# Ch 1 - Basic DE

- ODE $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 + \sin y = \cos x$

LDE $a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} \cdots a_n(x)y = b(x)$

- Normal Form

$\frac{dy}{dx} = f(x,y) \leftarrow$ Normal Form $\rightarrow V = e^{-\frac{1}{2}\int P\,dx}$

$q(x) = Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x)$

- General Sols Contain Constants

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# Ch-2 -

① Seperation of Variables

$\frac{dy}{dx} = g(x)h(y) \Rightarrow \frac{dy}{h(y)} = g(x)dx$

ⓐ Equations reducible to SoV

$f(x,y) = f(ax+by+c) = \frac{dy}{dx}$ $\quad ax+by+c = z$

$a + b\frac{dy}{dx} = \frac{dz}{dx}$

$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$

Use SoV

② Homogenous form

$\frac{dy}{dx} = f(x,y)$ $\quad f(tx,ty) = t^n f(x,y)$

↳ Homogenous of degree n

$Mdx + Ndy = 0$ if Homogenous then M and N have same degree

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(5) Exact forms

(i) $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ $\quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k \quad \frac{dy}{dx} = k$

(ii) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ $\quad a_1x + b_1y + c_1 = z$

(iii) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \rightarrow$ They Intersect $\quad X = x - h$ as substitution
$Y = y - k$

(6) If they ask, find $n$ so that it reduces to sov, that means that once you substitute, you do not have any weird term

# Exact DEs

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad M = \frac{\partial f}{\partial x} = f_x \quad N = \frac{\partial f}{\partial y} = f_y$

$\boxed{\therefore M_y = N_x \text{ for exactness}}$

$f = \int M dx + g(y)$ then $\frac{\partial f}{\partial y} = N = \frac{d}{dy}\left[\int M dx + g(y)\right]$

# Integrating factor

$\mu M dx + \mu N dy = 0 \qquad \frac{d}{dy}[\mu M] = \frac{d}{dx}[\mu N]$

$\mu \frac{dM}{dy} + M\frac{d\mu}{dy} = N\frac{d\mu}{dx} + \mu\frac{dN}{dy}$

$\mu\left[\frac{dN}{dx} - \frac{dM}{dy}\right] = M\frac{d\mu}{dy} - N\frac{d\mu}{dx}$

$\mu[N_x - M_y] = [M - N]\frac{d\mu}{dz}$

$\therefore \mu = \mu(z) = e^{\int \frac{N_x - M_y}{M-N} dz}$

$\mu = \mu(z)$ say $z = x+y$
$d\mu/dx = d\mu/dz \cdot dz/dx$
$= d\mu/dz = d\mu/dy$

## Methods for finding IF

(i) $\frac{M_y - N_x}{N} = f(x)$ $\quad IF = e^{\int f(x)dx}$

(ii) $\frac{N_x - M_y}{M} = f(y)$ $\quad IF = e^{\int f(y)dy}$

(iii) M and N are homogenous $\quad IF = \frac{1}{Mx + Ny}$ (Not Differential

(iv) $yf(xy)dx + xg(xy)dy = 0$

$\hookrightarrow IF = \frac{1}{Mx - Ny}$

## Inspection Methods

$d(xy) = xdy + ydx$ $\qquad d(\ln(xy)) = \frac{1}{xy}(xdy + ydx)$

$d(x/y) = ydx - xdy / y^2$

$d(y/x) = xdy - ydx / x^2$ $\qquad d\left(\ln \frac{x}{y}\right) = \frac{y}{x}\left[\frac{ydx - xdy}{y^2}\right]$

$d\left[\tan^{-1}\frac{y}{x}\right] = \frac{xdy - ydx}{x^2 + y^2}$

## LDE

- $\frac{dy}{dx} + P(x)y = Q(x)$ $\qquad$ Soln $\boxed{ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx}$
- $\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P(x) = Q(x)$ Bernoulli's Equation

## Reduction of Order

(i) y is missing

$f(x, y', y'') = 0 \qquad y' = p \quad y'' = \frac{dp}{dx}$

(ii) x is missing

$F(y, y', y'') = 0 \qquad \frac{dy}{dx} = p \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = y'' = \frac{dp}{dx} = p\frac{dp}{dy}$

Orthogonal Trajectories

$\frac{dy}{dx} \to \frac{-dx}{dy}$ and obtain DE

Wronskian Formula

$$\boxed{W = ce^{-\int P(t)dt}}$$

# Second Order DEs

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \qquad y = y_g + y_p$$

$y_p \to$ Particular Soln of $y'' + P(x)y' + Q(x)y = R(x)$

$\to$ No Constants

$y_g \to$ General Soln of Homogenous Part

$$y'' + P(x)y' + Q(x)y = 0$$

How to find G.Soln?

- By Observation, if we find one soln $y_1$, then $y_2 = v_1 y_1$, where

$$\boxed{v_1 = \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx}$$

- Use auxiliary Equation if it's an equation with constant Coefficients

$m^2 + mp + q = 0 \to$ Take roots $m_1, m_2$

(a) $m_1 \neq m_2$ $\quad c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(b) $m_1 = m_2$ $\quad c_1 e^{m_1 x} + x c_2 e^{m_1 x}$

(c) $m_1 = a + ib$, $m_2 = a - ib$ $\quad e^{ax}(c_1 \cos bx + c_2 \sin bx)$

Euler Cauchy's Equation

$x^2 y'' + Pxy' + Qy = 0$. Take $x = e^z$

If $\frac{Q' + 2PQ}{2Q^{3/2}}$ is const, take $z = \int \sqrt{Q(x)}\, dx$ i.e. if $y'' + P(x)y' + Q(x)y = 0$

$$\boxed{dz = \sqrt{Q(x)}\, dx}$$

How to Calculate $y_p$?

ⓐ Method of Undetermined Coefficients, We assume a Soln and then find Coefficients of this Soln

$$y_p'' + py_p' + qy_p = R(x)$$

(i) $R(x) = ce^{ax}$, then

$$y_p = Ace^{ax} \qquad A = \frac{1}{a^2+ap+q}$$

If $a$ is a simple root, $y_p = Axce^{ax}$

$$A = \frac{1}{2a+p}$$

If $a$ is a double root $y_p = Ax^2ce^{ax}$ where $a = \frac{1}{2}$

Now, Here $a$ is given in the question only.

(ii) $R(x) = C\sin bx$ or $D\cos bx$

Take $y_p = A\cos bx + B\sin bx$

If $y_p$ and $y_g$ are redundant multiply $y_g$ by $x$

(iii) $R(x)$ = Polynomial, then, we take $y = ax^n + bx^{n-1} \ldots$

(iv) $R(x) = e^x\cos x$, also similar only

We can superimpose various $R(x)$ Solutions

ⓑ Variation of Parameters

$$y_p = v_1y_1 + v_2y_2 \quad \text{where} \quad v_1 = -\int \frac{y_2R(x)}{W}dx$$

$$v_2 = \int \frac{y_1R(x)}{W}dx$$

# nth Order LDE & System of Equations

If $y^n = g(t, y, y' \ldots y^{n-1})$

$$y = y_1$$
$$y' = y_1' = y_2$$
$$y'' = y_2' = y_3$$
$$\vdots$$
$$y^n = y_{n-1}'$$

Then we solve the eqn, normally. For example, we have

$$y'' + by = 0$$

$$\left.\begin{aligned} y &= y_1 \\ y' &= y_2 = y_1' \\ y'' &= y_2' = y_1'' \end{aligned}\right\} \quad \begin{aligned} y_2 &= y_1' \quad \text{i.e. } \underline{y_1' = y_2} \\ y_2' &= \underline{-by_1} \end{aligned}$$

## Now System of Equations

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- First find the auxiliary eqn, by finding $\begin{vmatrix} A_1 - m & B_1 \\ A_2 & B_2 - m \end{vmatrix} = 0$

Then, if $m_1 \neq m_2$ then replace $m$ in the matrix and solve for $A, B$, where

$$\begin{vmatrix} a_1 - m & b \\ a_2 & b_2 - m \end{vmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

- If $m_1 = m_2 = m$, then

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} (a' + At)e^{mt} \\ (b' + Bt)e^{mt} \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} Ae^{mt} \\ Be^{mt} \end{pmatrix}$$

To find $a', b'$, we solve $\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} \begin{vmatrix} a' \\ b' \end{vmatrix} = \begin{vmatrix} A \\ B \end{vmatrix}$

- If we have $m = a \pm ib$, then soln

$$\begin{pmatrix} x_1 \\ y \end{pmatrix} = e^{at} \begin{bmatrix} c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt) \\ c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt) \end{bmatrix}$$

where, after solving we get $A = A_1 + iA_2$

$B = B_1 + iB_2$

Non Homogenous $\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ $\qquad dx/dt = a_1 x + b_1 y + h_1(t)$

$dy/dt = a_2 x + b_2 y + h_2(t)$

↳ Soln of General Eqn

# Power Series

ⓐ Ordinary Point, at that point the series $P(x)$ and $Q(x)$ are defined.
$y = \sum_{0}^{\infty} a_n (x-x_0)^n$ then find the recurrence relation
We ideally get two solns, one in terms of $a_0$ and other in terms of $a_1$

ⓑ Singular Point
↳ For regular Singular Point

$$\lim_{x \to x_0} (x-x_0) P(x) = P_0 < \infty$$

$$\lim_{x \to x_0} (x-x_0)^2 Q(x) = q_0 < \infty$$

↳ Otherwise irregular singular point

Step 1: Assume Series as $y = (x-x_0)^m \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$m(m-1) + P_0 m + q_0 = 0$$ Indicial Equation

Step 2: Then write max powers of all the terms

ⓐ $|m_1 - m_2| \notin \mathbb{Z}$ – has 2 LI Frobenius Power Series Solns
ⓑ When roots are equal. We will have only one Frobenius PS Soln

$$y_2 = v_1 y_1$$
$$v_1 = \int \frac{1}{y_2} e^{-\int P(x)\,dx}\,dx$$

ⓑ $|m_1 - m_2| \in \mathbb{Z}$, has one for sure → larger root
smaller depends

# Hypergeometric Solution

$$x(1-x)y'' + (c-(a+b+1)x)y' + (-ab)y = 0$$

$$c_1 F(a,b,c,x) + c_2 x^{1-c} F(a+1-c, b+1-c, 2-c, x)$$

Now to convert

$$(x-A)(x-B)y'' + (C+Dx)y' + Ey = 0$$

$$\boxed{t = \frac{x-A}{B-A}}$$

For ex $(x^2-1)y'' + (5x+4)y' + 4y = 0$ at $x=-1$

$$t = \frac{x+1}{1+1} = \frac{x+1}{2} \qquad x = 2t-1$$

$$dt = \frac{dx}{2} \qquad \frac{dy}{dx} = \frac{1}{2}\frac{dy}{dt} \qquad \frac{d^2y}{dx^2} = \frac{1}{4}\frac{d^2y}{dt^2}$$

$$(2t)(2t-2)\frac{y''}{4} + (5(2t)-5+4)\frac{1}{2}y' + 4y = 0$$

$$t(t-1)y'' + \left(5t - \frac{1}{2}\right)y' + 4y = 0 \qquad -ab = 4$$

$$a+b+1 = 5$$

$$c = 1/2 \qquad \cancel{a+b=0} \qquad \boxed{a=2 \quad b=+2}$$

$$a+b=4$$

# Sturm Conversion

$$y'' + P(x)y' + Q(x)y = 0 \qquad v = e^{-\frac{1}{2}\int P(x)dx}$$

$$u'' + q(x)u = 0 \qquad q(x) = Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x)$$

# Legendre's Polynomial

Term: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$ Using this we get various recurrence relations

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

Orthogonality $\int_{-1}^{1} P_m(x) P_n(x)\,dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$

Legendre Series converges to $\frac{1}{2}(f(x^-) + f(x^+))$ at discontinuous points

Polynomial $= \sum_{n=1}^{\infty} a_n \phi_n(x) \rightarrow a_n = \frac{1}{\alpha_n} \int_a^b f(x)\, \phi_n(x)\,dx$

where $\int_a^b \phi_m(x) \phi_n(x)\,dx = \begin{cases} 0 & m \neq n \\ \alpha_n & m = n \end{cases}$

Here $a_n = \frac{2n+1}{2} \int_{-1}^{1} P_n(x) f(x)\,dx$

- $P_n(-x) = (-1)^n P_n(x)$
- $(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$ Substitute Main Recurrence Relation
- $(n+1) P_n(x) = P'_{n+1}(x) - x P'_n(x)$ We get by Differentiation and Main Recurrence
- $n P_n(x) = x P'_n(x) - P'_{n-1}(x)$ We get by Diff
- $\int_{-1}^{1} P_n(x)\,dx = 0$

# Chebyshev's Polynomial

$$(1-x^2)y'' - xy' + p^2 y = 0$$

$$\hookrightarrow T_n(x) + T_{n-2}(x) = 2x\, T_{n-1}(x)$$

$$T_n(x) = F\left(n, -n, \frac{1}{2}, \frac{1-x}{2}\right)$$

$$T_n(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right]$$

$$T_n(x) = \cos[n \cos^{-1} x]$$

$$\int_0^{\pi} \cos m\theta \cos n\theta = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \pi/2 & m = n \neq 0 \end{cases}$$

$$\int_{-1}^{1} \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \pi/2 & m = n \neq 0 \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x) T_0(x)}{\sqrt{1-x^2}} dx$$

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_n(x)}{\sqrt{1-x^2}} dx$$

# Bessel's Polynomial

$$x^2 y'' + xy' + (x^2 - p^2) y = 0$$

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!(n+p)!}$$

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t} dt \quad p > 0 \; ; \; \Gamma(p+1) = p! = p\Gamma(p) \quad \text{if } p = n$$

For Bessel, we get two solns $J_p$ and $J_{-p}$ but if $p$ is an integer, then we get issues

$$y = c_1 J_p + c_2 J_{-p} \qquad p \notin \mathbb{Z}$$

$$\Gamma\left(\frac{1}{2}\right) = \Gamma\left(\frac{-1}{2}\right) = \sqrt{\pi}$$

$$\left(n + \frac{1}{2}\right)! = \Gamma\left(n + \frac{1}{2} + 1\right) = \left(n + \frac{1}{2}\right)\Gamma\left(n + \frac{1}{2}\right)$$

$$= \frac{2n+1}{2}\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n+1)(2n-1)\cdots 3 \cdot 1}{2 \cdot 2 \cdots 2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2n+1)!}{2^{2n+1} n!}\sqrt{\pi}$$

$$\frac{d}{dx} x^p J_p(x) = x^p J_{p-1}(x)$$

$$\frac{d}{dx} x^{-p} J_p(x) = x^{-p} J_{p+1}(x)$$

Reverse for Integral Formula.

$$J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$

$$J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$$

Orthogonal Property $\int_0^1 x J_p(\lambda_m x) J_p(\lambda_n x) = 0$ $\quad m \neq n$

$\frac{1}{2} J_{p+1}^2(\lambda_m)$ $\quad m = n$

$$a_m = \frac{2}{J_{p+1}^2(\lambda_m)} \int_0^1 x f(x) J_p(\lambda_m x)\, dx$$

Fourier Bessel Series

Also $J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left[\frac{x}{2}\right]^{2n}}{n!\,(n+\frac{1}{2})!} \sqrt{\frac{x}{2}}$

$$= \sum_{n=0}^{\infty} \frac{1}{\cancel{2^{2n}}} \frac{(-1)^n x^{2n}}{\cancel{n!}} \sqrt{\frac{x}{\cancel{2}}} \frac{\sqrt{2}}{(2n+1)!\sqrt{\pi}} \cancel{2^{2n+1}}\, \cancel{n!}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)}{n!\,(n-\frac{1}{2})!} = \sqrt{\frac{2}{\pi x}} \cos x$$

$\hookrightarrow \frac{2n!\sqrt{\pi}}{2^{2n} n!}$

Now

$z^2 w'' + z w' + (z^2 - p^2) w = 0 \rightarrow w = c_1 J_p(z) + c_2 J_{-p}(z)$

$\hookrightarrow x^2 y'' + (2c+1) x y' + \left(a^2 b^2 x^{2b} + (c^2 - p^2 b^2)\right) y = 0$

$\hookrightarrow z = a x^b$

$w = y x^c$

Now $\frac{d}{dx} x^p J_p(x) = x^p J_{p-1}(x)$ $\quad p x^{p-1} J_p(x) + x^p J_p'(x) = x^p J_{p-1}$

and we get two formulae

$By = \frac{1}{u} \frac{du}{dx}$ $\quad \frac{dy}{dx} + By^2 = Cx^m \longrightarrow \frac{d^2u}{dx^2} - BCx^m u$

Will give $p = \frac{1}{m+2}$ $\quad$ Compare this with Bess

ILATE Hint

| | + | − | + | − |
|---|---|---|---|---|
| D | $f(x)$ | $f'(x)$ | $f''(x)$ | 0 |
| I | $\int g(x)$ | $\iint g(x)$ | $\iiint g(x)$ | |

# Laplace

$$L[f(x)] = \int_0^\infty e^{-px} f(x)\,dx = F(p)$$

| | | |
|---|---|---|
| $f(x)=1$ | $F(p) = 1/p$ | |
| $f(x) = x$ | $F(p) = 1/p^2$ | $f(x) = e^{ax} = 1/p-a$ |
| $f(x) = x^n$ | $F(p) = n!/p^{n+1}$ | $f(x) = \sin ax = \dfrac{a}{p^2+a^2}$ |
| | | $f(x) = \cos ax = \dfrac{p}{p^2+a^2}$ |

$L[f(x)] = F(p)$ $\quad L[e^{-ax} f(x)] = F(p+a)$

## Properties of Laplace

Derivative $L[y''] = p^2 L[y] - p\,y(0) - y'(0)$

$L[y'] = p\,L(y) - y(0)$

NOTE:

$$\int_0^\infty \frac{f(x)}{x}\,dx = \int_0^\infty F(p)\,dp$$

Multiplication $L[x^n f(x)] = (-1)^n F^n(p)$

Division $L\left[\dfrac{f(x)}{x}\right] = \int_p^\infty F(p)\,dp$

Laplace & Laplace inverse both are linear

Integral $L\left[\int_0^x f(x)\,dx\right] = \dfrac{F(p)}{p}$

NOTE

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\,dx$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\,dx$$

## Applications to DE

Take laplace, solve, take inverse laplace

## Convolution

$$(f*g)x = \int_0^x f(x-t)\,g(t)\,dt = \int_0^x f(t)\,g(x-t)\,dt = (g*f)(x)$$

$$L[f*g] = L(f)\,L(g)$$

# Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos nx + b_n \sin nx \right]$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad \text{but } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

If $f(x)$ is even, then

$$b_n = 0$$

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx$$

is odd

$$a_n = a_0 = 0$$

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx$$

# Fourier Series for $-L \leq x \leq L$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

# Sturm-Liouville BVPs

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[\lambda(q(x)) + r(x)\right]y = 0 \rightarrow \text{Sturm form of}$$

$$p(x)y'' + q(x)y' + r(x)y = 0$$

$$y'' + \lambda y = 0$$

$\lambda < 0$

$\alpha = \sqrt{-\lambda}$

$\lambda = 0$

$y = c_1 + c_2 x$

$\lambda > 0$

$m^2 + \lambda = 0$

Solve and get non trivial solns

# Exact Second Order Homo LDE

$P(x)y'' + Q(x)y' + R(x)y = 0$ is exact iff $[P(x)y']' + [S(x)y]' = 0$

Condition for exactness

$$P''(x) - Q'(x) + R(x) = 0 \quad \text{but } S = Q - P'$$

Integrating Factor

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$$

$$[\mu(x)P(x)y']' + [S(x)y]' = 0$$

Comparing we get $\mu(x)P'(x) + \mu'(x)P(x) + S(x) = \mu(x)Q(x)$ & $S'(x) = R(x)\mu(x)$

To make any eqn self adjoint $P'(x) = Q(x)$

$$\ln P(x) = \int \frac{Q(x)}{P(x)}\,dx$$

$$P(x) = e^{\int \frac{Q}{P}dx}$$

# Wave & Heat Equation

ⓐ **Waves**

$$\alpha^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

$$\alpha = \sqrt{\frac{T}{m}}$$

$T \to$ Time period

$m \to$ mass/length

$y(0,t) = 0 \quad t \geq 0$

$y(L,t) = 0 \quad t \geq 0$

$y(x,0) = f(x)$

$y_t(x,0) = g(x)$

$$y(x,t) = u(x)v(t)$$

$$\alpha^2 u_{xx} v = u v_{tt}$$

$$\frac{u_{xx}}{u} = \frac{1}{\alpha^2}\frac{v_{tt}}{v} = -k$$

$$u_{xx} = -uk$$

$$u'' + ku = 0$$

$$u(0) = u(L) = 0$$

$$\begin{cases} k = \dfrac{n^2\pi^2}{L^2} & n \in N \\ u = c_n \sin \dfrac{n\pi x}{L} \end{cases}$$

$$v_{tt} = -\alpha^2 k v$$

$$v'' + \alpha^2 k v = 0$$

$$v'' + \frac{n^2\pi^2\alpha^2}{L^2} v = 0$$

$$v_n = c_n \cos \frac{n\pi a t}{L} + d_n \sin \frac{n\pi a t}{L}$$

$$y = uv = \sin \frac{n\pi x}{L}\left(b_n \cos \frac{n\pi a t}{L} + d_n \sin \frac{n\pi a t}{L}\right)$$

$$\boxed{y = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left[a_n \cos \frac{n\pi a t}{L} + b_n \sin \frac{n\pi a t}{L}\right]}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \longrightarrow a_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{2}{n\pi a}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

$x \quad 0$

$\frac{L}{n\pi}\cos\frac{n\pi x}{L} \quad -\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}$

classmate

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Q $\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \quad 0 \le x \le \pi \quad t \ge 0$

$y(0,t) = 0 \quad t \ge 0$

$y(\pi,t) = 0 \quad t \ge 0$

$y(x,0) = \frac{1}{\pi}x(\pi - x) \quad 0 \le x \le \pi = f(x)$

$y = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[a_n \cos\frac{n\pi a t}{L} + b_n \sin\frac{n\pi a t}{L}\right]$

$y_t(x,0) = 0 \quad 0 \le x \le \pi = g(x)$

But Here $b_n = \frac{2}{n\pi a}\int_0^L 0 \sin\frac{n\pi x}{L}\,dx = 0$

$a_n = \frac{2}{L}\int_0^L \frac{1}{\pi}x(\pi - x)\sin\frac{n\pi x}{L}\,dx$

$= \frac{2\pi}{\pi L}\int_0^L x\sin\frac{n\pi x}{L}\,dx - \frac{2}{\pi L}\int_0^L x^2 \sin\frac{n\pi x}{L}\,dx$

$= \frac{2}{L}\left[\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^L$

$- \frac{2}{\pi L}\left[\frac{-x^2 L}{n\pi}\cos\frac{n\pi x}{L} + \frac{2xL^2}{n^2\pi^2}\sin\frac{n\pi x}{L} + \frac{2L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}\right]_0^L$

$= \left[\frac{-2x}{n\pi}\cos\frac{n\pi x}{L} + \frac{2L}{n^2\pi^2}\sin\frac{n\pi x}{L} + \frac{2x^2}{n\pi^2}\cos\frac{n\pi x}{L} - \frac{4xL}{n^2\pi^3}\sin\frac{n\pi x}{L} - \frac{4L^2}{n^3\pi^4}\cos\frac{n\pi x}{L}\right]_0^{\pi}$

$L = \pi$

$a_n = \left[\frac{-2x}{n\pi}\cos nx + \cancel{\frac{2L}{n^2\pi^2}\sin nx} + \frac{2x^2}{n\pi^2}\cos nx - \cancel{\frac{4xL}{n^2\pi^3}\sin nx} - \frac{4}{n^3\pi^2}\cos nx\right]$

$= \cancel{-\frac{2\pi}{n\pi}\cos n\pi} + \cancel{\frac{2\pi^2}{n\pi^2}\cos n\pi} - \frac{4}{n^3\pi^2}\cos n\pi + \frac{4}{n^3\pi^2}$

$\boxed{a_n = \frac{4}{n^3\pi^2}\left(1 - (-1)^n\right)}$

$a_{2n} = 0$

$a_{2n-1} = \frac{8}{(2n-1)^3\pi^2}$

| | | | |
|---|---|---|---|
| + | − | + | |
| $x^2$ | $2x$ | $2$ | $0$ |
| $-\frac{L}{n\pi}\cos$ | $-\frac{L^2}{n^2\pi^2}\sin$ | $+\frac{L^3}{n^3\pi^3}\cos$ | |

(b) Heat

$$a^2 \frac{d^2 w}{dx^2} = \frac{dw}{dt} \qquad a = \sqrt{\frac{k}{c\rho}}$$

$w(x,0) = f(x) \quad 0 \le x \le L$

Initial Conditions

$$w(x,t) = u(x)\,v(t) \qquad w_{xx}(x,t) = u''(x)\,v(t)$$
$$w_t(x,t) = u(x)\,v'(t)$$

$$\frac{u''(x)}{u(x)} = \frac{1}{a^2}\frac{v'(t)}{v(t)} = -\lambda$$

$u'' + \lambda u = 0 \longrightarrow \lambda = \frac{n^2\pi^2}{L^2} \quad n \in N$

$v' + \lambda a^2 v = 0$

$$u = c \sin \frac{n\pi x}{L}$$

$$v' + \frac{n^2\pi^2 a^2}{L^2} v = 0$$

$$\ln v = -\frac{n^2\pi^2 a^2}{L^2} t$$

$$\boxed{v = e^{-\frac{n^2\pi^2 a^2}{L^2} t}}$$

$$\therefore y = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2 a^2}{L^2} t} \sin \frac{n\pi x}{L} \qquad c_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\,dx$$

If rods are kept at fixed temperatures $w_1$ & $w_2$, then

$$w(x,t) = u(x) + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2 a^2}{L^2} t} \sin \frac{n\pi x}{L}$$

$$u(x) = w_1 + (w_2 - w_1)\frac{x}{L}$$

$$b_n = \frac{2}{L}\int_0^L [f(x) - u(x)] \sin \frac{n\pi x}{L}\,dx$$