M2

① Linear Equations

Gauss Elimination
REF

Gauss Jordan Elimination
RREF
[REF with all zeroes above the 1s]

Ⓐ $AX = B$

$Rank(A) = Rank(A:B) = n$ Unique Soln

$Rank(A) = Rank(A:B) < n$ Infinite Solns

$Rank(A) \neq Rank(A:B)$ No Soln

Ⓑ $AX = 0$

$rank(A) = n$ Trivial Soln $X = 0$

$rank(A) < n$ Infinitely many solutions

* Rank:
No. of non-zero rows in REF of the matrix

Ⓒ Inverse of matrix

$AB = I_n$

$B = A^{-1}$

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② Eigenvalues & Eigenvectors

$AX = \lambda X \rightarrow (A - \lambda I)X = 0$

$|A - \lambda I| = 0$

we get Values, Put these, then we get matrices

Using these matrices, solve equations, get Eigen Vector Solution

$f(\lambda)$ is a eigenvalue for $f(A)$

③ Vector Spaces

AXIOMS

① $u+v \in R$ if $u \in R$ and $v \in R$

② cu is an element of u

③ $u+v = v+u$

④ $u+(v+w) = (u+v)+w$

⑤ There exists an element of V called zero vector $\rightarrow 0$

$u+0=u$

⑥ For every element u of $V \rightarrow -u$ such that

$u+(-u)=0$

⑦ $c(u+v) = cu+cv$

⑧ $(c+d)u = cu+du$

⑨ $c(du) = (cd)u$

⑩ $1(u) = u$

Subspaces: Pass through origin.

OR

Check Closure under addition & scalar multiplication

Linear Combination of Vectors

$$v = c_1 v_1 + c_2 v_2 \cdots c_n v_n$$

$$c_1 \neq 0 \quad c_2 \neq 0 \cdots c_n \neq 0$$

Linear ly Dependent
Det = 0

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Span

Let $v_1 \dots v_m$ be vectors in a vector space V. These vectors span V if every OTHER vector in V can be expressed as a linear combination of them.

Linear Independence of Vectors

Linearly Independent
Non zero determinant

$$c_1 v_1 + c_2 v_2 \dots c_m v_m = 0$$

- Linearly dependent if not all $c_1 \dots c_m$ are 0
- Linearly Independent if $c_1 = c_2 \dots c_m = 0$

→ Use REF and we get: $a = b = c = 0$ ← Constant

Basis

- A finite set of vectors is called a basis of V if set spans V and is linearly independent.
- All bases for a vectorspace V have same no. of vectors

$$c_1 v_1 + c_2 v_2 + c_3 v_3 \dots c_n v_n = 0$$

Only if $c_1 = c_2 \dots c_n = 0$

Dimension

Set of linearly independent vectors that are not zero in RREF

Rank

No of non-zero rows

Simplified Span method

We use RREF, and find those vectors that span. i.e. find basis

$$\begin{bmatrix} 2 & -2 & 3 & 5 & 5 \\ -1 & 1 & 4 & 14 & -8 \\ 4 & -4 & -2 & -14 & 18 \\ 3 & -3 & -1 & -9 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis → $(1, -1, 0, -2, 4)$
$(0, 0, 1, 3, -1)$

Independence test

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 1 & 5 & -7 \\ -2 & 4 & -8 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Take
$S = (1,3,2)\ (2,1,4)$
$(0,5,8)\ (1,-7,14)$

This shows that 3rd and fourth are LC of first and second.

$\therefore$ Basis $= (1,3,2), (2,1,4)$

Enlarge a LI set to basis

If we have $[2,0,4,-12], [0,-1,-3,9]$ of $V = \mathbb{R}^4$

We add $[2,0,4,-12]$ $[0,-1,-3,9]$ $[1,0,0,0]$ $[0,1,0,0]$
$[0,0,1,0]$ $[0,0,0,1]$

Then we reduce and get similar to Independence test

$[2,0,4,-12]$ $[1,0,0,0]$
$[0,-1,-3,9]$ $[0,0,1,0]$

Vectors in Column

Change in Basis

$U_S = P_B U_B$ B to S

$U_S = P_{B'} U_{B'}$ B' to S

Now for B to B'

$U = P_{B'}^{-1} P_B U_{B'}$

Orthonormal basis

$$V_B = \begin{bmatrix} V \cdot \bar{U}_1 \\ V \cdot \bar{U}_2 \\ V \cdots \bar{U}_n \end{bmatrix}$$

(4) Linear Transformation

(a) L is a LT from V to W if

(a) $L(u+v) = L(u) + L(v)$

(b) $L(\alpha u) = \alpha L(u)$

(b) Kernal → Set of vectors in V that give 0 in V, also called null space

Range → Set of Images

NOTE $T(x,y,z) = x, y, 0$

Kernal → $(0,0,z)$

Range → $(x, y, 0)$

(c) $T(0_V) = 0_V$

For matrix $A \rightarrow T(x) = Ax$

$T(x) = 0 \rightarrow$ Kernal → Solve eqns, get ans

For Range, spanned by column vectors

Take transpose, do RREF, and find which rows are non zero

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} 1,0 & 0,1 & 1,1 \\ s, & t, & s+t \end{matrix}$$

* Range of T is column space

(d) $\underset{\text{dim Range}}{\text{Rank}(T)} + \underset{\text{dim kernel}}{\text{nullity}(T)} = \dim \text{domain}(T)$

For $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \text{rank}(A) = 2$, $\dim(T) = 3$ $\quad \therefore$ nullity = kernal dim $= 3-2 = 1$

(e) Orthogonal complement of row space of A is null space of A and orthogonal complement of column space of A is null space of A^T

$\text{row}(A)_\perp = \text{null}(A) \qquad \text{col}(A)_\perp = \text{null}(A^T)$

Set of all vectors $\perp^r$ to each vector of row space

For $A \to v$ vector

$Av = 0$

(f) A linear transformation T is one-one if and only if the kernal is the zero vector. (To prove if LT is one to one)

(g) $T(x) = Ax$ is one to one only if and only A is nonsingular

(i) $|A| \neq 0$

(ii) $\ker(T) = 0$

(iii) Rank $= n$

If $\{w_1, \dots, w_m\}$ is a set of more than n vectors where n is basis $\rightarrow$ linearly dependent

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If $a = \begin{matrix} 1 & -2 & 5 & 7 \\ 0 & 1 & 9 & 8 \\ 0 & 0 & 1 & 3 \end{matrix}$

$\text{rank}(A) = 3$

$\dim(\text{domain})(T_a) = 4$

$\therefore \ker(T_a) = 1 \neq 0$

$\therefore$ Not One to One

$b = \begin{matrix} 2 & 0 & -1 \\ 3 & 4 & 2 \\ 0 & 7 & 5 \end{matrix}$

(h) $LT \rightarrow R^n \rightarrow R^n$ T is invertible if there exists a transformation that

$$S(T(u)) = u \quad \text{and} \quad T(S(u)) = u$$

If $T(x, y) = (3x + 4y, 5x + 7y)$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 4y \\ 5x + 7y \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$$

$$T^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7x - 4y \\ -5x + 3y \end{bmatrix}$$

(i) Transformations & Systems of Linear Equations

$$Ax = y \rightarrow T(x) = y$$

Solution means pre images of y
If y is not in range of T, no soln.

Non homogenous

$$Ax_1 = y \quad Ax_2 = y$$
$$A(x_1 + x_2) = y \rightarrow \underline{\underline{\text{Not LT}}}$$

$$x_1 + 2x_2 + 3x_3 = 11$$
$$-x_2 + x_3 = -2$$
$$x_1 + x_2 + 4x_3 = 9$$

$$\begin{matrix} 1 & 2 & 3 & 11 \\ 0 & -1 & 1 & -2 \\ 1 & 1 & 4 & 9 \end{matrix} \approx \begin{matrix} 1 & 0 & 5 & 7 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{matrix} \rightarrow \begin{matrix} x_1 + 5x_3 = 7 \\ x_2 - x_3 = 2 \end{matrix}$$

We get $(\underline{\underline{-5r+7}}, r+2, r)$

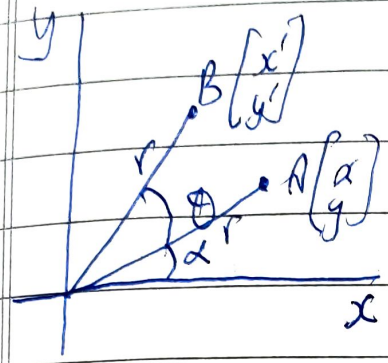
(j) Isomorphism
Let T be one-to-one, LT of V onto W, T is called an isomorphism. V and W are called isomorphic vector space

(k) Dilation & Contraction

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r\begin{bmatrix} x \\ y \end{bmatrix}$$

$r > 1$	$0 < r < 1$
dilation	Contraction

For reflection $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{bmatrix}x\\-y\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$



$\therefore T\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

In transformations

$T_1(x) = A_1 x \quad T_2(x) = A_2 x$

$T = T_2 \circ T_1 \therefore T(x) = A_2 A_1 x$

$T(x) = T_2(T_1(x))$

Q Determine matrix → reflection in y axis
→ rotation by $\pi/2$
→ dilation of 3

Find the image of $\begin{pmatrix}1\\2\end{pmatrix}$

$T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \cancel{\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}}$

$\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$= \begin{bmatrix}0 & -3\\3 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & 3\\3 & 0\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Therefore, Composition matrix is $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

For $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 6 \\ 3 \end{bmatrix}}}$

(i) Orthogonal matrix A

$$A^{-1} = A'$$

$T(u) = Au$ where A is an orthogonal matrix

(ii) Find eqn of line $y = 2x + 3$ under translation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 2x+3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x+2 \\ 2x+4 \end{bmatrix} = \begin{matrix} x' \\ y' \end{matrix}$$

Complex Analysis

① Limits Continuity & Differentiability

ⓐ Mostly use first principle

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

For differentiability, check if u_x, u_y, v_x, v_y are continuous, then use CR equations

$$u_x = v_y$$
$$u_y = -v_x$$

ⓑ For limits, can use sandwich and ε & δ form also

$$\lim_{x,y \to x_0, y_0} u(x,y) = u_0$$

$$|u - u_0| < \varepsilon$$
$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

ⓒ $\lim_{z \to z_0} f(z) = \infty$ iff $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

② Polar Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\left.\begin{aligned} r u_r &= -v_\theta \\ r v_r &= u_\theta \end{aligned}\right\} \text{ CR Eqns in Polar}$$

③ Analytic Functions

→ Differentiable

→ Aka entire functions if diff. everywhere

④ Harmonic functions

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

⑤ Exponential function

$$e^z = e^x e^{iy} = e^x \cos y + i e^x \sin y$$

$$|e^z| = e^x \quad \arg(e^z) = y + 2n\pi$$

⑥ Logarithmic function

$$e^w = z$$

$$w = \log z = \ln r + i(\theta + 2n\pi)$$

Generally

If they ask principle value, which is

$$\operatorname{Log} z = \ln r + i\theta$$

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

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7) Complex Exponents

$$z^c = e^{c\log z}$$

For ex: $i^{-2i} = e^{-2i\log i}$

$$\log i = \ln 1 + i(\pi/2 + 2n\pi)$$
$$= i(\pi/2 + 2n\pi)$$

$$e^{\pi + 4n\pi} \Rightarrow \exp[(4n+1)\pi]$$

$$-i^{i} = e^{i\log(-i)} = e^{i(\ln 1 + i(-\pi/2))} = \underline{e^{\pi/2}}$$

$$\frac{dc^z}{dz} = c^z \log z$$

8) Trigo

ⓐ $e^{ix} = \cos x + i\sin x \qquad e^{-ix} = \cos x - i\sin x$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Similarly, $e^{iz} = \cos z + i\sin z$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2}$$

ⓑ $\frac{de^z}{dz} = e^z \qquad \frac{de^{iz}}{dz} = ie^{iz}$

$$\frac{de^{-iz}}{dz} = -ie^{-iz}$$

(c) $\sinh y = \frac{e^{y} - e^{-y}}{2}$ $\cosh y = \frac{e^{y} + e^{-y}}{2}$

$$\boxed{\therefore \sin iy = i\sinh y \quad \cos iy = \cosh y}$$

(d) Now $\sin z = \sin x + iy = \sin x \cos iy + \sin iy \cos x$

$= \sin x \cosh y + i \sinh y \cos x$

$= \sin x \cosh y + i \sinh y \cos x$

$\cos z = \cos x \cosh y - i \sin x \sinh y$

Now $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$

(e) If say $z = 0$, $\sin z = 0$, $\sin^2 x + \sinh^2 y = 0$

$\hookrightarrow \sin x = 0 \rightarrow x = n\pi$

$\sinh y = 0 \rightarrow y = 0$

$\therefore \sin z = 0$ iff $z = n\pi$

$\cos z = -\sin\left(z - \frac{\pi}{2}\right)$

(f) $\cos z = 0$ iff $z = \frac{\pi}{2} + n\pi$

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(9) Hyperbolic functions.

(a) $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$ $\quad$ $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$

$\dfrac{d}{dz}\sinh z = \cosh z$ $\quad$ $\dfrac{d}{dz}\cosh z = \sinh z$

(b) $-i\sinh(iz) = \sin z$ $\quad$ $\cosh(iz) = \cos z$

$-i\sin(iz) = \sinh z$ $\quad$ $\cos(iz) = \cosh z$

(10) Inverse functions

$\cos^{-1} z = -i\log\left(z + i\sqrt{1-(z)^2}\right)$

$\sin^{-1} z = -i\log\left(iz + \sqrt{1-(z)^2}\right)$

$\tan^{-1} z = \dfrac{i}{2}\log\left(\dfrac{i+z}{i-z}\right)$

$\dfrac{d}{dz}\sin^{-1} z = \dfrac{1}{\sqrt{1-(z)^2}}$

$\sinh^{-1} z = \log\left(z + (z^2+1)^{1/2}\right)$

$\cosh^{-1} z = \log\left(z + (z^2-1)^{1/2}\right)$

(11) Integrals

(i) $w = u + iv$ function of t

$w'(t) = u'(t) + iv'(t)$

(ii) $\frac{d}{dt} z_0 w(t) = z_0 w'(t)$

(iii) $\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

(A) Contours

$z = x(t) + iy(t)$

$z'(t) = x'(t) + iy'(t)$ $\quad |z'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$

$T = \frac{z'(t)}{|z'(t)|}$ Unit tangent Vector

(B) Contour Integrals

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

We draw the curve, split into parts and integrate

(C) Modulii Inequality

If f is continuous on a smooth curve C and if $|f(z)| \leq M$ for all z on C, then

$$\left|\int_C f(z)\,dz\right| \leq ML$$

L → Length of C

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(D) Antiderivatives

An analytic function $F(z)$ is said to be an antiderivative of $f(z)$ (in domain D)

$$F'(z) = f(z)$$

(E) Suppose that a function f is analytic in a simply connected domain D and f' is continuous in D. Then for every simple closed contour C in D

$$\oint_C f(z)\,dz = 0$$

$$\int_C f(z)\,dz = \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt$$

$$= 0$$

Note $\int P\,dx + Q\,dy = \iint Q_x - P_y\,dA$

(F) $$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases}$$

(12) Cauchy Goursat Theorem

~~(13) Cauchy Integral Formula~~

~~Let~~

$$\cancel{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}}$$

$$\cancel{\int_C \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0)}$$

~~Extension to Cauchy Integral is~~

$$\cancel{f^{n}(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

~~nth derivative~~

NOTE:

If any of the following statements is true, then so are the others

(a) $f(z)$ has an antiderivative $F(z)$ throughout D

(b) the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any z_2

$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)$$

(c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero

12.1 Cauchy Goursat Theorem for Multiple Connected Domains

$$\int_C f(z)\,dz + \sum_{1}^{n}\int_{C_k} f(z)\,dz = 0$$

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(13) Cauchy Integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

Given $f(z)$ is analytic → Simple closed contour C and z_0 is interior point

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

OR

$$\boxed{f^{n}(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

→ nth derivative

- If a function $f(z)$ is analytic at a point z_0, then its derivatives of all orders are analytic there

(14) Morera's Theorem

If $\int_C f(z)\,dz = 0$ for every closed contour in D, then $f(z)$ is analytic throughout D

*Converse of Cauchy Goursat Integral Theorem

(15) Cauchy's Inequality

$$|f^{n}(z_0)| \leq \frac{n!\,M_R}{R^n} \quad n = 1, 2, \ldots$$

(16) Liouville's Theorem

If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

(17) $P(z) = a_0 + a_1 z - \dots a_n z^n$ has atleast one zero

(18) Series

(A) Revision:

Tests to check Convergence & Divergence

(a) Test of divergence

If $\lim_{n\to\infty} a_n \neq 0 \rightarrow \Sigma a_n$ is divergent

Else inconclusive

(b) Integral test for decreasing series

$\int_1^\infty a_n \, dn = l$ Σa_n is convergent

$\int_1^1 a_n \, dn = \infty$ Σa_n is divergent

(c) Comparision test

Given Σa_n, assume a Σb_n, if $a_n \leq b_n$ b_n is convergent

a_n is convergent

$a_n \geq b_n$ b_n is divergent

a_n is divergent

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(d) Ratio test

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ >1 Convergence

$=1$ inconclusive

<1 divergence

(e) Root test

$\lim_{n\to\infty} a_n^{1/n}$ <1 Convergence

$=1$ Inconclusive

>1 divergence

(f) Absolute Convergence $\to$ $\Sigma|a_n|$ is convergent

Conditional Convergence $\to$ $\Sigma|a_n|$ is divergent but Σa_n is not divergent.

(g) $\Sigma \frac{1}{n^p}$ diverges for $p \leq 1$

(h) Limit Comparison test

$\lim_{n\to\infty} \frac{a_n}{b_n} = l$

$l > 0$ Converge/Diverge together

$l = 0$ b_n converges a_n converges

$l = \infty$ b_n diverges, a_n diverges

(i) For a Power Series $\sum_0^\infty a_n (x - x_0)^n$ $\to$ $x_0 \to$ Centre of convergence

$R \to$ radius of Convergence

Interval of Convergence $(x_0 - R, x_0 + R)$

Radius can get from RATIO Test

(B) For all $|z|<1 \rightarrow$

$$\sum_{0}^{\infty} z^n = \frac{1}{1-z}$$

(C) Taylor's Theorem

Suppose $f(z)$ is analytic at each point of the disk $|z-z_0|<R_0$ centre z_0 and radius R_0, then

$$f(z) = \sum_{0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n \qquad n=0,1\ldots$$

(D) $\sum_{0}^{\infty} z^n = 1+z+z^2\ldots$

$= \frac{1}{1-z}$ $\quad |z|<1 \rightarrow$ Convergent

$\quad |z| \geq 1 \rightarrow$ Divergent

(E) $\sum_{0}^{\infty} \frac{z^n}{n!} \longrightarrow$ Ratio Test $\lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$

Ratio $<0 \rightarrow$ Convergent

(F) $e^z = 1+z+\frac{z^2}{2!}\ldots\ldots\frac{z^n}{n!}$

(G) $\sin z = \frac{e^{iz}-e^{-iz}}{2i} = \frac{1}{2i}\left[\sum_{0}^{\infty}(1-(-1)^{2n+1})\, i^{2n+1} \frac{z^{2n+1}}{(2n+1)!}\right.$

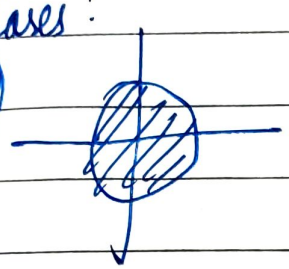
(H) We have 4 cases for questions based on laurents theorem

$$f(z) = \sum \frac{a_n}{(z-z_0)^n} + \sum b_i (z-z_0)^n$$

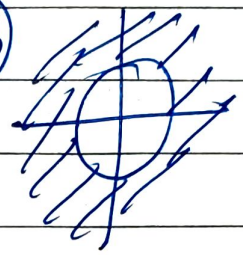
Both +ve & -ve terms

Now Cases:

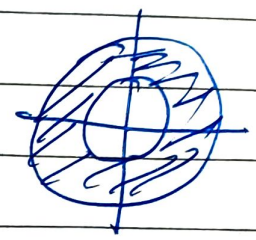
(a)



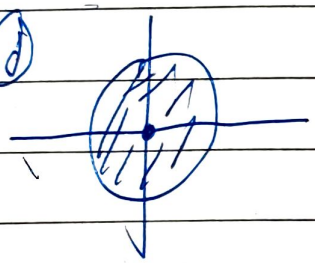
(b)



(c)



(d)



Just put the end case in the inequality and then use that to get infinite GP form

$$\frac{1}{1+z} = \sum_0^\infty (-1)^n z^n$$

$$\frac{1}{1-z} = \sum_0^\infty z^n$$

(I) Zeroes and Poles

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \longrightarrow \int_C f(z)\,dz = 2\pi i b_1$$

$$b_1 = \operatorname{Res}_{z=z_0} f(z)$$

Residue of f at the isolated singular point z_0

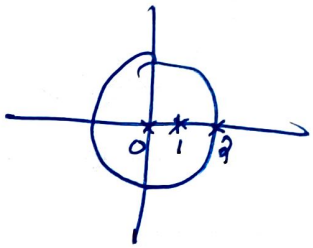
$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z)$$

↳ To use this we are finding integral over a region, where the curve C has a hole, where its not analytic.

(J) Cauchy's Residue theorem

$$\int_C f(z)\,dz = 2\pi i \sum_{1}^{n} \operatorname*{Res}_{z=z_n} f(z)$$

Ex: $\int_C \frac{5z-2}{z(z-1)} \Rightarrow C \to |z| = 2$



↳ Singularities $z=0, z=1$

$0 < |z| < 1$

$$\frac{5z-2}{z}\,\frac{-1}{1-z} = 5 - \frac{2}{z}(-1-z-z^2-\dots)$$

$B_1 = 2$

$0 < |z-1| < 1$

↓

We get $B_2 = 3$

$$\text{Ans} = 2\pi i(5) = 10\pi i$$

* Its easier to separate series sometimes

(K) Negative Powers → Principal Part

If we have only one −ve power (x^{-1}) → Simple Pole

n −ve power ⟶ pole of order n

Zero −ve power ⟶ Removable singular point

classmate

Date

Page

(L) Residues at Poles

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \qquad \operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{m-1}(z_0)}{(m-1)!}$$

(M) Let C be a simple closed contour, described in the sense. If a function is analytic inside and on C except for a finite number of singular points $z_k (k=1,2 \ldots n)$ inside C

$$\int_C f(z)\,dz = 2\pi i \sum_{z=z_k} \operatorname{Res} f(z)$$

(N) Residue at Infinity

$$\int_{C_0} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z) = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

→ Here its analytic everywhere except for a finite no. of singular points inside a region C

(O) Let p, q are analytic at $z=z_0$. $p(z_0) \neq 0$, $q(z) = 0$, $q'(z_0) \neq 0$

$\frac{p(z)}{q(z)}$ has simple pole at z_0

$$\operatorname{Res} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

(P) $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$ real rational function.

$$e^{i\theta} = z \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \left(z + \frac{1}{z}\right)\frac{1}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \left(z - \frac{1}{z}\right)\frac{1}{2i}$$

$\frac{1}{1} \begin{matrix} r^{-2} \\ \times -2 \end{matrix} = \frac{-2}{-2}$ $\quad -2, 1$

We get $d\theta = \frac{dz}{iz} \longrightarrow \int_C F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{iz}$

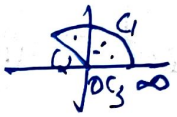
Ex: $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \oint_{|z|=1} \frac{1}{5 + \frac{4(z - 1/z)}{2i}} \frac{dz}{iz} = \oint_C \frac{dz}{2z^2 + 5iz - 2} = \frac{1}{2}\oint_C \frac{dz}{z^2 + \frac{5i}{2}z - 1}$

$= \frac{1}{2}\oint \frac{dz}{\left(z + \frac{i}{2}\right)(z + 2i)}$

Improper Integrals

Mostly we'll have integral 0 to ∞ or $-\infty$ to ∞

+ve singularities



Ensure all points are inside the Curve

Then $\int_{C_1} I + \int_{C_2} I = \int_C$ ← Use Cauchy Here

And then for C_1 use ML, find $\int_{C_2} I$