1 K-level learning implementation

1.1 $\mathbf{k(1,1)}$ (Standard EWA algorithm, Camerer & Ho 1999)

For a given player with k-level $k_x = 1$ and opponent (with strategies denoted by $\bar{x}_i$) of k-level $k_y = 1$ the map is given by

$$x_i(t+1) = \frac{x_i(t)^{1-\lambda} \exp\left[\beta \sum_j a_{ij} \bar{x}_j(t)\right]}{\sum_j x_j(t)^{1-\lambda} \exp\left[\beta \sum_l a_{jl} \bar{x}_l(t)\right]} \tag{1}$$

And similarly for the opponent. We find the 6×6 Jacobian for Rock-Paper-Scissors at the central fixed point ($x_i = \frac{1}{3}$):

$$\underline{\underline{J}} = (1-\lambda)\hat{\mathbf{1}} - \frac{1}{3}(1-\lambda)U + \frac{\beta}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where U is a 6×6 matrix with all elements equal to 1. This has distinct eigenvalues

$$\mu = (1-\lambda) \pm i\frac{\beta}{\sqrt{3}}$$

Figure 1 shows the region in $\lambda - \beta$ parameter space defined by the stability condition $|\mu| < 1$.

1.2 $\mathbf{k(2,2)}$

For a player with k-level $k_x = 2$ and opponent of k-level $k_y = 2$ the map is defined recursively

$$x_i(t+1) = \frac{x_i(t)^{1-\lambda} \exp\left[\beta \sum_j a_{ij} \bar{x}'_j(t+1)\right]}{\sum_j x_j(t)^{1-\lambda} \exp\left[\beta \sum_l a_{jl} \bar{x}'_l(t+1)\right]} \tag{2}$$

with

$$\bar{x}'_i(t+1) = \frac{\bar{x}_i(t)^{1-\lambda} \exp\left[\beta \sum_j a_{ij} x_j(t)\right]}{\sum_j \bar{x}_j(t)^{1-\lambda} \exp\left[\beta \sum_l a_{jl} x_l(t)\right]}$$

In this case, we find the 6×6 Jacobian at the RPS fixed point

$$\underline{\underline{J}} = (1-\lambda)\hat{\mathbf{1}} - \frac{1}{3}(1-\lambda)U + \frac{\beta}{3}(1-\lambda)\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \frac{\beta^2}{9}\begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{pmatrix}$$

where U is the 6×6 as defined previously. which has distinct eigenvalues

$$\mu = \left(1 - \frac{\beta^2}{3} - \lambda\right) \pm i\frac{\beta}{\sqrt{3}}(1-\lambda)$$

The stability condition $|\mu| < 1$ defines the region shown in figure 2.

1.3 $\mathbf{k(2,1)}$

For a player with k-level $k_x = 2$, with map given by equation (2) playing against an opponent with k-level $k_y = 1$, with the map given by equation (1), we find the Jacobian at the fixed point

$$\underline{\underline{J}} = (1-\lambda)\hat{\mathbf{1}} - \frac{1}{3}(1-\lambda)U + \frac{\beta}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & -(1-\lambda) & 1-\lambda \\ 0 & 0 & 0 & 1-\lambda & 0 & -(1-\lambda) \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$+ \frac{\beta^2}{9}\begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This has distinct eigenvalues

$$\mu = \left(1 - \frac{\beta^2}{6} - \lambda\right) \pm \frac{\beta}{6}\sqrt{\beta^2 - 12(1-\lambda)}$$

The eigenvalue with the minus sign and the condition $|\mu| < 1$ defines the region shown in figure 5.

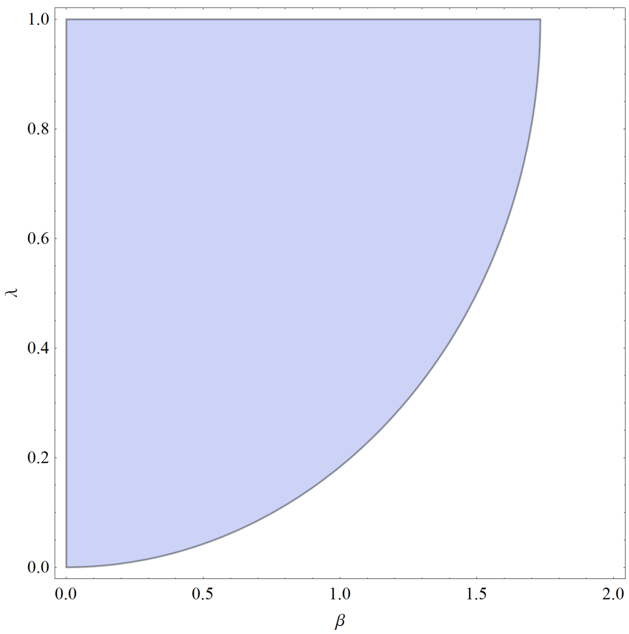

Figure 1: Stability region for $k(1,1)$ RPS at $x_i = \frac{1}{3}$.

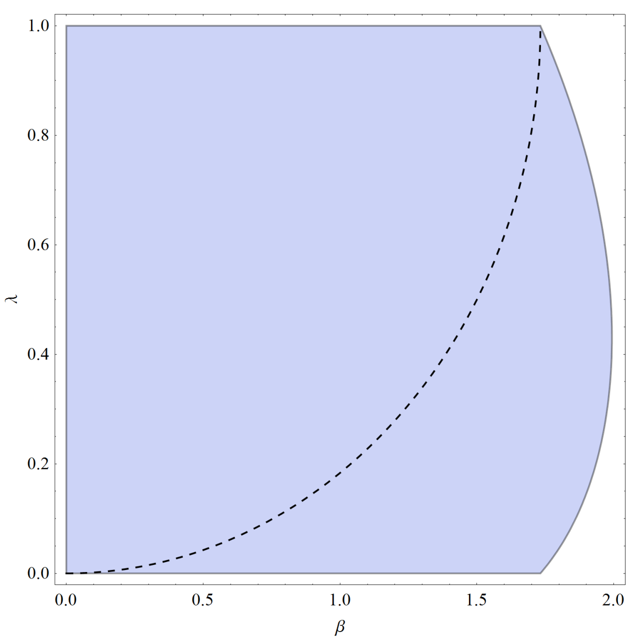

Figure 2: Stability region for $k(2,2)$ RPS at $x_i = \frac{1}{3}$, with the boundary of the corresponding region for $k(1,1)$ shown overlaid

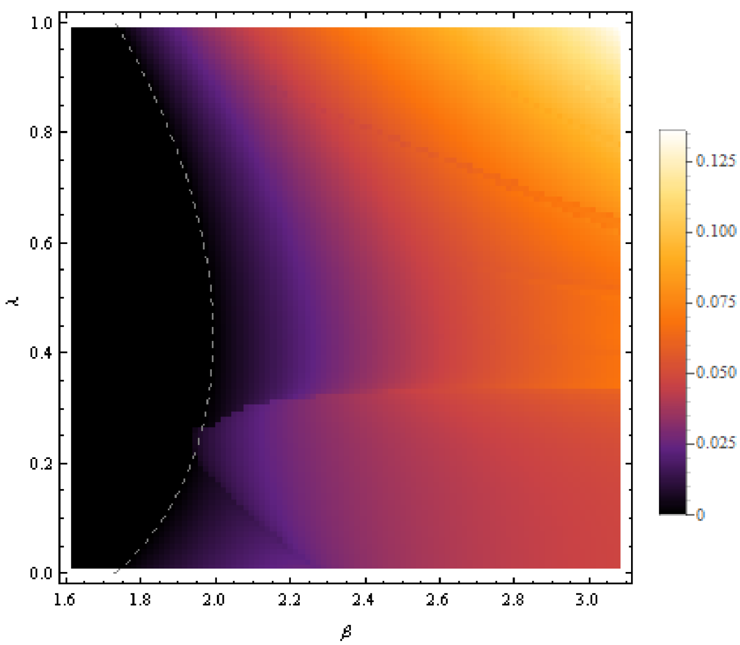

Figure 3: The average variance of strategy components x_i in rounds $6600 - 10000$ for $k(2,2)$ RPS, with the boundary of the analytic stability region (fig. 2) shown as a dashed line.

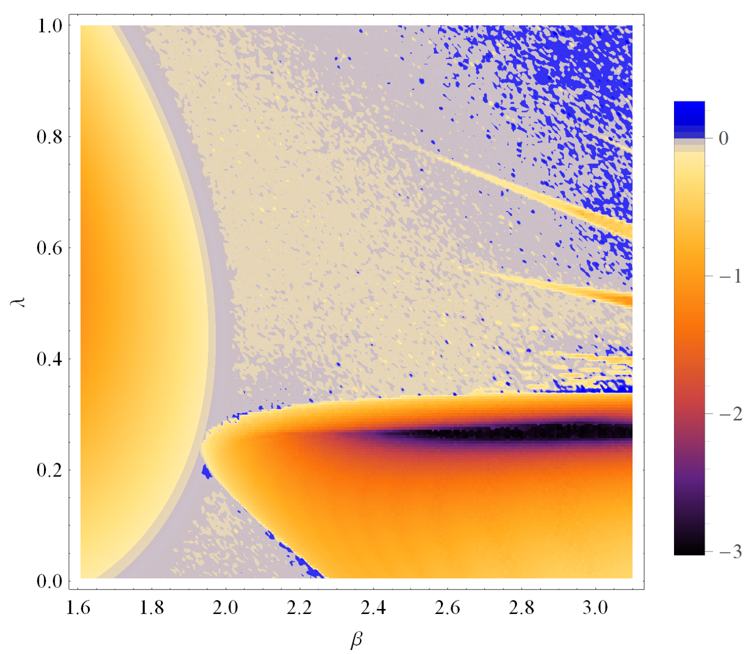

Figure 4: The largest Lyapunov exponent averaged over the attractor for $k(2,2)$ RPS

$$E(t+1) = \sum_{ij} x_i(t) \Pi_{ij} \bar{x}_j(t+1)$$

$$\frac{\partial E(t+1)}{\partial x_i(t)} = \sum_j \Pi_{ij} \bar{x}_j(t+1) \cdot \left\{ 1 + \beta \left(\sum_k \bar{\Pi}_{jk} x_k(t) - \sum_{lk} \bar{x}_l(t+1) \bar{\Pi}_{lk} x_k(t) \right) + \mathcal{O}(\beta^2) \right\}$$

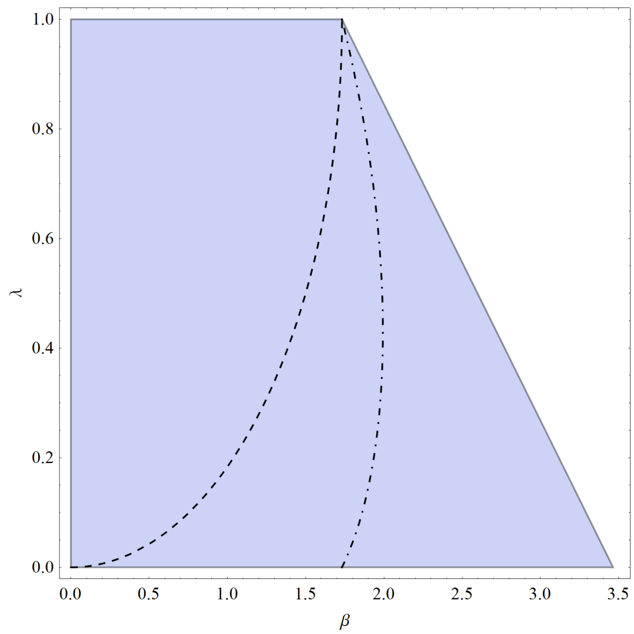

Figure 5: The stability region for $k(2,1)$, with boundaries of the corresponding regions for $k(2,2)$ and $k(1,1)$ overlaid.

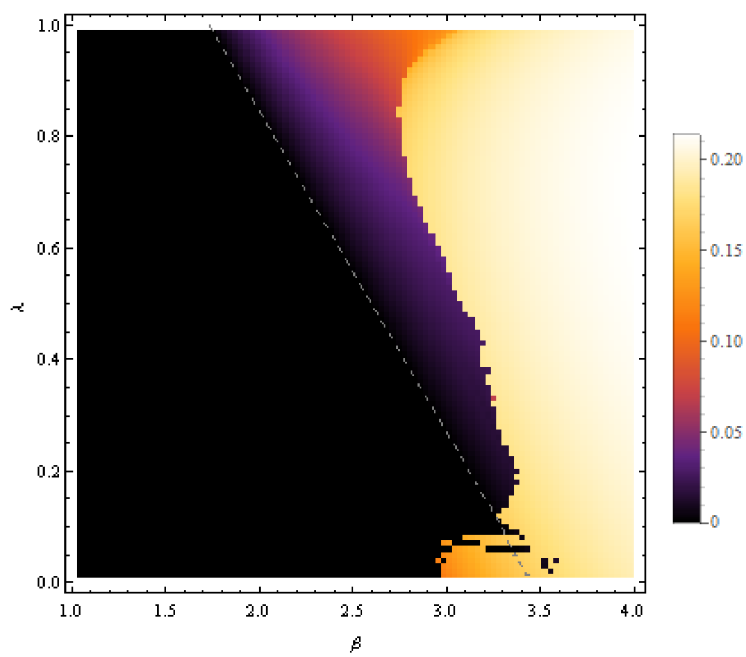

Figure 6: The average variance of strategy components x_i in rounds $6600 - 10000$ for $k(2,1)$ RPS for high β, with the boundary of the analytic stability region (fig. 5) shown as a dashed line.

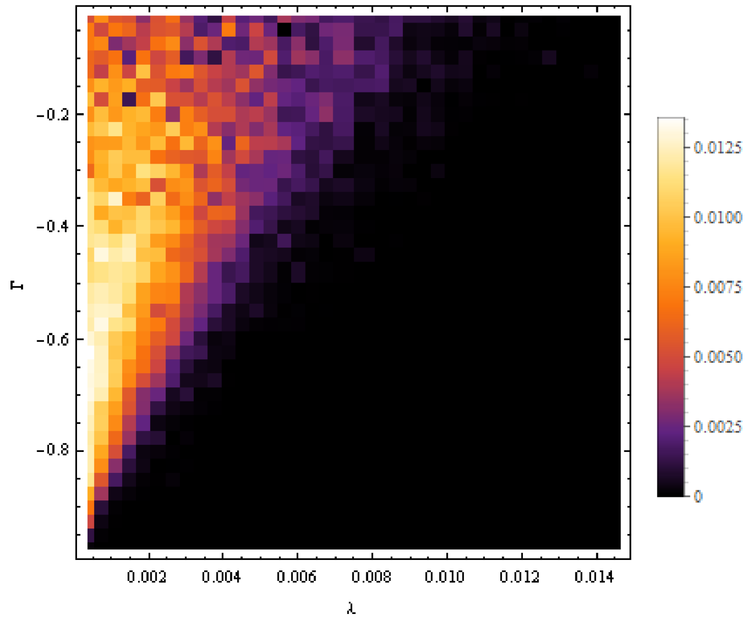

Figure 7: The average variance of strategy components x_i in rounds $6600 - 10000$ for $k(1,1)$ play in large random games (Galla, T., & Farmer, J. D. (2013). “Complex dynamics in learning complicated games”).

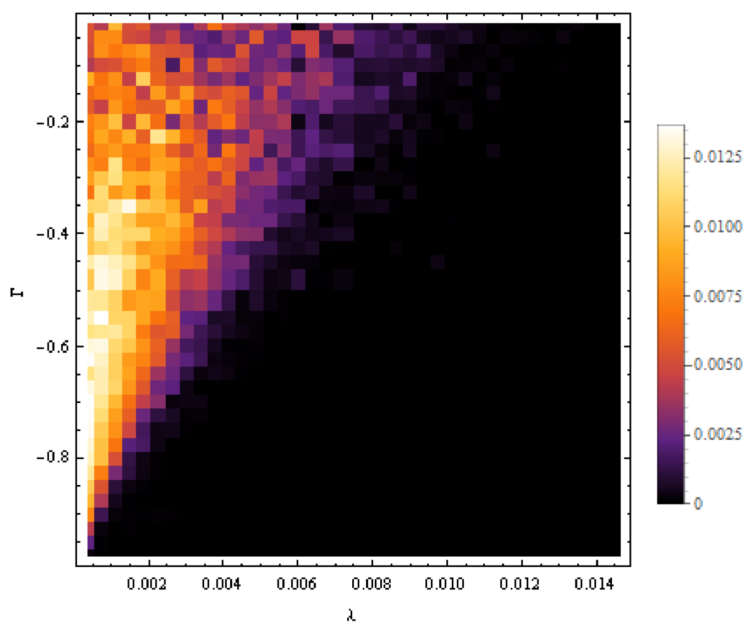

Figure 8: The average variance of strategy components x_i in rounds $6600 - 10000$ for $k(2,2)$ play in large random games.