Lecture I213E – Class 13

Discrete Signal Processing

Sakriani Sakti

JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY 1990

Course Materials

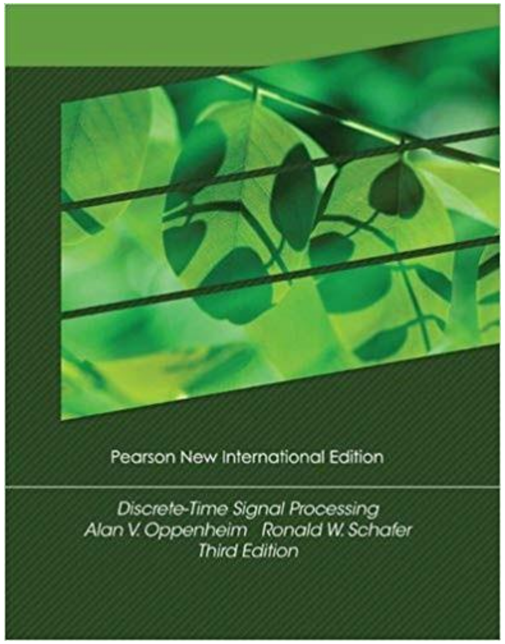
- **Materials**

→ Lecture notes will be uploaded before each lecture
https://jstorage-2018.jaist.ac.jp/s/PGXRrC7iFmN2FWo
Pass: dsp-i213e-2022
(Slide Courtesy of Prof. Nak Young Chong)

- **References**

→ Chi-Tsong Chen:
Linear System Theory and Design, 4th Ed., Oxford University Press, 2013.

→ Alan V. Oppenheim and Ronald W. Schafer:
Discrete-Time Signal Processing, 3rd Ed., Pearson New International Ed., 2013.



Related Courses & Prerequisite

- **Related Courses**
  - → I212 Analysis for Information Science
  - → I114 Fundamental Mathematics for Information Science

- **Prerequisite**
  - → None

Evaluation

▪ Viewpoint of evaluation

→ Students are able to understand:

- Basic principles in modeling and analysis of linear time-invariant systems
- Applications of mathematical methods and tools to different signal processing problems.

▪ Evaluation method

→ Homework, term project, midterm exam, and final exam

▪ Evaluation criteria

→ Homework/labs (30%), term project (30%)
midterm exam (15%), and final exam (25%)

Contact

- ## Lecturer

  → Sakriani Sakti

- ## TA

  Tutorial hours & Term project
  → WANG Lijun (s2010026)
  → TANG Bowen (s2110411)
  Homework
  → PUTRI Fanda Yuliana (s2110425)

- ## Contact Email

  → dsp-i213e-2022@ml.jaist.ac.jp

Schedule

- December 8th, 2022 – February 9th, 2023

- Lecture Course Term 2-2
  → Tuesday 9:00 — 10:40
  → Thursday 10:50 — 12:30

- Tutorial Hours
  → Tuesday 13:30-15:10

Schedule

Dec

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Jan

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	~~5~~	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	~~24~~	25	26	27	28
29	30	31				

Feb

Sun	Mon	Tue	Wed	Thu	Fri	Sat
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28				

Lecture:
Tuesday 9:00 — 10:40
Thursday 10:50 — 12:30

Additional Lecture:
Monday 17:10 — 18:50

Tutorial:
Tuesday 13:30 — 15:10

Course review &
term project evaluation
(on tutorial hours)

Midterm & final exam
Thursday 10:50 — 12:30

Syllabus

Class	Date	Lecture Course Tue 9:00 — 10:40 / Thr 10:50 — 12:30	Tutorial Hours Tue 13:30 — 15:10
1	12/08	Introduction to Linear Systems with Applications to Signal Processing	
2	12/13	State Space Description	○
3	12/15	Linear Algebra	
4	12/20	Quantitative Analysis (State Space Solutions) and Qualitative Analysis (Stability)	○
5	12/22	Discrete-time Signals and Systems	
X	01/05		
6	01/10	Discrete-time Fourier Analysis	
7	01/10*	Review of Discrete-time Linear Time-Invariant Signals and Systems (on Tutorial Hours)	
	01/12	Midterm Exam	
8	01/17	Sampling and Reconstruction of Analog Signals	○
9	01/19	z-Transform	
X	01/24		○
10	01/31	Discrete Fourier Transform	○
11	02/02	FFT Algorithms	
12	02/06	Implementation of Digital Filters	
13	02/07	Digital Signal Processors and Design of Digital Filters	
14	02/07*	Review of the Course and Term Project Evaluation (on Tutorial Hours)	
	02/09	Final exam	

Class 13
Digital Signal Processors and Design of Digital Filters

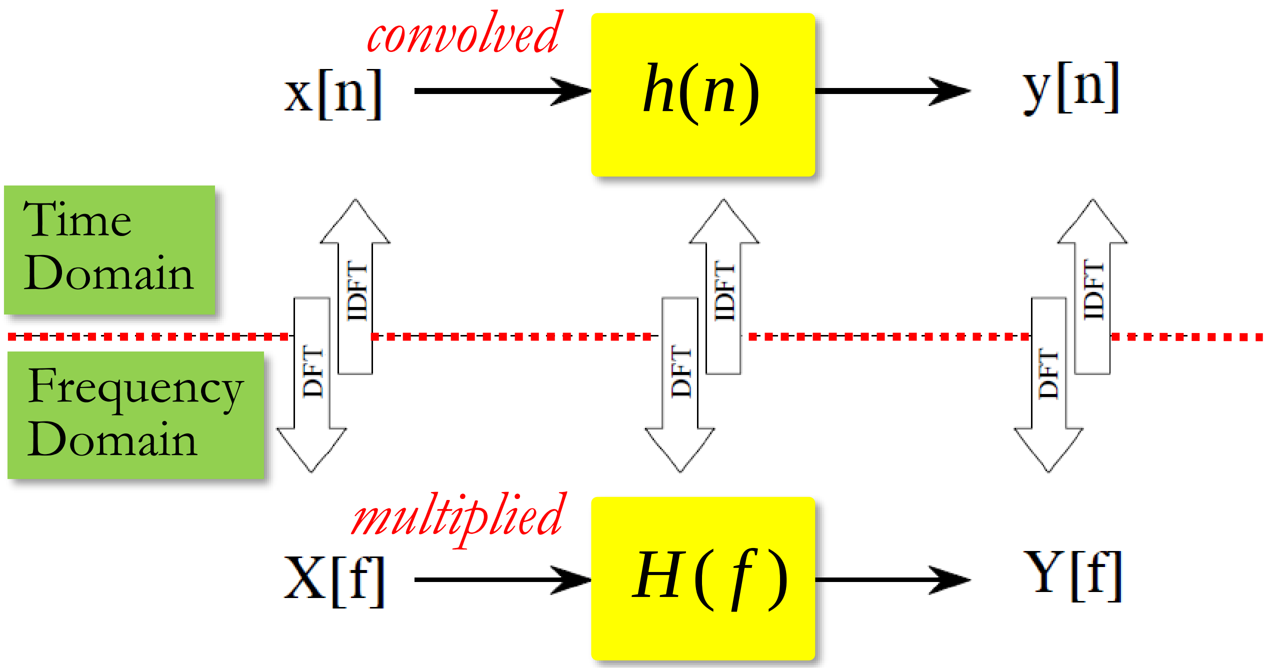
Review Frequency Response

LTI/LSI System

- **LTI/LSI**

$\rightarrow$ They are basically equivalent: the linear time invariant systems refers to an analog system and shift-invariant system refers to a discrete-time system.

$$\boldsymbol{y}[\boldsymbol{n}] = \boldsymbol{x}[\boldsymbol{n}] * \boldsymbol{h}[\boldsymbol{n}]$$

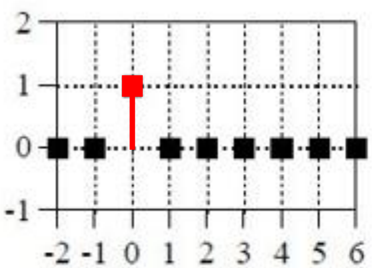


$$\boldsymbol{Y}[\boldsymbol{f}] = \boldsymbol{X}[\boldsymbol{f}]\boldsymbol{H}[\boldsymbol{f}]$$
$$\boldsymbol{Y}[\boldsymbol{\omega}] = \boldsymbol{X}[\boldsymbol{\omega}]\boldsymbol{H}[\boldsymbol{\omega}]$$

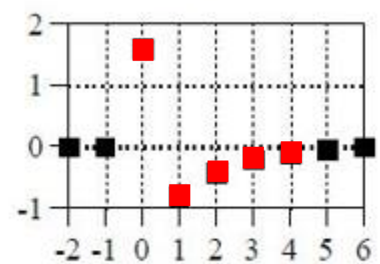
Impulse Input and Response

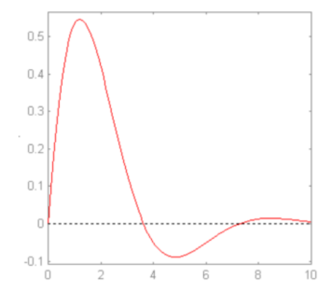
- **Impulse (Delta Function)**
  - → Impulse input
  - → Impulse response

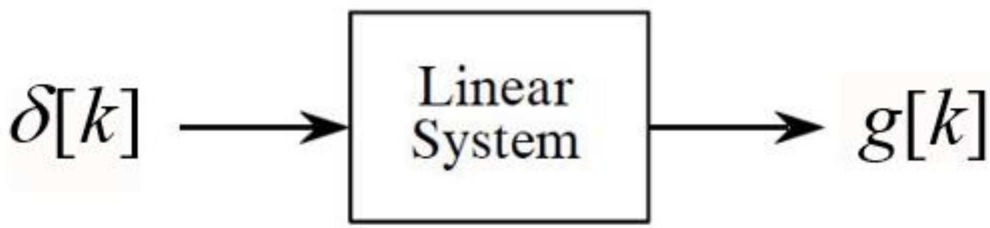
Delta Function



Impulse Response







The impulse response is the signal that exits a system when a delta function (unit impulse) is the input

Frequency Response

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$$

$$X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z)$$

Frequency response

$$\boxed{\begin{aligned} H(e^{j\omega}) &= H(z)\big|_{z=e^{j\omega}} \\ &= \underbrace{\left|H(e^{j\omega})\right|}_{\text{mag response}} \underbrace{\angle H(e^{j\omega})}_{\text{phase response}} \end{aligned}}$$

Phase delay
(a single frequency)

$$\boxed{-\frac{\angle H(e^{j\omega})}{\omega}}$$

Group delay
(multiple freq. components)

$$\boxed{-\frac{d}{d\omega}\angle H(e^{j\omega})}$$

$$y(n) = x(n-k)$$

$$Y(e^{j\omega}) = e^{-j\omega k} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j\omega k}$$

$$= \underbrace{1}_{mag} \cdot e^{j\overbrace{(-\omega k)}^{phase}}$$

$$H(e^{j\omega}) = R(e^{j\omega}) + jI(e^{j\omega})$$

$$\left|H(e^{j\omega})\right| = \sqrt{R^2(e^{j\omega}) + I^2(e^{j\omega})}$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{I(e^{j\omega})}{R(e^{j\omega})}\right)$$

Example

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$$

$$X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z)$$

<u>*Frequency response*</u>

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$
$$= \underbrace{|H(e^{j\omega})|}_{mag\ \ response}\ \underbrace{\angle H(e^{j\omega})}_{phase\ \ response}$$

$$h(n) = \delta(n-4)$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} \delta(n-4) z^{-4} = z^{-4}$$

$$H(e^{j\omega}) = (e^{j\omega})^{-4}$$
$$= \underbrace{1}_{mag} \cdot e^{j\overbrace{(-4\omega)}^{phase}}$$

<u>*Group delay*</u> *varies with freq.*

$$-\frac{d}{d\omega}(-4\omega) = 4$$

Filter Design

Filter Design

▪ Digital Filters

→ Used to implement frequency-selective operations.
→ Specifications are required in the frequency domain in terms of the desired magnitude and phase response of the filter.
→ Generally, a linear phase response in the passband is desirable.

FIR filters- exact linear phase

The impulse response (filter kernel) is directly specified in the design process. Making the filter kernel have left-right symmetry is all that is required.

IIR filters- a linear phase in the passband is not achievable.

The recursion coefficients are what is specified, not the impulse response. The impulse response of a recursive filter is not symmetrical between the left and the right, and therefore has a nonlinear phase.

We will consider magnitude-only specifications.

Filter Design

The design of a digital filter is carried out in three steps:

1. *Specifications*- *determined by the applications*

2. *Approximations*- *conversion of specifications into a filter description in a form of either a difference equation, or a system function* $H(z)$, *or an impulse response* $h(n)$.

3. *Implementation*- *hardware or through software on a computer*

$$y[n] = \sum_{m=0}^{M} b_m x(n-m) - \sum_{m=1}^{N} a_m y(n-m), \quad \boxed{H(e^{j\omega}) = \frac{\sum_{n=0}^{M} b_n e^{-j\omega n}}{1 + \sum_{n=1}^{N} a_n e^{-j\omega n}}}$$

Approximate any of the ideal frequency response characteristics with a system that has a frequency response by properly selecting the coefficients $\{a_m\}$ *and* $\{b_m\}$.

Filter Design: 1. Specifications

The magnitude specifications are given in one of two ways:

1. ***Absolute specifications*:**
   - *a set of requirements on the magnitude response function* $\left|H(e^{j\omega})\right|$
   - *generally used for FIR filters*

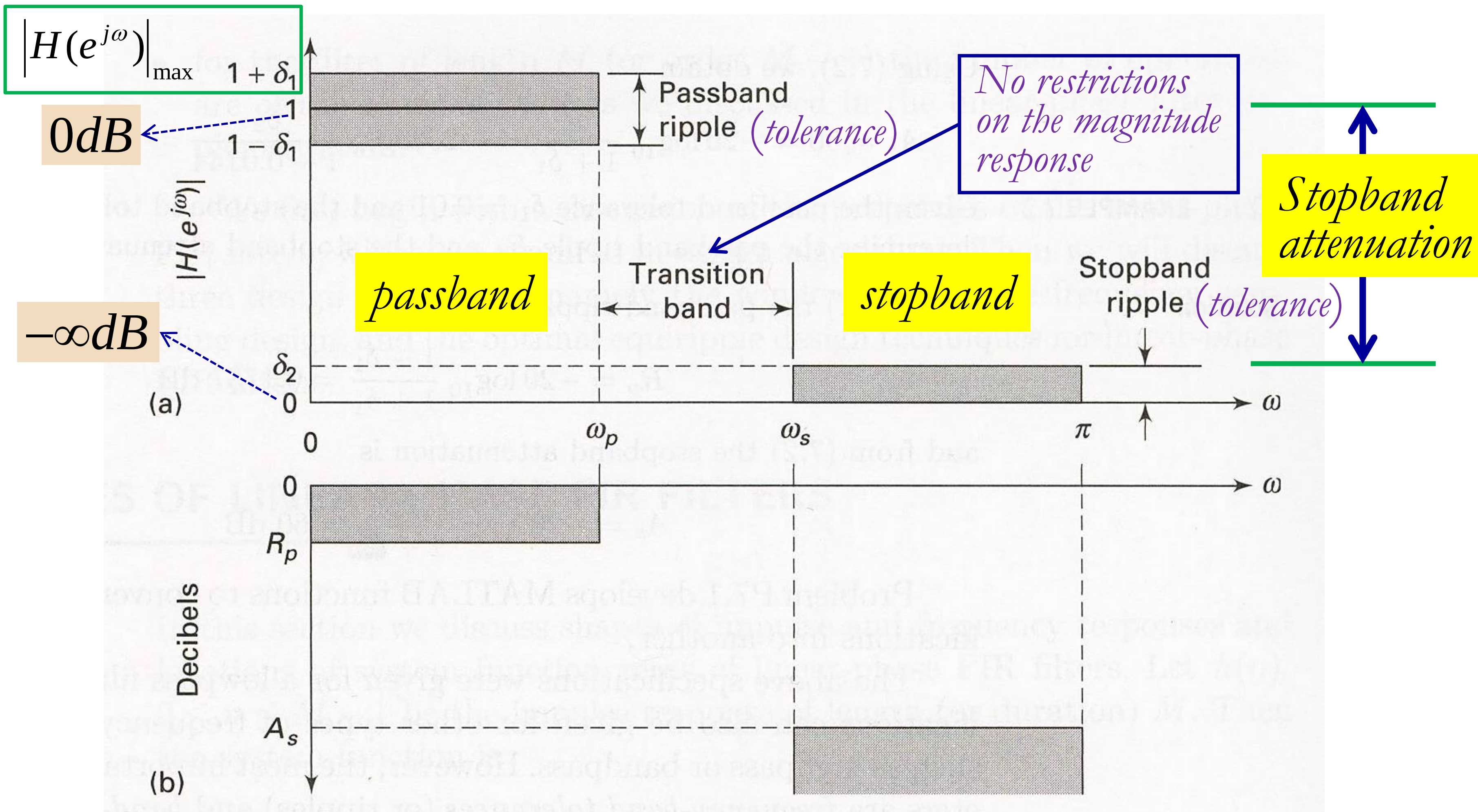
2. ***Relative specifications*:**
   - *requirements in decibels* (dB)

$$\text{dB scale} = -20\log_{10}\frac{\left|H(e^{j\omega})\right|}{\left|H(e^{j\omega})\right|_{\max}} \geq 0$$

   - *the most popular one in practice and is used for FIR and IIR filters*

Filter Design: 1. Specifications of LP FIR

Lowpass (FIR) *filter specifications:* (*a*) *Absolute* (*b*) *Relative*



Filter Design: 1. Specifications of LP FIR

Absolute specifications:

- *band* $[0, \omega_p]$ *is called the passband, and* δ_1 *is the tolerance (or ripple) that we are willing to accept in the ideal passband response*
- *band* $[\omega_s, \pi]$ *is called the stopband, and* δ_2 *is the corresponding tolerance (or ripple)*
- *band* $[\omega_p, \omega_s]$ *is called the transition band, and there are no restrictions on the magnitude response in this band*

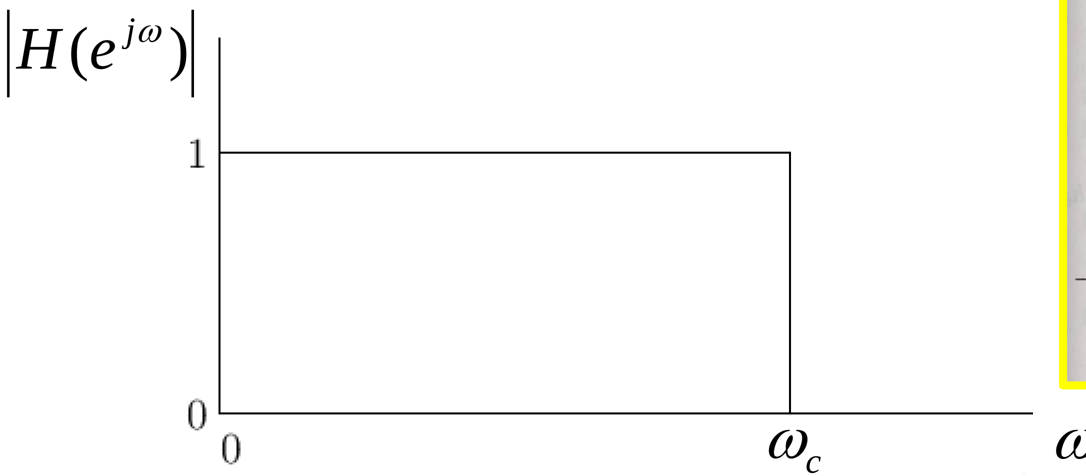
Relative specifications:

- R_p *is the passband ripple in dB*
- A_s *is the stopband attenuation in dB*

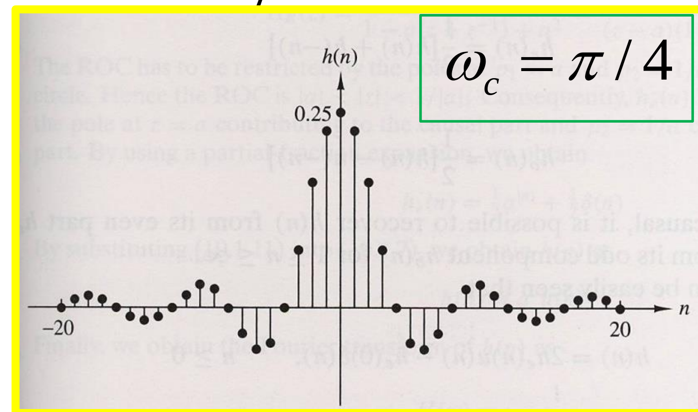
$$\boxed{\begin{aligned} R_p &= -20\log_{10}\frac{1-\delta_1}{1+\delta_1} > 0 \quad (\approx 0) \\ A_s &= -20\log_{10}\frac{\delta_2}{1+\delta_1} > 0 \quad (>> 1) \end{aligned}}$$

Filter Design: 1. Specifications of LP FIR

Amplitude Response Specification for the Ideal Low-pass Filter



cut-off frequency



$$h(n) = F^{-1}\left[H(e^{j\omega})\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi jn}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$

$$= \frac{\sin \omega_c n}{\pi n}$$

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < \omega < \pi \end{cases}$$

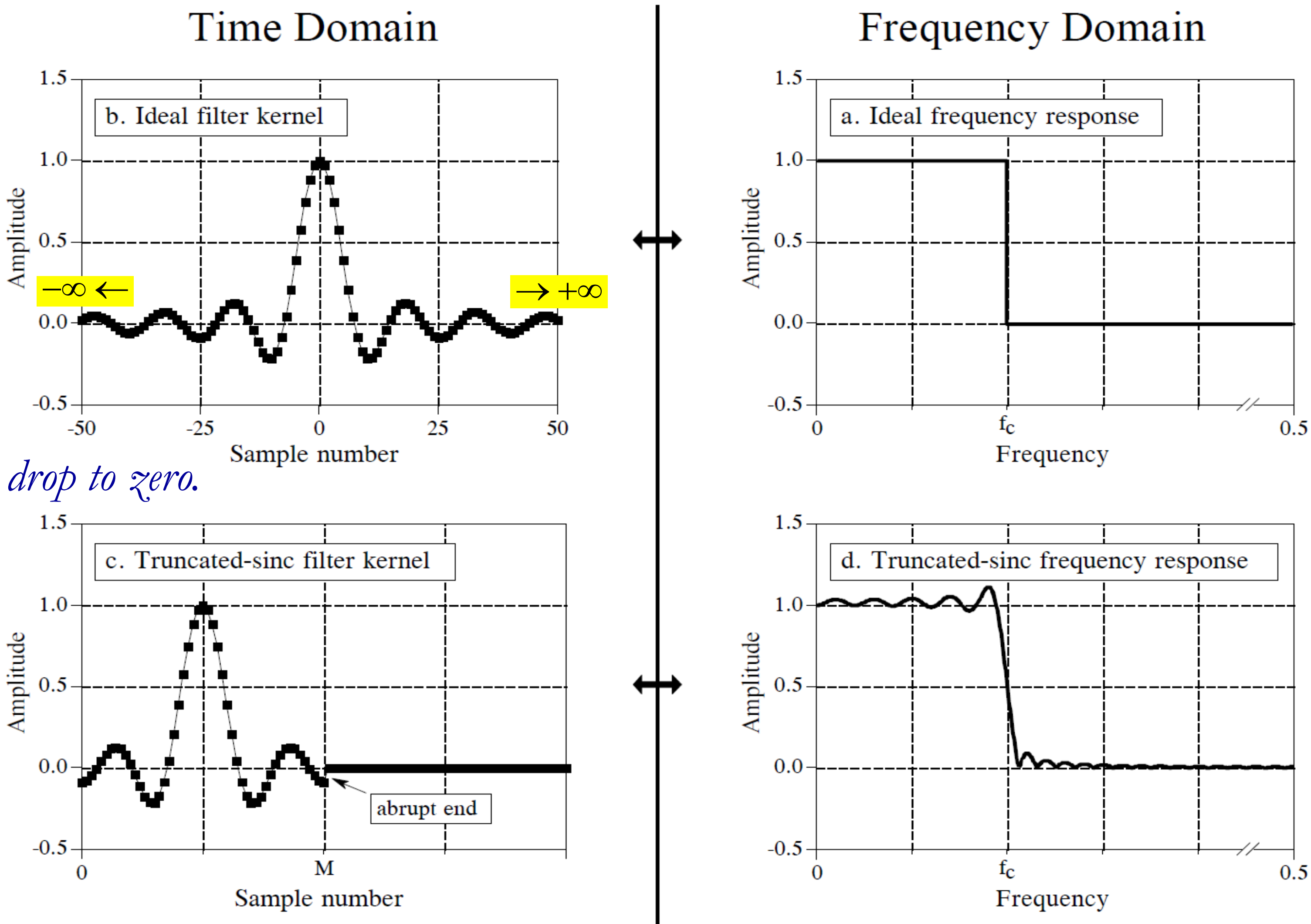
$$h(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin \omega_c n}{\omega_c n}, & n \ne 0 \end{cases}$$

Filter Design: 1. Specifications of LP FIR

FILTER USED FOR:	FILTER IMPLEMENTED BY: Convolution *Finite Impulse Response (FIR)*	FILTER IMPLEMENTED BY: Recursion *Infinite Impulse Response (IIR)*
Time Domain *(smoothing, DC removal)*	Moving average	Single pole
Frequency Domain *(separating frequencies)*	Windowed-sinc	Chebyshev
Custom *(Deconvolution)*	FIR custom	Iterative design

Filter Design: 1. Specifications of LP FIR

Windowed-Sinc Filter



values do not drop to zero.

Filter Design: 1. Specifications of LP FIR

Windowed-Sinc Filter

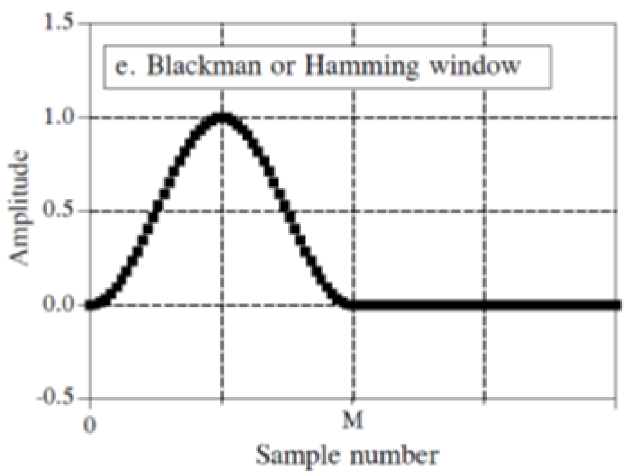
Multiplying by the Blackman (or Hamming) window

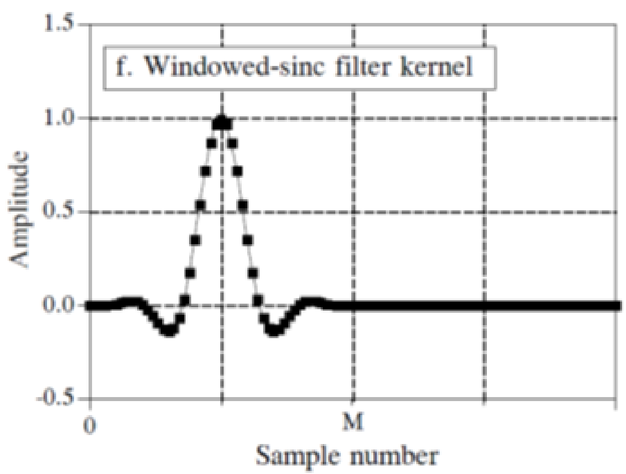
$$h(n) = \frac{\sin(2\pi f_c n)}{\pi n} = 2 f_c \frac{\sin(2\pi f_c n)}{2\pi f_c n}$$

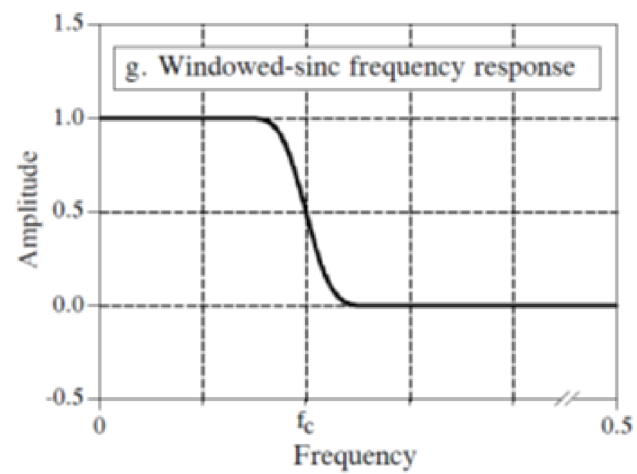
$$h(n) = 2 f_c \,\mathrm{sinc}\,(2 f_c n)$$

$$w(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) + 0.08\cos\left(\frac{4\pi n}{M-1}\right)$$

$$h(n) = \mathrm{sinc}\left(2 f_c\left(n - \frac{M-1}{2}\right)\right) w(n) \Big/ \sum_{n=0}^{M-1} h(n)$$







Filter Design: 2. Approximation of LP FIR

The most important design parameters are frequency-band tolerances (or ripples) and band-edge frequencies.

Problem Statement:

Design a lowpass filter (i.e., obtain its system function $H(z)$ or its difference equation) that has a passband $[0, \omega_p]$ with tolerance δ_1 (or R_p in dB) and a stopband $[\omega_s, \pi]$ with tolerance δ_2 (or A_s in dB).

Design and approximation of FIR digital filters

- *The phase response can be exactly linear.*
- *They are relatively easy to design since there are no stability problems.*
- *They are efficient to implement.*
- *The DFT can be used in their implementation.*

Filter Design: 2. Approximation of LP FIR

Linear-phase frequency-selective FIR filters

Advantages of a linear-phase response:

- *design problem contains only real arithmetic and not complex arithmetic.*
- *linear-phase filters provide no delay distortion and only a fixed amount of delay.*
- *for the filter of length* M (*or order* M -1) *the number of operations are of the order of* $M/2$.

Filter Design: 2. Approximation of LP FIR

An FIR filter of length M with input x(n) and output y(n)

$$y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_{M-1} x(n-M+1)$$

$$= \sum_{m=0}^{M-1} b_m x(n-m)$$

$$y(n) = \sum_{m=0}^{M-1} h(m) x(n-m)$$

the convolution of
the impulse response of the system and the input

$$H(z) = \sum_{m=0}^{M-1} h(m) z^{-m}$$

can also be characterized by its system function

$\boxed{\begin{array}{l}\text{A polynomial of degree } M-1 \text{ in the variable } z^{-1}. \\ \text{The roots of this polynomial constitute the zeros of the filter.}\end{array}}$

Properties of Linear Phase FIR Filter

Properties of Linear Phase FIR Filter

- **Linear Phase FIR Filter**

The delay through the filter being the same at all frequency (no phase distortion).

- **Properties of Linear Phase FIR Filter**

Shapes of

1) *Impulse responses*
2) *Frequency responses*
3) *Locations of system function zeros*

Properties of Linear Phase FIR Filter

Properties of Linear Phase FIR Filter

1) *Impulse responses*
2) *Frequency responses*
3) *Locations of system function zeros*

$\boxed{h(n),}\ 0 \le n \le M-1$ *impulse response of length* M

the system function

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n} = z^{-(M-1)} \sum_{n=0}^{M-1} h(n) z^{M-1-n}$$

$(M\text{-}1)$ *trivial poles at* $z=0$

$\boxed{(M\text{-}1)\ \textit{zeros}}$

the frequency response function

$$\boxed{H(e^{j\omega})} = \sum_{n=0}^{M-1} h(n) e^{-j\omega n}, \quad -\pi < \omega \le \pi$$

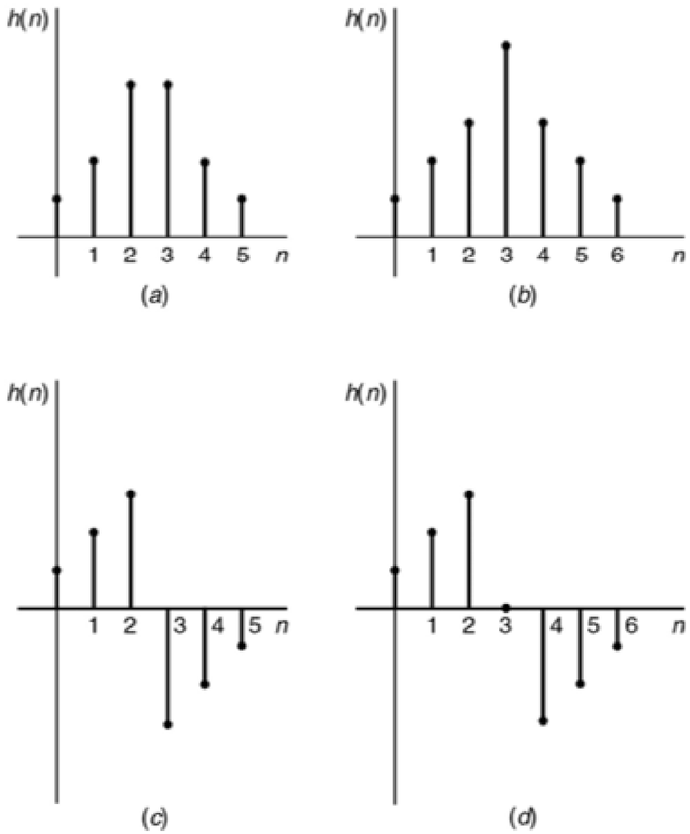
Properties: 1. Impulse Response

Impulse Response: Symmetry - Antisymmetry

An FIR filter has linear phase if its impulse response satisfies the (symmetry and antisymmetry) condition

FIR II: even length, symmetric

FIR I: odd length, symmetric



FIR IV: even length, antisymmetric

FIR III: odd length, antisymmetric

Note for this case that h[M/2]=0

Properties: 1. Impulse Response

▪ Impulse Response: Symmetry - Antisymmetry

An FIR filter has linear phase if its impulse response satisfies the (symmetry and antisymmetry) condition

$$h(n) = \pm h(M-1-n), \quad n = 0,1,\ldots,M-1$$

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}$$

$$= z^{-(M-1)/2}\left\{ h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{(M-3)/2} h(n)\left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2}\right]\right\}, \quad M \ \boxed{odd}$$

$$= z^{-(M-1)/2} \sum_{n=0}^{(M/2)-1} h(n)\left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2}\right], \quad M \ \boxed{even}$$

$$z^{-(M-1)} H(z^{-1}) = \pm H(z)$$

the roots of $H(z)$ *are identical to the roots of* $H(z^{-1})$.

Example

A 5-point moving average filter

$$\ldots,0,0,\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},0,0,\ldots$$

$$h(n) = \frac{1}{5}\{\underset{\uparrow}{1},1,1,1,1\}$$

$$= \frac{1}{5}\left(\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)\right)$$

$$H(z) = \frac{1}{5}\left(z^{0} + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right)$$

$$H(e^{j\omega}) = \frac{1}{5}\left(e^{j0} + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega}\right)$$

$$= \frac{1}{5}\left(e^{j2\omega} + e^{j\omega} + e^{j0} + e^{-j\omega} + e^{-j2\omega}\right)e^{-j2\omega}$$

<u>*Group delay*</u> $\quad -\frac{d}{d\omega}(-2\omega) = 2$

Properties: 1. Impulse Response

- **Impulse Response: Phase Delay & Group Delay**

Phase response is $\theta(\mathrm{w}) = \angle H(e^{j\omega}) = \beta - \alpha\omega, \quad -\pi < \omega \leq \pi$

$\therefore$ Phase Delay $\tau_p = \frac{-\theta(\omega)}{\omega} = \frac{-(-\alpha\omega)}{\omega} = \alpha$, which is a constant.

$\therefore$ Group Delay $\tau_g = \frac{-d}{d\omega}\theta(\omega) = \frac{-d}{d\omega}(-\alpha\omega) = \alpha$, which is a constant.

$$\Downarrow$$

$$h(n) = h(M-1-n); \quad \beta = 0, \alpha = \frac{M-1}{2}, 0 \leq n \leq M-1$$

symmetric impulse response

$$h(n) = -h(M-1-n); \quad \beta = \pm\pi/2, \alpha = \frac{M-1}{2}, 0 \leq n \leq M-1$$

antisymmetric impulse response

$$y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_1 x(n-M+2) + b_0 x(n-M+1)$$
$$= b_0\left[x(n) + x(n-M+1)\right] + b_1\left[x(n-1) + x(n-M+2)\right] + \cdots$$

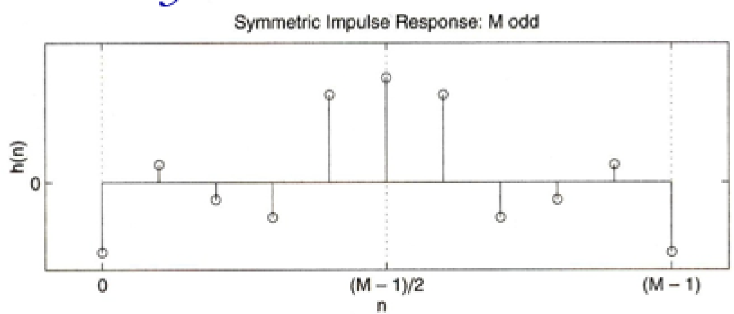
Properties: 1. Impulse Response

- **Impulse Response: Phase Delay & Group Delay**

$$\text{Linear-phase constraint } \angle H(e^{j\omega}) = -\alpha\omega, \quad -\pi < \omega \le \pi$$

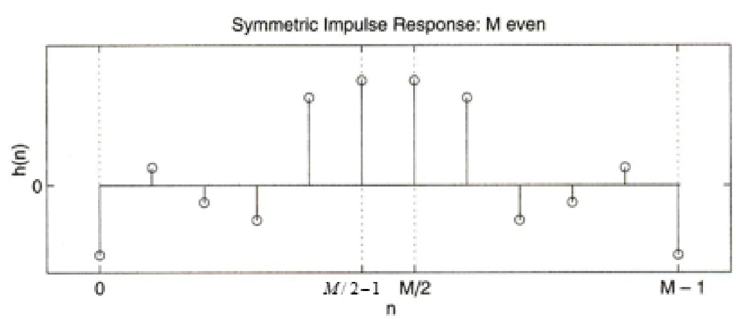
$$h(n) = h(M-1-n), \quad 0 \le n \le M-1 \quad \text{with} \quad \alpha = \frac{M-1}{2}$$

must be symmetric



$$M \text{ odd: } \alpha = \frac{M-1}{2} \text{ integer}$$

symmetric about α



$$M \text{ even: } \alpha = \frac{M-1}{2} \text{ not an integer}$$

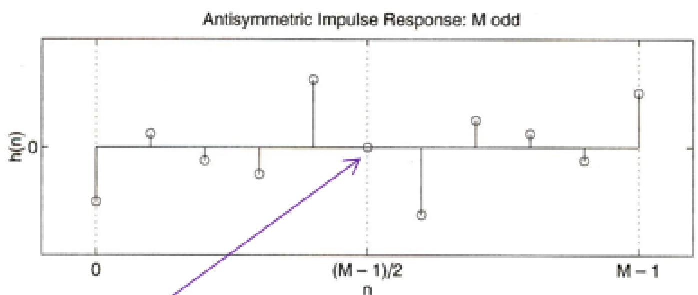
Properties: 1. Impulse Response

▪ Impulse Response: Phase Delay & Group Delay

Linear-phase constraint $\angle H(e^{j\omega}) = \beta - \alpha\omega$

$$h(n) = -h(M-1-n), \quad 0 \le n \le M-1 \quad \textit{with} \quad \alpha = \frac{M-1}{2}, \beta = \pm\frac{\pi}{2}$$

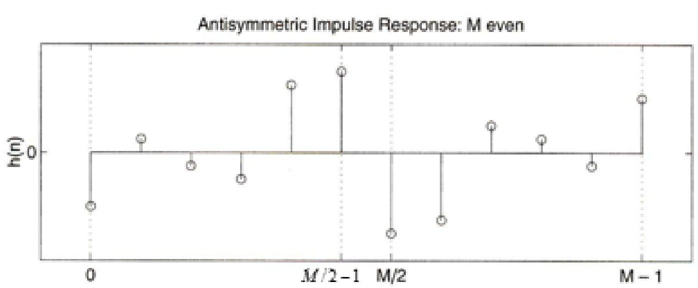
antisymmetric



$$M\ \textit{odd:}\ \alpha = \frac{M-1}{2}\ \textit{integer}$$

Must necessarily be equal to zero

$$h((M-1)/2) = 0$$



$$M\ \textit{even:}\ \alpha = \frac{M-1}{2}\ \textit{not an integer}$$

Properties: 2. Frequency Response

- **Frequency Response**

$$H(e^{j\omega}) = \boxed{H_r(\omega)} e^{j(\beta-\alpha\omega)}; \quad \beta = \pm\frac{\pi}{2}, \alpha = \frac{M-1}{2}$$

an <u>amplitude</u> (not a <u>magnitude</u>) response function

<u>both positive and negative</u>   <u>positive real</u>

The phase response associated with the magnitude response
is a discontinuous function,
while the associated with the amplitude response
is a continuous linear function.

Properties: 2. Frequency Response

▪ Frequency Response

Example *the impulse response* $h(n) = \left\{ \underset{\uparrow}{1}, 1, 1 \right\}$

the frequency response

$$H(e^{j\omega}) = \sum_{0}^{2} h(n) e^{-j\omega n} = 1 + e^{-j\omega} + e^{-j2\omega} = \left\{ e^{j\omega} + 1 + e^{-j\omega} \right\} e^{-j\omega}$$

$$= \{1 + 2\cos\omega\} e^{-j\omega}$$

the magnitude and the phase responses

$$\left| H(e^{j\omega}) \right| = \left| 1 + 2\cos\omega \right|, \quad 0 < \omega \le \pi$$

$$\angle H(e^{j\omega}) = \begin{cases} -\omega, & 0 < \omega < 2\pi/3 \\ \pi - \omega, & 2\pi/3 < \omega < \pi \end{cases}$$

$\cos\omega$ *can be both positive and negative*

piecewise linear

Properties: 2. Frequency Response

Frequency Response

Example *the impulse response* $h(n)=\left\{\underset{\uparrow}{1},1,1\right\}$

the frequency response

$$H(e^{j\omega})=\sum_{0}^{2}h(n)e^{-j\omega n}=1+e^{-j\omega}+e^{-j2\omega}=\left\{e^{j\omega}+1+e^{-j\omega}\right\}e^{-j\omega}$$

$$=\{1+2\cos\omega\}e^{-j\omega}$$

the amplitude and the corresponding phase responses

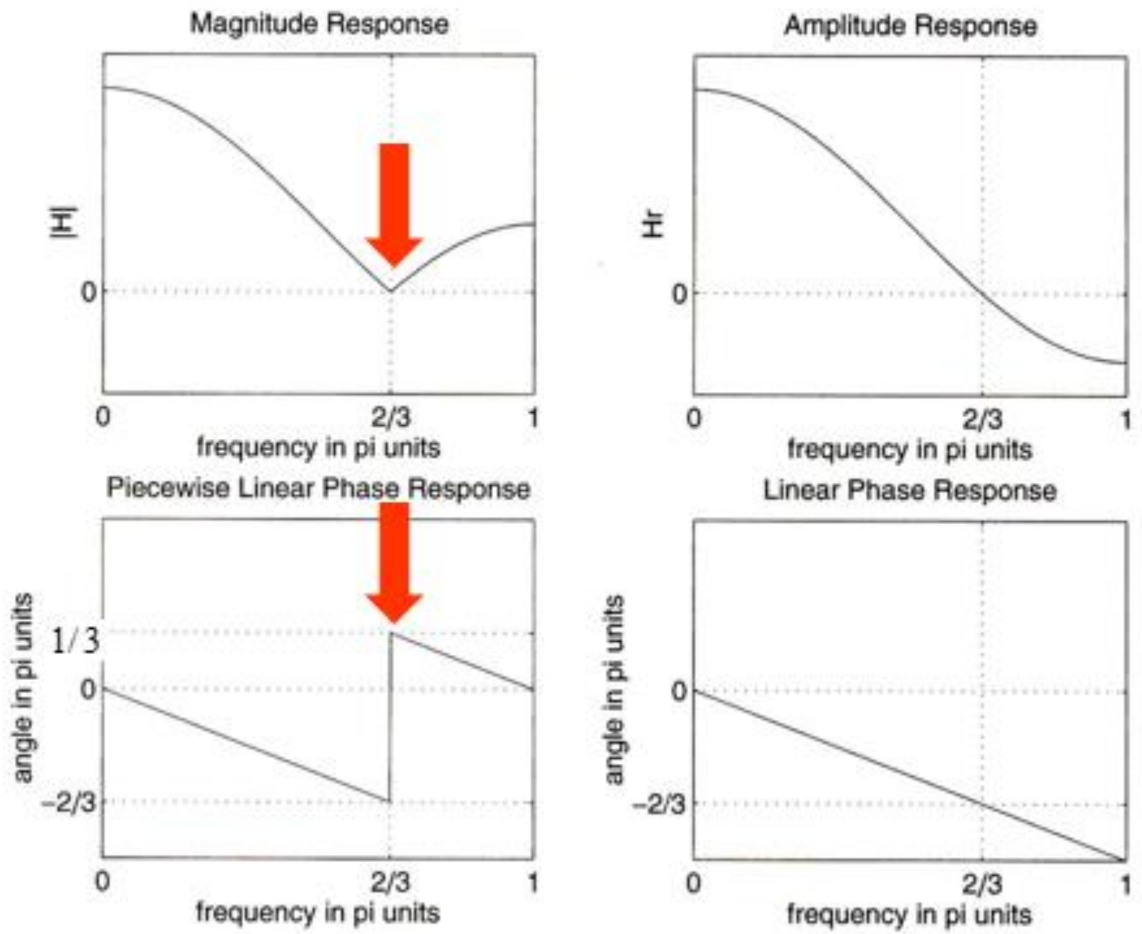
$H_r(\omega)=1+2\cos\omega,$

$$-\pi<\omega\le\pi$$

$\angle H(e^{j\omega})=-\omega,$ *truly linear*

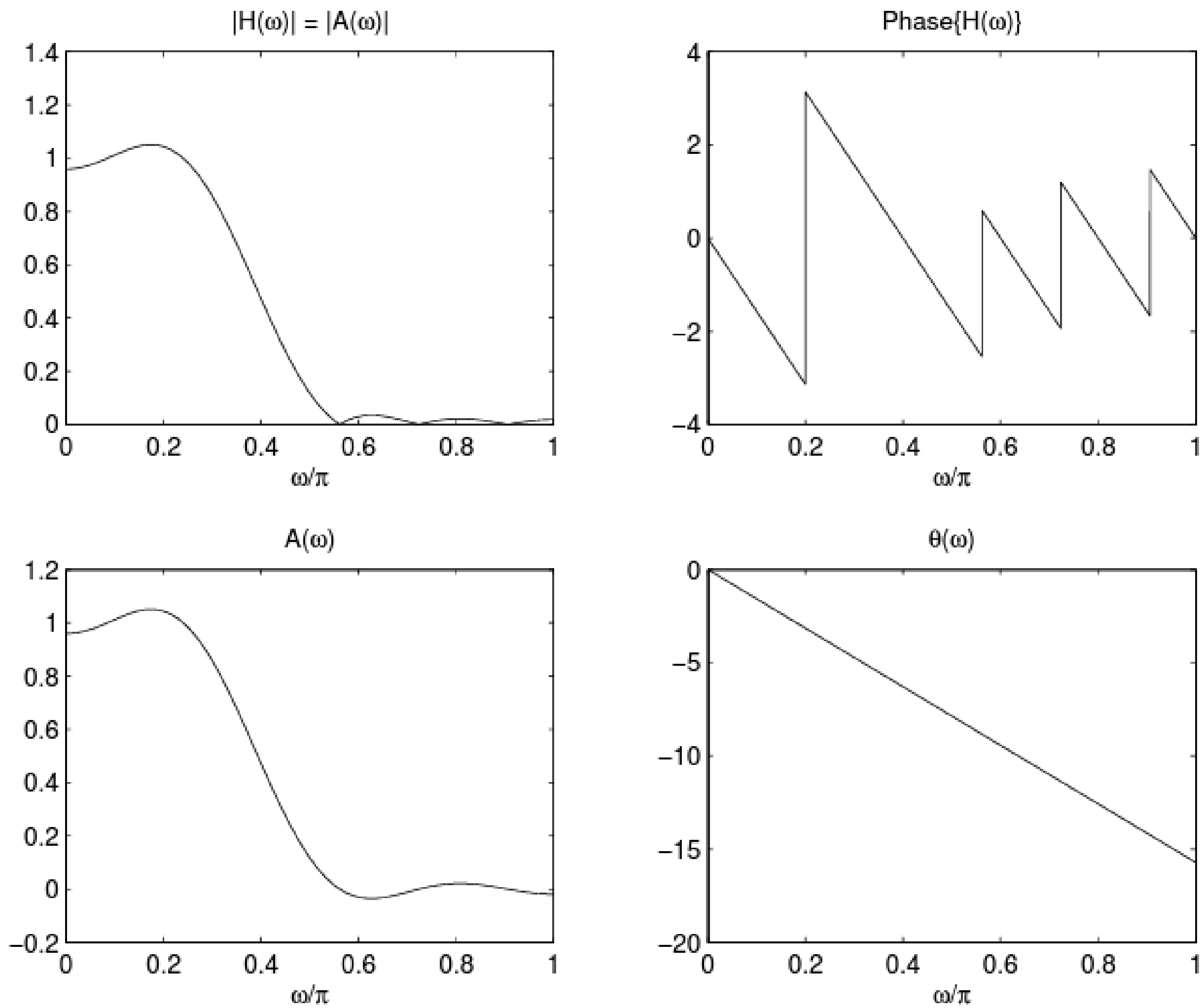
Properties: 2. Frequency Response

- **Frequency Response**



Properties: 2. Frequency Response

- **Another Example:**



Type-1 *linear-phase* **FIR** *filter: Symmetrical impulse response*, **M** *odd*

$$H(e^{j\omega}) = \left[\sum_{n=0}^{(M-1)/2} a(n)\cos\omega n\right] e^{-j\omega(M-1)/2}$$

$\beta = 0, \quad \alpha = (M-1)/2$ *integer*

$h(n) = h(M-1-n), \quad 0 \le n \le M-1$

$$a(0) = h\left(\frac{M-1}{2}\right):$$ *the middle sample*

$$a(n) = 2h\left(\frac{M-1}{2} - n\right), \quad 1 \le n \le \frac{M-3}{2}$$

$$H_r(\omega) = \sum_{n=0}^{(M-1)/2} a(n)\cos\omega n$$

Type-1 LPF coefficients

Order of Hr

```
function [Hr,w,a,L] = Hr_Type1(h);
```

Amplitude Response

Type-1 LPF impulse response

frequencies between [0 pi] *over which* Hr *is computed*

Type-2 linear-phase **FIR** *filter: Symmetrical impulse response,* **M** *even*

$\beta = 0, \quad \alpha = (M-1)/2$ *not an integer*

$h(n) = h(M-1-n), \quad 0 \le n \le M-1$

$$H(e^{j\omega}) = \left[\boxed{\sum_{n=1}^{M/2} b(n)\cos\left\{\omega\left(n-\frac{1}{2}\right)\right\}} \right] e^{-j\omega(M-1)/2}$$

$$b(n) = 2h\left(\frac{M}{2}-n\right), \quad n = 1,2,\ldots,\frac{M}{2}$$

$$H_r(\omega) = \sum_{n=1}^{M/2} b(n)\cos\left\{\omega\left(n-\frac{1}{2}\right)\right\}$$

Type-2 LPF coefficients

Order of Hr

function [Hr,w,b,L] = Hr_Type2(h);

$$\boxed{H_r(\pi) = \sum_{n=1}^{M/2} b(n)\cos\left\{\pi\left(n-\frac{1}{2}\right)\right\} = 0}$$

Type-2 LPF impulse response

Cannot use this type for highpass or bandstop filters

Amplitude Response

frequencies between [0 pi] *over which* Hr *is computed*

Type-3 linear-phase **FIR** *filter: Antisymmetrical impulse response,* **M** *odd*

$\beta = \pi/2, \quad \alpha = (M-1)/2$ *integer*

$h(n) = -h(M-1-n), \quad 0 \le n \le M-1$

$h((M-1)/2) = 0$

$$H(e^{j\omega}) = \left[\boxed{\sum_{n=1}^{(M-1)/2} c(n)\sin\omega n}\right] e^{j\left[\frac{\pi}{2} - \left(\frac{M-1}{2}\right)\omega\right]}$$

$$c(n) = 2h\left(\frac{M-1}{2} - n\right), \quad n = 1,2,\ldots,\frac{M-1}{2}$$

$$H_r(\omega) = \sum_{n=1}^{(M-1)/2} c(n)\sin\omega n$$

Type-3 LPF coefficients

Order of Hr

```
function [Hr,w,c,L] = Hr_Type3(h);
```

Amplitude Response

Type-3 LPF impulse response

frequencies between [0 pi] *over which* Hr *is computed*

$H_r(\omega) = 0 \quad at \quad \omega = 0, \pi$

$e^{j\pi/2} = j, \quad jH_r(\omega)$ *purely imaginary*

Not suitable for designing a lowpass filter or a highpass filter.

Suitable for approximating ideal digital Hilbert transformer and differentiators.

Type-4 linear-phase **FIR** *filter: Antisymmetrical impulse response,* **M** *even*

$\beta = \pi/2, \quad \alpha = (M-1)/2$ *not an integer*

$h(n) = -h(M-1-n), \quad 0 \le n \le M-1$

$$H(e^{j\omega}) = \left[\sum_{n=1}^{M/2} d(n) \sin\left\{ \omega \left(n - \frac{1}{2} \right) \right\} \right] e^{j\left[\frac{\pi}{2} - \left(\frac{M-1}{2} \right) \omega \right]}$$

$$d(n) = 2h\left(\frac{M}{2} - n \right), \quad n = 1, 2, \ldots, \frac{M}{2}$$

$$H_r(\omega) = \sum_{n=1}^{M/2} d(n) \sin\left\{ \omega \left(n - \frac{1}{2} \right) \right\}$$

Type-4 LPF coefficients

Order of Hr

function [Hr,w,d,L] = Hr_Type4(h);

Amplitude Response

Type-4 LPF impulse response

frequencies between [0 pi] *over which* Hr *is computed*

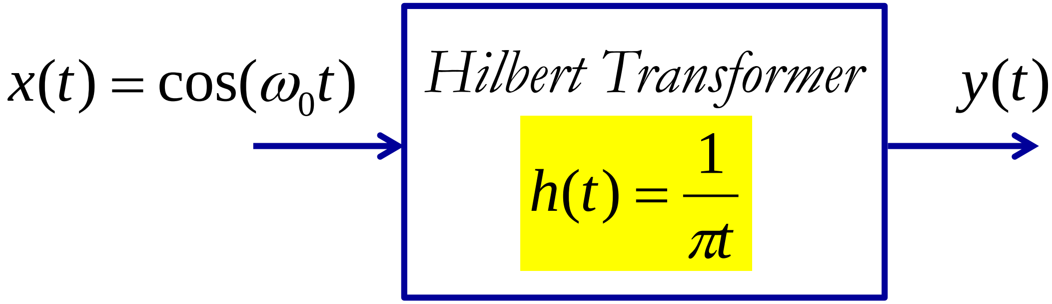
$$H_r(\omega) = 0 \quad at \quad \omega = 0$$
$$e^{j\pi/2} = j$$

Suitable for designing digital Hilbert transformer and differentiators.

Hilbert Transformer

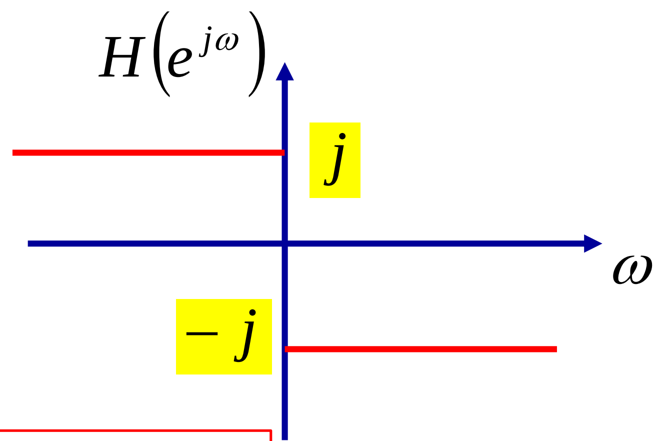
$$y(t) = x(t) * h(t)$$

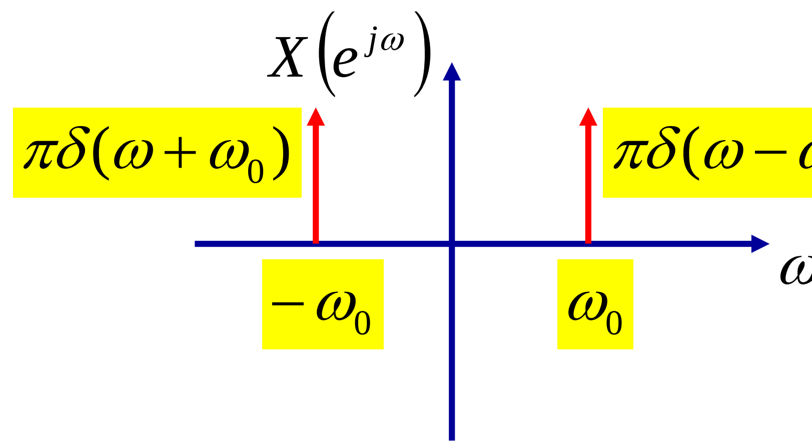
$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) \cdot H\left(e^{j\omega}\right)$$

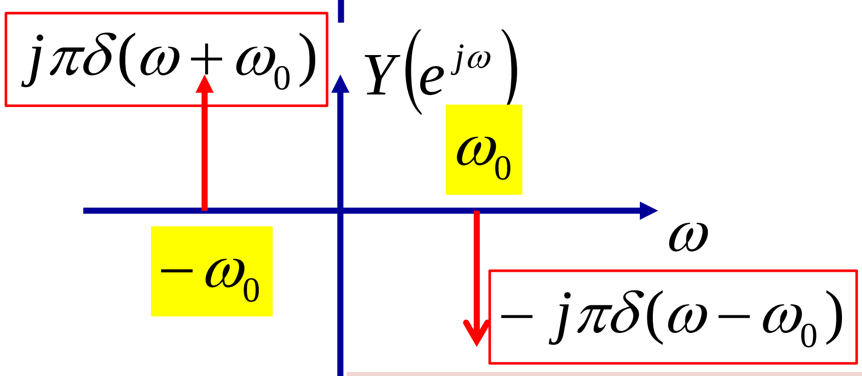


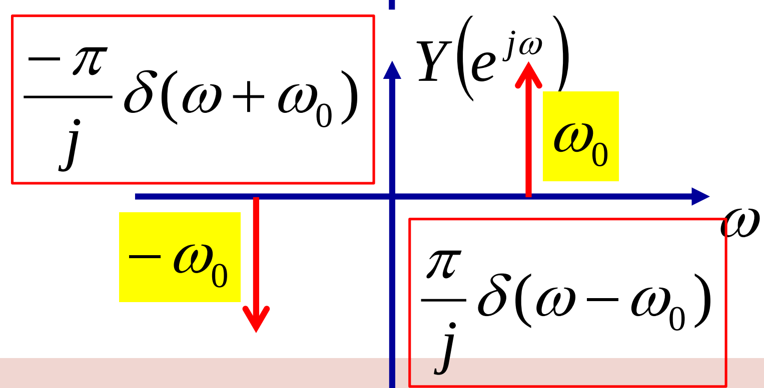
$$H\left(e^{j\omega}\right) = -j\,\mathrm{sgn}(\omega)$$

$$X\left(e^{j\omega}\right) = \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$









$$Y\left(e^{j\omega}\right) = \frac{\pi}{j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right] \rightarrow y(t) = \sin(\omega_0 t)$$

Differentiator

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})e^{j\omega t}d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} X(e^{j\omega})e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})\frac{d}{dt}\left(e^{j\omega t}\right)d\omega$$

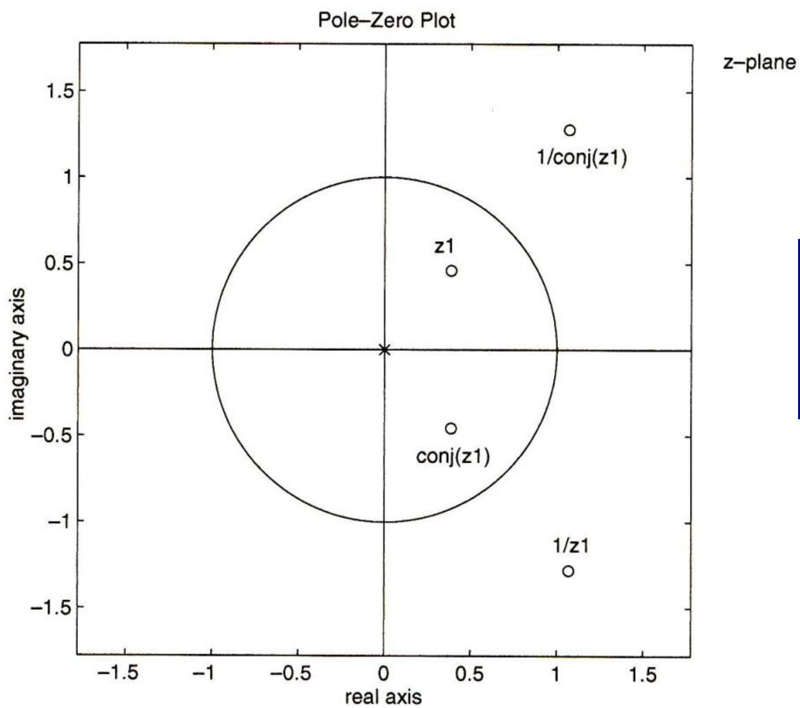
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{X(e^{j\omega})\left(j\omega\right.}\left.e^{j\omega t}\right)d\omega$$

$$x(t) \xleftrightarrow{F.T.} X\left(e^{j\omega}\right)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{F.T.} j\omega X\left(e^{j\omega}\right)$$

Properties: 3. Zero Locations

Zeros possess certain symmetries due to the symmetry constraints on $h(n)$.



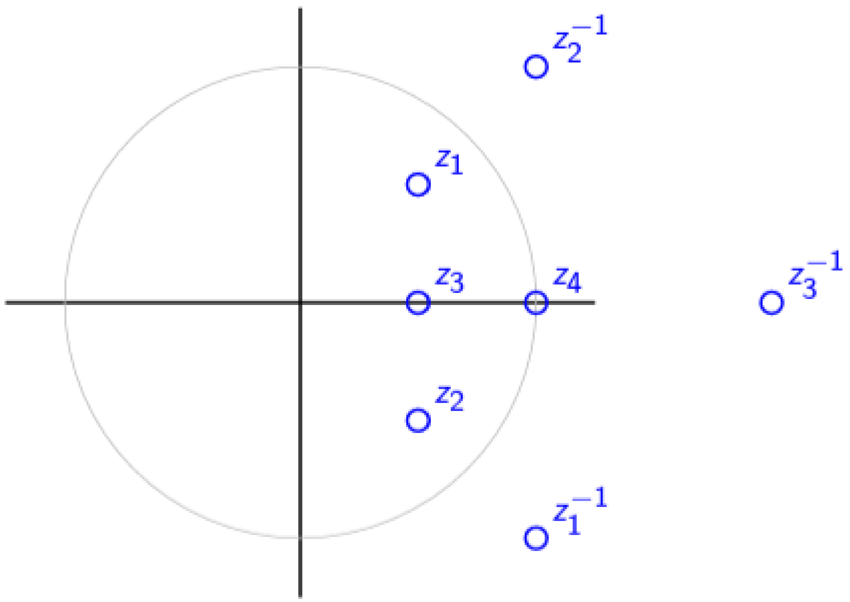
Zeros not on $|z|=1$ *are conjugate reciprocal pairs.*

A general zero constellation is a quadruplet

$$\boxed{re^{j\theta},\ \frac{1}{r}e^{j\theta},\ re^{-j\theta},\ \frac{1}{r}e^{-j\theta}}$$

Properties: 3. Zero Locations

$H(z)$ and $H(z^{-1})$ must therefore be zero for the same values of z. If z is a zero then z^{-1} must also be a zero.



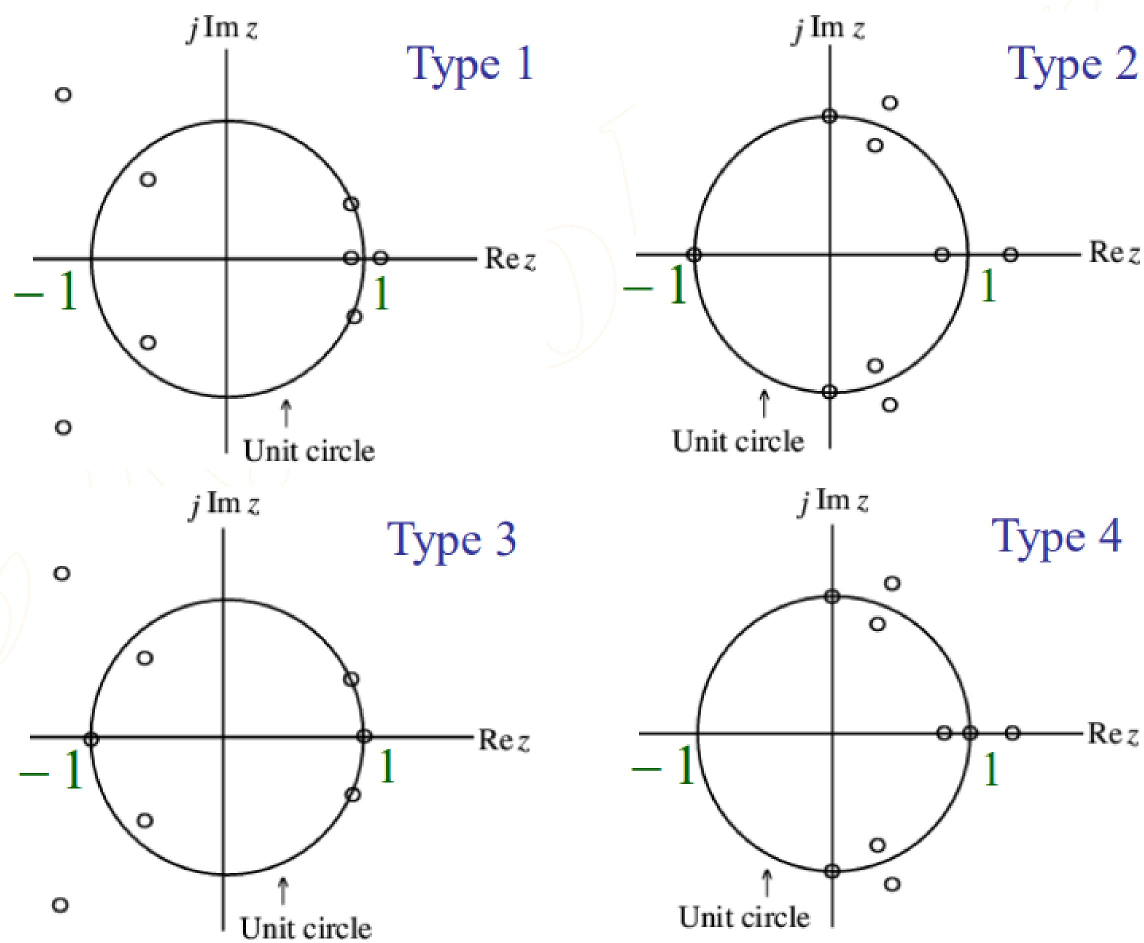
Properties: 3. Zero Locations

Type 1 FIR filter: Either an even number or no zeros at z = 1 and z=-1

Type 2 FIR filter: Either an even number or no zeros at z = 1, and an odd number of zeros at z=-1

Type 3 FIR filter: An odd number of zeros at z = 1 and and z=-1

Type 4 FIR filter: An odd number of zeros at z = 1, and either an even number or no zeros at z=-1



The presence of zeros at z=±1 leads to some limitations on the use of these linear-phase transfer functions for designing frequency-selective filters

Properties: 3. Zero Locations

Example $h(n)=\left\{\underset{\uparrow}{-4},1,-1,-2,5,6,5,-2,-1,1,-4\right\}$

$M=11$, *odd*, $h(n)$ *is symmetric about* $\alpha=(11-1)/2=5$

$\boxed{Type\text{-}1}$

$$a(0)=h(\alpha)=h(5)=6,\quad a(1)=2h(5-1)=10,\quad a(2)=2h(5-2)=-4$$

$$a(3)=2h(5-3)=-2,\quad a(4)=2h(5-4)=2,\quad a(5)=2h(5-5)=-8$$

$$H_r(\omega)=a(0)+a(1)\cos\omega+a(2)\cos 2\omega+a(3)\cos 3\omega+a(4)\cos 4\omega+a(5)\cos 5\omega$$

$$=6+10\cos\omega-4\cos 2\omega-2\cos 3\omega+2\cos 4\omega-8\cos 5\omega$$

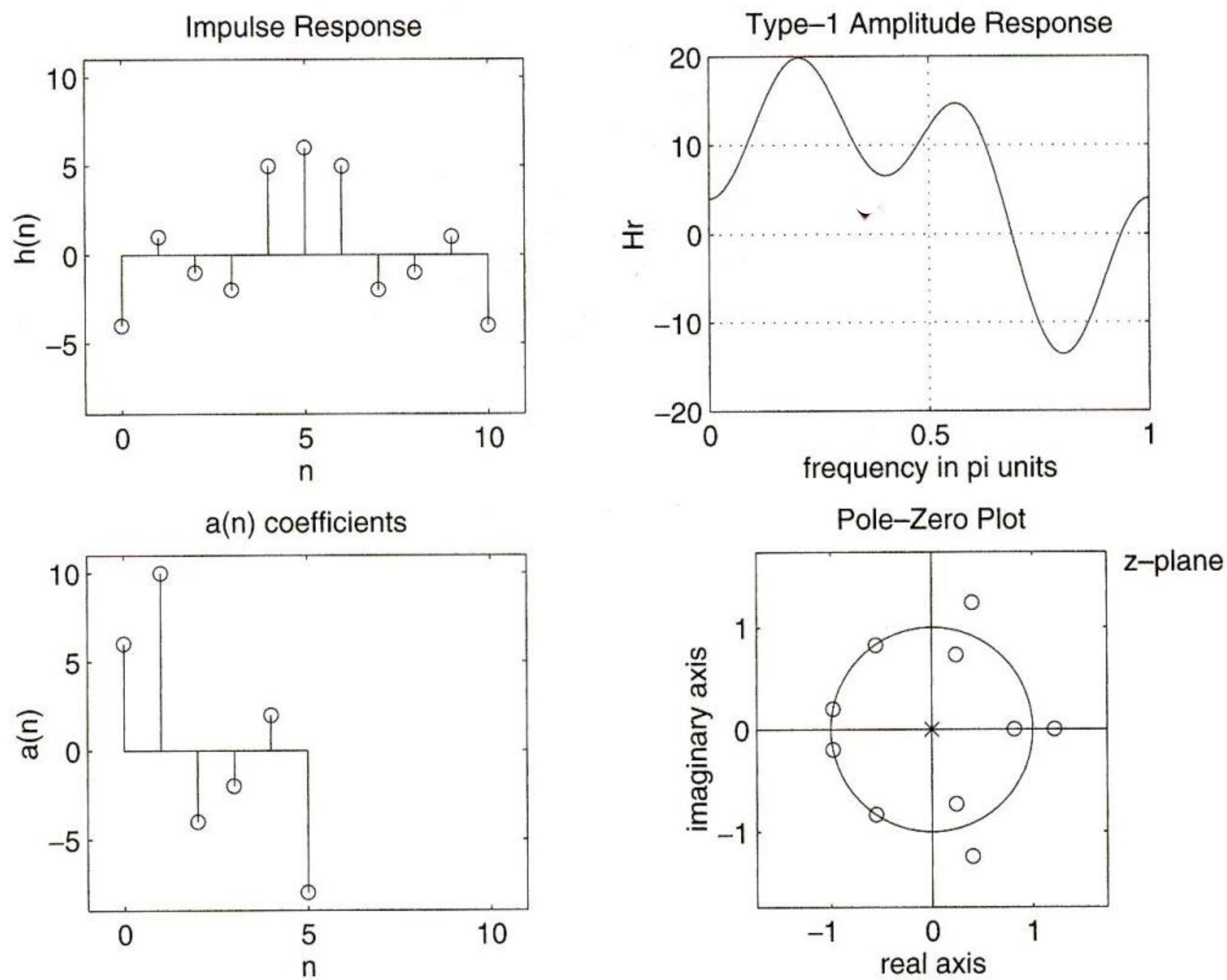
Properties: 3. Zero Locations

Example

```
>> h = [-4,1,-1,-2,5,6,5,-2,-1,1,-4]:
>> M = length(h); n = 0:M-1;
>> [Hr,w,a,L] = Hr_Type1(h);
>> a,L
a = 6    10    -4    -2     2    -8
L = 5
>> amax = max(a)+1; amin = min(a)-1;
>> subplot(2,2,1); stem(n,h); axis([-1 2*L+1 amin amax])
>> xlabel('n'); ylabel('h(n)'); title('Impulse Response')
>> subplot(2,2,3); stem(0:L,a); axis([-1 2*L+1 amin amax])
>> xlabel('n'); ylabel('a(n)'); title('a(n) coefficients')
>> subplot(2,2,2); plot(w/pi,Hr); grid
>> xlabel('frequency in pi units'); ylabel('Hr')
>> title('Type-1 Amplitude Response')
>> subplot(2,2,4); pzplotz(h,1)
```

Properties: 3. Zero Locations

Example



Window Design Techniques

Window Design Techniques

- Choose a proper ideal frequency-selective filter (which always has a non-causal, infinite-duration impulse response)
- Then truncate (or window) its impulse response to obtain a linear-phase and causal FIR filter.

Select an appropriate windowing function and appropriate ideal filter

α *sample delay*

shift in the positive n *direction or delay*

cutoff frequency

$$H_d(e^{j\omega}) = \begin{cases} 1 \cdot \boxed{e^{-j\alpha\omega}}, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

A unity-magnitude gain
Linear-phase characteristics over its passband
Zero response over its stopband

Window Function

a window function

$$h(n) = h_d(n)\boxed{w(n)},$$

$$w(n) = \begin{cases} some \quad symmetric \quad function \quad w.r.t. \\ \alpha \quad over \quad 0 \le n \le M-1 \\ \\ 0, \qquad\qquad otherwise \end{cases}$$

Depending on how to define w(n), *we obtain different window designs. For example,*

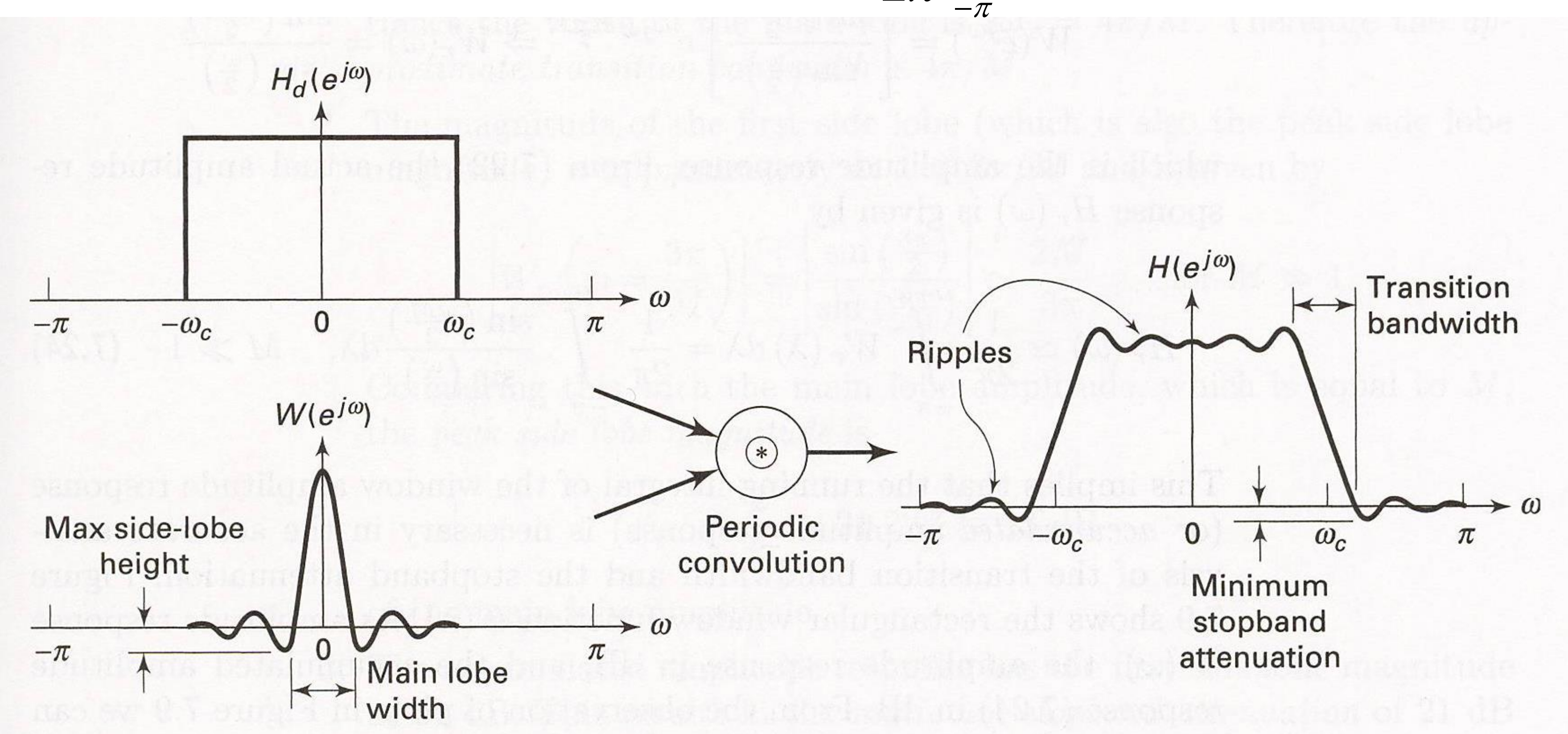
$$w(n) = \begin{cases} 1, & 0 \le n \le M-1 \\ 0, & otherwise \end{cases} = R_M(n)$$

the rectangular window

Window Function

In the frequency domain the causal FIR *filter response* $H(e^{j\omega})$ *is given by the periodic convolution of* $H_d(e^{j\omega})$ *and the window response* $W(e^{j\omega})$;

$$H(e^{j\omega}) = H_d(e^{j\omega}) \circledast W(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j\lambda}) H_d\left(e^{j(\omega-\lambda)}\right) d\lambda$$



Window Design

Observations:

1. *Since the window* $w(n)$ *has a finite length equal to* M*, its response has a peaky main lobe whose width is proportional to* $1/M$*, and has side lobes of smaller heights.*

2. *The periodic convolution produces a smeared version of the ideal response* $H_d(e^{j\omega})$.

3. *The main lobe produces a transition band in* $H(e^{j\omega})$ *whose width is responsible for the transition width. This width is proportional to* $1/M$*. The wider the main lobe, the wider will be the transition width.*

4. *The side lobes produce ripples that have similar shapes in both the passband and stopband.*

Window Design

Basic Window Design Idea:

For the given filter specifications choose the filter length M *and a window function* $w(n)$ *for the narrowest main lobe width and the smallest side lobe attenuation possible.*

$$\boxed{\textit{passband tolerance} = \textit{stopband tolerance}}$$

Rectangular Window

Rectangular Window: simplest but worst

$$w(n)=\begin{cases}1, & 0\le n\le M-1\\ 0, & otherwise\end{cases}$$

frequency response *amplitude response*

$$W(e^{j\omega})=\left[\frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right]e^{-j\omega\frac{M-1}{2}}\Rightarrow W_r(\omega)=\frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

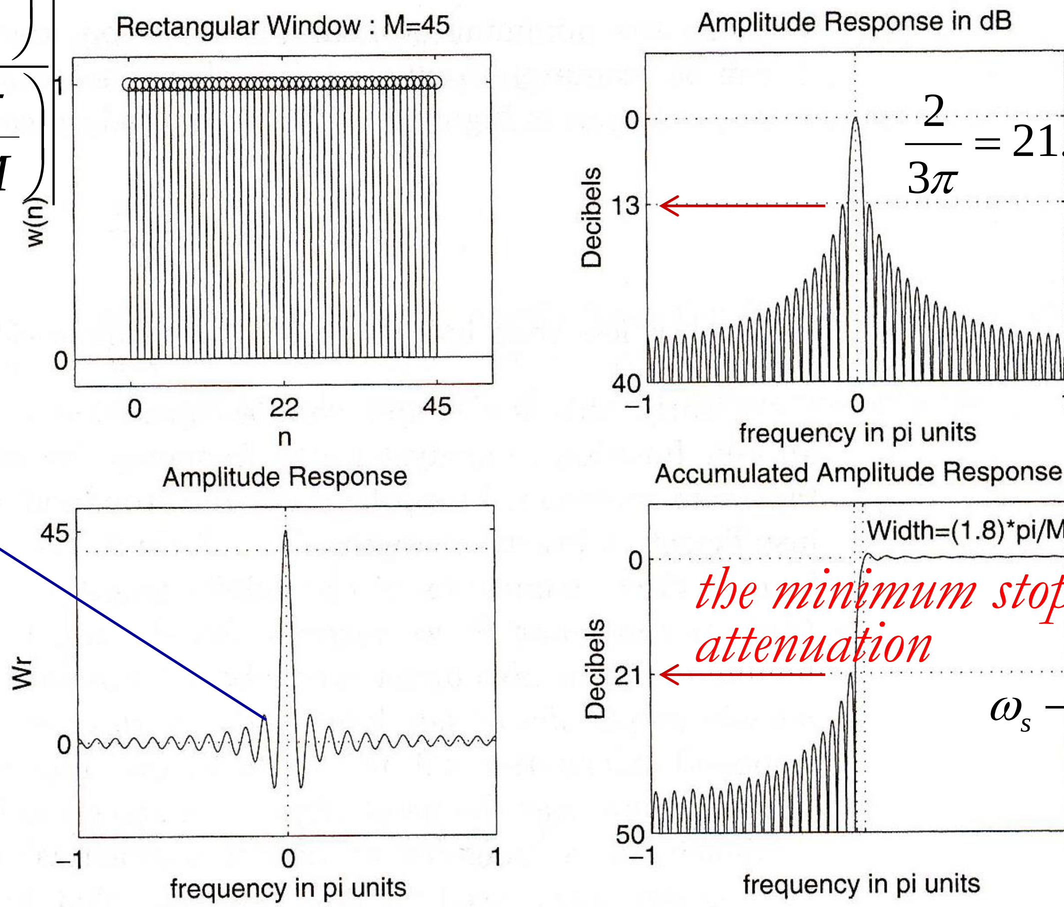
actual amplitude response

$$H_r(\omega)\approx\frac{1}{2\pi}\int_{-\pi}^{\omega+\omega_c}W_r(\lambda)d\lambda=\frac{1}{2\pi}\int_{-\pi}^{\omega+\omega_c}\frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}d\lambda,\quad M>>1$$

Rectangular Window

$$\left|W_r\left(\omega=\frac{3\pi}{M}\right)\right| = \left|\frac{\sin\left(\frac{3\pi}{2}\right)}{\sin\left(\frac{3\pi}{2M}\right)}\right|$$

$$\approx \frac{2M}{3\pi}, \qquad for\ M >> 1$$



$$\frac{2}{3\pi} = 21.22\% \equiv 13dB$$

the minimum stopband attenuation

$$\omega_s - \omega_p = \frac{1.8\pi}{M}$$

$W_r(\omega)$ *has the first zero at* $\omega = \omega_1$.

$$\frac{\omega_1 M}{2} = \pi$$

the width of the main lobe

$$2\omega_1 = 4\pi / M$$

the approximate transition bandwidth

w=boxcar(M)

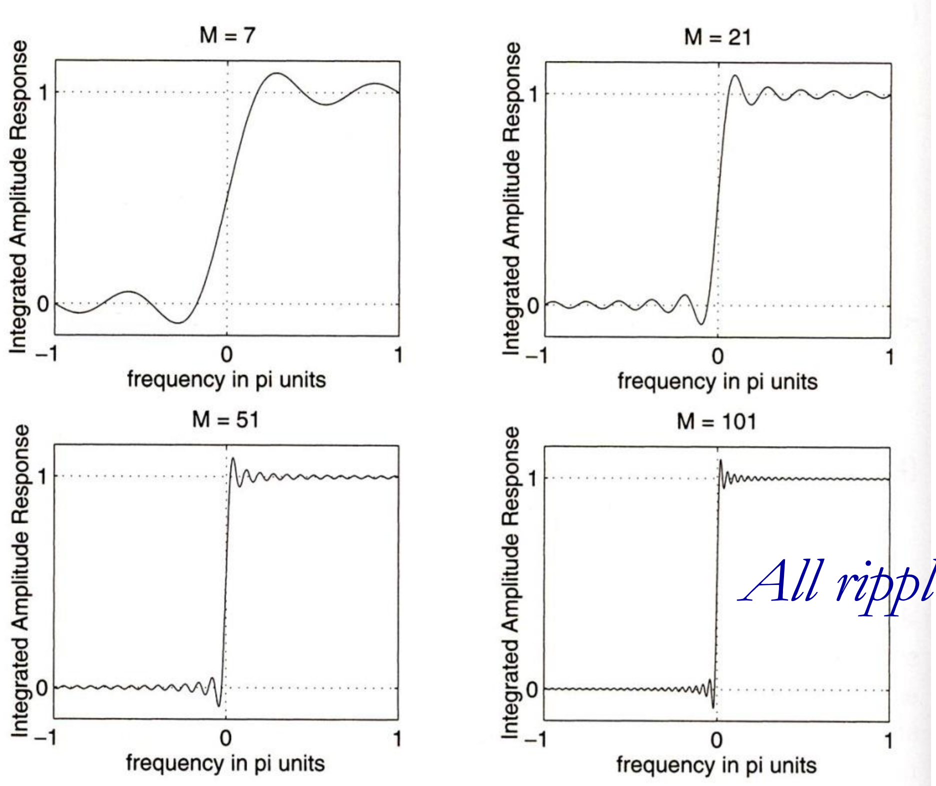
Rectangular Window

$h_d(n)w(n)$

Infinitely long, not absolutely summable, filter becomes unstable

Abrupt transition from 1 *to* 0

Gibbs Phenomenon:

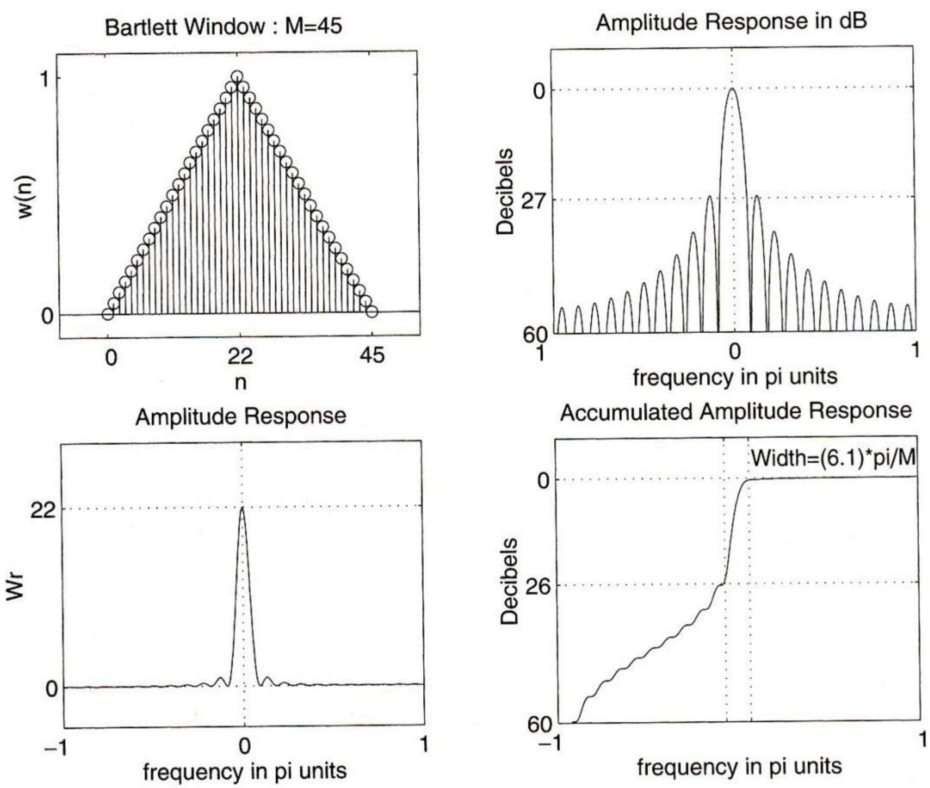


All ripples bunch up near the band edges.

If we increase **M**, *the width of each side lobe will decrease, but the area under each lobe will remain constant. Therefore the relative amplitudes of side lobes will remain constant, and the minimum stopband attenuation will remain at* 21dB.

Barlett Window

$$w(n)=\begin{cases}\dfrac{2n}{M-1}, & 0\le n\le \dfrac{M-1}{2}\\[2ex] 2-\dfrac{2n}{M-1}, & \dfrac{M-1}{2}\le n\le M-1\\[2ex] 0, & otherwise\end{cases}$$



a more gradual transition in the form of a triangular window

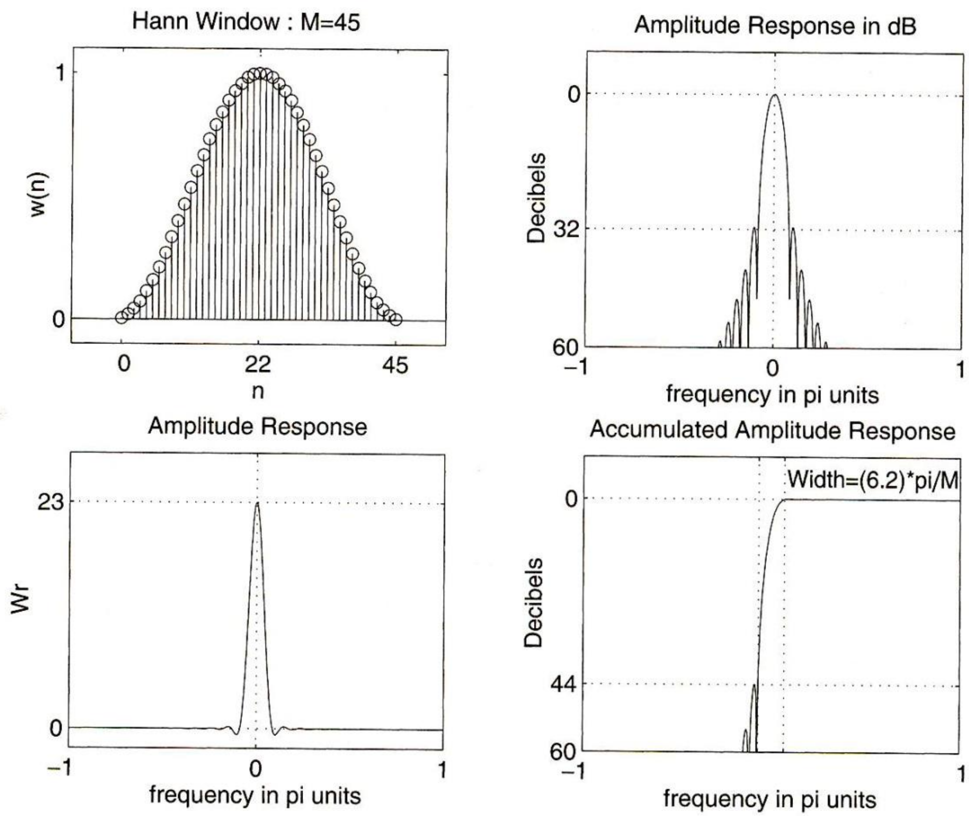
w=bartlett(M)

The Gibbs phenomenon results from the fact that the rectangular window has a sudden transition from 0 *to* 1 *(or* 1 *to* 0*).*

Hann Window

$$w(n) = \begin{cases} 0.5\left[1 - \cos\left(\dfrac{2\pi n}{M-1}\right)\right], & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

a raised cosine window function

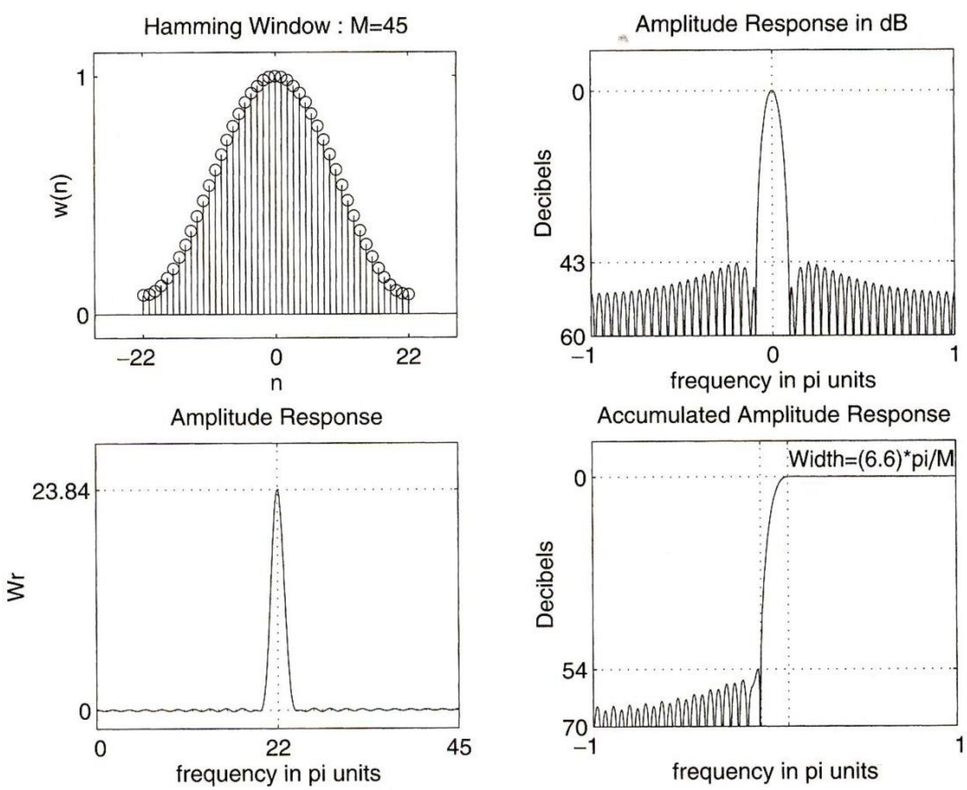


w=hann(M)

Hamming Window

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\dfrac{2\pi n}{M-1}\right), & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

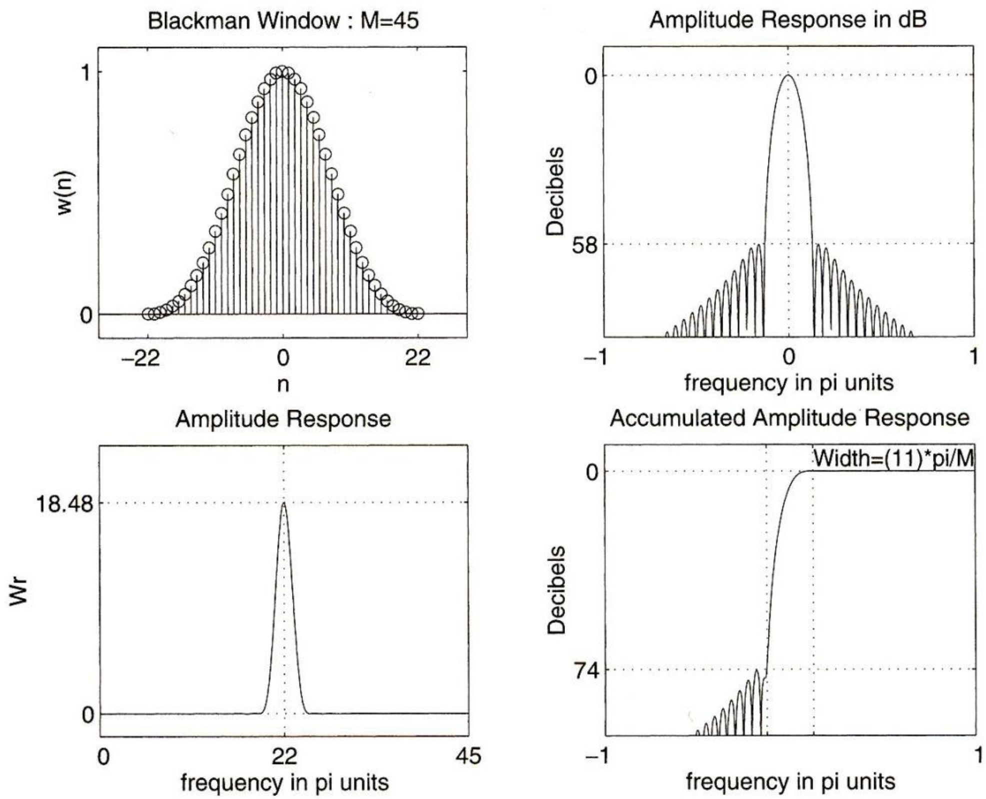
a small amount of discontinuity



w=hamming(M)

Blackman Window

$$w(n) = \begin{cases} 0.42 - 0.5\cos\left(\dfrac{2\pi n}{M-1}\right) + 0.08\cos\left(\dfrac{4\pi n}{M-1}\right), & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$



a second harmonic term

w=blackman(M)

Summary

Summary of Commonly Used Window Function Characteristics

Window Name	Transition Width $\Delta\omega$ Approximate	Transition Width $\Delta\omega$ Exact Values	Min. Stopband Attenuation
Rectangular	$\frac{4\pi}{M}$	$\frac{1.8\pi}{M}$	21 dB
Bartlett	$\frac{8\pi}{M}$	$\frac{6.1\pi}{M}$	25 dB
Hann	$\frac{8\pi}{M}$	$\frac{6.2\pi}{M}$	44 dB
Hamming	$\frac{8\pi}{M}$	$\frac{6.6\pi}{M}$	53 dB
Blackman	$\frac{12\pi}{M}$	$\frac{11\pi}{M}$	74 dB

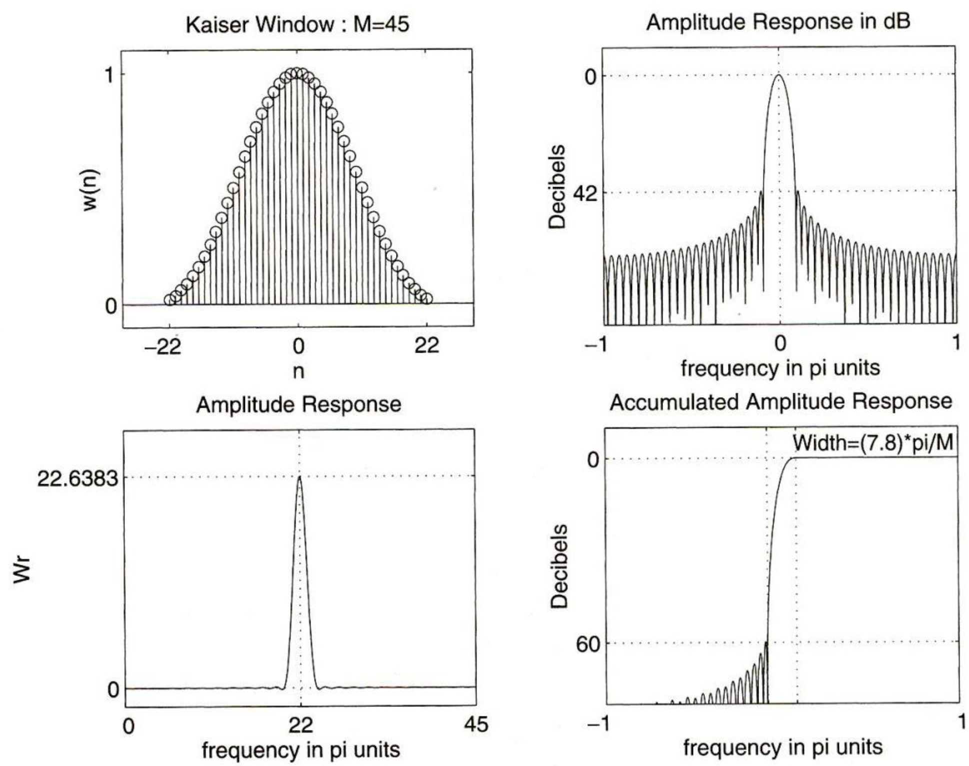
Kaiser Window

adjustable window function

$$w(n) = \frac{I_0\left[\beta\sqrt{1-\left(1-\frac{2n}{M-1}\right)^2}\right]}{I_0[\beta]}, \quad 0 \le n \le M-1$$

$$I_0(x) = 1 + \sum_{k=0}^{\infty}\left[\frac{(x/2)^k}{k!}\right]^2$$

modified zero-order Bessel function



w=kaiser(M,beta)

The parameter β *controls the minimum stopband attenuation* A_s *and can be chosen to yield different transition widths for near-optimum* A_s.

Kaiser Window

$$\textit{transition width} = \Delta\omega = \omega_s - \omega_p$$

$$\textit{Filter length } M \approx \frac{A_s - 7.95}{2.285\Delta\omega} + 1$$

$$\textit{Parameter } \beta = \begin{cases} 0.1102(A_s - 8.7), & A_s \ge 50 \\ 0.5842(A_s - 21)^{0.4} + 0.07886(A_s - 21), & 21 < A_s < 50 \end{cases}$$

```
function hd = ideal_lp(wc,M);
% Ideal LowPass filter computation
% --------------------------------------
% [hd] = ideal_lp(wc,M)
%  hd = ideal impulse response between 0 to M-1
% wc = cutoff frequency in radians
%  M = length of the ideal filter
%
alpha = (M-1)/2; n = [0:1:(M-1)];
m = n - alpha; fc = wc/pi; hd = fc*sinc(fc*m);
```

Display the frequency-domain plots of digital filters: freqz

freqz_m*: returns the magnitude response in absolute as well as in relative* dB *scale, the phase response, and the group delay response.*

```
function [db,mag,pha,grd,w] = freqz_m(b,a);
% Modified version of freqz subroutine
% ------------------------------------------
% [db,mag,pha,grd,w] = freqz_m(b,a);
%    db = Relative magnitude in dB computed over 0 to pi radians
% mag = Absolute magnitude computed over 0 to pi radians
% pha  = Phase response in radians over 0 to pi radians
% grd  =  Group delay over 0 to pi radians
%     w = 501 frequency samples between 0 to pi radians
%     b = numerator polynomial of H(z)     (for FIR: b=h)
%     a = denominator polynomial of H(z)  (for FIR: a=[1])
%
[H,w] = freqz(b,a,1000,'whole');
      H = (H(1:1:501))'; w = (w(1:1:501))';
   mag = abs(H); db = 20*log10((mag+eps)/max(mag));
   pha  = angle(H); grd = grpdelay(b,a,w);
```

Digital Filter Design

Design Lowpass Filter

Design a digital FIR *lowpass filter with the following specifications:*

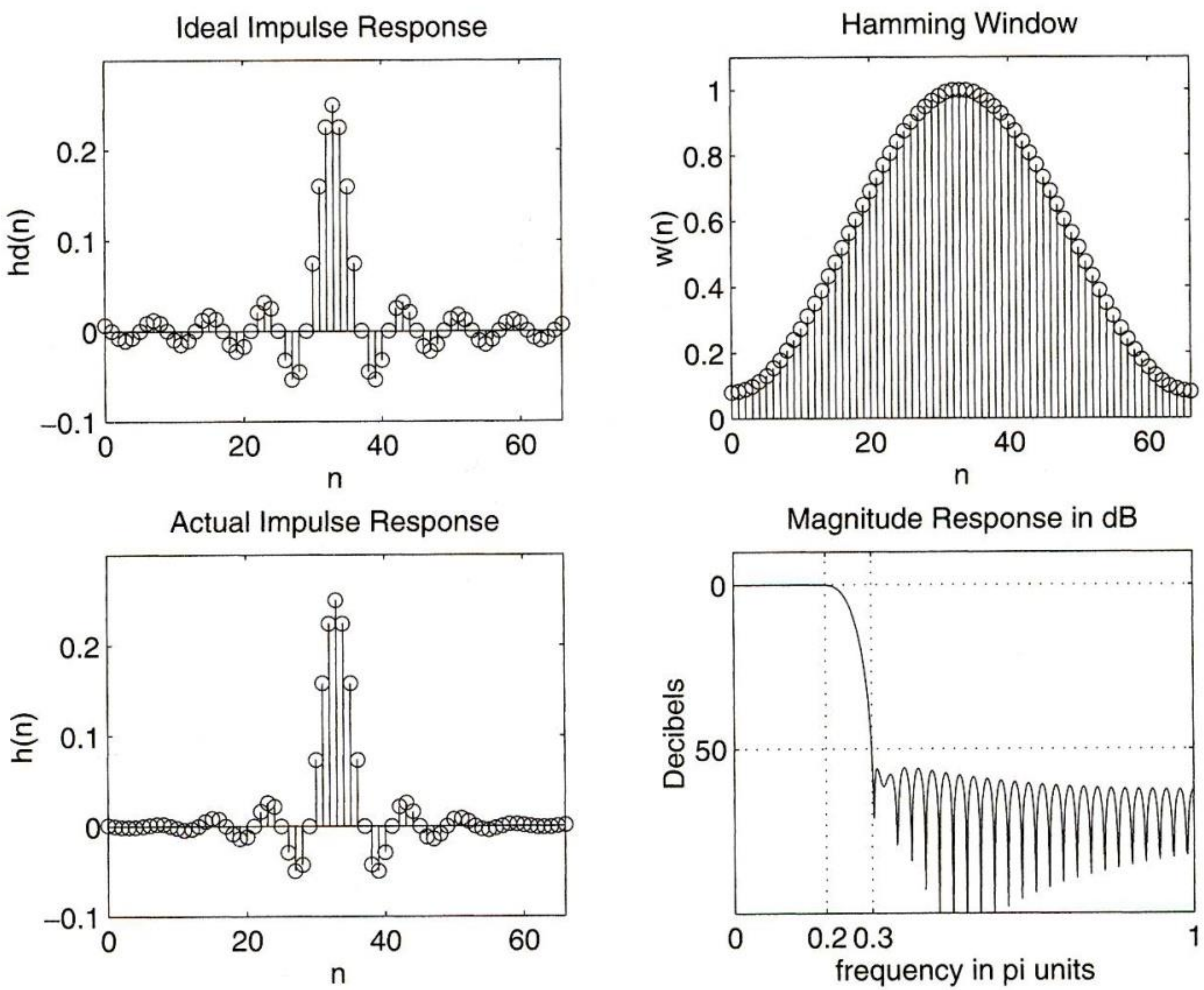
$$\omega_p = 0.2\pi, \quad R_p = 0.25dB$$

$$\omega_s = 0.3\pi, \quad A_s = 50dB$$

Choose an appropriate window function and determine the impulse response and provide a plot of the frequency response of the designed filter.

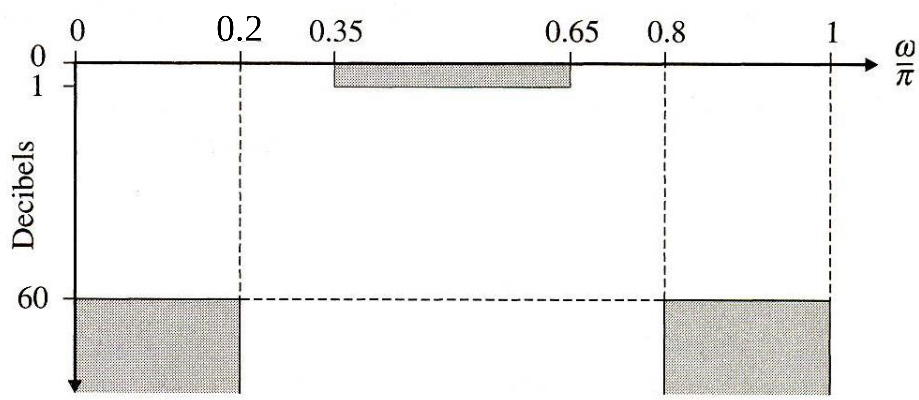
```
>> wp = 0.2*pi; ws = 0.3*pi; tr_width = ws – wp;
>> M = ceil(6.6*pi/tr_width) + 1
M = 67
>> n=[0:1:M-1];
>> wc = (ws+wp)/2; % Ideal LPF cutoff frequency
>> hd = ideal_lp(wc,M); w_ham = (hamming(M))’; h = hd .* w_ham;
>> [db,mag,pha,grd,w] = freqz_m(h,[1]); delta_w = 2*pi/1000;
>> Rp = -(min(db(1:1:wp/delta_w+1))); % Actual Passband Ripple
Rp = 0.0394
>> As = -round(max(db(ws/delta_w+1:1:501))) % Min Stopband Attenuation
As = 52
% plots
```

Design Example

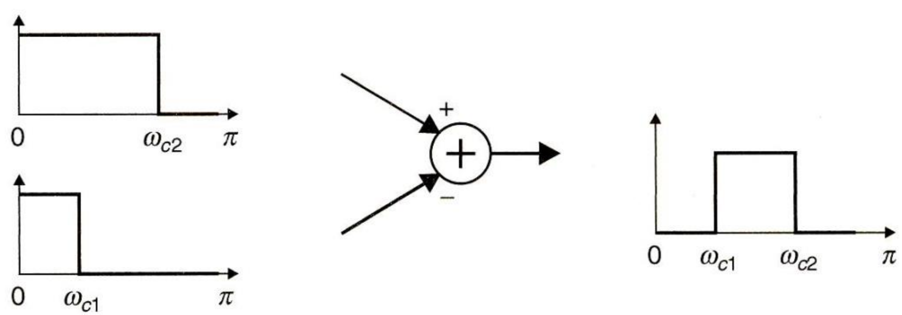


Design Bandpass Filter

Let us design the following digital bandpass filter.



$$\omega_{1s} = 0.2\pi, \quad A_s = 60dB$$
$$\omega_{1p} = 0.35\pi, \quad R_p = 1dB$$
$$\omega_{2p} = 0.65\pi, \quad R_p = 1dB$$
$$\omega_{2s} = 0.8\pi, \quad A_s = 60dB$$

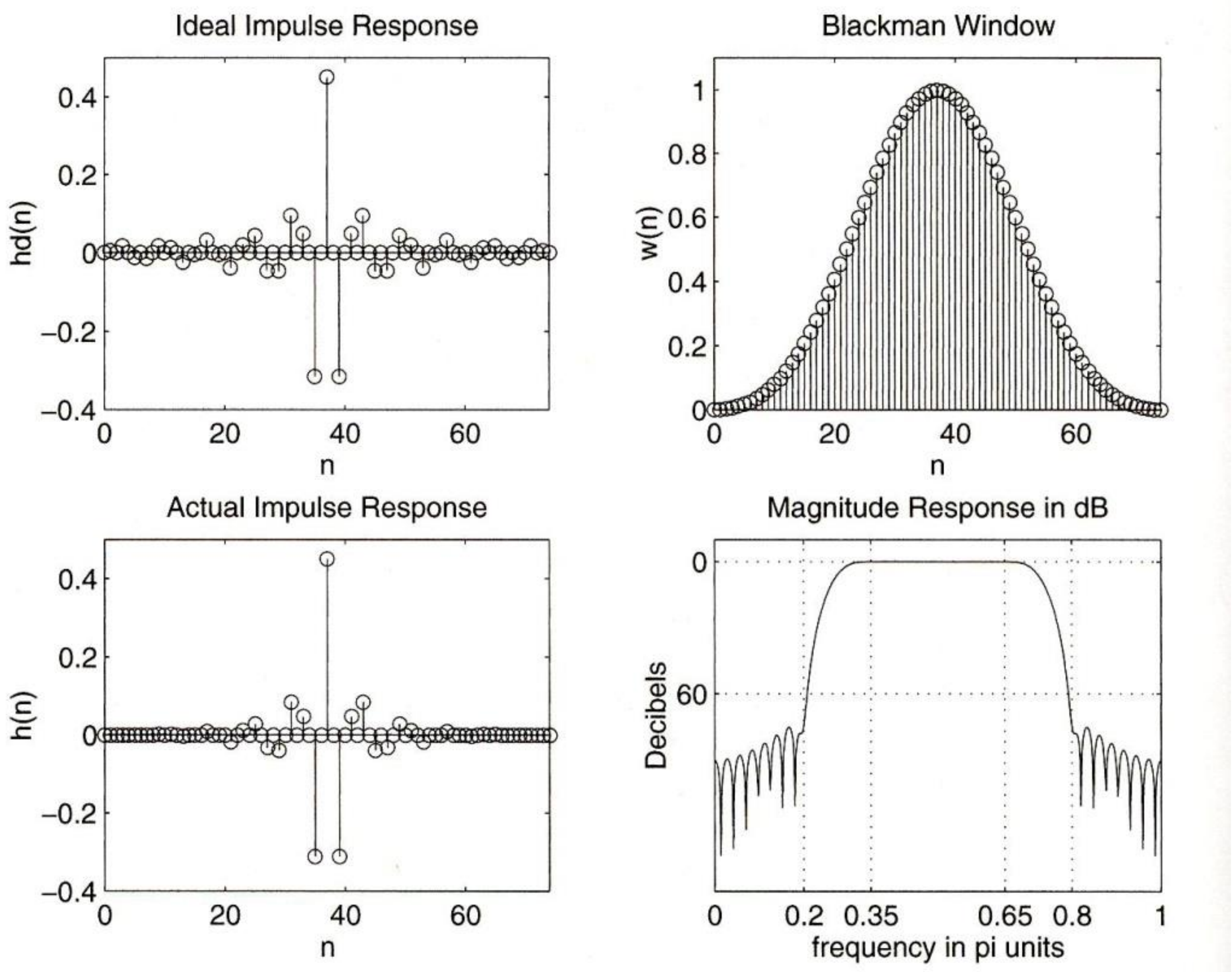


Ideal bandpass filter from two lowpass filters

Design Bandpass Filter

```
>> ws1 = 0.2*pi; wp1 = 0.35*pi; wp2 = 0.65*pi; ws2 = 0.8*pi; As =60;
>> tr_width = min((wp1-ws1),(ws2-wp2)); M = ceil(11*pi/tr_width) + 1
M = 75
>> n=[0:1:M-1]; wc1 = (ws1+wp1)/2; wc2 = (wp2+ws2)/2;
>> hd = ideal_lp(wc2,M) – ideal_lp(wc1,M);
>> w_bla = (blackman(M))’; h = hd .* w_bla;
>> [db,mag,pha,grd,w] = freqz_m(h,[1]); delta_w = 2*pi/1000;
>> Rp = -(min(db(wp1/delta_w+1:1:wp2/delta_w)); % Actual Passband Ripple
Rp = 0.0030
>> As = -round(max(db(ws2/delta_w+1:1:501))) % Min Stopband Attenuation
As = 75
% plots
```

Design Bandpass Filter



Design Hilbert Transformer

Design a length-25 digital Hilbert transformer using a Hann window.

An all-pass filter that imparts a 90° phase shift on the signal at its input.

$$H_d(e^{j\omega}) = \begin{cases} -je^{-j\alpha\omega}, & 0 < \omega < \pi \\ je^{-j\alpha\omega}, & -\pi < \omega < 0 \end{cases}$$

After inverse transformation,

$$h_d(n) = \begin{cases} \dfrac{2}{\pi} \dfrac{\sin^2 \pi(n-\alpha)/2}{n-\alpha}, & n \neq \alpha \\ 0, & n = \alpha \end{cases}$$

```
>> M = 25; alpha = (M-1)/2; n = 0:M-1;
>> hd = (2/pi)*(sin((pi/2)*(n-alpha)).^2)./(n-alpha)); hd(alpha+1)=0;
>> w_han = (hann(M))'; h = hd .* w_han; [Hr,w,P,L] = Hr_Type3(h);
% plots
```

SP Toolbox fir1

- h = fir1(N,wc)

  *N*th-*order* (*N* = *M*-1) *lowpass FIR filter, returns the impulse response in vector* h.

  a Hamming-window based, linear phase design with normalized cutoff frequency

  wc *between* 0 *and* $\boxed{1.}$ π rad/sample

  wc = [wc1 wc2] *bandpass filter with passband cutoffs* wc1 wc2

  wc *a multi-element vector*, a *multiband filter with cutoffs given in* wc

- h = fir1(N,wc,'ftype')

  'high' *a highpass filter with cutoff frequency* Wn

  'stop' *a bandstop filter* Wc = [wc1 wc2], *stop frequency range*

  'DC-1' *the first band of a multiband filter a passband*

  'DC-0' *the first band of a multiband filter a stopband*

- h = fir1(N,wc,'ftype', window), h = fir1(N,wc,window)

  uses the vector window *of length N*+1 (*default: Hamming window*)

*To design **FIR** filters using the Kaiser window,*

```
[N,wc,beta,ftype] = kaiserord(f,m,ripple);
```

Window order N
Cutoff frequency vector wc
Parameter β *in* beta
Filter type ftype

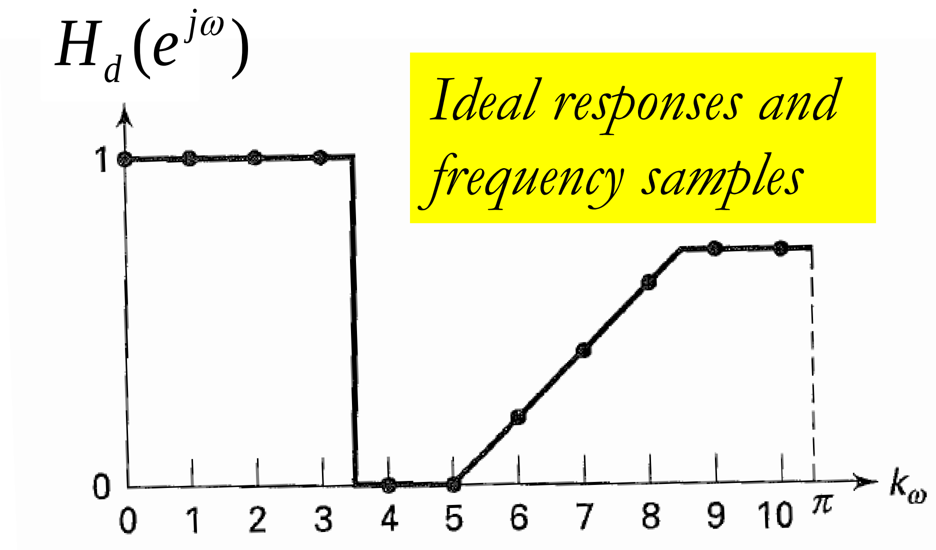
f *a vector of normalized band edges*
m *a vector specifying the desired amplitude on the bands defined by* f
The length of f *is twice the length of* m, *minus* 2; *does not contain* 0 *or* 1
The vector ripple *specifies tolerances in each band (not in decibels)*

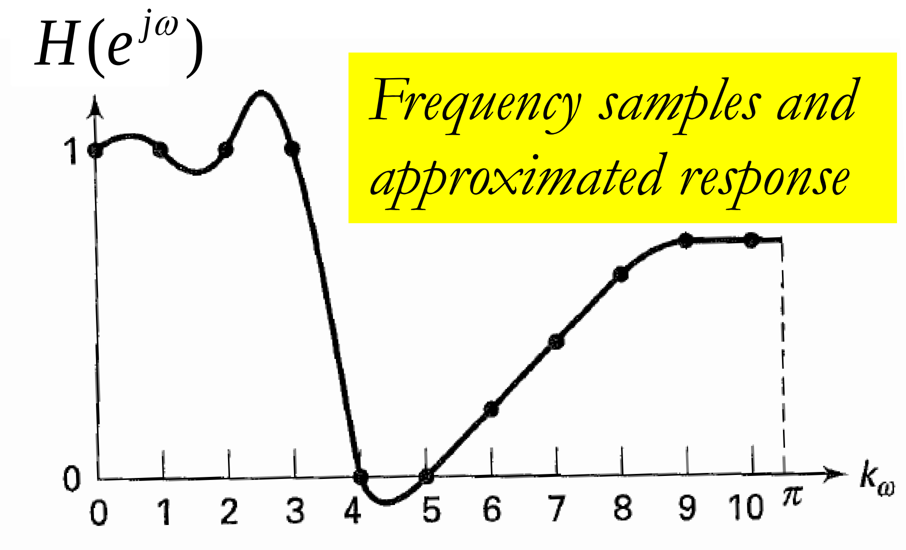
Using the estimated parameters,
Kaiser window array can be computed and used in the fir1 *function.*

Frequency Sampling Design Techniques

Frequency Sampling Design Techniques:

Given the ideal lowpass filter $H_d(e^{j\omega})$, *choose the filter length* M *and then sample* $H_d(e^{j\omega})$ *at equispaced frequencies between* 0 *and* 2π. *The actual response* $H(e^{j\omega})$ *is the interpolation of the samples* $H(k)$.





The system function can be obtained from the samples of the frequency response.

$$H(z)=\sum_{n=0}^{M-1}h(n)z^{-n}=\frac{1-z^{-M}}{M}\sum_{k=0}^{M-1}\frac{H(k)}{1-z^{-1}e^{j2\pi k/M}}$$

$$H(e^{j\omega})=\frac{1-e^{-j\omega M}}{M}\sum_{k=0}^{M-1}\frac{H(k)}{1-e^{-j\omega}e^{j2\pi k/M}}$$

$$H(k)=H\left(e^{j2\pi k/M}\right)=\begin{cases}H(0), & k=0\\ H^*(M-k), & k=1,\ldots,M-1\end{cases}$$

(M-point *DFT*)

For a linear-phase **FIR** *filter,* *(impulse response)*

$$h(n)=\pm h(M-1-n),\quad n=0,1,\ldots,M-1$$

+	*Type*-1 & *Type*-2
−	*Type*-3 & *Type*-4

Example

Design an **FIR** *filter using the frequency sampling approach.*

$$\omega_p = 0.2\pi, \quad R_p = 0.25dB$$

$$\omega_s = 0.3\pi, \quad A_s = 50dB$$

Naïve Design Method $H(k) = H_d(e^{j2\pi k/M}), \quad k = 0, \ldots, M-1$

No constraints on the approximation error

$$M = 20 \qquad \omega_p = 0.2\pi = \frac{2\pi}{20}2, \quad \omega_s = 0.3\pi = \frac{2\pi}{20}3$$

3 *samples* $[0 \le \omega \le \omega_p]$, 7 *samples* $[\omega_s \le \omega \le \pi]$

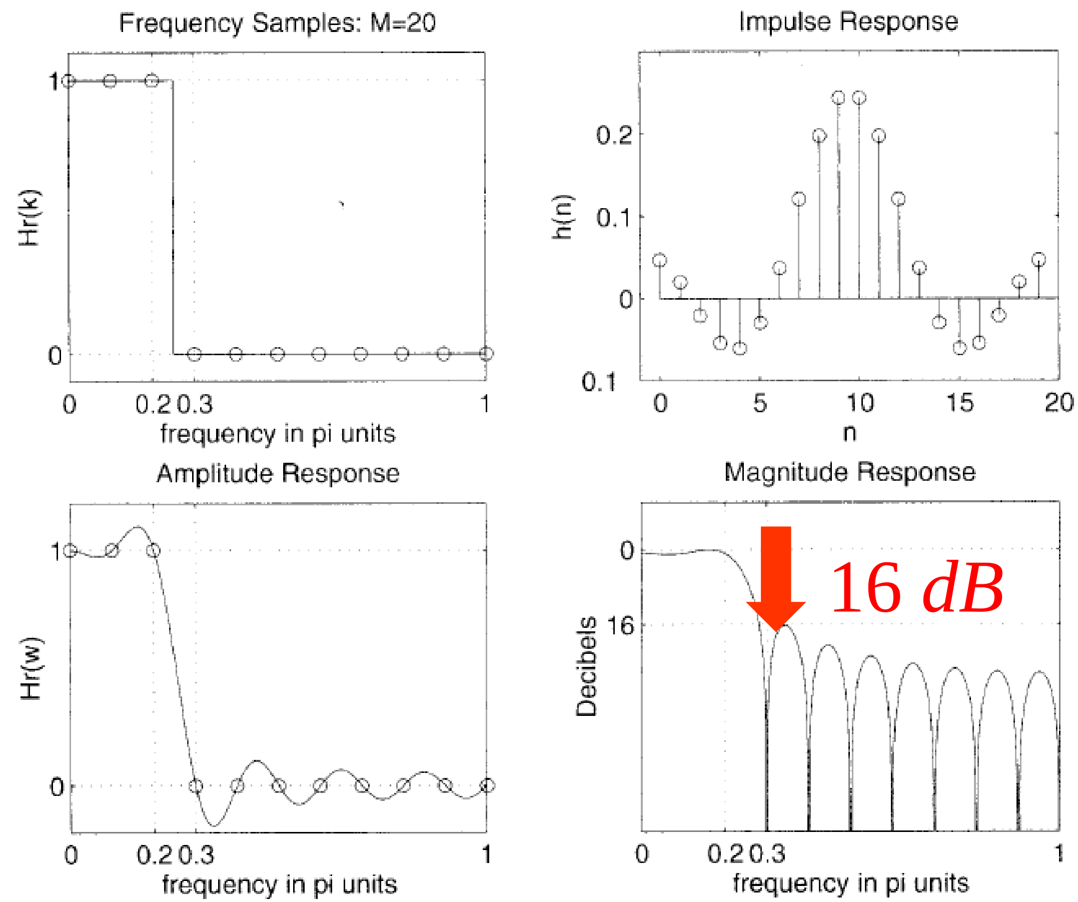
$$H_r(k) = \left[1, \quad 1, \quad 1, \quad \underbrace{0, \ldots, 0,}_{15 \ zeros} \quad 1, \quad 1 \right]$$

$$M = 20$$

$$\alpha = \frac{20-1}{2} = 9.5$$

$$\angle H(k) = \begin{cases} -9.5\dfrac{2\pi}{20}k = -0.95\pi k, & 0 \le k \le 9 \\ 0.95\pi(20-k), & 10 \le k \le 19 \end{cases}$$

```
>> M = 20; alpha = (M-1)/2; l = 0:M-1; wl = (2*pi/M)*l;
>> Hrs = [1,1,1,zeros(1,15),1,1]; %Ideal Amp Res sampled
>> Hdr = [1,1,0,0]; wd1 = [0,0.25,0.25,1]; %Ideal Amp Res for plotting
>> k1 = 0:floor((M-1)/2); k2 = floor((M-1)/2)+1:M-1;
>> angH = [-alpha*(2*pi)/M*k1, alpha*(2*pi)/M*(M-k2)];
>> H = Hrs.*exp(j*angH); h = real(ifft(H,M));
>> [db,mag,pha,grd,w] = freqz_m(h,1); [Hr,ww,a,L] = Hr_Type2(h);
% plots
```



Optimum Design Method (*to have more attenuation*)

Increase **M** (=40)
Make the transition band samples free – *we vary their values to obtain the largest attenuation for the given* **M** *and the transition width.*

$$\omega_1 = \frac{2\pi}{40}, \quad k = 5, k = 40 - 5 = 35$$

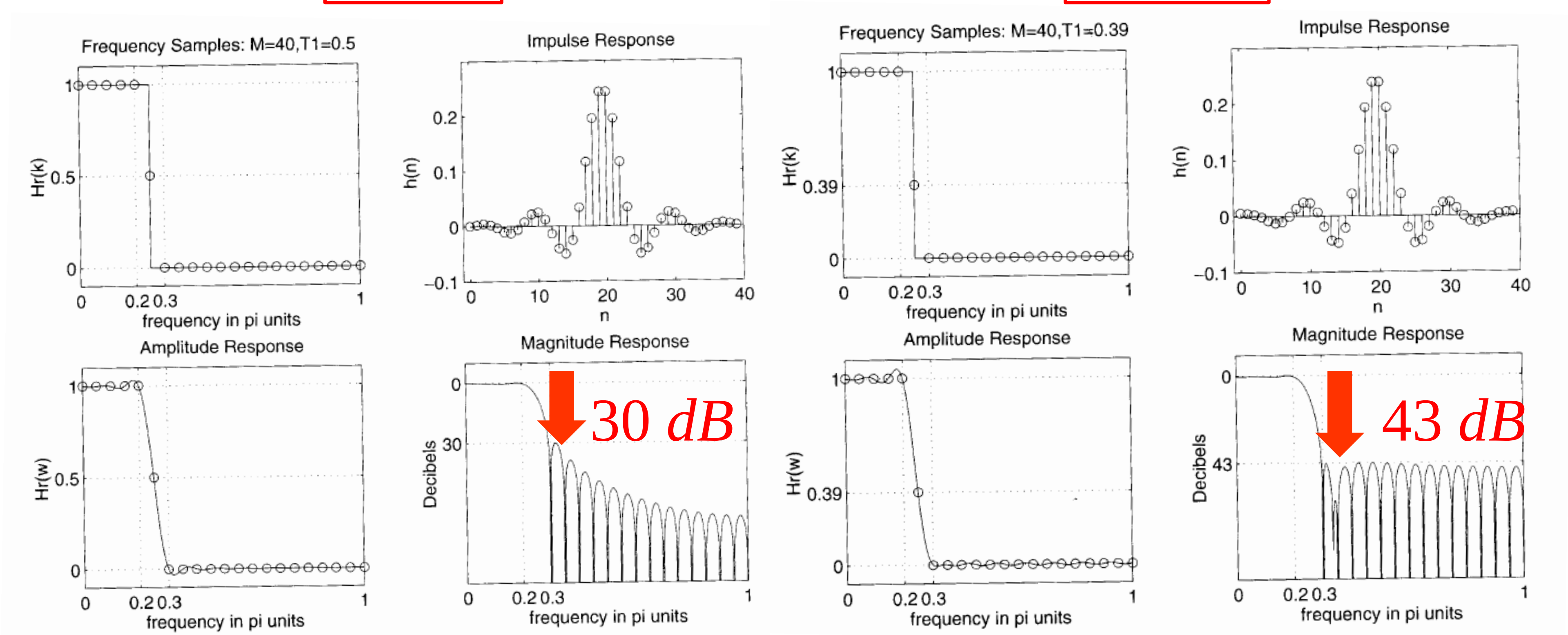
$$H_r(k) = \left[1,1,1,1,1, T_1, \underbrace{0, \ldots, 0}_{29 \ \ zeros}, T_1, 1,1,1,1 \right]$$

$$\alpha = \frac{40-1}{2} = 19.5$$

$$\angle H(k) = \begin{cases} -19.5 \dfrac{2\pi}{40} k = -0.975\pi k, & 0 \le k \le 19 \\ 0.975\pi(40-k), & 20 \le k \le 39 \end{cases}$$

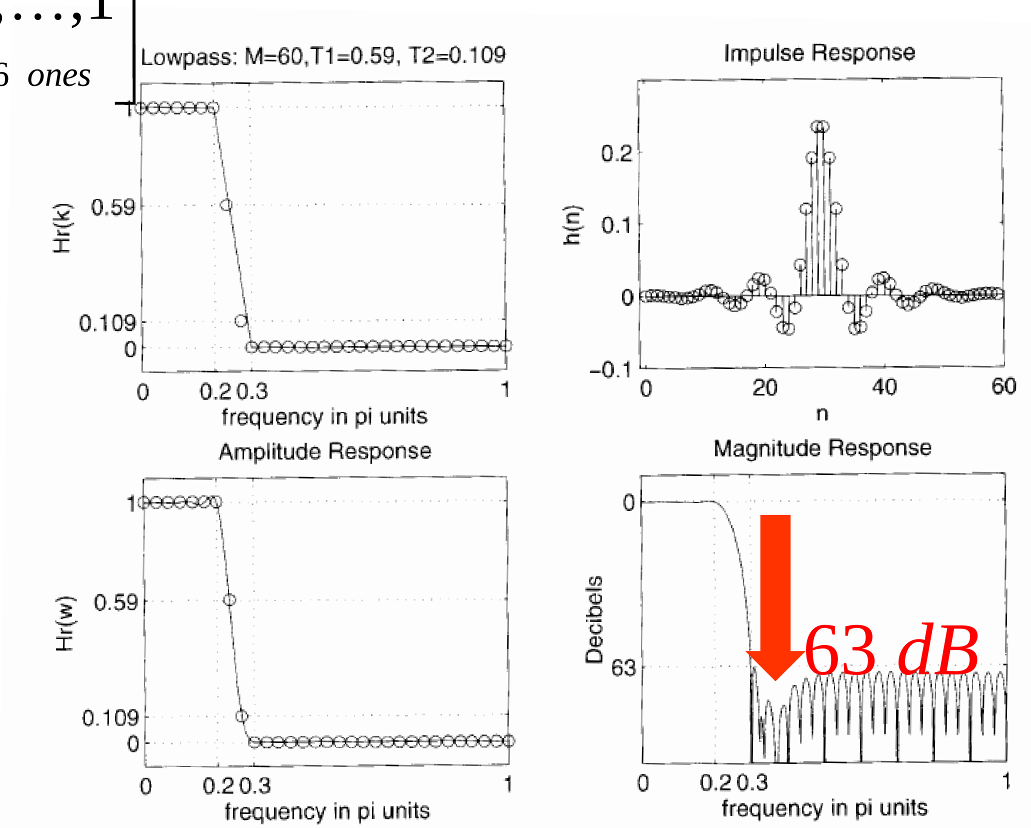
```
% T1=0.5
>> M = 40; alpha = (M-1)/2;
>> Hrs = [ones(1,5), 0.5, zeros(1,29), 0.5, ones(1,4)];
>> k1 = 0:floor((M-1)/2); k2 = floor((M-1)/2)+1:M-1;
>> angH = [-alpha*(2*pi)/M*k1, alpha*(2*pi)/M*(M-k2)];
>> H = Hrs.*exp(j*angH); h = real(ifft(H,M));
```

$T_1 = 0.5$ $\qquad\qquad$ $T_1 = 0.39$

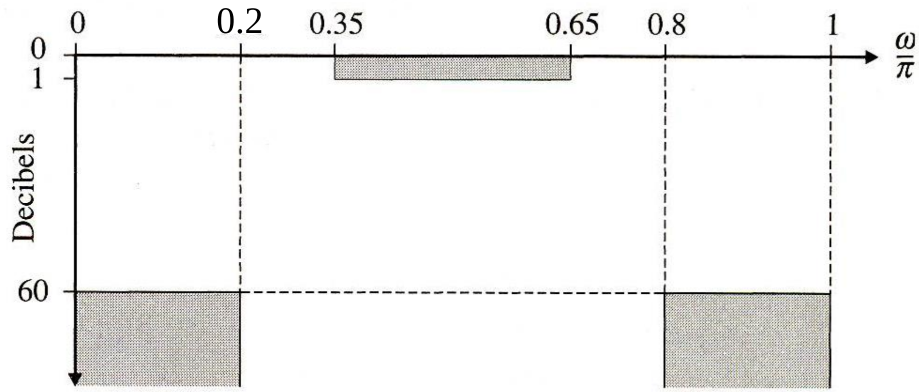


```
>> M = 60; alpha = (M-1)/2; l = 0:M-1; wl = (2*pi/M)*l;
>> Hrs = [ones(1,7), 0.5925, 0.1099, zeros(1,43), 0.1099, 0.5925, ones(1,6)];
>> Hdr = [1, 1, 0, 0]; wdl = [0, 0.2, 0.3, 1];
>> k1 = 0:floor((M-1)/2); k2 = floor((M-1)/2)+1:M-1;
>> angH = [-alpha*(2*pi)/M*k1, alpha*(2*pi)/M*(M-k2)];
>> H = Hrs.*exp(j*angH); h = real(ifft(H,M));
>> [db,mag,pha,grd,w] = freqz_m(h,1);  [Hr,ww,a,L] = Hr_Type2(h);
```

$$H_r(k) = \left[\underbrace{1,\ldots,1}_{7\ \ ones}, T_1, T_2, \underbrace{0,\ldots,0}_{43\ \ zeros}, T_2, T_1, \underbrace{1,\ldots,1}_{6\ \ ones}\right]$$



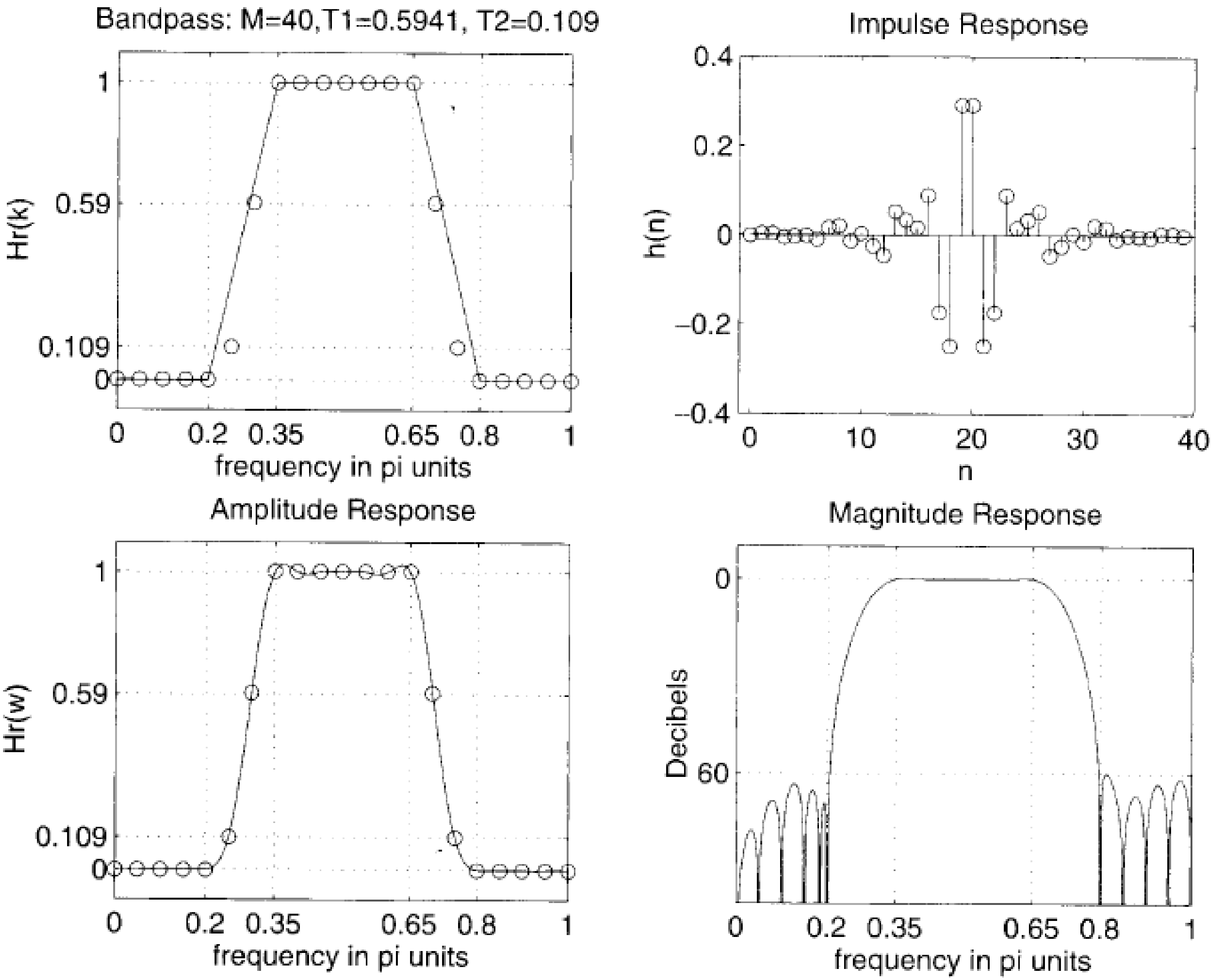
Example



$$\omega_{1s} = 0.2\pi, \quad A_s = 60dB$$
$$\omega_{1p} = 0.35\pi, \quad R_p = 1dB$$
$$\omega_{2p} = 0.65\pi, \quad R_p = 1dB$$
$$\omega_{2s} = 0.8\pi, \quad A_s = 60dB$$

$$H_r(k) = \left[\underbrace{0,\ldots,0}_{5\ \ zeros}, T_1, T_2, \underbrace{1,\ldots,1}_{7\ \ ones}, T_2, T_1, \underbrace{0,\ldots,0}_{9\ \ zeros}, T_1, T_2, \underbrace{1,\ldots,1}_{7\ \ ones}, T_2, T_1, \underbrace{0,\ldots,0}_{4\ \ zeros} \right]$$

```
>> M = 40; alpha = (M-1)/2; l = 0:M-1; wl = (2*pi/M)*l;
>> T1 = 0.109021; T2 = 0.59417456;
>> Hrs = [zeros(1,5), T1, T2, ones(1,7), T2, T1, zeros(1,9), T1, T2, ones(1,7), T2, T1,
zeros(1,4) ];
>> Hdr = [0, 0, 1, 1, 0, 0]; wdl = [0, 0.2, 0.35, 0.65, 0.8, 1];
>> k1 = 0:floor((M-1)/2); k2 = floor((M-1)/2)+1:M-1;
>> angH = [-alpha*(2*pi)/M*k1, alpha*(2*pi)/M*(M-k2)];
>> H = Hrs.*exp(j*angH); h = real(ifft(H,M));
>> [db,mag,pha,grd,w] = freqz_m(h,1);  [Hr,ww,a,L] = Hr_Type2(h);
```


Bandpass: M=40,T1=0.5941, T2=0.109
1
0.59
0.109
0
Hr(k)
0
0.2
0.35
0.65
0.8
1
frequency in pi units
Impulse Response
0.4
0.2
0
−0.2
−0.4
h(n)
0
10
20
30
40
n
Amplitude Response
1
0.59
0.109
0
Hr(w)
0
0.2
0.35
0.65
0.8
1
frequency in pi units
Magnitude Response
0
60
Decibels
0
0.2
0.35
0.65
0.8
1
frequency in pi units

SP Toolbox fir2 (frequency sampling + window)

- h = fir2(N,f,m)

  *N*th-*order* (*N* = *M*-1) *lowpass* ***FIR*** *filter, returns the impulse response in vector* h.

  The desired magnitude response of the filter is supplied in vectors f *and* m.

  f *between* 0 *and* 1.

  m *samples of the desired magnitude response at the values specified in* f.

  The desired frequency response is then interpolated onto a dense, evenly spaced grid of 512. *naïve design method*

- h = fir2(N,f,m,window)

  uses the vector window *of length* *N*+1 (*default: Hamming window*)

- h = fir2(N,f,m,npt), h = fir2(N,f,m,npt,window)

  specifies the number of points, npt, *for the grid onto which* fir2 *interpolates the frequency response.* (*default:* 512)

  Does not implement the classic optimum frequency sampling method.

Optimal Equiripple Design Techniques

Optimal Equiripple Design Technique:

The last two techniques

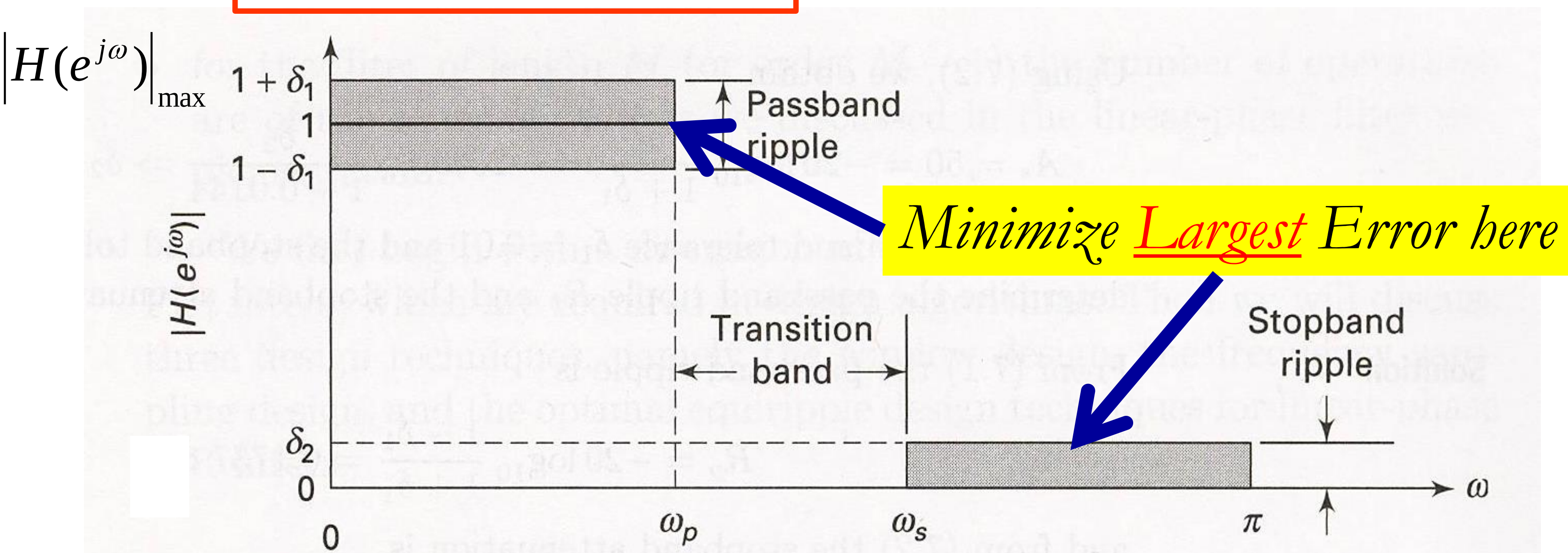
- *Cannot specify the band frequencies* ω_p *and* ω_s *precisely in the design.*
- *Cannot specify both* δ_1 *and* δ_2 *ripple factors simultaneously.*

$$\boxed{\delta_1 = \delta_2,} \qquad \boxed{\delta_2}$$

- *The approximation error is not uniformly distributed over the band intervals.*

$$\boxed{\min_{over\ \ coeff.} \left[\max_{\omega \in S} \left| E(\omega) \right| \right]}$$

$$\boxed{\begin{aligned} &1-\delta_1 \le \left|H(e^{j\omega})\right| \le 1+\delta_1 \\ &0 \le \left|H(e^{j\omega})\right| \le \delta_2 \end{aligned}} \Rightarrow W(\omega) = \begin{cases} \delta_2 / \delta_1, & \omega \in [0, \omega_p] \\ 1, & \omega \in [\omega_s, \pi] \end{cases}$$

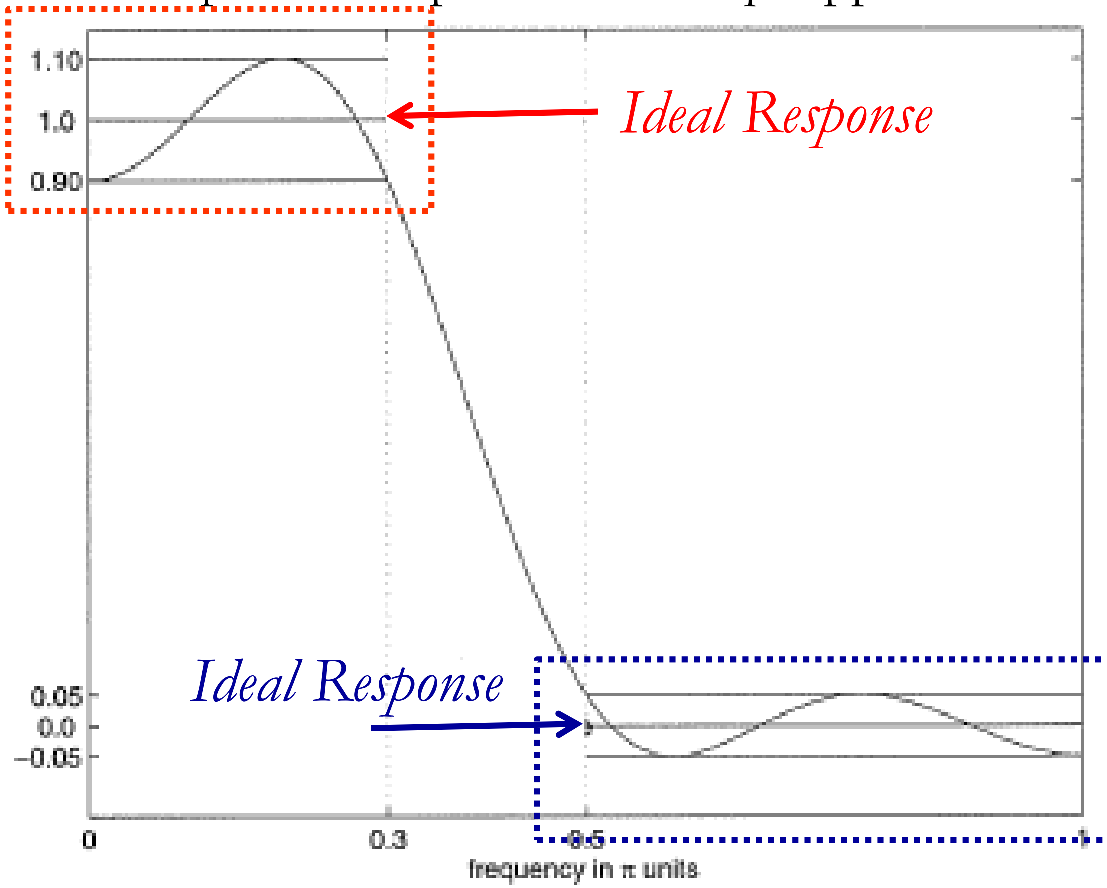


$$E(\omega) \triangleq W(\omega)\left[H_{dr}(\omega) - H_r(\omega)\right],$$

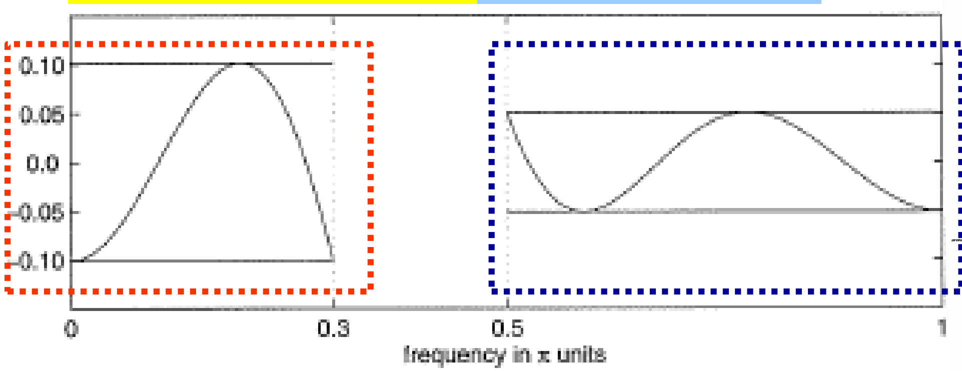
$$\omega \in S \triangleq \left[0, \omega_p\right] \cup \left[\omega_s, \pi\right]$$

Amplitude Response of an Equiripple Filter

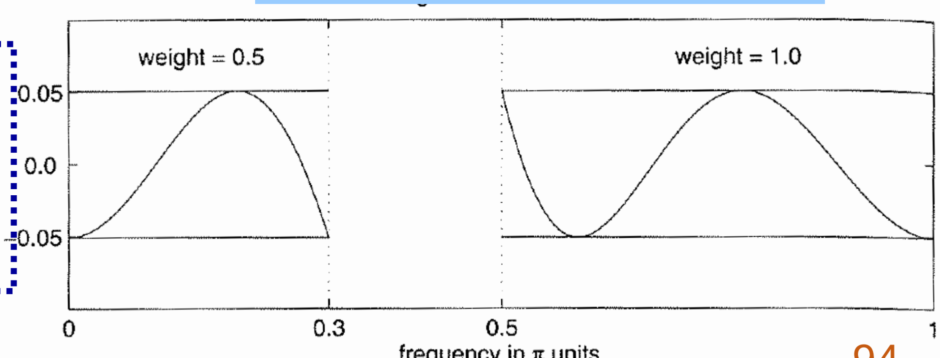
$H_r(\omega)$



$[H_{dr}(\omega) - H_r(\omega)]$ Error Function



Weighted Error Function



Type-1 $H_r(\omega) = \boxed{1} \cdot \boxed{\sum_{n=0}^{(M-1)/2} a(n)\cos\omega n}$

Type-2 $H_r(\omega) = \boxed{\cos(\omega/2)} \boxed{\sum_{n=0}^{(M-1)/2} \tilde{b}(n)\cos\omega n}$

Type-3 $H_r(\omega) = \boxed{\sin(\omega)} \boxed{\sum_{n=0}^{(M-3)/2} \tilde{c}(n)\cos\omega n}$

Type-4 $H_r(\omega) = \boxed{\sin(\omega/2)} \boxed{\sum_{n=0}^{(M-1)/2} \tilde{d}(n)\cos\omega n}$

$$H_r(\omega) = \boxed{Q(\omega)} P(\omega)$$

Common Form

$$P(\omega) = \boxed{\sum_{n=0}^{L} \alpha(n)\cos\omega n}$$

$$E(\omega) = W(\omega)\left[H_{dr}(\omega) - Q(\omega)P(\omega)\right]$$

$$= W(\omega)Q(\omega)\left[\frac{H_{dr}(\omega)}{Q(\omega)} - P(\omega)\right], \quad \omega \in S$$

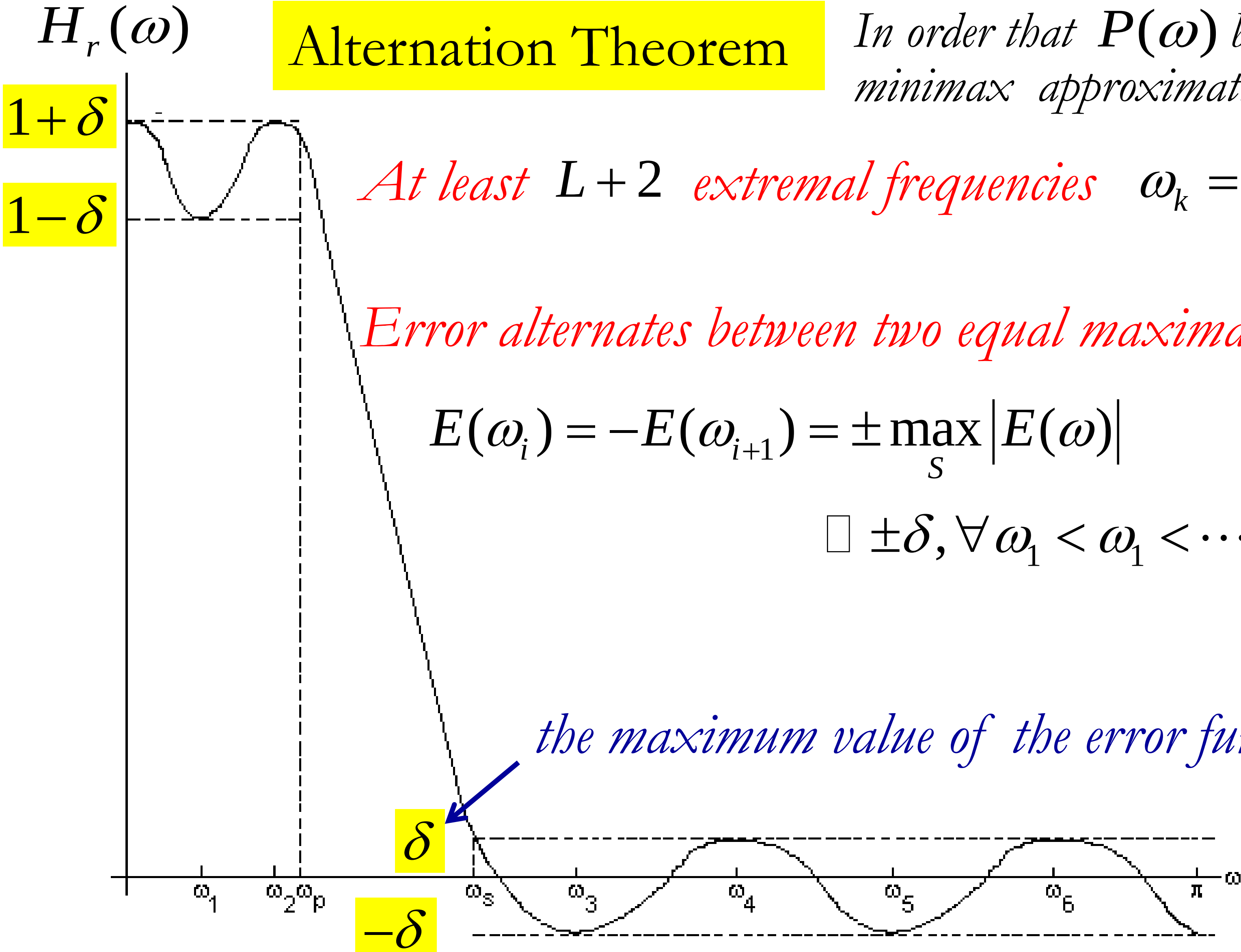
$$\hat{W}(\omega) \triangleq W(\omega)Q(\omega), \quad \hat{H}_{dr}(\omega) \triangleq \frac{H_{dr}(\omega)}{Q(\omega)}$$

$$\boxed{E(\omega) = \hat{W}(\omega)\left[\hat{H}_{dr}(\omega) - P(\omega)\right], \quad \omega \in S}$$

A common form for all four cases!

Alternation Theorem

In order that $\mathbf{P}(\omega)$ *be the unique minimax approximation to* $\mathbf{H}_{dr}(\omega)$



At least $L+2$ *extremal frequencies* $\omega_k = 1, 2, \cdots, L+2$

Error alternates between two equal maxima and minima

$$E(\omega_i) = -E(\omega_{i+1}) = \pm \max_S \left| E(\omega) \right|$$

$$\square \ \pm\delta, \forall\, \omega_1 < \omega_1 < \cdots < \omega_{L+2} \in S$$

the maximum value of the error function $E(\omega)$

Find extremal frequencies ω_k, *choose* $\alpha(n)$ *to minimize error at* ω_k.

$$P(\omega) = \sum_{n=0}^{L} \alpha(n) \cos \omega n$$

$$\cos(2\omega) = 2\cos^2(\omega) - 1$$

$$\cos(3\omega) = 4\cos^3(\omega) - 3\cos(\omega)$$

$$\vdots \quad = \quad \vdots$$

$$P(\omega) = \sum_{n=0}^{L} \alpha'(n) \cos^n \omega$$

L-th order polynomial

can have at most $L-1$ *local maxima and minima in the open interval* $0 < \omega < \pi.$

$$\omega = 0, \quad \omega = \pi \quad \omega = \omega_p, \quad \omega = \omega_s$$

at most $L+3$ *extremal frequencies in* $E(\omega)$ *for the unique, best approximation of the ideal lowpass filter.*

Chebyshev Polynomial

$$\boxed{\cos(n\omega) = T_n(\cos\omega)}$$

$$\begin{aligned}
\cos\big((n+1)\omega\big) &= \cos(n\omega + \omega) \\
&= \cos(n\omega)\cos\omega - \sin(n\omega)\sin\omega \\
&= \cos(n\omega)\cos\omega + \cos(n\omega)\cos\omega - \cos(n\omega)\cos\omega - \sin(n\omega)\sin\omega \\
&= 2\cos(n\omega)\cos\omega - \big(\cos(n\omega)\cos\omega + \sin(n\omega)\sin\omega\big) \\
&= 2\cos(n\omega)\cos\omega - \cos(n\omega - \omega) \\
&= 2\cos\omega\cos(n\omega) - \cos\big((n-1)\omega\big)
\end{aligned}$$

$$\boxed{\begin{aligned}
&= 2\cos\omega T_n(\cos\omega) - T_{n-1}(\cos\omega) \\
&= T_{n+1}(\cos\omega)
\end{aligned}}$$

$$\cos(0\omega) = 1 = T_0(\cos\omega)$$

$$\cos(1\omega) = \cos\omega = T_1(\cos\omega)$$

$$\begin{aligned}
\cos(2\omega) &= 2\cos\omega T_1(\cos\omega) - T_0(\cos\omega) \\
&= 2\cos^2\omega - 1
\end{aligned}$$

$$\begin{aligned}
\cos(3\omega) &= 2\cos\omega T_2(\cos\omega) - T_1(\cos\omega) \\
&= 2\cos\omega\left(2\cos^2\omega - 1\right) - \cos\omega \\
&= 4\cos^3\omega - 3\cos\omega
\end{aligned}$$

$$
\boxed{
\begin{aligned}
&(-1)^k \delta = \hat{W}(\omega_k)\left[\hat{H}_{dr}(\omega_k) - H_r(\omega_k)\right], \quad k = 0,1,\cdots,L+1 \\
&H_r(\omega_k) + \frac{(-1)^k \delta}{\hat{W}(\omega_k)} = \hat{H}_{dr}(\omega_k), \quad k = 0,1,\cdots,L+1
\end{aligned}
}
$$

$$
\begin{bmatrix}
1 & \cos\omega_0 & \cos 2\omega_0 & \cdots & \cos(L\omega_0) & \dfrac{1}{\hat{W}(\omega_0)} \\
1 & \cos\omega_1 & \cos 2\omega_1 & \cdots & \cos(L\omega_1) & \dfrac{-1}{\hat{W}(\omega_1)} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & \cos\omega_{L+1} & \cos 2\omega_{L+1} & \cdots & \cos(L\omega_{L+1}) & \dfrac{(-1)^{L+1}}{W(\omega_{L+2})}
\end{bmatrix}
\boxed{\begin{bmatrix}
\alpha(0) \\ \alpha(1) \\ \vdots \\ \vdots \\ \alpha(L) \\ \delta
\end{bmatrix}}
=
\begin{bmatrix}
\hat{H}_{dr}(\omega_0) \\ \hat{H}_{dr}(\omega_1) \\ \vdots \\ \vdots \\ \vdots \\ \hat{H}_{dr}(\omega_{L+1})
\end{bmatrix}
$$

The Parks-McClellan Algorithm

1. Make an initial guess for the L+2 extremal frequencies:

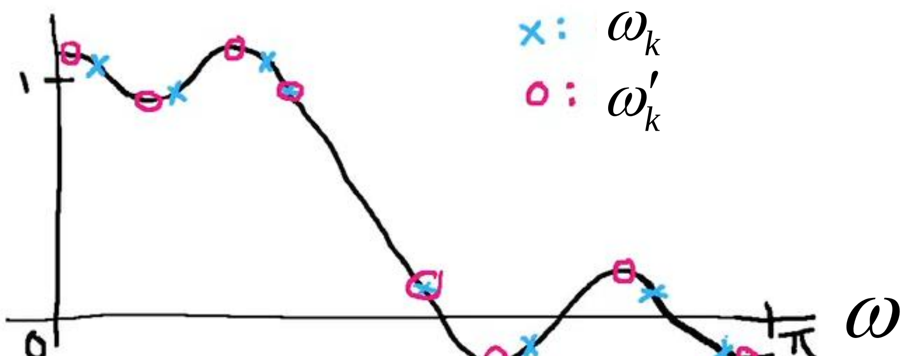
$$\omega_k = \omega_0, \omega_1, \ldots, \omega_{L+1}$$

2. Find $\alpha(n), \delta$ to satisfy the alternation criterion.

3. Compute the true alternation frequencies ω_k' for $\alpha(n)$.

$$P(\omega) = \sum_{n=0}^{L} \alpha(n) \cos \omega n$$

4. If $\omega_k' \neq \omega_k$, set $\omega_k = \omega_k'$, and go back to Step 2.



The Parks-McClellan Algorithm (MATLAB)

designs N-th order (M=N+1) FIR digital filter

```
[h] = firpm(N,f,m)
[h] = firpm(N,f,m,weights)
[h] = firpm(N,f,m,ftype)
[h] = firpm(N,f,m,weights,ftype)
```

h: *filter coefficients or impulse response*

f: *band-edge frequencies*

m: *desired magnitude response*

weights: *weighting function in each band*

ftype: *'differentiator' or 'hilbert'*

[N,f0,m0,weights] = firpmord(f,m,delta); (SP Toolbox)

estimates the filter order N and other parameters

f0: *normalized frequency band edges*, f: *normalized band edges*

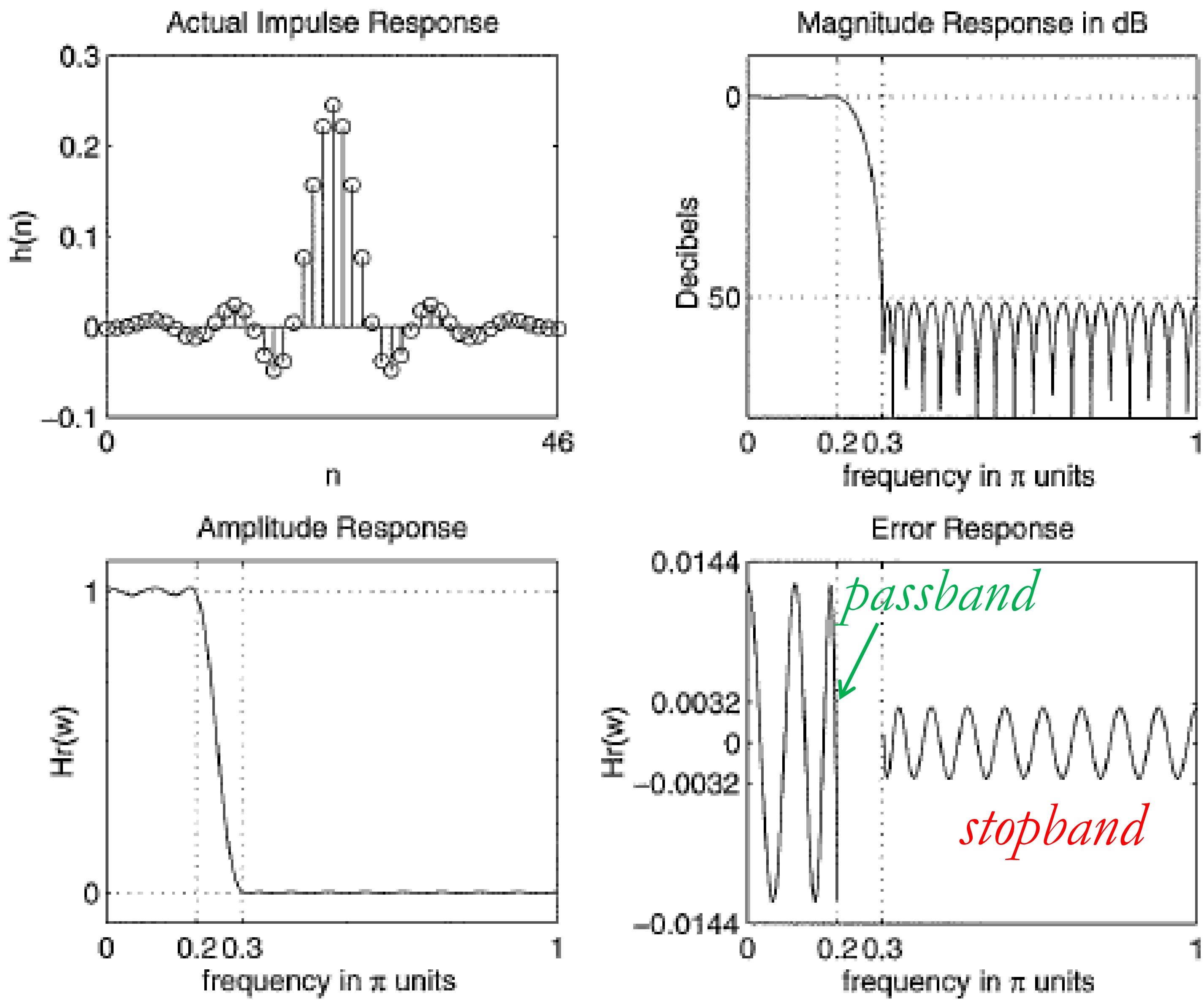
m0: *amplitude response*, m: *desired amplitude on the bands*

delta: *tolerances in each band*

Lowpass filter design

```
>> wp = 0.2*pi; ws = 0.3*pi; Rp = 0.25; As = 50;
>> [delta1,delta2] = db2delta(Rp,AS);
>> [N,f,m,weights] = firpmord([wp,ws]/pi,[1,0],[delta1,delta2]);
>> h = firpm(N,f,m,weights);
>> [db,mag,pha,grd,w] = freqz_m(h,[1]);
>> delta_w = 2*pi/1000; wsi=ws/delta_w+1; wpi = wp/delta_w;
>> Asd = -max(db(wsi:1:501))
Asd = 47.8404
>> N = N+1
N = 43
>> h = firpm(N,f,m,weights); [db,mag,pha,grd,w] = freqz_m(h,[1]);
>> Asd = -max(db(wsi:1:501))
Asd = 48.2131
>> N = N+1
N = 44
>> h = firpm(N,f,m,weights); [db,mag,pha,grd,w] = freqz_m(h,[1]);
>> Asd = -max(db(wsi:1:501))
Asd = 48.8689
.
.
                    Asd = 51.0857
>> M = N+1
M =47
```

Stop this iterative procedure when the computed stopband attenuation exceeds the given A_s.


Actual Impulse Response
h(n)
0.3
0.2
0.1
0
−0.1
0
46
n
Magnitude Response in dB
Decibels
0
50
0
0.2
0.3
1
frequency in π units
Amplitude Response
Hr(w)
1
0
0
0.2
0.3
1
frequency in π units
Error Response
Hr(w)
0.0144
0.0032
0
−0.0032
−0.0144
0
0.2
0.3
1
frequency in π units
passband
stopband

Digital Signal Processors

How DSPs (Digital Signal Processors) are different from other types of microprocessors

	Traditional microprocessors	DSPs
	Data Manipulation	Math Calculation
Typical Applications	Word processing, database management, spread sheets, operating systems, etc.	Digital Signal Processing, motion control, scientific and engineering simulations, etc.
Main Operations	data movement ($A \rightarrow B$) value testing (*If* $A=B$ *then*…)	addition ($A+B=C$) multiplication ($AXB=C$)

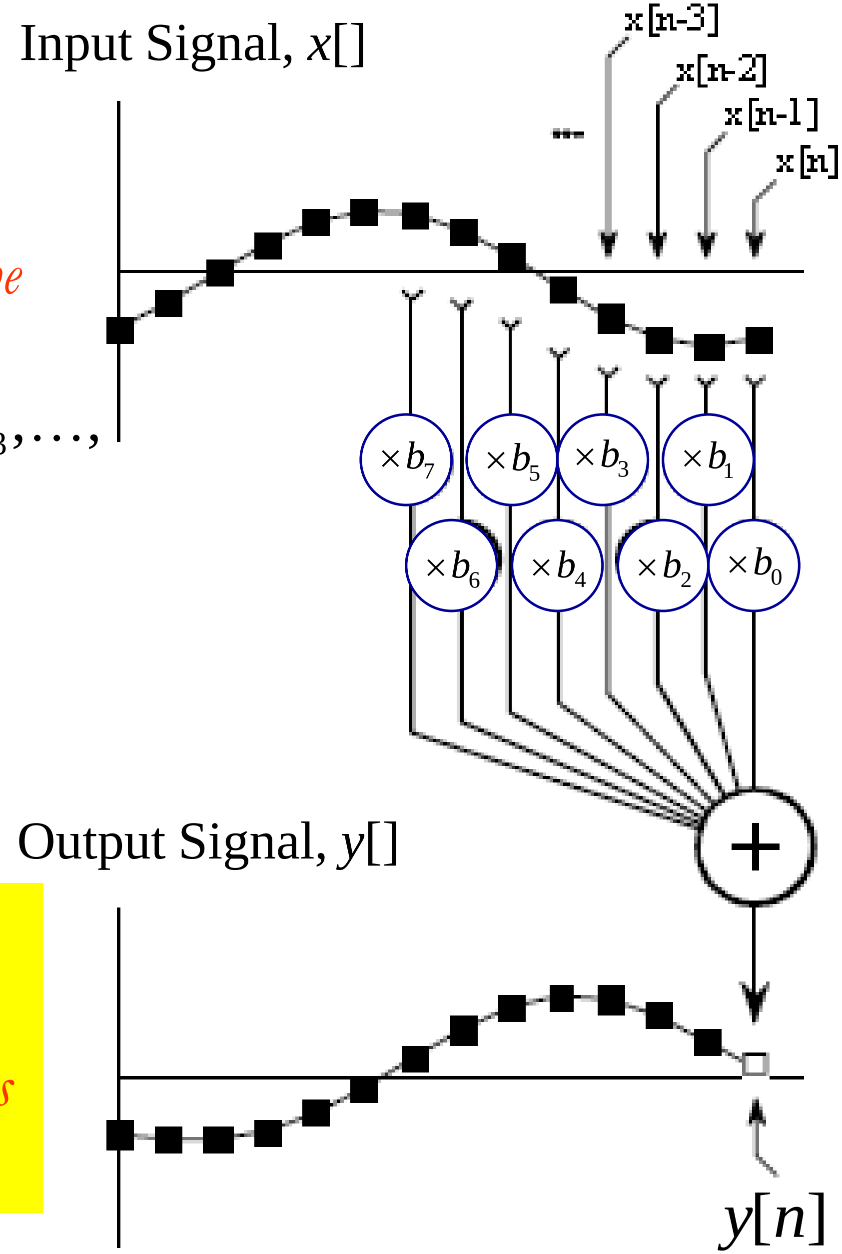
Storing and sorting information

FIR digital filter

Each sample in the *output signal*, $y[n]$, is found by multiplying *samples from the input signal*, $x[n], x[n-1], x[n-2], \ldots,$ by the *filter kernel coefficients*, $b_0, b_1, b_2, b_3, \ldots,$ and summing the products.

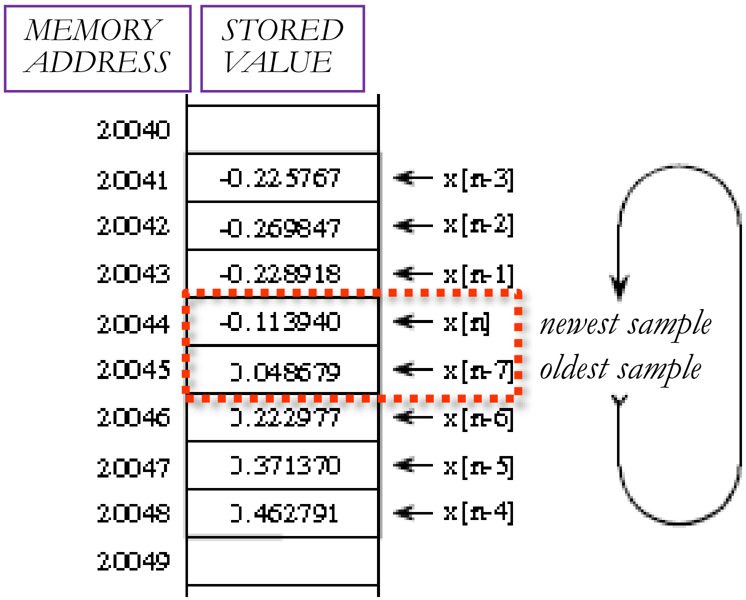
$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3] + b_4 x[n-4] + \cdots$$

The execution speed of most DSP algorithms is limited almost completely by the number of multiplications and additions required.

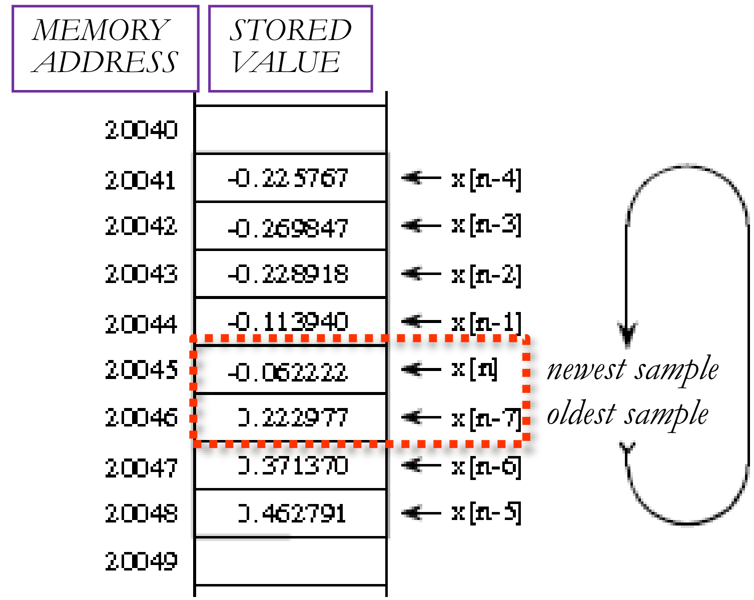


Real-time processing: an eight sample circular buffer

- store the most recent values of a continually updated signal.



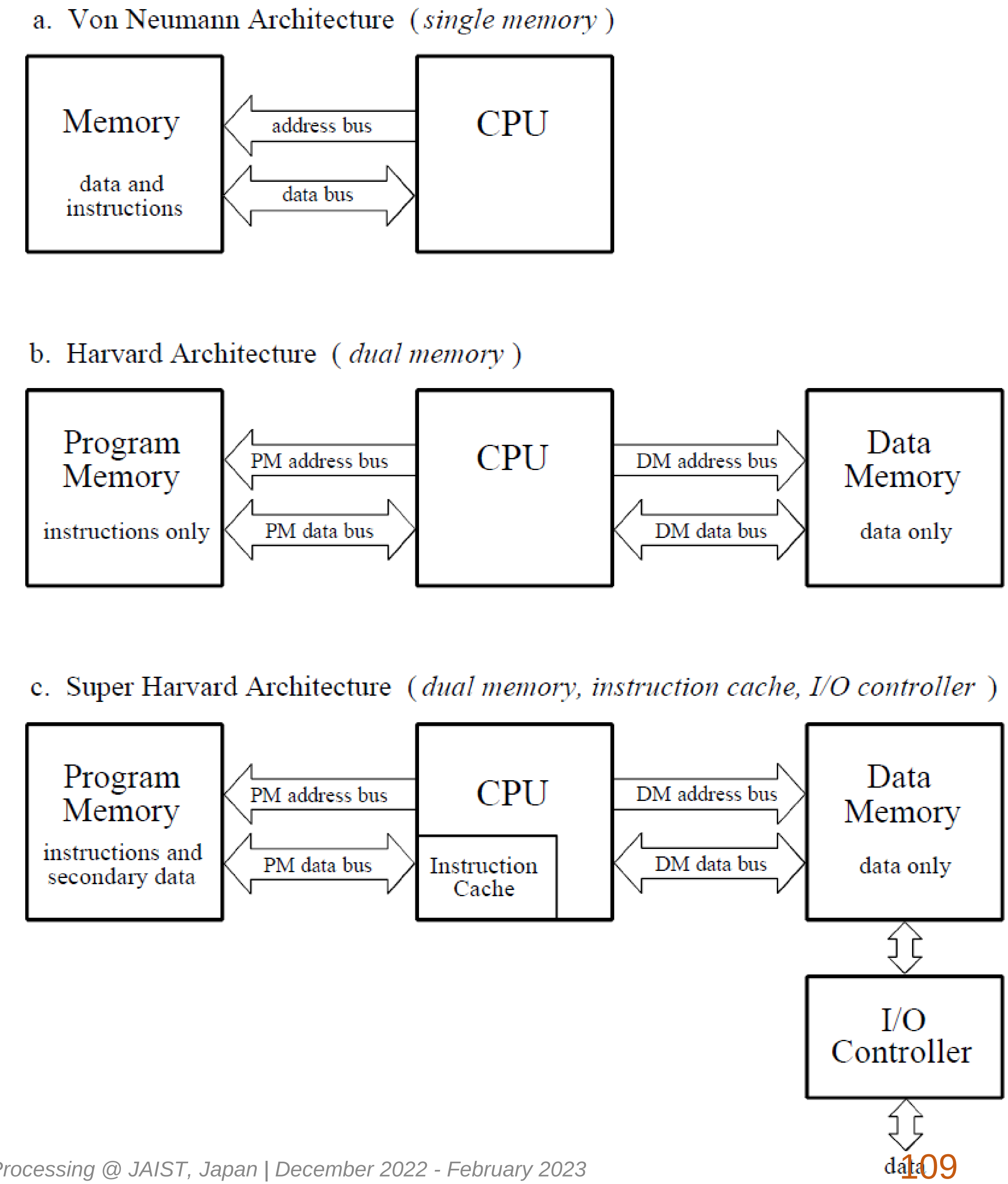
Circular buffer at some instant

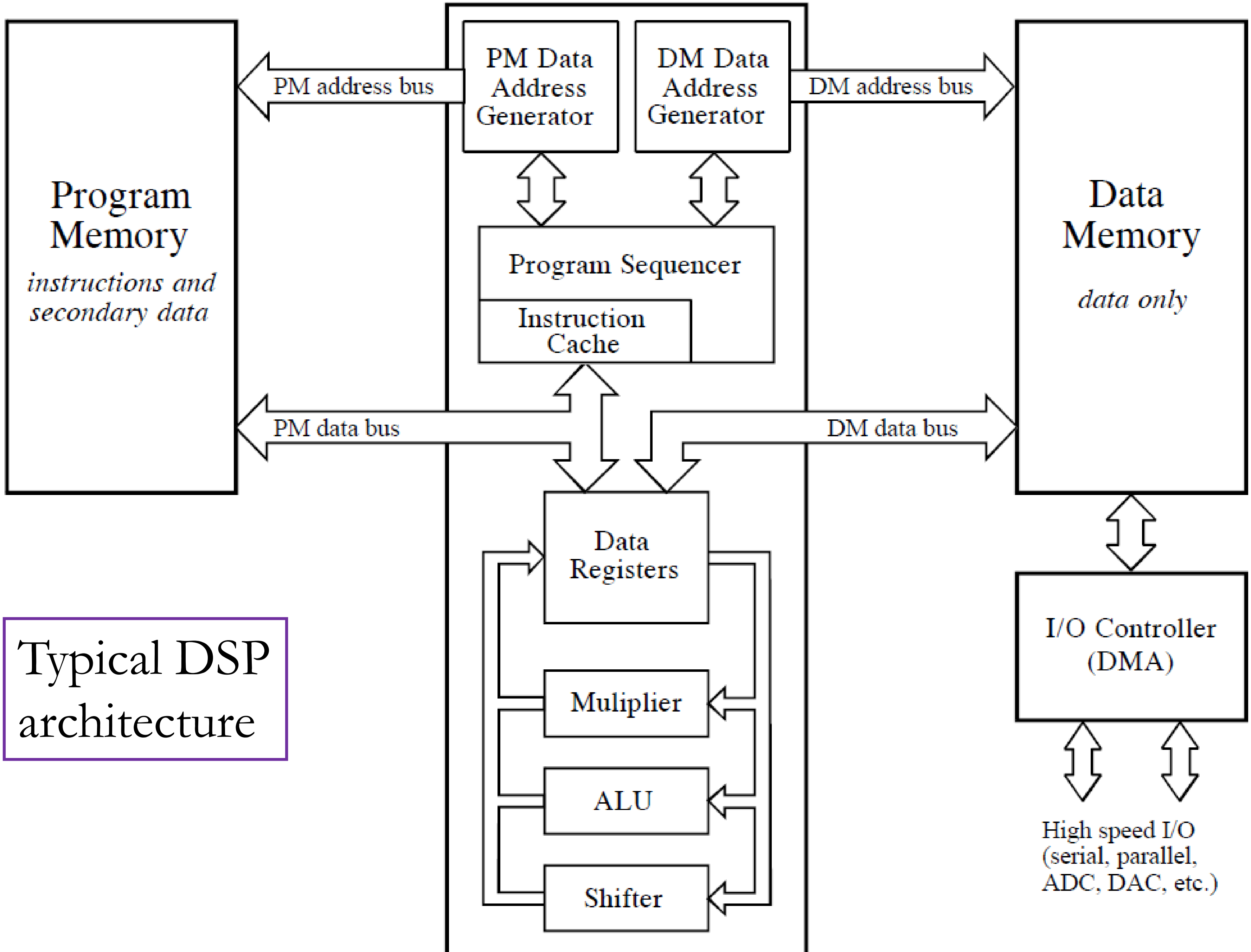


Circular buffer after next sample

We must have access to a certain number of the most recent samples from the input. These samples must be stored in memory and continually updated as new samples are acquired.

Microprocessor architecture





Typical DSP architecture

Thank you

JAIST
JAPAN
ADVANCED INSTITUTE OF
SCIENCE AND TECHNOLOGY
1990