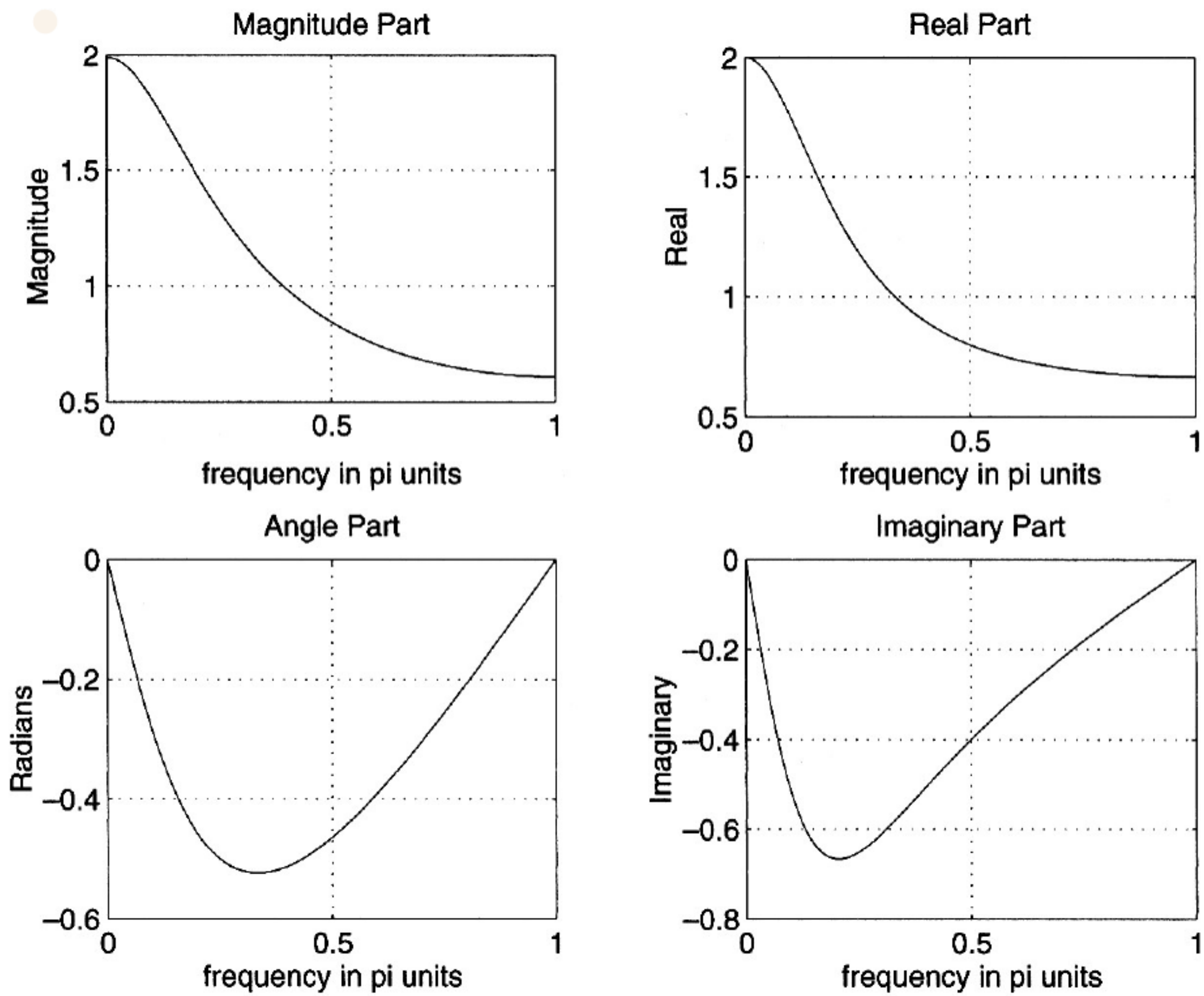


Example using MATLAB (1)

Example: Evaluate $X(e^{j\omega}) = \dfrac{e^{j\omega}}{e^{j\omega} - 0.5}$ at 501 equispaced points between $[0, \pi]$.

```
>> w = [0:1:500]*pi/500; % [0, pi] axis divided into 501 points.
>> X = exp(j*w) ./ (exp(j*w) – 0.5*ones(1,501));
>> magX = abs(X); angX = angle(X); realX = real(X); imagX = imag(X);
>> subplot(2,2,1); plot(w/pi,magX); grid
>> xlabel('frequency in pi units'); title('Magnitude part'); ylabel('Magnitu
>> subplot(2,2,2); plot(w/pi,realX); grid
>> xlabel('frequency in pi units'); title('Real part'); ylabel('Real')
>> subplot(2,2,3); plot(w/pi, angX); grid
>> xlabel('frequency in pi units'); title('Angle part'); ylabel('Radians')
>> subplot(2,2,4); plot(w/pi,imagX); grid
>> xlabel('frequency in pi units'); title('Imaginary part'); ylabel('Imaginar
```

Example using MATLAB (1)

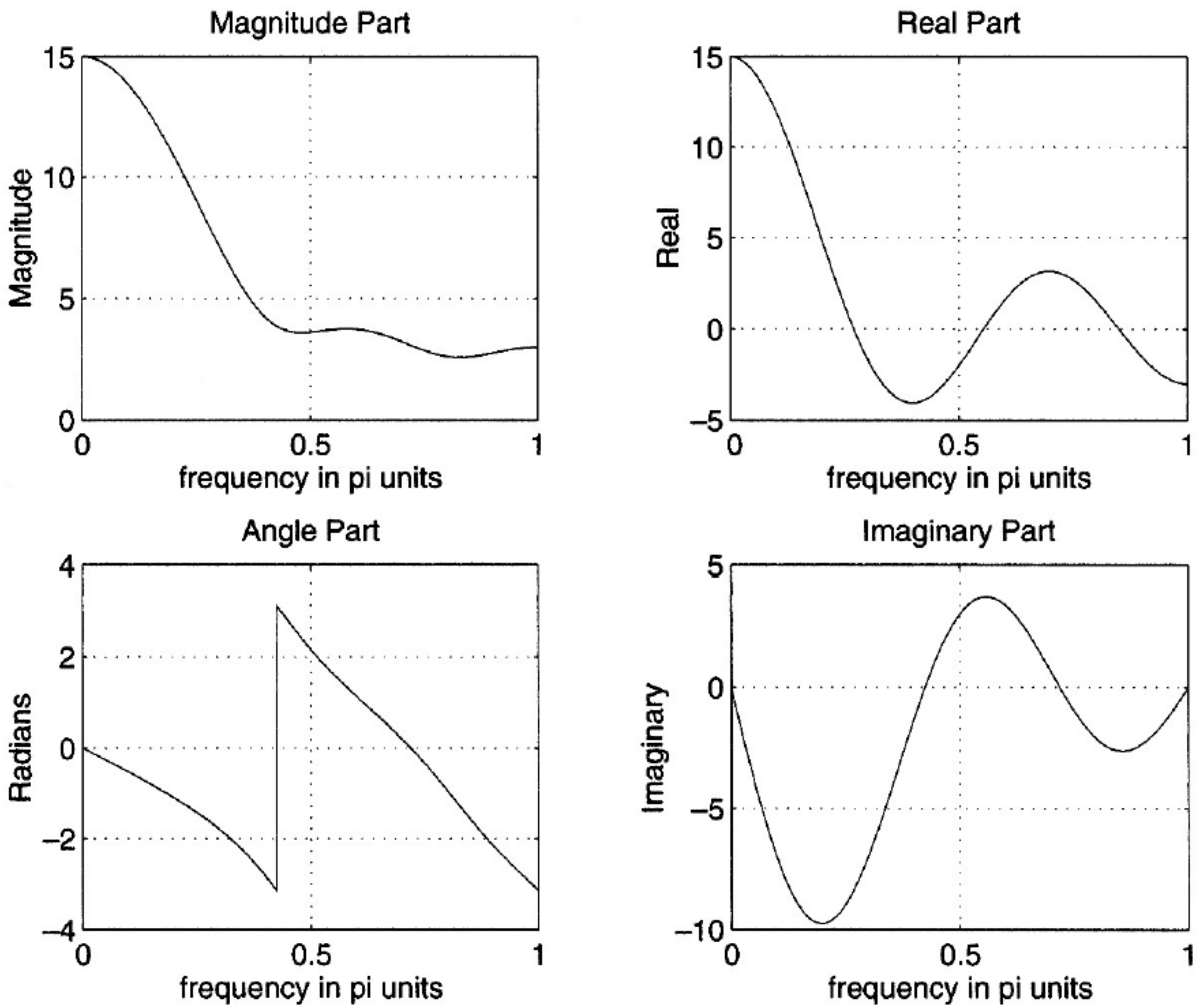


Example using MATLAB (2)

Example: Compute the DTFT of $x(k) = \{1, \underset{\uparrow}{2}, 3, 4, 5\}$ at 501 equispaced frequencies between $[0, \pi]$.

```
>> k = -1:3; x = 1:5; i = 0:500; w = (pi/500)*i;
>> X = x * exp(-j*pi/500)) .^ (k'*i);
>> magX = abs(X); angX = angle(X); realX = real(X); imagX = imag(X);
>> subplot(2,2,1); plot(n/500,magX); grid
>> xlabel('frequency in pi units'); title('Magnitude part'); ylabel('Magnitu
>> subplot(2,2,2); plot(n/500,realX); grid
>> xlabel('frequency in pi units'); title('Real part'); ylabel('Real')
>> subplot(2,2,3); plot(n/500, angX); grid
>> xlabel('frequency in pi units'); title('Angle part'); ylabel('Radians')
>> subplot(2,2,4); plot(n/500,imagX); grid
>> xlabel('frequency in pi units'); title('Imaginary part'); ylabel('Imaginar
```

Example using MATLAB (2)



Periodicity & Symmetry

ω *is a real variable between* $-\infty$ *and* ∞

Periodicity *The* DTFT $X(e^{j\omega})$ *is periodic in* ω *with period* 2π.

$$X(e^{j\omega}) = X(e^{j[\omega+2\pi]}) \qquad \boxed{\omega \in [0,2\pi],[-\pi,\pi], etc.}$$

Symmetry *For real-valued* $x(k)$, $X(e^{j\omega})$ *is conjugate symmetric.*

$$X(e^{-j\omega}) = X^*(e^{j\omega}) \qquad \boxed{\omega \in [0,\pi]}$$

$$\mathrm{Re}[X(e^{-j\omega})] = \mathrm{Re}[X(e^{j\omega})], \quad \mathrm{Im}[X(e^{-j\omega})] = -\mathrm{Im}[X(e^{j\omega})]$$

$$|X(e^{-j\omega})| = |X(e^{j\omega})|, \quad \angle X(e^{-j\omega}) = -\angle X(e^{j\omega})$$

Example of Periodicity

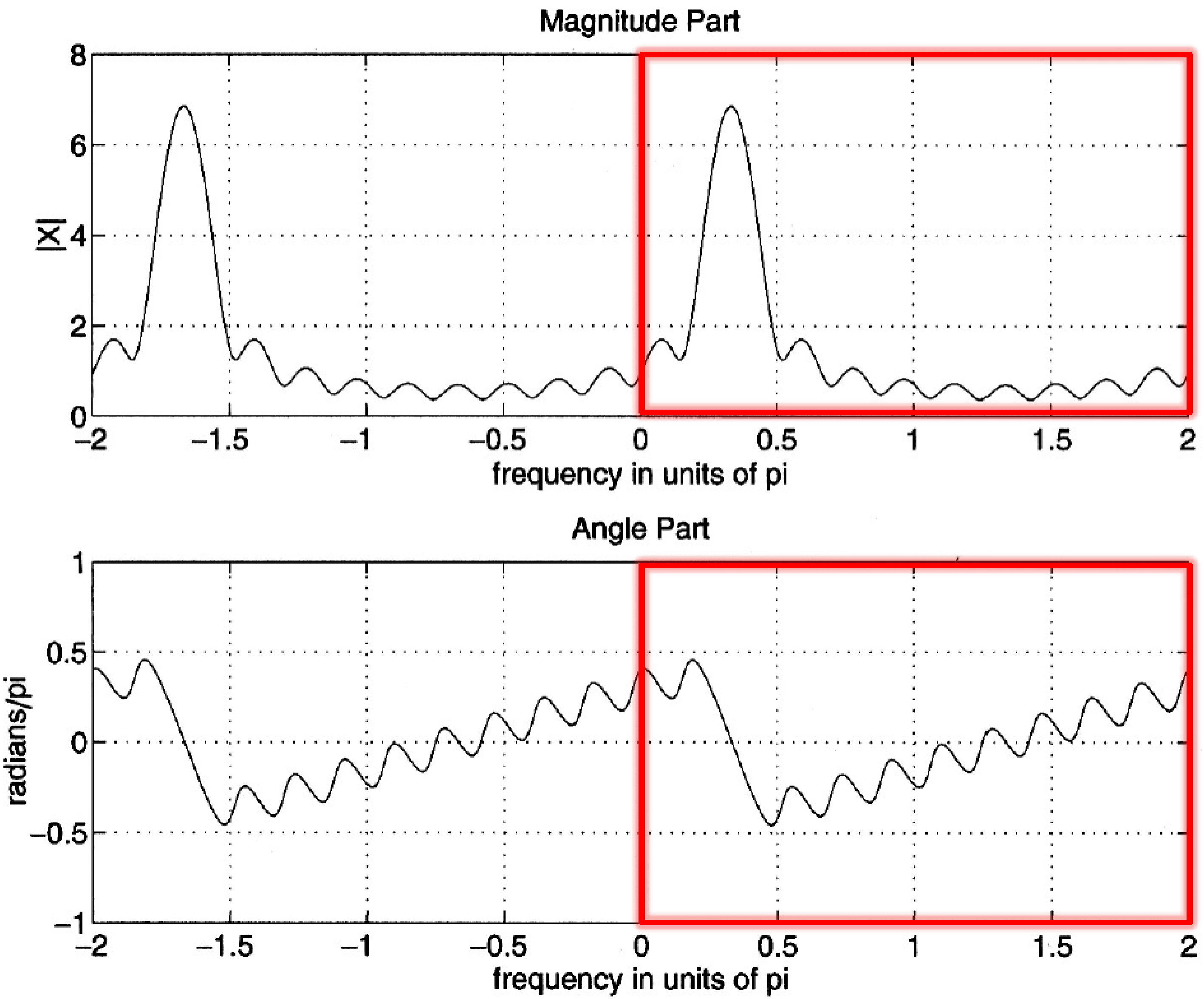
Example: $x(k) = (0.9\exp(j\pi/3))^k, \quad 0 \le k \le 10$

Determine $X(e^{j\omega})$ and investigate its periodicity.

```
>> k = 0:10; x = (0.9*exp(j*pi/3)) .^k;
>> i = -200:200; w = (pi/100)*i;
>> X = x * (exp(-i*pi/100)) .^ (k'*i);
>> magX = abs(X); angX = angle(X);
>> subplot(2,1,1); plot(w/pi,magX); grid
>> xlabel('frequency in pi units'); ylabel('|X|')
>> title('Magnitude part')
>> subplot(2,1,2); plot(w/pi, angX/pi); grid
>> xlabel('frequency in pi units'); ylabel('radians/pi')
>> title('Angle part')
```

$[-2\pi, 2\pi]$

Example of Periodicity



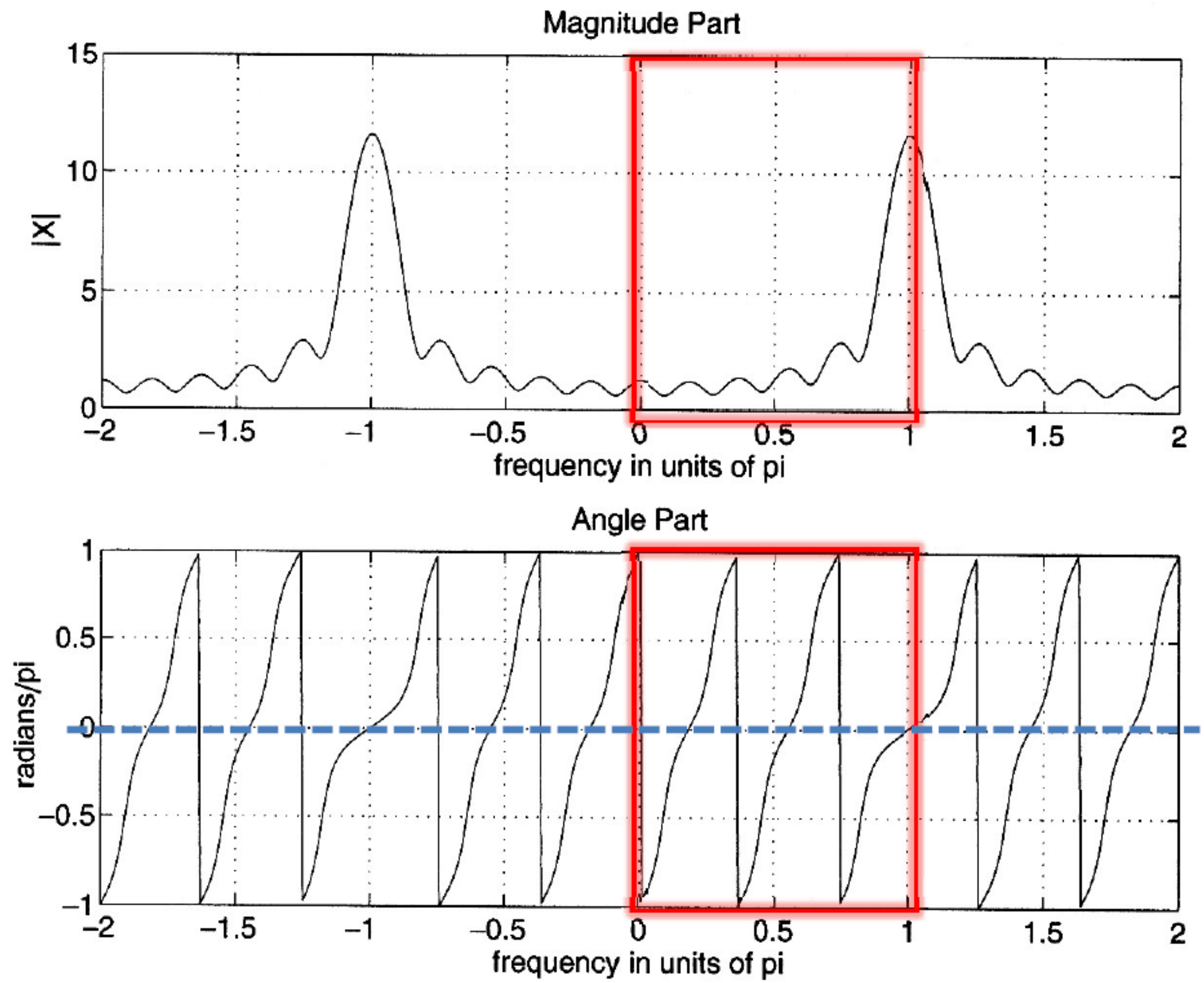
Example of Symmetry

Example: $x(k) = (-0.9)^k, \quad -5 \leq k \leq 5$

Investigate the conjugate-symmetry property of its DTFT.

```
>> k = -5:5; x = (-0.9) .^k;
>> i = -200:200; w = (pi/100)*i; X = x * (exp(-i*pi/100)) .^ (n'*i);
>> magX = abs(X); angX = angle(X);
>> subplot(2,1,1); plot(w/pi,magX); grid; axis([-2,2,0,15])
>> xlabel('frequency in pi units'); ylabel('|X|')
>> title('Magnitude part')
>> subplot(2,1,2); plot(w/pi, angX/pi); grid; axis([-2,2,-1,1])
>> xlabel('frequency in pi units'); ylabel('radians/pi')
>> title('Angle part')
```

Example of Symmetry



Not only periodic, but also conjugate-symmetric

Properties of DTFT

Properties of DTFT

1. Linearity

$$F[\alpha x_1(k) + \beta x_2(k)] = \alpha F[x_1(k)] + \beta F[x_2(k)]$$

2. Time shifting

$$F[x(k-m)] = X(e^{j\omega})e^{-j\omega m}$$

3. Frequency shifting

$$F[x(k)e^{j\omega_0 k}] = X(e^{j(\omega-\omega_0)})$$

4. Conjugation

$$F[x^*(k)] = X^*(e^{-j\omega})$$

Properties of DTFT

5. Folding

$$F[x(-k)] = X(e^{-j\omega})$$

6. Symmetries in real sequence

$$x(k) = x_e(k) + x_o(k)$$

$$F[x_e(k)] = \mathrm{Re}[X(e^{j\omega})]$$

$$F[x_o(k)] = j\,\mathrm{Im}[X(e^{j\omega})]$$

7. Convolution

$$F[x_1(k) * x_2(k)] = F[x_1(k)]F[x_2(k)] = X_1(e^{j\omega})X_2(e^{j\omega})$$

Properties of DTFT

8. Multiplication

$$F[x_1(k)\cdot x_2(k)] = F[x_1(k)] \circledast F[x_2(k)]$$

$$\equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})X_2(e^{j(\omega-\theta)})d\theta$$

9. Energy

$$\mathrm{E}_x = \sum_{-\infty}^{\infty}|x(k)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega$$

$$= \int_0^{\pi} \boxed{\frac{|X(e^{j\omega})|^2}{\pi}} d\omega$$

$\Phi_x(\omega)$ *Energy density spectrum of* $x(k)$

$[\omega_1, \omega_2]$

$\downarrow$

$$\int_{\omega_1}^{\omega_2} \Phi_x(\omega)d\omega,$$

$$0 \le \omega_1 < \omega_2 \le \pi$$

Properties of DTFT (1): Linearity

$$F[\alpha x_1(k)+\beta x_2(k)]=\alpha F[x_1(k)]+\beta F[x_2(k)]$$

$$X(e^{j\omega})=\sum_{k=-\infty}^{\infty}x[k]e^{-j\omega k}$$

$$x[k]=\alpha_1 x_1[k]+\alpha_2 x_2[k]$$

$$X(e^{j\omega})=\sum_{k=-\infty}^{\infty}[\alpha_1 x_1[k]+\alpha_2 x_2[k]]e^{-j\omega k}$$

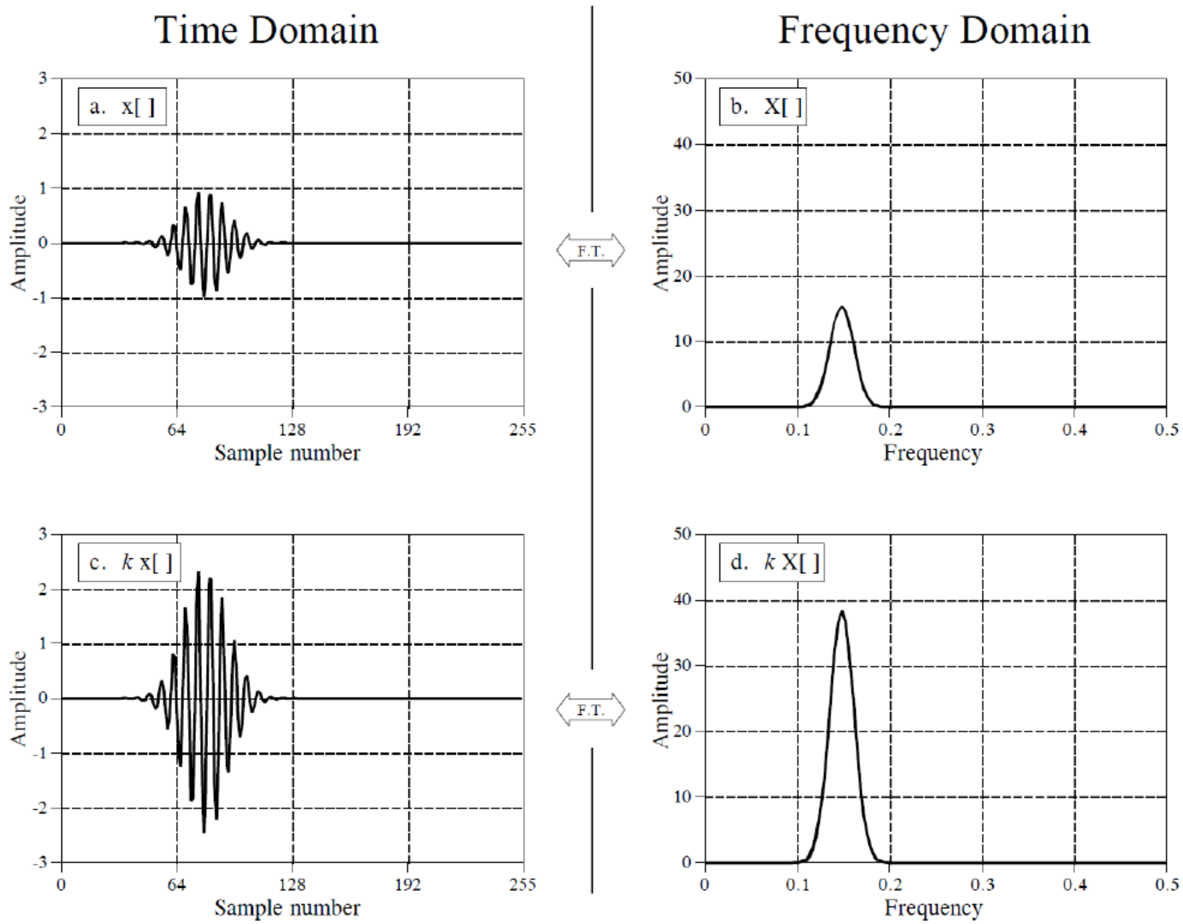
$$=\sum_{k=-\infty}^{\infty}\alpha_1 x_1[k]e^{-j\omega k}+\sum_{k=-\infty}^{\infty}\alpha_2 x_2[k]e^{-j\omega k}$$

$$=\alpha_1\sum_{k=-\infty}^{\infty}x_1[k]e^{-j\omega k}+\alpha_2\sum_{k=-\infty}^{\infty}x_2[k]e^{-j\omega k}$$

$$=\alpha_1 X_1(e^{j\omega})+\alpha_2 X_2(e^{j\omega})$$

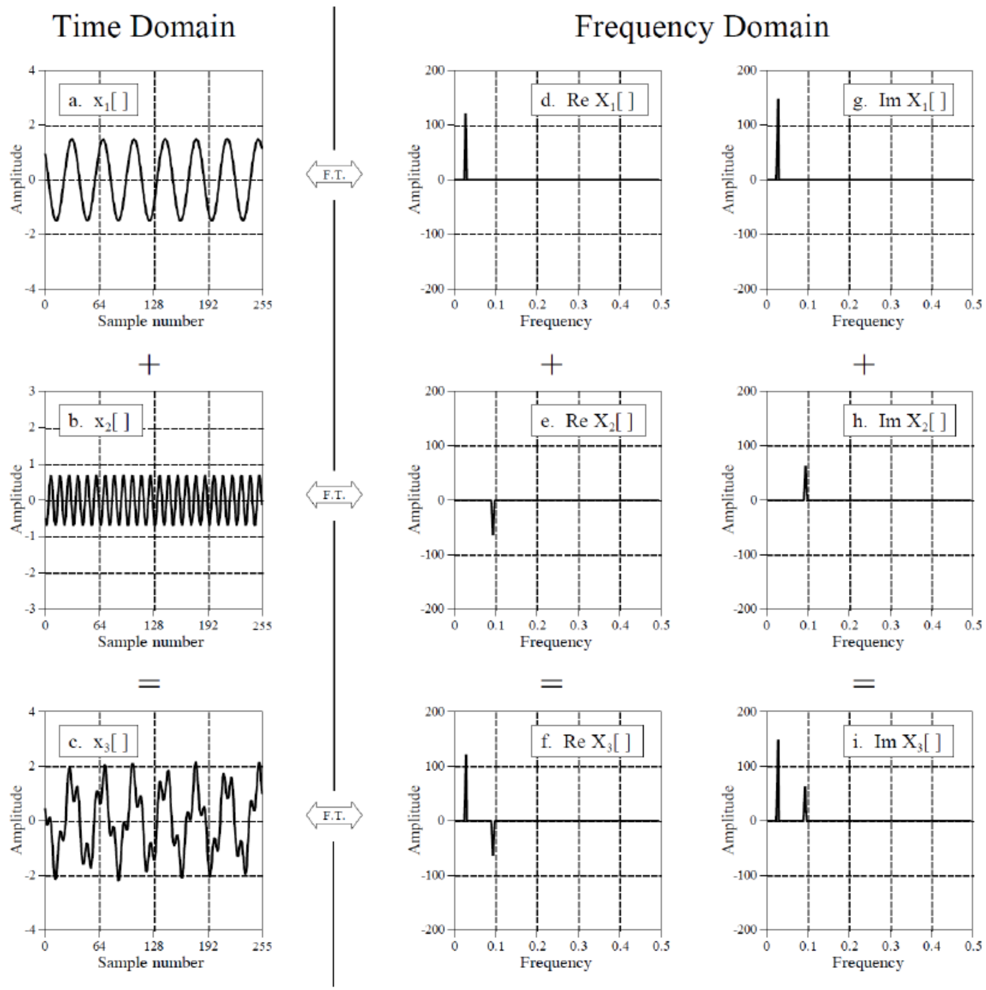
Linearity: Homogeneity & Additivity

Homogeneity of the Fourier transform



Linearity: Homogeneity & Additivity

Additivity of the Fourier transform



Example: MATLAB

Verify the *linearity* property (*using real-valued finite duration sequences*)

$x_1(k), x_2(k)$: random sequences uniformly distributed between $[0,1]$ over $0 \le k \le 10$.

```
>> x1 = rand(1,11); x2 = rand(1,11); k = 0:10;
>> alpha = 2; beta = 3; i = 0:500; w = (pi/500)*i;
>> X1 = x1 * (exp(-j*pi/500)).^(k'*i);  % DTFT of x1
>> X2 = x2 * (exp(-j*pi/500)).^(k'*i);  % DTFT of x2
>> x = alpha*x1 + beta*x2;    % Linear combination of x1 & x2
>> X = x * (exp(-j*pi/500)).^(k'*i);      % DTFT of x
>> % verification
>> X_check = alpha*X1 + beta*X2;  % Linear combination of X1 & X2
>> error = max(abs(X-X_check))        % Difference
error =
   7.1054e-015
```

Properties of DTFT (2): Time Shifting

$$F[x(k-m)] = X(e^{j\omega})e^{-j\omega m}$$

$$x[k] = x[k-m]$$

$$\sum_{k=-\infty}^{\infty} x[k-m]e^{-j\omega k}, \quad k-m=n$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega(n+m)}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j\omega m}$$

$$= X(e^{j\omega})e^{-j\omega m}$$

Example: MATLAB

Verify the *sample shift* property:

$x(k)$: a random sequence uniformly distributed between $[0,1]$ over $0 \le k \le 10$.

$y(k) = x(k-2)$

```
>> x = rand(1,11); k = 0:10;
>> i = 0:500; w = (pi/500)*i;
>> X = x * (exp(-j*pi/500)).^(k'*i);  % DTFT of x
>> % signal shifted by two samples
>> y = x; n = k+2;
>> Y = y * (exp(-j*pi/500)).^(n'*i);  % DTFT of y
>> % verification
>> Y_check = (exp(-j*2).^w).*X;      % Multiplication by exp(-j2w)
>> error = max(abs(Y-Y_check))       % Difference
error =
  5.7737e-015
```

Properties of DTFT (3): Frequency Shifting

$$F[x(k)e^{j\omega_0 k}] = X(e^{j(\omega-\omega_0)})$$

$$x[k] = x[k]e^{j\omega_0 k}$$

$$\sum_{k=-\infty}^{\infty} x[k]e^{j\omega_0 k}e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j(\omega-\omega_0)k}$$

$$= X(e^{j(\omega-\omega_0)})$$

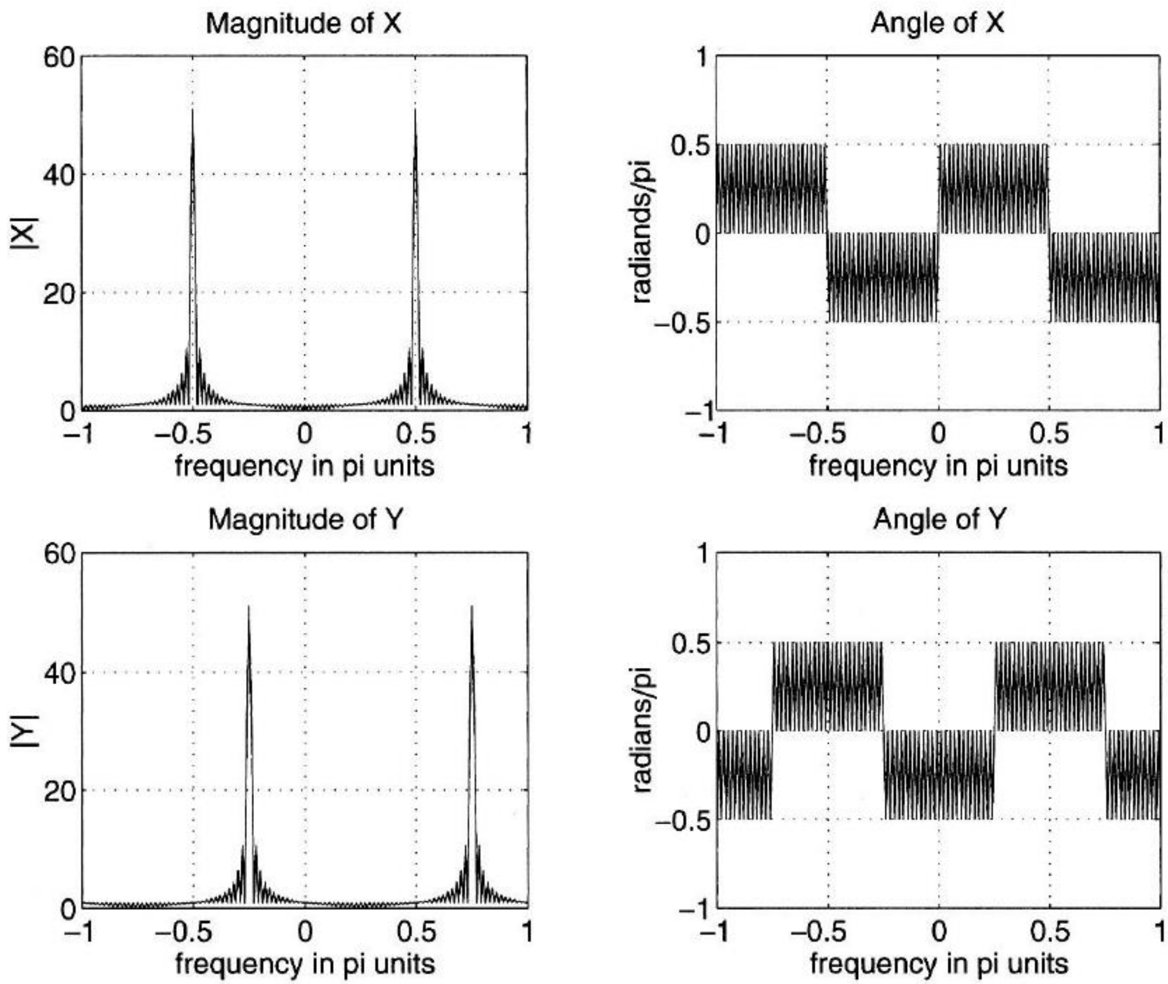
Example: MATLAB

Verify the *frequency shift* property:

$$x(k) = \cos(\pi k / 2), \quad 0 \le k \le 100, \quad y(k) = e^{j\pi k/4} x(k)$$

```
>> k = 0:100; x = cos(pi*k/2);
>> i = -100:100; w = (pi/100)*i;        % frequency between –pi and +pi
>> X = x * (exp(-j*pi/100)).^(k'*i);  % DTFT of x
%
>> y = exp(j*pi*k/4).*x;              % signal multiplied by exp(j*pi*k/4)
>> Y = y * (exp(-j*pi/100)).^(k'*i);  % DTFT of y
% Graphical verification
>> subplot(2,2,1); plot(w/pi,abs(X)); grid; axis([-1,1,0,60])
>> subplot(2,2,2); plot(w/pi,angle(X)/pi); grid; axis([-1,1,-1,1])
>> subplot(2,2,3); plot(w/pi,abs(Y)); grid; axis([-1,1,0,60])
>> subplot(2,2,4); plot(w/pi,angle(Y)/pi); grid; axis([-1,1,-1,1])
```

Example: MATLAB



$X(e^{j\omega})$ *is indeed shifted by* $\pi/4$ *in both magnitude and angle.*

Properties of DTFT (4): Conjugation

$$F[x^*(k)] = X^*(e^{-j\omega})$$

$$x[k] \rightleftarrows X(e^{j\omega}), \quad x^*[k] \rightleftarrows X^*(e^{-j\omega})$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$\boxed{x^*[k] = x[k]} \quad \textit{real}$$

$$X^*(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x^*[k]e^{j\omega k}$$

$$X^*(e^{-j\omega}) = \sum_{k=-\infty}^{\infty} x^*[k]e^{-j\omega k}$$

$$X^*(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x^*[k]e^{j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j(-\omega)k}$$

$$= X(e^{-j\omega})$$

Example: MATLAB

Verify the *conjugation* property:

$x(k)$: a complex-valued random sequence over $-5 \le k \le 10$ with real and imaginary parts uniformly distributed between $[0,1]$.

```
>> k = -5:10; x = rand(1,length(k)) + j*rand(1,length(k));
>> i = -100:100; w = (pi/100)*i;        % frequency between –pi and +pi
>> X = x * (exp(-j*pi/100)).^(k'*i);  % DTFT of x
% conjugation property
>> y = conj(x));                        % signal conjugation
>> Y = y * (exp(-j*pi/100)).^(k'*i);  % DTFT of y
% verification
>> Y_check = conj(fliplr(X));          % conj(X(-w))
>> error = max(abs(Y-Y_check))        % Difference
error =
      0
```

Properties of DTFT (5): Folding

$$F[x(-k)] = X(e^{-j\omega})$$

$$x[k] = x[-k]$$

$$\sum_{k=-\infty}^{\infty} x[-k]e^{-j\omega k}, \quad -k = m$$

$$= \sum_{m=\infty}^{-\infty} x[m]e^{-j\omega(-m)}$$

$$= \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega(-m)}$$

$$= \sum_{m=-\infty}^{\infty} x[m]e^{-j(-\omega)m}$$

$$= X(e^{-j\omega})$$

Example: MATLAB

Verify the *folding* property:

$x(k)$: a random sequence over $-5 \le k \le 10$ uniformly distributed between $[0,1]$.

```
>> k = -5:10; x = rand(1,length(k));
>> i = -100:100; w = (pi/100)*i;        % frequency between –pi and +pi
>> X = x * (exp(-j*pi/100)).^(k'*i);  % DTFT of x
% folding property
>> y = fliplr(x); n = -fliplr(k);         % signal folding
>> Y = y * (exp(-j*pi/100)).^(n'*i);  % DTFT of y
% verification
>> Y_check = fliplr(X);                    % X(-w)
>> error = max(abs(Y-Y_check))         % Difference
error =
      0
```

Properties of DTFT (6):
Symmetry in Real Sequence

$$x(k) = x_e(k) + x_o(k)$$

$$F[x_e(k)] = \mathrm{Re}[X(e^{j\omega})]$$

$$F[x_o(k)] = j\,\mathrm{Im}[X(e^{j\omega})]$$

Example: MATLAB

Verify the *symmetry* property of real signals.

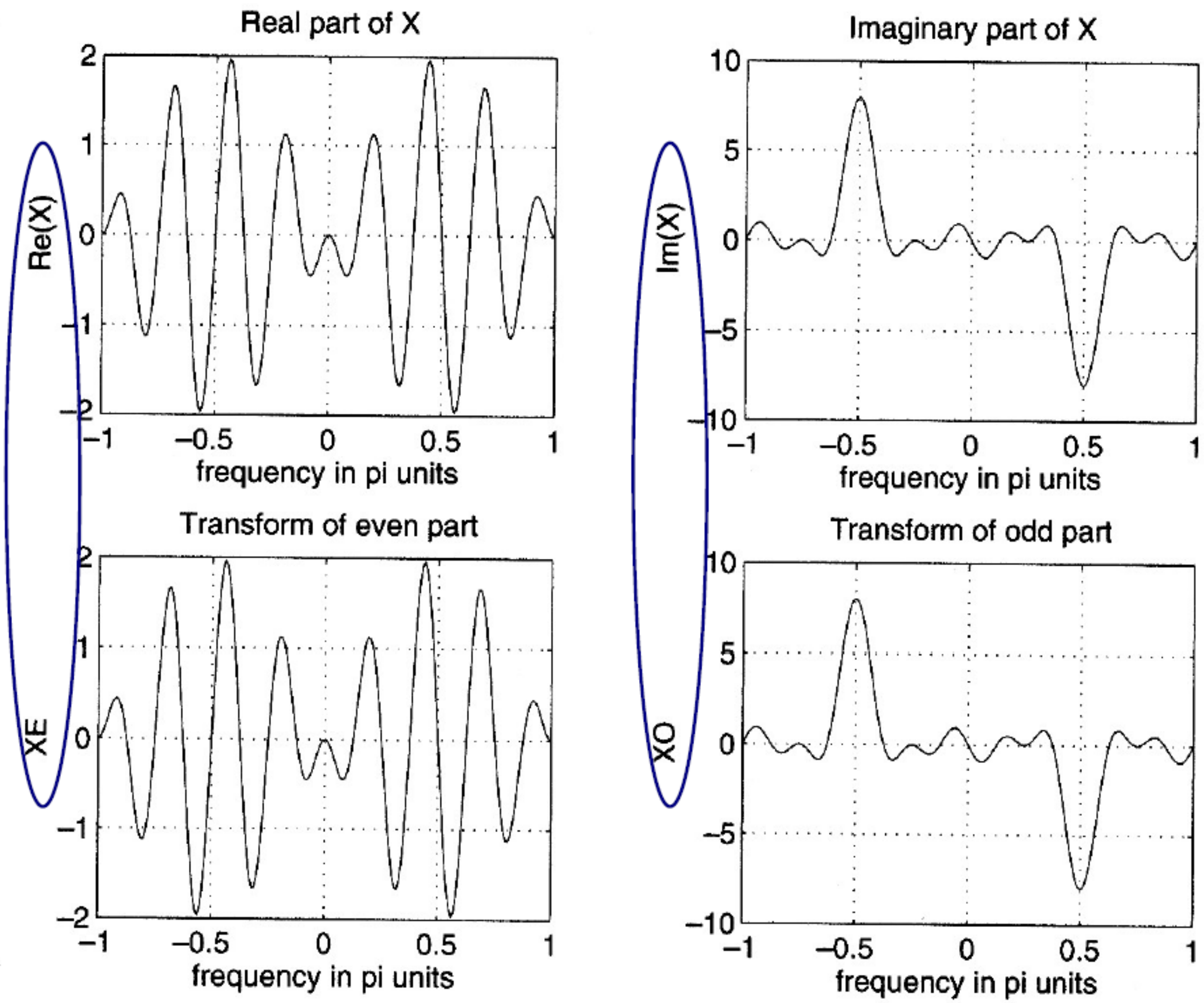
$$x(k) = \sin(\pi k / 2), \quad -5 \le k \le 10$$

```
>> k = -5:10; x = sin(pi*k/2);
>> i = -100:100; w = (pi/100)*i;        % frequency between –pi and +pi
>> X = x * (exp(-j*pi/100)).^(k'*i);  % DTFT of x
% signal decomposition
>> [xe,xo,m] = evenodd(x,k);            % even and odd parts
>> XE = xe * (exp(-j*pi/100)).^(m'*i);  % DTFT of xe
>> XO = xo * (exp(-j*pi/100)).^(m'*i);  % DTFT of xo
% verification
>> XR = real(X);                        % real part of X
>> error1 = max(abs(XE-XR))             % Difference
error1 =
  1.8974e-019
>> XI = imag(X);                        % imag part of X
>> error2 = max(abs(XO-j*XI))           % Difference
error2 =
  1.8033e-019
```

Example: MATLAB

```
% graphical verification
>> subplot(2,2,1); plot(w/pi,XR); grid; axis([-1,1,-2,2])
>> xlabel('frequency in pi units'); ylabel('Re(X)');
>> title('Real part of X')
>> subplot(2,2,2); plot(w/pi,XI); grid; axis([-1,1,-10,10])
>> xlabel('frequency in pi units'); ylabel('Im(X)');
>> title('Imaginary par of X')
>> subplot(2,2,3); plot(w/pi,real(XE)); grid; axis([-1,1,-2,2])
>> xlabel('frequency in pi units'); ylabel('XE');
>> title('Transform of even part')
>> subplot(2,2,4); plot(w/pi,imag(XO)); grid; axis([-1,1,-10,10])
>> xlabel('frequency in pi units'); ylabel('XO');
>> title('Transform of odd part')
```

Example: MATLAB



Properties of DTFT (7): Convolution

$$F[x_1(k) * x_2(k)] = F[x_1(k)]F[x_2(k)] = X_1(e^{j\omega})X_2(e^{j\omega})$$

$$x[k] = x_1[k] * x_2[k]$$

$$\sum_{k=-\infty}^{\infty} \left(x_1[k] * x_2[k]\right) e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_1[n] x_2[k-n] e^{-j\omega k}$$

$$= \sum_{n=-\infty}^{\infty} x_1[n] \sum_{k=-\infty}^{\infty} x_2[k-n] e^{-j\omega k}, \quad k - n = m$$

$$= \sum_{n=-\infty}^{\infty} x_1[n] \sum_{m=-\infty}^{\infty} x_2[m] e^{-j\omega(m+n)}$$

$$= \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\omega n} \sum_{m=-\infty}^{\infty} x_2[m] e^{-j\omega m} = X_1(e^{j\omega})X_2(e^{j\omega})$$

Properties of DTFT (8): Multiplication

$$F[x_1(k)\cdot x_2(k)] = F[x_1(k)] \circledast F[x_2(k)]$$

$$x_1[k] \rightleftharpoons X_1(e^{j\omega}),\quad x_2[k] \rightleftharpoons X_2(e^{j\omega})$$

$$\sum_{k=-\infty}^{\infty}\left(x_1[k]\cdot x_2[k]\right)e^{-j\omega k}$$

$$=\sum_{k=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(e^{j\theta})e^{j\theta k}d\theta\right)x_2[k]e^{-j\omega k}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(e^{j\theta})d\theta\left(\sum_{k=-\infty}^{\infty}x_2[k]e^{-j(\omega-\theta)k}\right)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

Properties of DTFT (9): Energy

$$E_x = \sum_{-\infty}^{\infty} |x(k)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Parseval's relation for discrete-time aperiodic signals with finite energy

$$\varepsilon_x = \sum_{k=-\infty}^{\infty} x(k)x^*(k) = \sum_{k=-\infty}^{\infty} x(k)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})e^{-j\omega k} d\omega\right]$$

energy in temporal space

$$\varepsilon_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\left[\sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

energy in spectral space

$$= \int_0^{\pi} \frac{|X(e^{j\omega})|^2}{\pi} d\omega$$ (for *real* sequences using *even symmetry*)

Properties of DTFT (9): Energy

$$E_x = \sum_{-\infty}^{\infty} |x(k)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Parseval's relation for periodic signals with finite power

$$P_x = \frac{1}{T}\int_0^T |x(t)|^2 dt, \quad |x(t)|^2 = x(t)x^*(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}, \quad x^*(t) = \sum_{k=-\infty}^{\infty} C_k^* e^{-jk\omega_0 t}$$

$$P_x = \frac{1}{T}\int_0^T x(t)\left[\sum_{k=-\infty}^{\infty} C_k^* e^{-jk\omega_0 t}\right] dt$$

$$= \sum_{k=-\infty}^{\infty} C_k^* \left[\frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt\right]$$

$$= \sum_{k=-\infty}^{\infty} C_k^* C_k = \sum_{k=-\infty}^{\infty} |C_k|^2$$

$$x = a + jb, \quad x^* = a - jb$$

$$xx^* = a^2 - j^2 b^2$$

$$= a^2 + b^2$$

$$|x| = \sqrt{a^2 + b^2}$$

$$|x|^2 = a^2 + b^2$$

Frequency Domain Representation of LTI Systems

Review: Eigenvector and Eigenvalue

Definition

A *real* or *complex number* λ is called an *eigenvalue* of the $n \times n$ real matrix A if there exists a *nonzero* vector x such that $Ax = \lambda x$. Any *nonzero* vector x satisfying $Ax = \lambda x$ is called a (right) *eigenvector* of A associated with eigenvalue λ.

How to find eigenvector and eigenvalue?

$$Ax = y \longrightarrow Ax = \lambda x$$

$$Ax = \lambda I x \longrightarrow (A - \lambda I)x = 0$$

$$\Delta(\lambda) = \det(A - \lambda I)$$ *The characteristic polynomial of* A

Every root of $\Delta(\lambda)$ *is eigenvalue*

The null space of $A - \lambda I$ *is called the eigenspace of* A *associated with eigenvalue* λ.

Eigenfunction of LTI System

- For a LTI system $\mathcal{H}$ with impulse response $h(t)$,

$$e^{j\omega t} \xrightarrow{\mathcal{H}} H(j\omega)e^{j\omega t},$$

  where ω is a real constant and

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

- That is, $e^{j\omega t}$ is an ***eigenfunction*** of a LTI system and $H(j\omega)$ is the corresponding ***eigenvalue***.
- We refer to $H(j\omega)$ as the **frequency response** of the system $\mathcal{H}$.

Frequency Domain Representation of LTI System

Response to a complex exponential $e^{j\omega_0 k}$

$$e^{j\omega_0 k} \longrightarrow \boxed{g(k)} \longrightarrow g(k) * e^{j\omega_0 k}$$

$$y(k) = g(k) * e^{j\omega_0 k} = \sum_{-\infty}^{\infty} g(m) e^{j\omega_0 (k-m)}$$

$$= \left[\sum_{-\infty}^{\infty} g(m) e^{-j\omega_0 m} \right] e^{j\omega_0 k}$$

$$= \left[F[g(k)] \Big|_{\omega=\omega_0} \right] e^{j\omega_0 k}$$

The Fourier transform representation is the most useful signal representation for LTI systems.

Frequency Response

The DTFT of an impulse response is called the *Frequency Response* (or *Transfer Function*) of an LTI system and is denoted by

$$G(e^{j\omega}) \equiv \sum_{-\infty}^{\infty} g(k) e^{-j\omega k}$$

We can represent the system by

Response of the system at frequency ω_0

Input exponential sequence

$$x(k) = e^{j\omega_0 k} \longrightarrow \boxed{G(e^{j\omega})} \longrightarrow y(k) = G(e^{j\omega_0}) \times e^{j\omega_0 k}$$

$$\sum_i A_i e^{j\omega_i k} \qquad\qquad \sum_i A_i G(e^{j\omega_i}) e^{j\omega_i k}$$

Frequency Response

$$x[k] = e^{j\omega k} \longrightarrow \boxed{G(e^{j\omega})} \longrightarrow y[k] = \underline{G(e^{j\omega})}e^{j\omega k}$$

independent of k

$$G(e^{j\omega}) = \left|G(e^{j\omega})\right| e^{j\measuredangle G(e^{j\omega})}$$

: polar form

$$\Downarrow$$

$$y[k] = \left|G(e^{j\omega})\right| e^{j\measuredangle G(e^{j\omega})} e^{j\omega k}$$

$$= \left|G(e^{j\omega})\right| e^{j\left(\omega k + \measuredangle G(e^{j\omega})\right)}$$

Amplitude of the input sinusoid changed by $\left|G(e^{j\omega})\right|$ *: magnitude response*

Phase of the input sinusoid changed by $\measuredangle G(e^{j\omega})$ *: phase response*

Frequency Response

$$x[k] = \alpha_1 e^{j\omega_1 k} + \alpha_2 e^{j\omega_2 k} \quad \Rightarrow \quad y[k] = \alpha_1 G(e^{j\omega_1})e^{j\omega_1 k} + \alpha_2 G(e^{j\omega_2})e^{j\omega_2 k}$$

Relationship to impulse response

$$y[k] = \sum_{n=-\infty}^{\infty} g[n]x[k-n] = \sum_{n=-\infty}^{\infty} g[n]e^{j\omega(k-n)}$$

$$= e^{j\omega k} \underbrace{\sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}}_{G(e^{j\omega})}$$

$$g[k] \xleftrightarrow{DTFT} G(e^{j\omega})$$

Example:

$$g[k] = \delta[k] + 2\delta[k-1] + 3\delta[k-2],$$

$$x[k] = \underset{\substack{\uparrow \\ \omega = 0}}{-2} + 3\cos\left(\frac{\pi}{4}k + \frac{\pi}{3}\right) + 10\cos\left(\frac{3\pi}{4}k - \frac{\pi}{5}\right)$$

Determine the frequency response $G(e^{j\omega})$ of this system.

$$G(e^{j\omega}) = \sum_{-\infty}^{\infty} g[k]e^{-j\omega k} = 1 + 2e^{-j\omega} + 3e^{-j2\omega}$$

Determine the steady-state output $y[k]$.

Example:

$$G(e^{j\omega}) = 1 + 2e^{-j\omega} + 3e^{-j2\omega}$$

$$G(e^{j\omega})\Big|_{\omega=0} = 1 + 2 + 3 = \boxed{6}$$

$$G(e^{j\omega})\Big|_{\omega=\frac{\pi}{4}} = 1 + 2e^{-j\frac{\pi}{4}} + 3e^{-j\frac{\pi}{2}} = 1 + 2(0.7071 - j0.7071) + 3(-j)$$

$$= 2.414 - j4.414 = 5.031e^{j(-1.070)}$$

$$G(e^{j\omega})\Big|_{\omega=\frac{3\pi}{4}} = 1 + 2e^{-j\frac{3\pi}{4}} + 3e^{-j\frac{3\pi}{2}} = 1 + 2(-0.7071 - j0.7071) + 3(j)$$

$$= -0.4142 - j1.586 = 1.639e^{j1.826}$$

$$y[k] = -2 \cdot 6 + 3 \cdot 5.031\cos\left(\frac{\pi}{4}k + \frac{\pi}{3} - 1.070\right)$$

$$+ 10 \cdot 1.639\cos\left(\frac{3\pi}{4}k - \frac{\pi}{5} + 1.826\right)$$

Lab #6.1 (1 pt.): Due Jan 17

Determine the frequency response $G(e^{j\omega})$ of a system characterized by $g(k) = (0.A)^k u(k)$. Plot the magnitude and phase response.

Use Your ID: sGFEDCBA

Frequency Response of Difference Equations

$$y(k)+\sum_{m=1}^{N} a_m y(k-m)=\sum_{n=0}^{M} b_n x(k-n)$$

$$G(e^{j\omega})e^{j\omega k}+\sum_{m=1}^{N} a_m G(e^{j\omega})e^{j\omega(k-m)}=\sum_{n=0}^{M} b_n e^{j\omega(k-n)}$$

$$G(e^{j\omega})=\frac{\sum_{n=0}^{M} b_n e^{-j\omega n}}{1+\sum_{m=1}^{N} a_m e^{-j\omega m}}$$

Example

$$y[k]+\frac{1}{2}y[k-1]=x[k], \quad x[k]=\left(\frac{1}{2}\right)^k u[k]$$

Determine the frequency response $G(e^{j\omega})$ of this system.

$$Y(e^{j\omega})+\frac{1}{2}e^{-j\omega}Y(e^{j\omega})=X(e^{j\omega})$$

$$G(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

Example

What is the response of the system?

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[k]e^{-j\omega k} = \sum_{0}^{\infty}\left(\frac{1}{2}\right)^k e^{-j\omega k} = \sum_{0}^{\infty}\left(\frac{1}{2}e^{-j\omega}\right)^k$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega}) = G(e^{j\omega})X(e^{j\omega}) = \frac{1}{1+\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$= \frac{1}{2}\frac{1}{1+\frac{1}{2}e^{-j\omega}} + \frac{1}{2}\frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$y[k] = \frac{1}{2}\left(-\frac{1}{2}\right)^k u[k] + \frac{1}{2}\left(\frac{1}{2}\right)^k u[k]$$

Lab #6.2 (1 pt.): Due Jan 17

An LTI system is specified by the difference equation

$$y(k) = (0.A)y(k-1) + x(k)$$

a. Determine $G(e^{j\omega})$.

b. Calculate and plot the steady-state response $y_{ss}(k)$ to

$$x(k) = \cos(0.05\pi k)u(k).$$

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Thank you

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