Tema V.A. continue

Ex 1.

A. $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Leftrightarrow \int_0^{\pi} A\sin x\,dx = 1 \Leftrightarrow \int_0^{\pi} A(-\cos x)'\,dx = 1$

$\Leftrightarrow A\cdot(-\cos x)\Big|_0^{\pi} = 1 \Leftrightarrow A = \frac{1}{2}$

B. $P(x < \frac{\pi}{3}) = \int_{-\infty}^{\pi/3} f(x)\,dx = \int_0^{\pi/3} \frac{1}{2}\sin x\,dx = -\frac{\cos x}{2}\Big|_0^{\pi/3} = 1/4$

$P(x < \pi/4 \mid x > \pi/6) = \frac{P(\pi/6 < x < \pi/4)}{P(x > \pi/6)} =$

$= \frac{\int_{\pi/6}^{\pi/4} f(x)\,dx}{\int_{\pi/6}^{\pi} f(x)\,dx} = \frac{-\frac{1}{2}\cdot\cos x\Big|_{\pi/6}^{\pi/4}}{-\frac{1}{2}\cdot\cos x\Big|_{\pi/6}^{\pi}} = \frac{\cos\pi/4 - \cos\pi/6}{\cos\pi - \cos\pi/6} =$

$= \frac{\sqrt{2}/2 - \sqrt{3}/2}{-1 - \sqrt{3}/2} = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + 2}$

C. $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\pi} x\cdot\frac{\sin x}{2}\,dx = \frac{-1}{2}\int_0^{\pi} x\cdot(\cos x)'\,dx =$

$\frac{-1}{2} x\cos x\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos x\,dx = \frac{\pi}{2} + \frac{1}{2}\sin x\Big|_0^{\pi} = \frac{\pi}{2}$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\pi} x^2\frac{\sin x}{2}\,dx = \frac{1}{2}\int_0^{\pi} x^2(-\cos x)'\,dx =$

$= \frac{1}{2}x^2(-\cos x)\Big|_0^{\pi} - \frac{1}{2}\int_0^{\pi} 2x(-\cos x)\,dx = \frac{\pi^2}{2} + \int_0^{\pi} x\cos x\,dx =$

$= \frac{\pi^2}{2} + \int_0^{\pi} x(\sin x)'\,dx = \frac{\pi^2}{2} + x\sin x\Big|_0^{\pi} - \int_0^{\pi}\sin x\,dx = \frac{\pi^2}{2} + 2$

$Var(X) = E(X^2) - E(X)^2 = \frac{\pi^2}{2} + 2 - \frac{\pi^2}{4} = \frac{\pi^2}{4} + 2$

$\Rightarrow$ abaterea medie patratică a lui X este $\sqrt{\frac{\pi^2}{4} + 2}$

d. $F(x) = \int_{-\infty}^{x} f(t)\, dt$

$$\int_0^x f(t)\,dt = \int_0^x \frac{\sin t}{2}\,dt = \int_0^x \left(\frac{-\cos t}{2}\right)' dt = \frac{-\cos t}{2}\Big|_0^x = \frac{1}{2} - \frac{\cos x}{2}$$

$$\Rightarrow F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2} - \frac{\cos x}{2}, & x \geq 0, x \leq \pi \\ 1, & x > \pi \end{cases}$$

Ex 2.

a. $$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Leftrightarrow \int_0^1 k(e^x + e^{-x})\,dx = 1 \Leftrightarrow$$

$$\Leftrightarrow \int_0^1 k\left((e^x)' - (e^{-x})'\right) dx = 1 \Leftrightarrow k(e^x - e^{-x})\Big|_0^1 = 1$$

$$\Leftrightarrow k(e - 1 + 1 - e^{-1}) = 1 \Leftrightarrow k = \frac{1}{e - e^{-1}} = \frac{e}{e^2 - 1}$$

b. $$\int_0^x f(x)\,dx = \int_0^x k(e^t + e^{-t})\,dt = k(e^t - e^{-t})\Big|_0^x = k\cdot(e^x - e^{-x})$$

$\Rightarrow$ Funcția de repartiție a lui X este

$$F(x) = \begin{cases} 0, & x < 0 \\ k\cdot(e^x - e^{-x}), & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

$$P\left(x < \frac{1}{2} \mid x > \frac{1}{4}\right) = \frac{P\left(\frac{1}{4} < X < \frac{1}{2}\right)}{P\left(X > \frac{1}{4}\right)} =$$

$$= \frac{F(1/2) - F(1/4)}{1 - F(1/4)} = \frac{k(e^{1/2} - e^{-1/2} - e^{1/4} + e^{-1/4})}{1 - k(e^{1/4} - e^{-1/4})}$$

$$c.\ E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\cdot k\cdot(e^x + e^{-x})\,dx =$$

$$= k\cdot\int_0^1 x(e^x - e^{-x})'\,dx = k\cdot x\cdot(e^x - e^{-x})\Big|_0^1 - k\int_0^1 e^x - e^{-x}\,dx$$

$$= k(e - \tfrac{1}{e}) - k\cdot(e^x + e^{-x})\Big|_0^1 = k(e - \tfrac{1}{e} - e - \tfrac{1}{e} + 2)$$

$$= k\left(2 - \tfrac{2}{e}\right)$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 k(e^x + e^{-x})\,dx = k\cdot\int_0^1 x^2\cdot(e^x - e^{-x})'\,dx$$

$$= k\cdot x^2(e^x - e^{-x})\Big|_0^1 - 2k\int_0^1 x(e^x + e^{-x})'\,dx =$$

$$= k(e - \tfrac{1}{e}) - 2k\cdot x(e^x + e^{-x})\Big|_0^1 + 2k\int_0^1 e^x + e^{-x}\,dx =$$

$$= k(e - \tfrac{1}{e}) - 2k(e + \tfrac{1}{e}) + 2k(e^x - e^{-x})\Big|_0^1 =$$

$$= \cancel{(e - \tfrac{1}{e})\cdot 3k + 2k(e - \tfrac{1}{e}) = 5k(e - \tfrac{1}{e})}$$

$$= k\cdot e - \frac{k}{e} - 2ke - \frac{2k}{e} + 2k(e - \tfrac{1}{e}) = k\left(e - \frac{5}{e}\right)$$

$$Var(X) = E(X^2) - E(X)^2 = k\left(e - \frac{5}{e}\right) - k^2\left(4 + \frac{4}{e^2} - \frac{8}{e}\right)$$

$$= \frac{e}{e^2-1}\left(e - \frac{5}{e} - \frac{e}{e^2-1}\left(4 + \frac{4}{e^2} - \frac{8}{e}\right)\right) =$$

$$= \frac{e}{e^2-1}\cdot\frac{\cancel{e}\cdot(e^2-1)\cdot e^2 - 5\cancel{e}(e^2-1) - 4e^{2} - 4\cancel{e} + 8e^{\cancel{2}}}{(e^2-1)\cdot \cancel{e^2}}$$

$$= \frac{e}{e^2-1}\,\frac{e^4 + e^2(-1-5-4) + 8e + 1}{e^2-1} = \frac{e^5 - 10e^3 + 8e^2 + e}{(e^2-1)^2}$$

Ex 3.

a. $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Leftrightarrow \int_0^1 k\cdot x^{a-1}(1-x)^{b-1}\,dx = 1$

$$\Leftrightarrow k\cdot \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)} = 1 \Leftrightarrow k = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}$$

b. $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = k\cdot\int_0^1 x^{a}(1-x)^{b-1}\,dx = k\cdot\frac{\Gamma(a+1)\cdot\Gamma(b)}{\Gamma(a+b+1)} =$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}\cdot\frac{a\cdot\Gamma(a)\cdot\Gamma(b)}{(a+b)\,\Gamma(a+b)} = \frac{a}{a+b}$$

$$E(X^2) = k\int_0^1 x^{a+1}(1-x)^{b-1}\,dx = k\cdot\frac{\Gamma(a+2)\,\Gamma(b)}{\Gamma(a+b+2)} = \frac{a(a+1)}{(a+b)(a+b+1)}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{a(a+1)}{(a+b)(a+b+1)} - \frac{a^2}{(a+b)^2} = \frac{a(a+1)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)} =$$

$$= \frac{a(a^2+ab+a+b-a^2-ab-a)}{(a+b)^2(a+b+1)} = \frac{ab}{(a+b)^2(a+b+1)}$$

$$m_r(X) = \int_{-\infty}^{\infty} x^r f(x)\,dx = \int_0^1 k\cdot x^{a+r-1}(1-x)^{b-1}\,dx =$$

$$= k\cdot\frac{\Gamma(a+r)\,\Gamma(b)}{\Gamma(a+b+r)} = \frac{\Gamma(a+b)}{\Gamma(a+b+r)}\cdot\frac{\Gamma(a+r)}{\Gamma(a)} = \frac{(a+r-1)\cdots a}{(a+b+r-1)\cdots(a+b)}$$

c. $(a,b) = (2,3)$

$$\Rightarrow f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}\,x(1-x)^2 = \frac{24}{2}\,x(1-x)^2 = 12x(1-x)^2$$

Fie $F(x)$ funcția de repartiție.

$$\int_0^x f(t)\,dt = \int_0^x 12t(1-t)^2\,dt = \int_0^x 12t+12t^3-24t^2\,dt =$$

$$= \int_0^x (6t^2)' + (3t^4)' - (8t^3)'\,dt = 6t^2+3t^4-8t^3\Big|_0^x = 3x^4-8x^3+6x^2$$

$$\Rightarrow F(x) = \begin{cases} 0, & x<0 \\ 3x^4-8x^3+6x^2, & 0\le x\le 1 \\ 1, & x>1 \end{cases}$$

$$P\left(X < \tfrac{1}{2}\right) = F(1/2) = \frac{3}{2^4} - \frac{8}{2^3} + \frac{6}{4} = \frac{3-16+24}{16} = \frac{11}{16}$$

$$P\left(X > \tfrac{1}{3}\right) = 1 - F(1/3) = 1 - \frac{3}{3^4} - \frac{8}{3^3} + \frac{6}{3^2} = \frac{1-8+18}{3^3} = \frac{11}{27}$$

$$P(X \le 1/2 \mid X > 1/4) = \frac{P(1/4 < X \le 1/2)}{P(X > \frac{1}{4})} = \frac{F(1/2) - F(1/4)}{1 - F(1/4)} = \ldots$$

4. $f(x) = \begin{cases} k \cdot x \cdot e^{-\frac{x^2}{2a^2}}, & x \ge 0 \\ 0, & x \le 0 \end{cases}$

a.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Leftrightarrow \int_0^{\infty} k \cdot x \cdot e^{\frac{-x^2}{2a^2}}\,dx = 1$$

$$\Leftrightarrow \int_0^{\infty} k \cdot x \cdot \frac{-2a^2}{2x}\left(e^{\frac{-x^2}{2a^2}}\right)'dx = 1 \Leftrightarrow -ka^2 e^{\frac{-x^2}{2a^2}}\Big|_0^{\infty} = 1$$

$$\Leftrightarrow +ka^2 = 1 \Rightarrow k = +\frac{1}{a^2}$$

b.

$$\int_0^x f(t)\,dt = \int_0^x \frac{x}{a^2} e^{\frac{-t^2}{2a^2}}\,dt = \int_0^x \frac{t}{a^2}\cdot\left(-\frac{2t}{2a^2}\right)\cdot\left(\frac{-2a^2}{2t}\right)\cdot e^{\frac{-t^2}{2a^2}}\,dt$$

$$= \int_0^x \frac{t}{a^2}\left(-\frac{2a^2}{2t}\right)\left(e^{\frac{-t^2}{2a^2}}\right)'dt = -e^{\frac{-t^2}{2a^2}}\Big|_0^x = 1 - e^{\frac{-x^2}{2a^2}}$$

$$\Rightarrow F(x) = \begin{cases} 0, & x \le 0 \\ 1 - e^{\frac{-x^2}{2a^2}}, & x > 0 \end{cases}$$

d.

$$P(X < 2a) = F(2a) = 1 - e^{\frac{-4a^2}{2a^2}} = 1 - e^{-2} = 1 - \frac{1}{e^2}$$

$$P(X > a) = 1 - F(a) = 1 - e^{\frac{-a^2}{2a^2}} = 1 - e^{-\frac{1}{2}} = 1 - \frac{1}{\sqrt{e}}$$

$$P(X \le 4a \mid X > 2a) = \frac{P(2a < X < 4a)}{P(X > 2a)} = \frac{F(4a) - F(2a)}{1 - F(2a)} = \ldots$$

c.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} k x^2 e^{\frac{-x^2}{2a^2}}\,dx = -\int_0^{\infty} x \left(e^{\frac{-x^2}{2a^2}}\right)'dx$$

$$= -x e^{\frac{-x^2}{2a^2}}\Big|_0^{\infty} + \int_0^{\infty} e^{\frac{-x^2}{2a^2}}\,dx$$

$$\lim_{x\to\infty} x e^{\frac{-x^2}{2a^2}} = \lim e^{\ln x - \frac{x^2}{2a^2}} = 0$$

Pt. $\int_0^{\infty} e^{\frac{-x^2}{2a^2}}\,dx$ facem $t = \frac{x}{a\sqrt{2}} \Rightarrow dt = \frac{dx}{a\sqrt{2}} \Leftrightarrow dx = dt \cdot a\sqrt{2}$

$$\int_0^{\infty} e^{\frac{-x^2}{2a^2}}\,dx = \int_0^{\infty} e^{-t^2} a\sqrt{2}\,dt = \sqrt{\tfrac{\pi}{2}}\,a$$

$$\Rightarrow E(X) = \sqrt{\tfrac{\pi}{2}}\,a$$

$$E(X^2) = \int_0^\infty k\, x^3 e^{\frac{-x^2}{2a^2}}\, dx = -\int_0^\infty x^2 \left(e^{\frac{-x^2}{2a^2}}\right)' dx =$$

$$= -x^2 e^{\frac{-x^2}{2a^2}} \Big|_0^\infty + 2\int_0^\infty x e^{\frac{-x^2}{2a^2}}\, dx =$$

$$= \int_0^\infty (-2a^2) \left(e^{\frac{-x^2}{2a^2}}\right)' dx = -2a^2 e^{\frac{-x^2}{2a^2}} \Big|_0^\infty = 2a^2$$

$$\Rightarrow Var(X) = E(X^2) - E(X)^2 = 2a^2 - \frac{\pi}{2} a^2 = a^2 \left(2 - \frac{\pi}{2}\right)$$

5. $f_n(x) = \begin{cases} p \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{3}}, & x > 0 \\ 0, & x < 0 \end{cases}$

a. $I = \int_{-\infty}^{\infty} f_n(x)\, dx = 1 \Leftrightarrow \int_0^\infty p\, x^{\frac{n}{2}-1} e^{-\frac{x}{3}}\, dx = 1.$

Fie $t = \frac{x}{3} \Leftrightarrow dt = \frac{dx}{3} \Leftrightarrow dx = 3\,dt$

$$\Rightarrow I = \int_0^\infty p\,(3t)^{\frac{n}{2}-1} e^{-t}\, 3\,dt = 3^{\frac{n}{2}} p \int_0^\infty t^{\frac{n}{2}-1} e^{-t}\, dt = 3^{\frac{n}{2}} p\, \Gamma(n/2)$$

$$\Rightarrow p = \frac{1}{3^{n/2}\, \Gamma(n/2)} = \begin{cases} \frac{1}{3^{n/2} (n/2 - 1)!}, & n \text{ par} \\ \frac{1}{3^{n/2} (n/2 - 1) \cdots \frac{3}{2} \cdot \sqrt{\pi}}, & n \text{ impar} \end{cases}$$

b. $X + Y = Z$

$$f(z) = \int_0^z f_n(t) \cdot f_m(z-t)\, dt$$

$$= \int_0^z p_n \cdot p_m \cdot e^{-\frac{t}{3} + \frac{(-z+t)}{3}}\, t^{\frac{n}{2}-1} (z-t)^{\frac{m}{2}-1}\, dt$$

$$= p_n \cdot p_m \cdot e^{-\frac{z}{3}} \int_0^z t^{n/2-1} (z-t)^{\frac{m}{2}-1}\, dt$$