Double Pendulum

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Abstract

The project presents an analysis of an unusual mechanical system, that of a double pendulum. For the purpose of understanding the nature and behavior of a double pendulum, its equations of motion were solved numerically. The resulting solutions were then used to generate multiple graphs and simulations. Throughout the project's implementation, the emphasis was placed on understanding as well as showing why the double pendulum is highly sensitive to its initial conditions, and why that sensitivity leads to chaotic behavior. To better represent the sensitive nature of the system, two double pendulums were generated and then compared. The resulting graphs presented in the project show how the smallest changes made to the initial conditions of these pendulums can result in drastically different and unpredictable behaviors.

Contents

1 Introduction

> "*The things that really change the world, according to Chaos theory, are the tiny things. A butterfly flaps its wings in the Amazonian jungle, and subsequently a storm ravages half of Europe.*"
>
> — Neil Gaiman, *Good Omens*

When we read the above quote from the great novelist Neil Gaiman, it left us wondering if what he said was indeed true, as it does seem to be quite a bizarre read at first. Neil Gaiman is trying to imply that the world around us is inter connected by very small little things or events. This apparent connection is so unpredictable & erratic that even the mere flapping of a butterfly's wings in one part of the world can start a chain of events that can eventually result in a storm in a totally different part of the world.

So this got us thinking - "What actually is the phenomenon behind this said connection?" After a lot of literature surveying and many a youtube watching sessions, we finally came across one of the most intriguing studies in the field of Physics - **'Chaos Theory'**.

Chaos Theory is the study of dynamical systems whose apparently random states of disorder and irregularities are actually governed by underlying patterns and deterministic laws that are highly sensitive to initial conditions. Thus, giving rise to a very unusual connection between randomness and predictability. Chaos Theory looks at certain systems that are very sensitive and tries to work out the 'chaotic patterns' associated with them. In any sensitive system, even a small change can make the system behave differently.

It didn't take a lot of time before we realised this is the topic that we want to base our project on as this particular field of Physics made an immediate connection with us.

During our literature surveying we came across a wonderful book on this topic - **Chaos: Making a New Science** by **James Gleick** [1].

The book tells about the history of the field of Chaos and explains how even the studying of simple systems such as the pendulums, lead to the development of this field. The book also says that the behaviours of some special simple pendulum systems could lead to complexity and unpredictability.

"*But unpredictability was not the reason physicists and mathematicians began taking pendulums seriously again in the sixties and seventies. Unpredictability was only the attention-grabber. Those studying chaotic dynamics discovered that the disorderly behavior of simple systems acted as a creative process. It generated complexity: richly organized patterns, sometimes stable and sometimes unstable, sometimes finite and sometimes infinite, but always with the fascination of living things. That was why scientists played with toys.*"

— James Gleick, *Chaos: Making a New Science*

This is why we chose a particularly special system - the **'Double Pendulum'** as the focus of our project. A double pendulum is an extension to the simple pendulum with the major difference being that the bob of the first simple pendulum acts as the center/hook [better word needed] of the second simple pendulum. The double pendulum, even when left undamped, shows a very erratic behaviour in its system as the time progresses.

Explaining this erratic behaviour or at least understanding the source of it is the main aim of this project. For that purpose, we numerically solved the equations of motion for the double pendulum and then used the solution to generate two different double pendulum with "practically identical" set of initial conditions. These two pendulums were then compared through the use of many graphs which intend to show that our system is very sensitive to its initial conditions.

Plan of the report

- Theory: Explains the double pendulum system and the differential equations which governs it
- Methodolgy: Shows the algorithm and the numerical method used to solve the system's first order differential equations
- Analysis & Results: Analsyses the two double pendulums and compares them by the use of a number of graphs and simulations.
- Summary & Conclusions: Summarises the results of our project and provides some information on the factors of chaos

2 Theory

2.1 Problem Statement - The Double Pendulum

We aim to understand non-linearity of the nature which gives rise to chaotic behavior of physical world through a focused study of dynamics of a Double Pendulum [refer Figure 1].

A double pendulum is an arrangement of two plane pendulums attached to each other, such that one plane pendulum acts as a driving force for the other.

2.2 Equations of Motion for a Double Pendulum

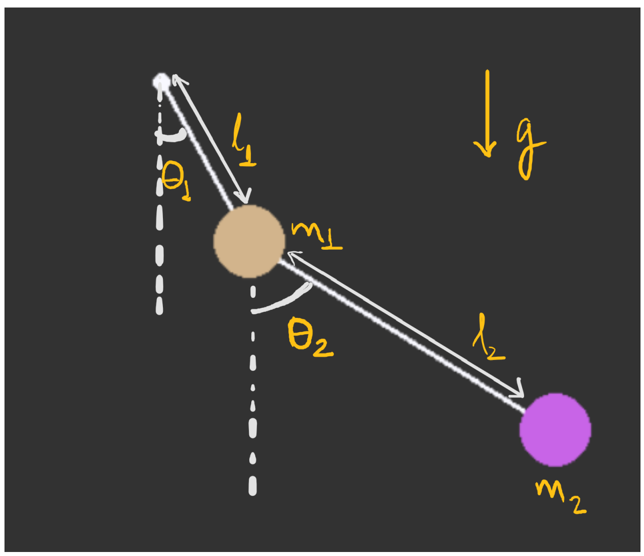


Figure 1: Double Pendulum

Equations of motion [check Appendix B] for full derivation] for a double pendulum are as follows,

$$\ddot{\theta}_1 = \frac{-g(2m_1 + m_2)\sin\theta_1 - gm_2\sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2)\, m_2(\dot{\theta}_2^2 l_2 + \dot{\theta}_1^2 l_1 \cos(\theta_1 - \theta_2))}{l_1(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))} \tag{a}$$

and,

$$\ddot{\theta}_2 = \frac{2\sin(\theta_1 - \theta_2)\,(\dot{\theta}_1^2 l_1(m_1 + m_2) + g(m_1 + m_2)\cos\theta_1 + \dot{\theta}_2^2 l_2 m_2 \cos(\theta_1 - \theta_2))}{l_2(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))} \tag{b}$$

It is clear that equations (a) and (b) are second order nonlinear differential equations. [title= Symbols Used]

θ_1 = Angle of top bob
θ_2 = Angle of bottom bob
ω_1 = Angular velocity of top bob
ω_2 = Angular velocity of bottom bob
m_1 = Mass of top bob
m_2 = Mass of bottom bob
l_1 = Length of string attached to top bob
l_2 = Length of string attached to bottom bob
g = Acceleration due to gravity
$\dot{o}$ = First time derivative of the physical quantity 'o'
$\ddot{o}$ = Second time derivative of the physical quantity 'o'

2.3 Analytical Solution

We can convert (a) and (b) into four first order differential equations.
Substituting,

$$\dot{\theta}_1 = \omega_1 \quad \text{(i)} \qquad \dot{\theta}_2 = \omega_2 \quad \text{(ii)}$$

Here, ω_1 = angular velocity of top bob, and, ω_2 = angular velocity of bottom bob
$\therefore$, (a) and (b) can be rewritten as,

$$\dot{\omega}_1 = \frac{-g(2m_1 + m_2)\sin\theta_1 - gm_2\sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2)\, m_2(\omega_2^2 l_2 + \omega_1^2 l_1 \cos(\theta_1 - \theta_2))}{l_1(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))} \tag{iii}$$

$$\dot{\omega}_2 = \frac{2\sin(\theta_1 - \theta_2)\,(\omega_1^2 l_1(m_1 + m_2) + g(m_1 + m_2)\cos\theta_1 + \omega_2^2 l_2 m_2 \cos(\theta_1 - \theta_2))}{l_2(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))} \tag{iv}$$

Equations (i), (ii), (iii), and (iv) are first order nonlinear differential equations.

Though equations (iii) and (iv) are first order differential equations, the nonlinear terms and the time dependency of coefficients (in the equations) makes it complicated to find an analytical solution. Furthermore, generally solutions of such equations, if exists [2], can not be written in a closed form [3]. Thus, any sufficiently 'good' numerical method can be used to compute the numerical solution for given system of equations.

3 Methodology

3.1 Tools & Algorithm

- The computational part of our project was carried out in **Python**.
- For plotting, **GNU Plot** was used.
- For the numerical solution of our system, **Runge-Kutta 4th order method** was used.

Algorithm 1 Runge-Kutta Order 4

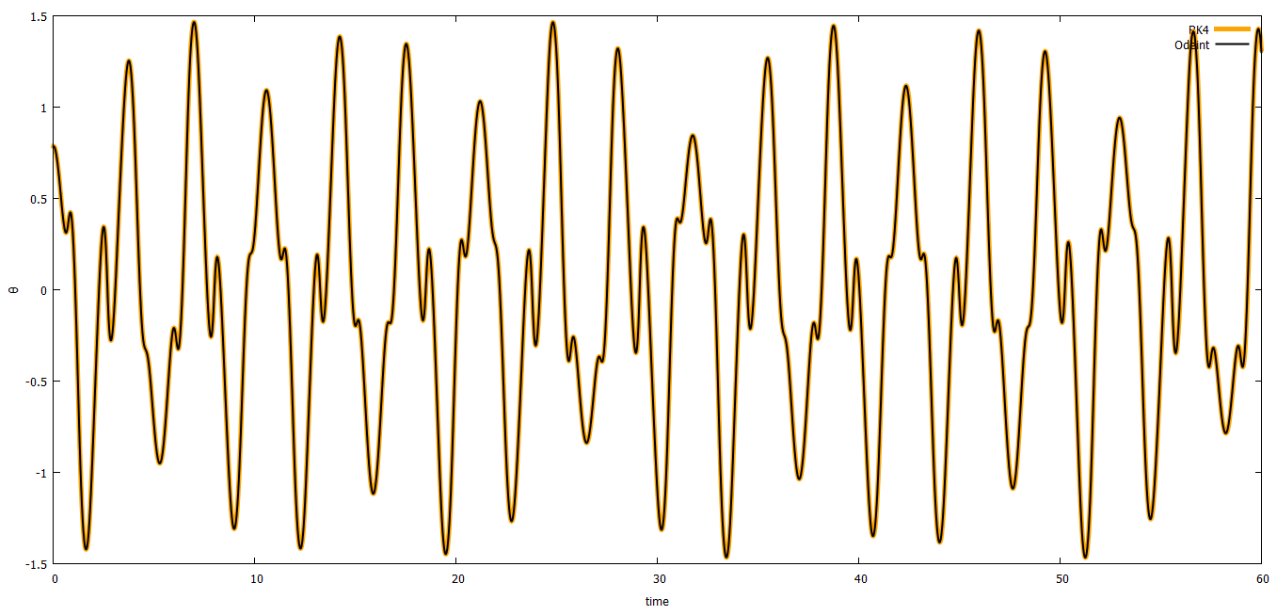
```
function RK4(func, y0, x, consts)                              ▷ Define the RK4 function
    h = (x[-1] - x[0])/len(x)                                  ▷ Define the step size
    yarr = [ ]                                                 ▷ Define initial values
    yin = y0
    xin = [x[0]]
    for i in range( len(x) ) do:
        yarr.append(yin)                                       ▷ Initialize y value array
        k1 = [h * ele for ele in func(yin, xin, consts)]
        yn = [e1 + e2/2 for (e1, e2) in zip(yin, k1)]
        xn = [e1 + h/2 for e1 in xin]
        k2 = [h * ele for ele in func(yn, xn, consts)]
        yn = [e1 + e2/2 for (e1, e2) in zip(yin, k2)]
        k3 = [h * ele for ele in func(yn, xn, consts)]
        yn = [e1 + e2 for (e1, e2) in zip(yin, k3)]
        xn = [e1 + h for e1 in xin]
        k4 = [h * ele for ele in func(yn, xn, consts)]
        yf = [ini_y + (e1 + 2 * (e2 + e3) + e4) / 6 for (ini_y,e1,e2,e3,e4) in zip(yin, k1, k2,
k3, k4)]                                                       ▷ Formula for RK4
        yin = yf
        xin = [e1 + h/2 for e1 in xn]
    end for
    yarr = np.array(yarr).reshape(-1,4)                        ▷ Reshape the result in [n, 4] Matrix
    return (yarr)                                              ▷ Compute values of y-variable
end function
```

3.2 Numerical Method

We created a function named $func$ where the differential equations (i), (ii), (iii), and (iv) were defined.
In the $RK-4$ function, y0 contains the initial conditions of θ_1, θ_2, ω_1, and ω_2 .This was then used to numerically solve the equations.
For loop in the RK-4 algorithm iterated over a time interval of 0 to 60 seconds divided into 6000 nodal points and thus, appended the calculated values into $yarr$ which was finally reshaped into a 6000×4 matrix.

3.3 Inbuilt Functions

We compared the RK-4 function with the inbuilt (python) function - 'ODEINT' from Scipy's Integrate Package.
The 'θ vs time' graph below compares the solutions obtained from RK-4 and ODEINT.



Initial conditions of the system: $\theta_1 = 45°$, $\theta_2 = 100°$, $\omega_1 = 0$ rad/s, $\omega_2 = 0$ rad/s

Figure 2: Comparison of the Numerical Solutions from RK-4 and Scipy's ODEINT

It is evident from the overlap of the solutions that, the solution approximated by RK-4 is sufficiently close to that approximated by the inbuilt function.

4 Analysis of Numerical Results

To observe the chaotic behavior of the double pendulum system, solutions for two independent double pendulum systems with 'practically identical' set of initial conditions were obtained. To interpret the results for general case in a comprehensible manner, let us first revisit the case for small angles.

4.1 Case for Small Angles

Following set of initial conditions was used to compute the results for Small Angles:

- **For Pendulum 1 :** $\theta_1 = 15°$, $\theta_2 = 10°$, $\omega_1 = 0$ rad/s and, $\omega_2 = 0$ rad/s
- **For Pendulum 2 :** $\theta_1 = 15°$, $\theta_2 = 10.0001°$, $\omega_1 = 0$ rad/s and, $\omega_2 = 0$ rad/s

It can be observed from Figure [3] that θ_1 and θ_2 for both the pendulums oscillate with a single frequency and small variations in the initial conditions of both the pendulums does not give rise to unpredictable or erratic behavior.

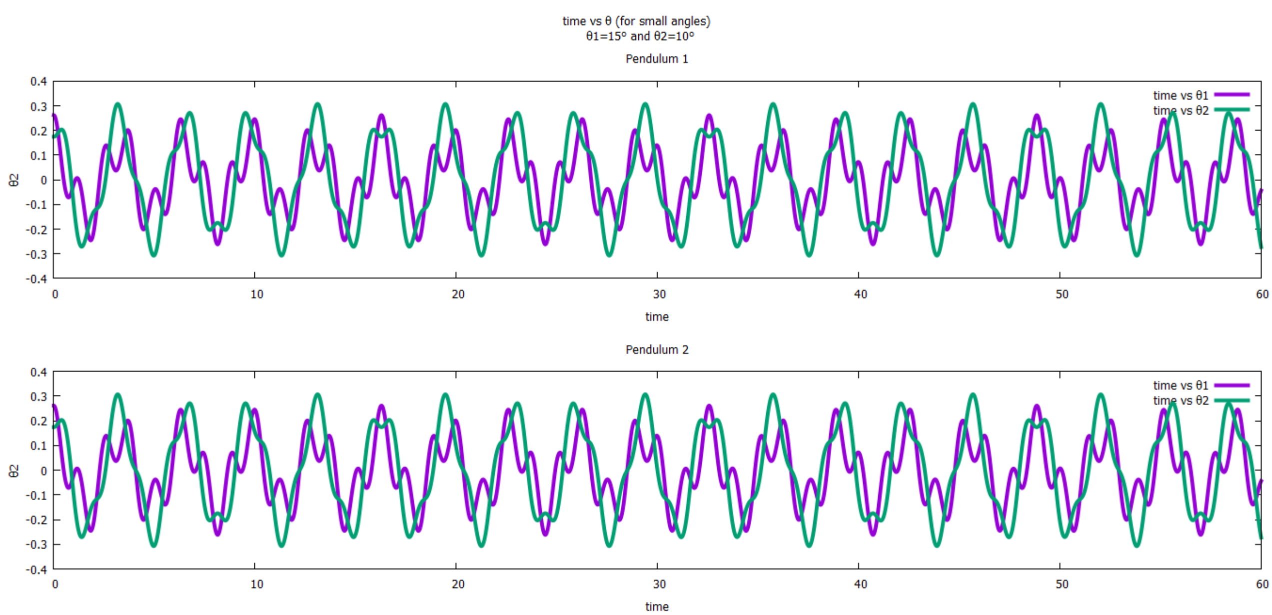


Figure 3: 'θ vs time' graphs for each pendulum (small angles)

Same observations can be made from observing figure Figure [4], where ω_1 and ω_2 of pendulum 1 are identical to that of ω_1 and ω_2 of pendulum 2 respectively, even if initial conditions were varied slightly.

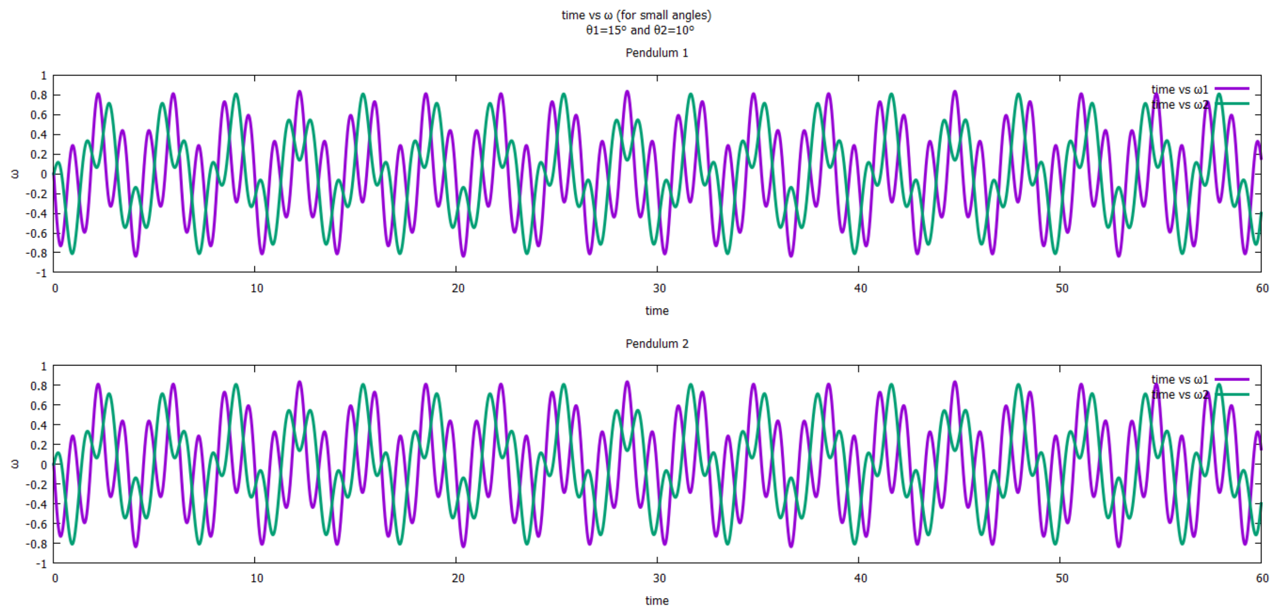


Figure 4: ‘ω vs time’ graphs for each pendulum (small angles)

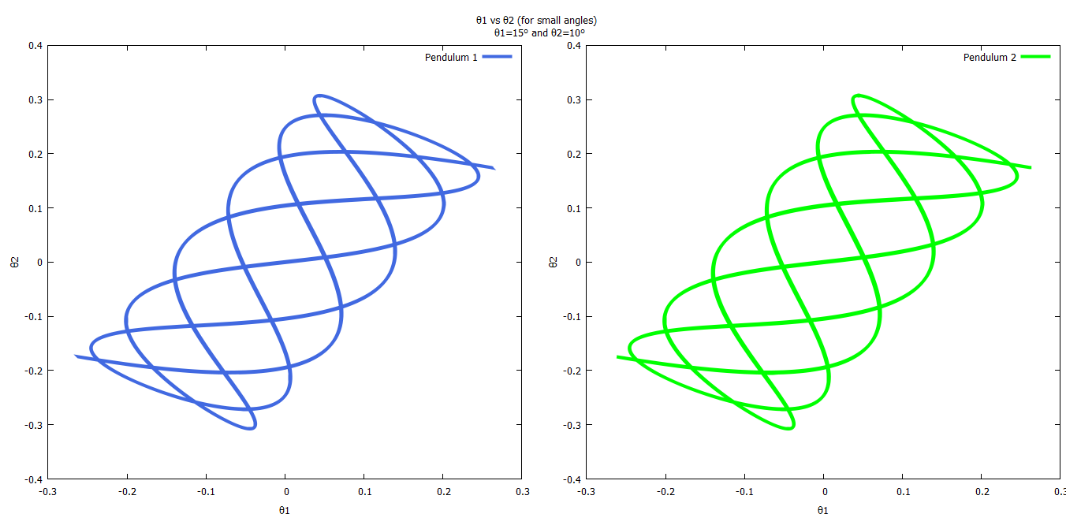


Figure 5: ‘θ_1 vs θ_2’ graphs for each pendulum (small angles)

Figure [5] shows a Lissajous figure with $a = 7$ and $b = 3$, thus supporting the fact that in this case, the double pendulum behaves like a linear system, more precisely, to that of a coupled pendulum [cite].

4.2 General Case

Following set of initial conditions was used to compute the results for General Case:

- **For Pendulum 1 :** $\theta_1 = 45°$, $\theta_2 = 200°$, $\omega_1 = 0$ rad/s and, $\omega_2 = 0$ rad/s
- **For Pendulum 2 :** $\theta_1 = 45°$, $\theta_2 = 200.0001°$, $\omega_1 = 0$ rad/s and, $\omega_2 = 0$ rad/s

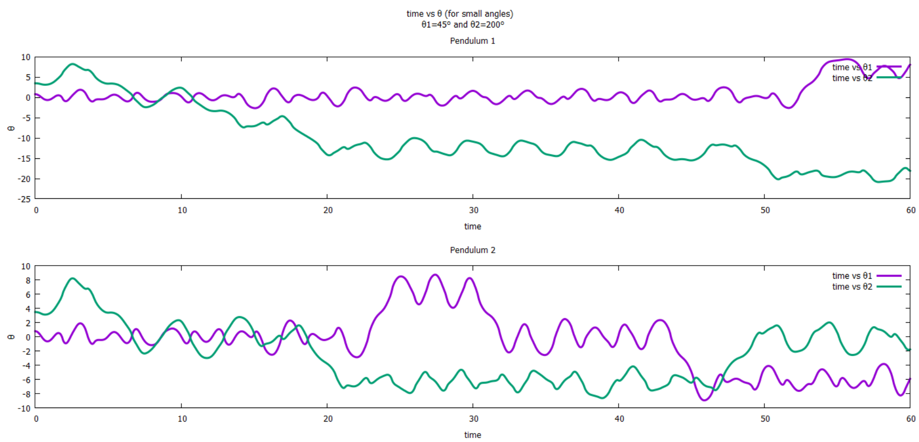


Figure 6: 'θ vs time' graphs for each pendulum (large angles)

It is clear from the results above that the θ_1 and θ_2 of Pendulum 1 evolve very differently in time when compared to θ_1 and θ_2 of Pendulum 2 respectively, even though the initial conditions were almost identical. This is very different from Figure [3].
This is the primary evidence of chaos in the system.

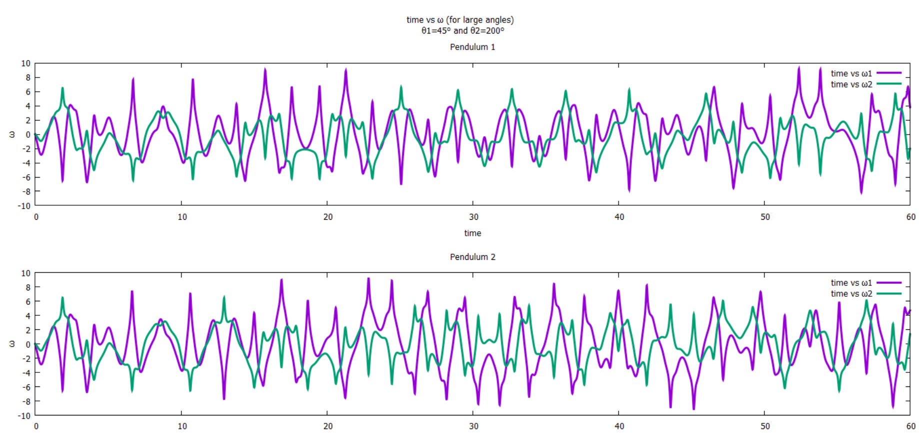


Figure 7: 'ω vs time' graphs for each pendulum (large angles)

ω_1 and ω_2 show a similar behavior as θ. One can compare Figure [7] & Figure [4], and observe that the behavior is totally different.
In Figure [7] they begin to evolve differently after $\sim$5 seconds.

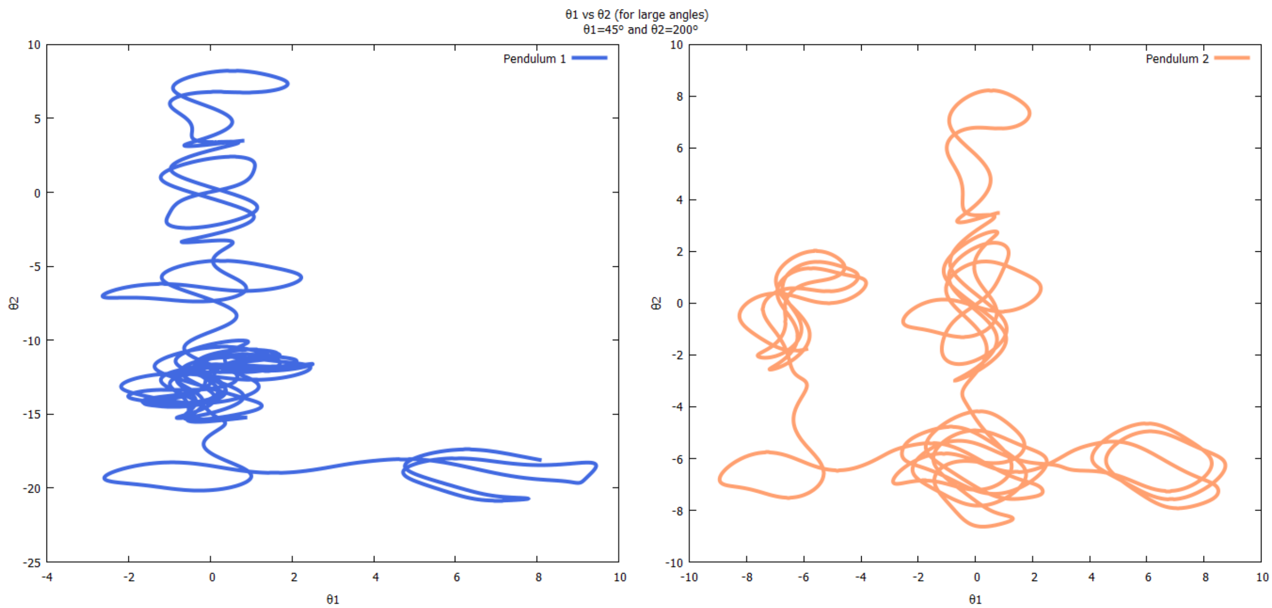


Figure 8: Phase Plots for each pendulum (large angles)

Figure [8] also depicts the high sensitivity to initial conditions - benchmark of chaos. Upon comparing this graph with Figure [5], it is clear that at large angles, chaotic behavior can be seen evolving much more rapidly and thus, making it impossible to make any *long-term predictions*.

4.3 Phase Space Plots and Dominant Cycles

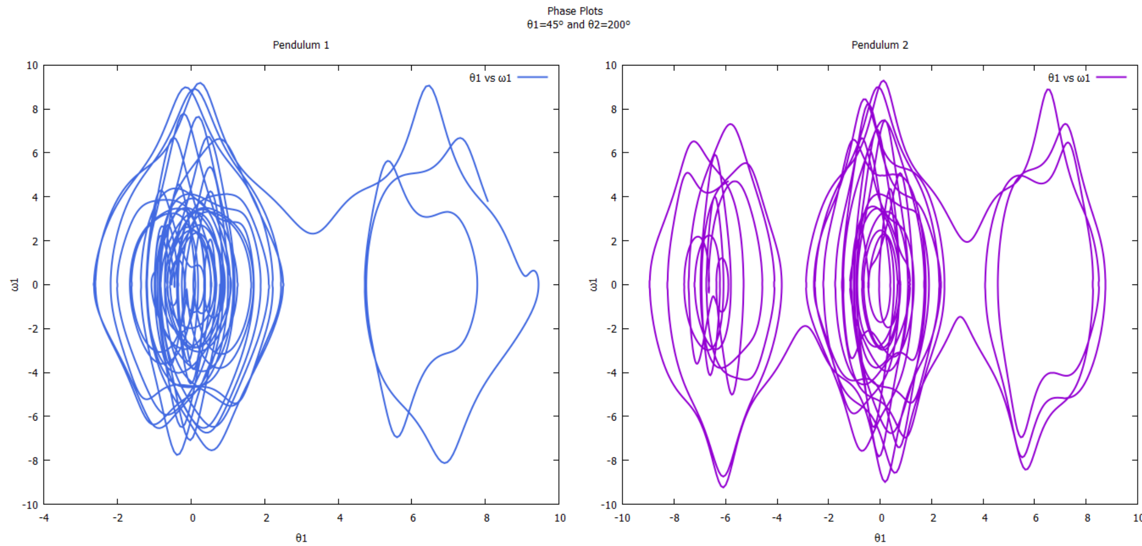


Figure 9: Phase Space plot of First Bob

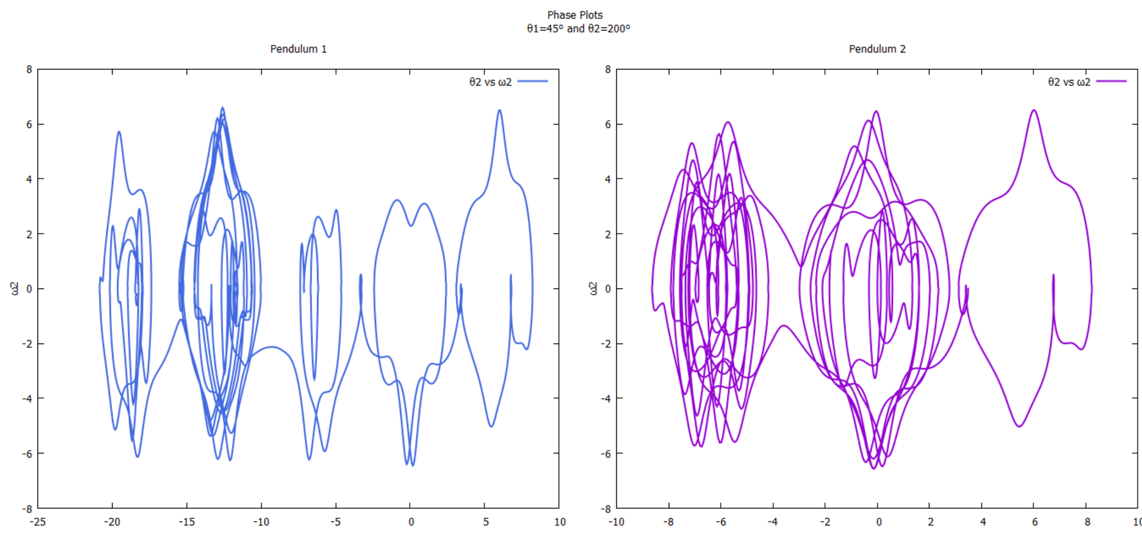


Figure 10: Phase Space plot of Second Bob

In Figure [9], dominant cycles in the phase plot of the first bob completely change for pendulum 1 and pendulum 2. A similar behavior can be seen in Figure [10]. This behavior is observed because the solutions lie on some invariant set but not on an attractor [4]. Thus, solutions' phase spaces change even with a small change in initial conditions.

4.4 Simulating the Double Pendulum

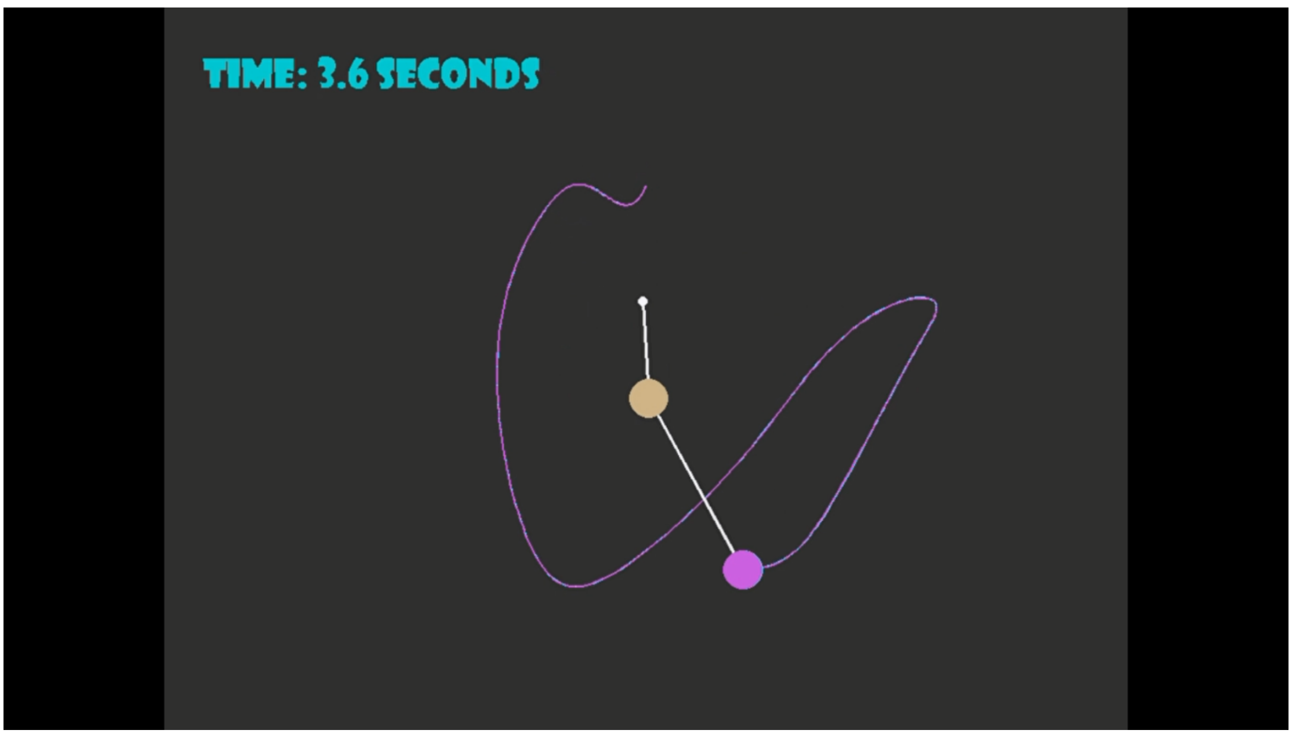
After obtaining the solutions for the 4 first order differential equations of each pendulum, we plugged them into 4 trajectory equations,

$$x_1 = l_1 \sin\theta_1 \qquad y_1 = -l_1 \cos\theta_1$$

$$x_2 = x_1 + l_2 \sin\theta_2 \qquad y_2 = y_1 - l_2 \cos\theta_2$$

in order to obtain the x and y coordinates of both the pendulums. These coordinates were then appended into 2 separate data files. Using the 'Pygame' animation package in Python [refer here for the script used], we simulated the two pendulums with the help of these data files.

Simulation of the two double pendulums



In order to run the above video in the same pdf file, you need to open the pdf file on 'Foxit PDF Reader.' After which you need to click on the image to start playing it.

A popup will appear asking you to trust this document to use optional features in which you need to either select - 'trust one time' or 'trust always' before you can run the video.

The above simulation basically depicts how the two double pendulums will move in an x-y space after the initial conditions are given to them.
The eventual separation between the paths taken by the two pendulums illustrates the fact that the two systems will evolve differently even though the initial conditions provided to them differ by an order of just 10^{-3}.

<u>Trajectory graphs of the two double pendulums</u>

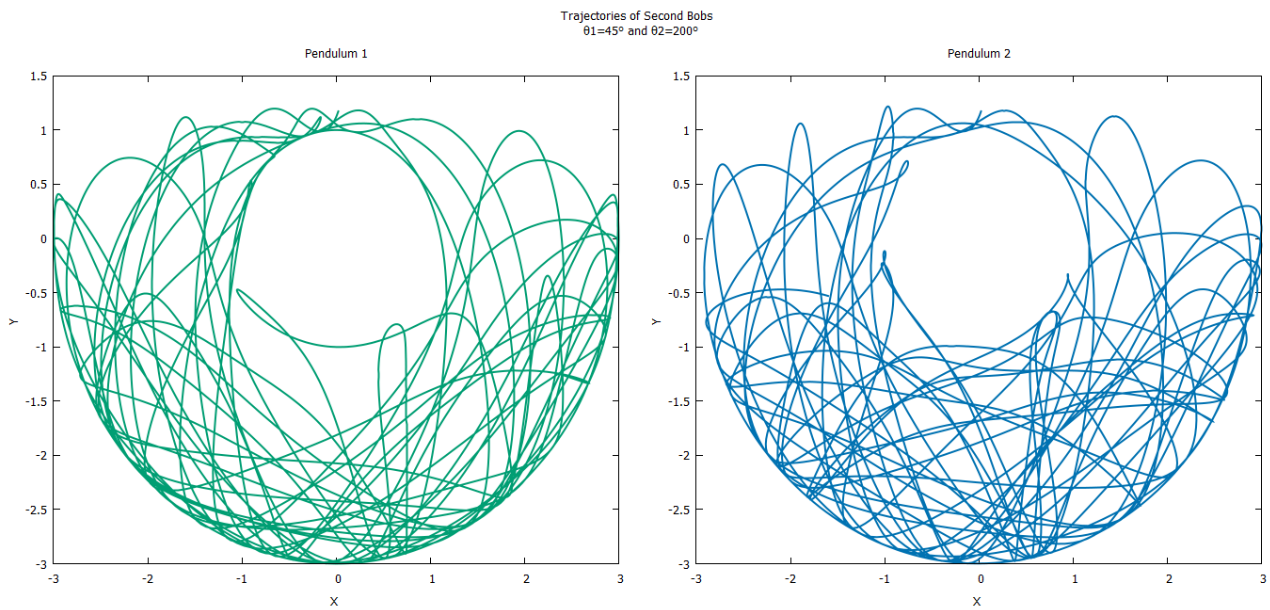


Figure 11: Trajectory plots for each pendulum

Figure [11] shows the trajectories or the paths taken by the lower bobs of each pendulum in the x-y space.
The difference in the trajectory graphs again points out the fact that both pendulums are evolving differently even with their initial conditions being "almost" same.
Thus, these graphs give another proof for the presence of the chaotic nature in this system.

5 Summary

After analyzing the results, one can be fairly satisfied with the fact that predicting the long-term behavior of a double pendulum (chaotic system) is impossible. At lower energies, double pendulum behaves like a coupled pendulum and no significant differences could be observed when initial conditions were varied by a small amount.

At higher energies (large angles), even smallest of variations in the initial conditions can lead to totally different results as time evolves. Though the systems initially follow similar trajectories but with evolution of time, trajectories converge to some invariant set with a finite number of dominant cycles in the phase space.

Thus, it can concluded with certainty that the double pendulum system shows chaotic behavior at higher energies.

While pursuing this project, we learned a lot of new, advanced and interesting physics and mathematics of chaos theory. We learned about various methods of checking the state (chaotic or not) of the system. While programming, we found some things to be too 'complex' to simulate. But in a nutshell, this project introduced us to a fascinating and active branches of Complex Dynamics and Chaos Theory.

We plan to continue learning about the subject even after completion of this project and produce the results which we could not produce while pursuing this project.

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Appendices

Appendix A Programs

1. Python script used for Calculation of Results
2. Python script used to create Simulation of Double Pendulum
3. GNU Plot scripts used for Plotting Graphs
4. Source code for the .tex file used for embedding a video in the pdf file

Appendix B Derivation

We employ Lagrangian Mechanics to derive the equation of motions for our Double Pendulum.

$$x_1 = l_1 \sin\theta_1,\ y_1 = -l_1 \cos\theta_1$$

$$x_2 = x_1 + l_2 \sin\theta_2,\ y_2 = y_1 - l_2 \cos\theta_2$$

Kinetic energy (K) of the system is :

$$K = \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) = \frac{1}{2}(m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2)) \tag{1}$$

and, Potential Energy (U) of the system is :

$$U = m_1 y_1 g + m_2 y_2 g \tag{2}$$

Lagrangian (L) of the system is defined as

$$L = K - U$$

Now, putting (1) and (2) in the above equation, we get :

$$L = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - (m_1 y_1 g + m_2 y_2 g) \tag{3}$$

$$L = \frac{m_1}{2}(l_1^2\dot{\theta}_1^2 + 2gl_1cos\theta_1) + \frac{m_2}{2}(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + 2g(l_1cos\theta_1 + l_2\cos\theta_2))$$

$$L = \frac{(m_1+m_2)}{2}l_1^2\dot{\theta}_1^2 + \frac{m_2}{2}l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + (m_1+m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2 \tag{4}$$

Now, using Euler-Lagrange Equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \tag{5}$$

Differentiating (4) w.r.t to θ_1 and θ_2 respectively, we get :

$$\frac{\partial L}{\partial\theta_1} = -m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1+m_2)gl_1\sin\theta_1 \tag{6}$$

and,

$$\frac{\partial L}{\partial\theta_1} = m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2gl_2\sin\theta_2 \tag{7}$$

Now, differentiating (4) w.r.t to $\dot{\theta}_1$ and $\dot{\theta}_2$ respectively, we get :

$$\frac{\partial L}{\partial\dot{\theta}_1} = (m_1+m_2)l_1^2\dot{\theta}_1 + m_2l_1l_2\dot{\theta}_2\cos(\theta_1 - \theta_2) \tag{8}$$

and,

$$\frac{\partial L}{\partial\dot{\theta}_2} = m_2l_2^2\dot{\theta}_2 + m_2l_1l_2\dot{\theta}_2\cos(\theta_1 - \theta_2) \tag{9}$$

Upon putting (6) and (8) in (5) and solving, we get :

$$(m_1+m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2l_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + (m_1+m_2)\,g\sin\theta_1 = 0. \tag{10}$$

similarly, putting (7) and (9) in (5) and solving, we get :

$$l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin\theta_2 = 0. \tag{11}$$

(10) and (11) form the system of Nonlinear Second Order Differential Equations of motion for our Double Pendulum.

Upon rearranging, (10) and (11) will result into (a) and (b).

Appendix C Contribution of team mates

Contribution of Shivani Nashier

- In Formulation of the problem: *Literature surveying and deriving trajectory equations*
- In Programming: *Gnuplot programming*
- In Plotting Graphs: *Making gnuplot scripts for each graph*
- In Report Writing: *Writing the beamer presentation*

Contribution of Samarth Jain

- In Formulation of the problem: *Literature surveying and deriving equations using Lagrangian mechanics*
- In Programming: *Python programming and simulation*
- In Plotting Graphs: *Making simulation and trajectory graphs*
- In Report Writing: *Helping in both project report and beamer presentation writing*

Contribution of Sarthak Jain

- In Formulation of the problem: *Literature surveying and converting second order DEs into first order DEs*
- In Programming: *Python programming*
- In Plotting Graphs: *Making phase space graphs*
- In Report Writing: *Writing the project report and helping in beamer presentation writing*