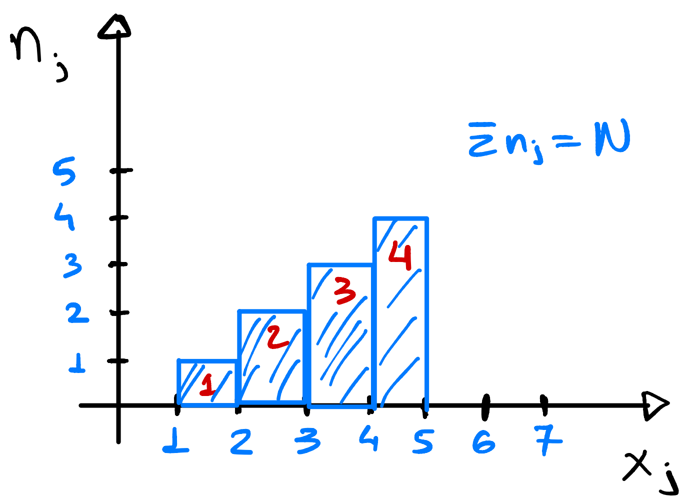
Ιστογραμματα – Κατανομες

Δειγμα δεδομενων $x_i = \{3,2,4,1,2,3,4,3,4,4\}$



$$\sum_{i=0}^{N-1} 1 = N = \underbrace{1+1+\dots 1}_{N} = 10$$

$$\frac{1}{N}\sum_i x_i = \frac{1}{10}(3+2+\dots+4) = \bar{x} = \hat{\mu}$$

$$\frac{1}{N}\sum_i (x_i-\bar{x})^2 = s_N^2$$

$$\frac{1}{N-1}\sum_i (x_i-\bar{x})^2 = s^2$$

$$s^2 = \frac{N}{N-1} s_N^2$$

Ιστογραμμα $(n_j, x_j) \longrightarrow j$ bins

$$\begin{pmatrix} 1, & 1 \\ 2, & 2 \\ 3, & 3 \\ 4, & 4 \end{pmatrix}$$

$$\sum_j n_j = N = 1+2+3+4 = 10$$

$$\frac{1}{N}\sum n_j x_j = \frac{1}{10}[1\cdot 1 + 2\cdot 2 + 3\cdot 3 + 4\cdot 4] = \frac{30}{10} = 3$$

$$\frac{1}{N}\sum n_j (x_j-\bar{x})^2 = \frac{1}{10}\Big[1(1-3)^2 + 2\cdot(2-3)^2 + 3(3-3)^2 + 4(4-3)^2\Big] = \frac{4+2+0+4}{10} = 1$$

$$\lim_{N\to\infty}\frac{n_j}{N} = f_j \equiv f(x_j) \qquad X \sim f$$

Τυχαιες Μεταβλητες

$X(\omega_i) = x_i$ $\qquad X =$ συναρτηση

π.χ. 1 κερμα $\quad \Omega = \{\kappa, \Gamma\}$

$$X(\kappa) = +1 = x_1$$
$$X(\Gamma) = -1 = x_2$$

f = συναρτηση πιθανοτητας

$f(x) = P(X=x)$ (PMF)

F = συναρτηση κατανομης

$F(x) = P(X \leq x)$ (CDF)

διακριτη ΤΜ

f = συναρτηση πυκνοτητας πιθανοτητας

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx \quad \text{(PDF)}$$

$P(X=x) = \ ?$

F αθροιστικη συναρτηση πυκνοτητας πιθανοτητας (CDF)

ή συναρτηση κατανομης

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

$$f(x) = \frac{d}{dx}\big(F(x)\big)$$

συνεχης ΤΜ

Αναμενομενες τιμες

$$E[X] \equiv E(X) \equiv \langle X \rangle \equiv \int_{-\infty}^{+\infty} x\, f(x)\,dx = \mu \quad \left(\sum_i x_i p_i = \mu\right)$$

μεση τιμη

$$E[(X-\mu)^2] \equiv Var(X) = \langle (X-\mu)^2 \rangle \equiv \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx = \sigma^2$$

rms , standard deviation , τυπικη αποκλιση $\sqrt{Var(X)} = \sigma$

$$Var(X) = \langle (X-\mu)^2 \rangle = \langle x^2 \rangle - 2\langle x \rangle \mu + \mu^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$\langle x \rangle \equiv \mu$

$$Var(aX) = a^2\, Var(X)$$

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

$$Var(\bar{x}) = Var\left(\frac{1}{N}\sum_i X\right) = \frac{1}{N^2}\sum_i Var(X) = \frac{1}{N^2}\sum_i \sigma^2 = \frac{\sigma^2}{N^2}\sum_i 1 = \frac{\sigma^2}{N}$$

$$\sigma_{\bar{x}} = \sigma/\sqrt{N}$$

Παραδειγμα

$$x_i = \{1, 2, 3, 4, 5, 6\}$$
$$p_i = \left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$$

$\langle x \rangle =$;

$\sigma^2 =$;

$$\longrightarrow \quad \sum x_i p_i = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5$$

$$\mu^2 + \sigma^2 = \frac{1}{6}\sum x_i^2 = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

$$\mu^2 + \sigma^2 = \langle x^2 \rangle$$
$$\langle x \rangle^2 + \sigma^2 = \langle x^2 \rangle$$

$$\sigma^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = 2.916$$
$$\sigma = (2.916)^{0.5}$$

Εκτιμητες (δειγμα δεδομενων)

x_i {δειγμα δεδομενων}

$$\bar{x} = \hat{\mu} = \frac{1}{N}\sum x_i \qquad \lim_{N\to\infty} \hat{\mu} = \mu$$

$$\bar{x} = \hat{\mu} = \frac{1}{N}\sum_j n_j x_j$$

$$\overline{x^2} = \frac{1}{N}\sum_i x_i^2 = \frac{1}{N}\sum_j n_j x_j^2 \qquad \bar{f} = \frac{1}{N}\sum f(x_i)$$

$$s^2 \equiv \frac{1}{N-1}\sum_i (x_i - \bar{x})^2 \qquad s_N^2 \equiv \frac{1}{N}\sum_i (x - \bar{x})^2$$

$$s_\mu^2 \equiv \frac{1}{N}\sum_i (x_i - \mu)^2$$

$$\langle s_N^2 \rangle = E\left[(X - \bar{x})^2\right] = \ldots = \frac{N-1}{N}\sigma^2$$

$$= E[x^2] - E[\bar{x}]^2 = E[x^2] - E[x]^2 + E[\bar{x}]^2 - E[\bar{x}]^2$$

$$E[\bar{x}] = E[x]$$

$$E[x^2] - E[x]^2 - \left(E[\bar{x}]^2 - E[\bar{x}]^2 \right) = Var(X) - Var(\bar{X})$$

$$= Var(X) - \frac{Var(X)}{N} = \frac{N}{N}\sigma^2 - \frac{\sigma^2}{N} = \frac{N-1}{N}\sigma^2 = E[s_N]$$

αρα $\langle s_n^2 \rangle \neq \sigma^2$ (αλλα $\frac{N-1}{N}\sigma^2$)

συνεπως ο $s^2 \equiv \frac{N}{N-1} s_N^2$ ειναι ενας "καλος" → μη προκατειλημμενος εκτιμητης του σ^2 με $\langle s^2 \rangle = \sigma^2$

Linear Congruential Method

γραμμικος μετασχηματισμος ισοδυναμιας υπολοιπου

$$x_{n+1} = (c \cdot x_n + a) \bmod (N_{max})$$

$\kappa \bmod (\lambda)$ = υπολοιπο ακεραιας διαιρεσης
$\kappa \% \lambda$

$7 \% 3 = 1$

$$\begin{array}{r|l} 7 & 3 \\ \hline -6 & 2 \\ \hline 1 & \end{array}$$

παραδειγμα :

$$x_{n+1} = 16807 \cdot x_n \bmod (2^{31} - 1)$$

... 32 bits

$$X \sim \text{unif}(0, 2^{31} - 1)$$

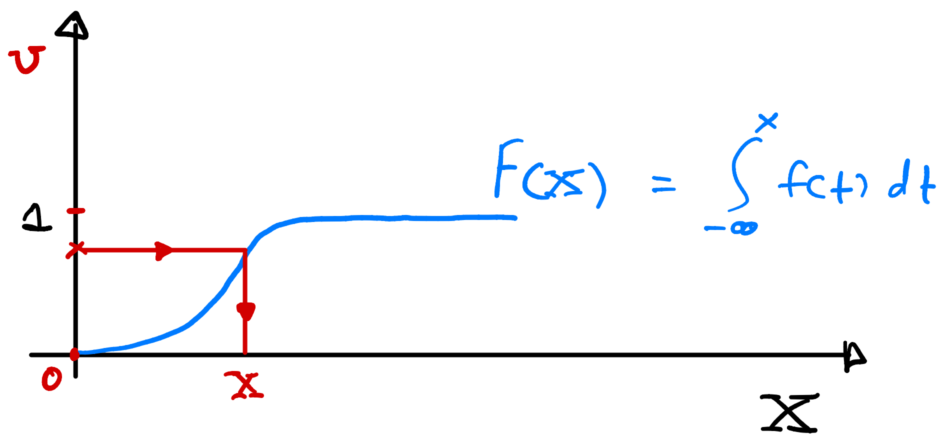
Παραγωγη τυχαιων αριθμων $X \sim f(x)$

a) μεθοδος αντιστροφου μετασχηματισμου

$$F(x) = P(X \leq x)$$

$$X = F^{-1}(U) \quad \text{με} \quad U \sim unif(0,1)$$

$$P(X \leq x) = P(F^{-1}(U) \leq x) = P(U \leq F(x)) = \int_0^{F(x)} 1\, dU = F(x)$$



1) $X \sim unif(a,b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ \emptyset & x \notin [a,b] \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_a^x \frac{1}{b-a}\,dt = \frac{x-a}{b-a} \quad (\text{με } x \leq b)$$

$$\boxed{F^{-1}(x) = (b-a)x + a}$$

$$F(F^{-1}(x)) = x \qquad F^{-1}(F(x)) = x$$

$$F(F^{-1}(x)) = \frac{(b-a)x + a - a}{b-a} = x$$

$$U \sim unif(0,1) \qquad X \sim unif(-1,9)$$

$$X = (9-(-1)) \cdot U + (-1)$$
$$= 10U - 1$$

b) $$f(x) = \begin{cases} c \cdot x & x \in [0,1) \\ 0 & x \notin [0,1) \end{cases}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1 \rightarrow \int_0^1 c \cdot x\,dx \Rightarrow c = 2$$

$$f(x) = \begin{cases} 2x & x \in [0,1] \\ 0 & x \notin [0,1) \end{cases}$$

$$F(x) = \int_0^x 2t\,dt = x^2$$

$$F^{-1}(x) = \sqrt{x}$$

$$X = \sqrt{U}$$

c) $$f(x) = \begin{cases} c \cdot e^x & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$

προσδιοριζω πρωτα το c

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_a^b c \cdot e^t\,dt = 1 \Rightarrow c = (e^b - e^a)^{-1}$$

$$F(x) = \int_a^x \frac{e^t}{e^b - e^a}\,dt = \frac{e^x - e^a}{e^b - e^a}$$

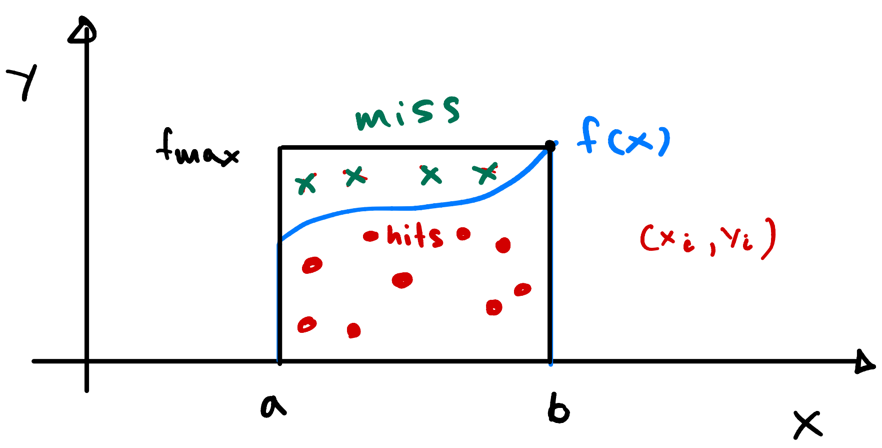
$$\boxed{F^{-1}(x) = \ln\left[x[e^b - e^a] + e^a\right]}$$

$$F^{-1}(F(x)) = \ln\left[\frac{e^x - e^a}{e^b - e^a}[e^b - e^a] + e^a\right] = \ln e^x = x$$

$$X = \ln\left[U[e^b - e^a] + e^a\right]$$

b) Δειγματοληψία απόρριψης (accept/reject)

hit or miss



$a \le x \le b$

$f(x) \le f_{max}$

$x_i \sim unif(a, b)$

$y_i \sim unif(0, f_{max})$

αν $y_i \le f(x_i)$ κρατάμε το $x_i \to \{x_1, x_2, \ldots\}$

$X \sim f(x)$

επόμενη φορά : $X \sim c e^x$ $\quad x \in [0, 5]$

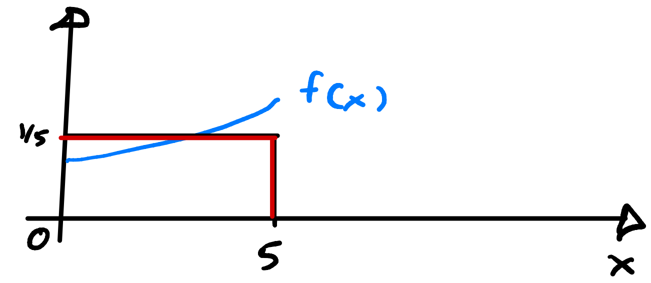
a) με αντίστροφο μετασχηματισμό

b) με accept / reject

$$X \sim f(x) = \begin{cases} c \cdot e^x & \text{για } x \in [0,5] \\ 0 & \text{για } x \notin [0,5] \end{cases}$$

$$\int_{-\infty}^{+\infty} f(x)\, dx = 1 \Rightarrow \int_0^5 c e^x\, dx = 1 \Rightarrow c = (e^5 - 1)^{-1}$$

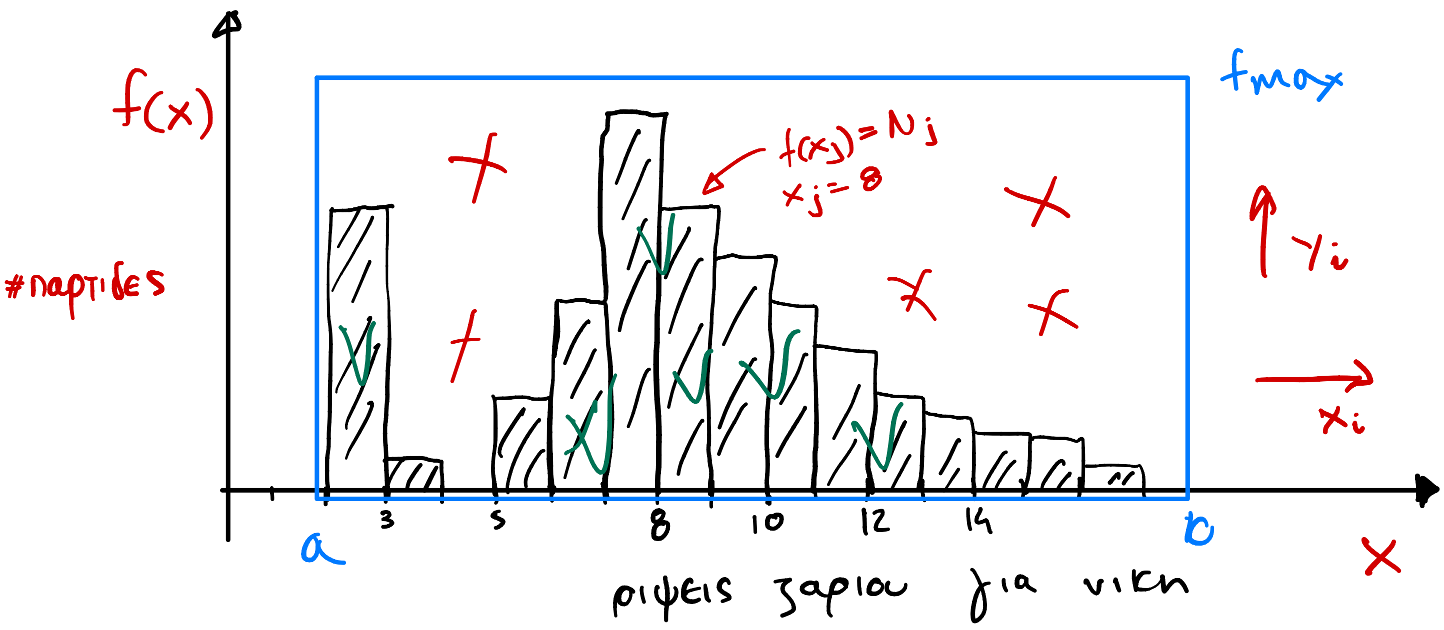
$$f(x) = \frac{e^x}{e^5 - 1}$$

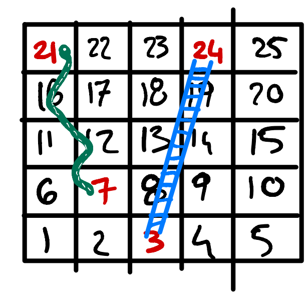


$$F^{-1}(x) = \ln\left[x[e^5 - e^0] + e^0\right]$$

$$X = \ln\left[U[e^5 - e^0] + e^0\right]$$

Φιδακι





$\{y_i\ x_i\}$

"εξομοιωση" αν φτιαξουμε γεννητρια X με $X \sim f(x)$

Αθροισμα ΤΜ

$X = X_1 + X_2$

$X_1 \sim unif(0,1)$

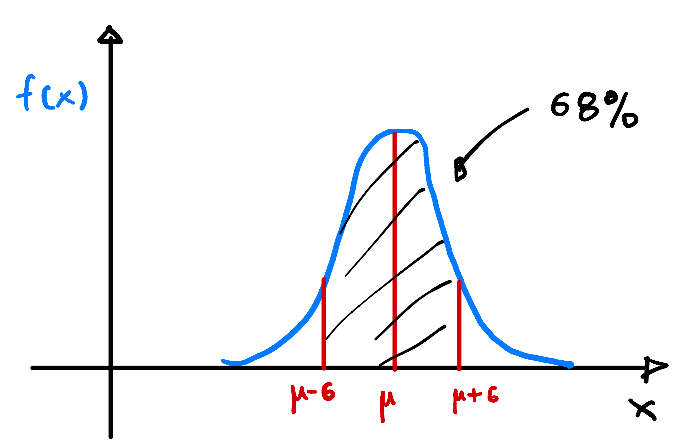
$X_2 \sim unif(0,1)$

$X \sim ?$

Θεωρημα Κεντρικου Οριου :

$X = \sum_i^N x_i \qquad x_i \sim f_i$

$$\lim_{N\to\infty} X \sim N\left(\mu = \sum \mu_i \,,\ \sigma^2 = \sum \sigma_i^2\right) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



για N ανεξαρτητες μετρησεις της ιδιας ποσοτητας

$$\bar{X} = \frac{1}{N}\sum_i^N x_i \qquad \boxed{f_i \sim f} \text{ ιδια } \forall i$$

$$\lim_{N\to\infty} \bar{X} \sim N\left(\mu = \mu_i \,,\ \sigma_{\bar{x}}^2 = \sigma^2/N\right)$$

$$\mu_i = \langle x_i \rangle = \mu = \langle \bar{x} \rangle = \langle \tfrac{1}{N}\sum_i x_i \rangle$$

$$\sigma_i = \langle x_i^2 \rangle - \mu^2 \equiv \sigma \rightsquigarrow \sigma_{\bar{x}} = \sigma/\sqrt{N}$$

Ραδιενεργες διασπασεις

$p = \lambda \cdot \Delta t$ πιθανοτητα διασπασης ενος πυρηνα στο $[t, t+\Delta t]$

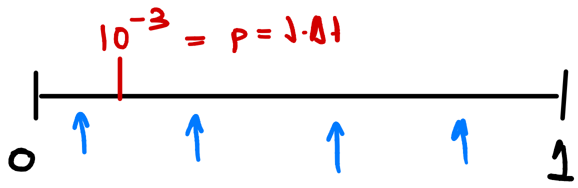
Υποθεσεις :

a) καθε διασπαση ειναι ανεξαρτητη

b) $p =$ σταθ (οχι μνήμη , ανεξαρτητη του χρονου)

$N(t) =$;

Στοχαστική εξομοίωση για $N_0 = 30$ $\quad \lambda = 10^{-3}\, s^{-1}$ $\quad \Delta t = 1s$



Πιθανότητα επιβίωσης ενός πυρήνα για

$\Delta t : \quad 1 - \lambda \Delta t$

$2\Delta t : \quad (1 - \lambda \Delta t)^2$

$\vdots$

$m\Delta t : \quad \left(1 - \lambda \frac{t}{m}\right)^m$ $\qquad$ έστω $m\Delta t = t$

$$\lim_{m\to\infty} \left(1 - \frac{\lambda t}{m}\right)^m = e^{-\lambda t} \qquad\qquad e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Εάν έχουμε N_0 $\rightsquigarrow$ $N(t) = N_0 P(t) = N_0 e^{-\lambda t}$

$$N(t + \Delta t) = N(t)\,(1 - \lambda \Delta t)$$

$$N(t+\Delta t) - N(t) = -\lambda N \cdot \Delta t \equiv -\mu \quad \approx \text{σταθερό}$$

$$P(\Delta N) = \frac{\mu^{\Delta N}}{(\Delta N)!} e^{-\mu} \quad \xrightarrow[\mu \gg 20-30]{} \quad P(\Delta N) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\Delta N - \mu)^2 / 2\mu^2}$$

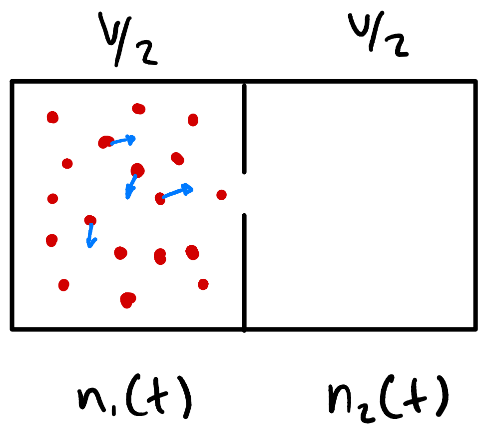
$$\boxed{\Delta N \ll N}$$

Poisson ως το όριο της διωνυμικής

$$\lim_{\substack{N\to\infty \\ \Delta t \to 0 \\ \mu = \text{σταθ}}} P(\Delta N, N, p = \lambda\Delta t, q = 1 - p) = \frac{N!}{\Delta N!\,(N - \Delta N)!} (\lambda \Delta t)^{\Delta N} \cdot (1 - \lambda\Delta t)^{N - \Delta N}$$

$$\simeq \frac{\mu^{\Delta N}}{(\Delta N)!} e^{-\mu}$$

Σωματίδια σε κουτί :



$n_1 + n_2 = N$

$\lambda_i dt$ = πιθανότητα το σωματίδιο "i" να περάσει την τρύπα στο χρονικό διάστημα $[t, t+\Delta t]$

$\lambda_i = \lambda$ = ίδια

$$dn_1 = n_2 \lambda dt - n_1 \lambda dt = (n_2 - n_1)\lambda dt$$

$$dn_1 = (N - 2n_1)\lambda dt$$

$$-\frac{1}{2} d(N - 2n_1) = (N - 2n_1)\lambda dt$$

$$\int \frac{d(N-2n_1)}{N-2n_1} = -2\lambda dt$$

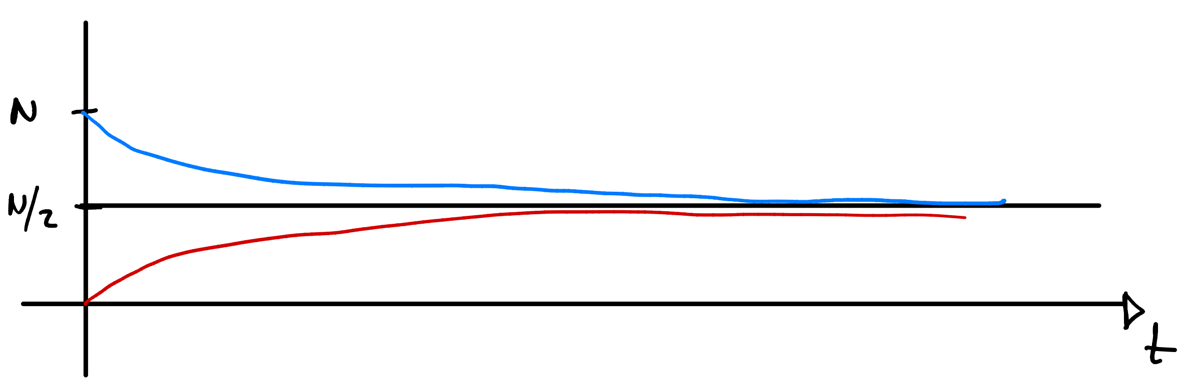
$$N - 2n_1 = c \cdot e^{-2\lambda t}$$

$$n_1(t) = \frac{1}{2}\left(N - ce^{-2\lambda t}\right)$$

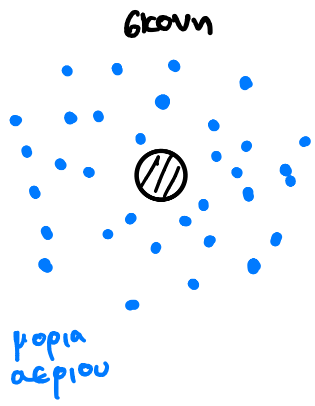
$$n_1(0) = N \Rightarrow c = -N$$

$$n_1(t) = \frac{N}{2}\left(1 + e^{-2\lambda t}\right)$$

$$n_2(t) = N - n_1(t)$$



Κίνηση Brown



Θεωρία Einstein

$$\langle x^2 \rangle = 2Dt = \sigma^2$$

μέση τετραγωνική απομάκρυνση ενός σωματιδίου Brown

$$\langle x \rangle = 0$$

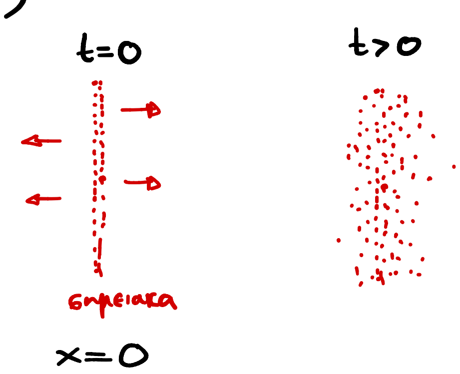
1-D

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} D$$

$$\rho(x,t) = \frac{N}{\sqrt{4\pi D t}} e^{-x^2/4Dt}$$

Διάχυση σταγόνας κρέμας στον καφέ

(1-D)



(τυχαίος περίπατος random walk)

$$s = \begin{cases} +\ell & 50\% \\ -\ell & 50\% \end{cases}$$

$$\langle s \rangle = (-\ell)(0{,}5) + (+\ell)(0{,}5)$$

$$\langle s^2 \rangle = (-\ell)^2(0{,}5) + (+\ell)^2(0{,}5) = \ell^2$$

Απόσταση ενός σωματιδίου Brown μετά από N βήματα :

$$x = \sum_i^N s_i$$

Θεώρημα κ.ο. για $x \gg \ell$

$$\lim_{N\to\infty} P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

$$\langle x \rangle = \sum_i \langle s_i \rangle = N \cdot 0 = 0$$

$$\langle x^2 \rangle = \sum_i \langle s_i^2 \rangle = \sum_i \langle s^2 \rangle = N\ell^2$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = N\ell^2$$

$$\sigma = \sqrt{N}\,\ell$$

Αποσταση ενος σωματιδιου Brown μετα απο $N=3$ βηματα:

εστω $\begin{cases} p \equiv \text{πιθανοτητα για βημα δεξια}: & s=+\ell \\ q \equiv \text{———//———} \quad \text{αριστερα}: & s=-\ell \end{cases}$

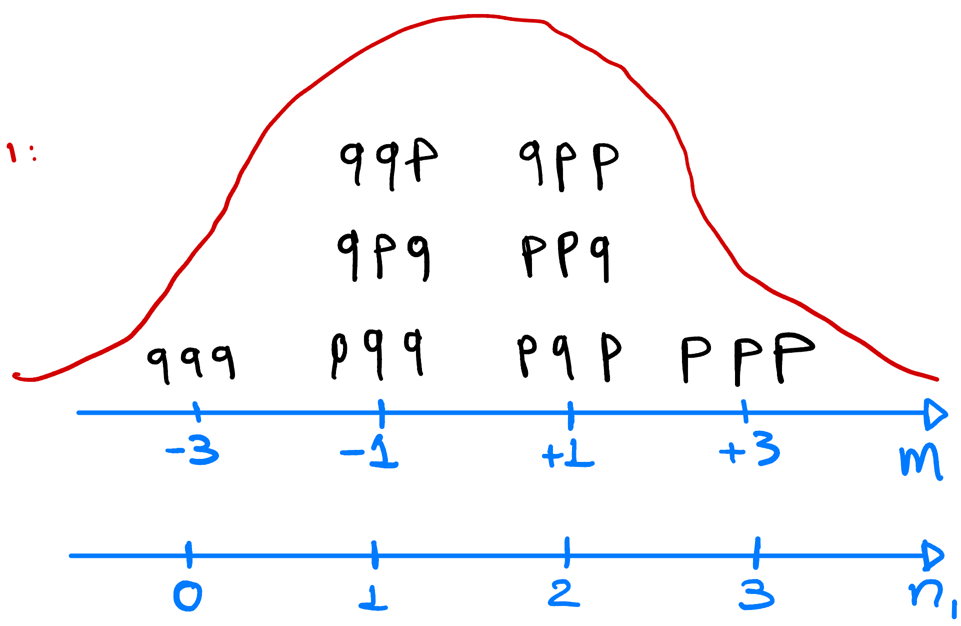
$$X = \sum_{i}^{3} S_i = S_1 + S_2 + S_3$$

$$X = \{-3\ell, -\ell, \underbrace{\ell, 3\ell}_{\Delta m^{min}=2}\} = m\ell = (n_1 - n_2)\ell$$

$$-N \leq m \leq N$$

$$\binom{N}{n_1} = \frac{N!}{n_1!\, n_2!} \quad \text{με } n_1 + n_2 = N$$

τροποι:



$$P(X=-3\ell) = \binom{3}{0} p^0 q^3 \qquad P(X=+\ell) = \binom{3}{2} p^2 q^1$$

$$P(X=-\ell) = \binom{3}{1} p^1 q^2 \qquad P(X=+3\ell) = \binom{3}{3} p^3 q^0$$

Αποσταση ενος σωματιδιου Brown μετα απο N βηματα:

$$P(n_1, n_2, N) = \frac{N!}{n_1!\, n_2!} p^{n_1} q^{n_2} \qquad \langle n_1 \rangle = N \cdot p$$

με $n_1 + n_2 = N$

$$x = m \cdot \ell = (n_1 - n_2)\ell = (2n_1 - N)\ell$$

$$Var(n_1) = Npq$$

$$Var(m) = Var(2n_1 - N) = 4Var(n_1) = 4Npq$$

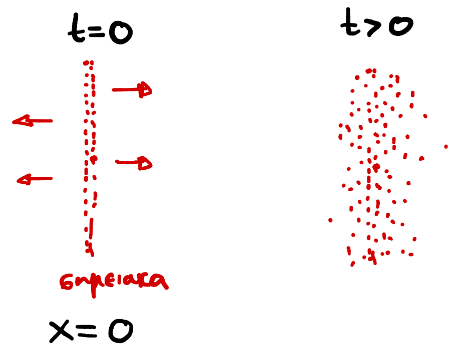
$$Var(x) = Var(m\ell) = \ell^2 Var(m) = 4\ell^2 Npq$$

$$\langle m \rangle = 0$$

για $p = q = 1/2$ $\qquad Var(x) = \ell^2 N \rightarrow \sigma = \sqrt{N} \cdot \ell$

Εφαρμογη : $M = 1000$ σωματιδια Brown μετα απο $N = 200$ βηματα

x Array $\rightarrow [1 \times M]$ θεσεις των M μετα απο N βηματα



Εφαρμογη : Τροχια $M = 1$ σωματιδιο Brown συναρτησει N

$$x = \sum_i^N s_i \rightarrow [1 \times 1]$$

$$\text{path} \rightarrow [\,x(N=0),\ x(N=1), \dots x(N)\,] \rightarrow [1 \times N]$$

Ασκηση : 2D τυχαιος περιπατος

$$r = x + y j \qquad s \begin{cases} 1 \\ -1 \\ j \\ -j \end{cases} \text{(ισοπιθανα)}$$

a) $\langle r \rangle = ?$

b) $\langle r^2 \rangle - \langle r \rangle^2 = ?$

1-Δ

$$\sigma_x = \sqrt{N} \ell$$

2-Δ

$\sigma_r > \sqrt{N} \ell$: A

$\sigma_r < \sqrt{N} \ell$: B

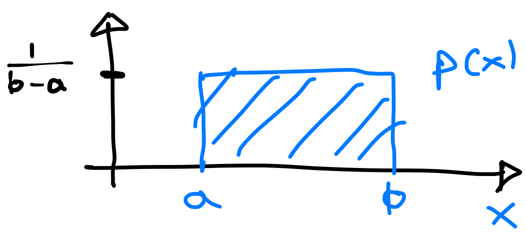
$\sigma_r = \sqrt{N} \ell$: Γ

Μιγαδικοι στην Python:

```
x = 1 + 2j
y = 2 + 2j
z = x + y
print(z.real , z.imag , abs(z))
>>  3    4    5
```

Απλοϊκό MC (crude MC)

$$I = \int_a^b f(x)\,dx = \int_a^b \frac{f(x)}{p(x)}\,p(x)\,dx$$



$$\int_{-\infty}^{+\infty} p(x)\,dx = 1$$

όταν
$\hookrightarrow p(x)$ = ομοιομορφη pdf $= \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ \phi & x \notin [a,b] \end{cases}$

απλοϊκό (crude MC)

$$I = (b-a)\int_a^b f(x)\,\frac{1}{b-a}\,dx = (b-a)\int_{-\infty}^{+\infty} f(x)\,p_x(x)\,dx$$

$$E[f(x)] = \int_{-\infty}^{+\infty} f(x)\,p_x(x)\,dx = \langle f \rangle$$

$$I = (b-a)\cdot\langle f \rangle$$

$$\hat{I} = (b-a)\,\frac{1}{n}\sum_{i=1}^{n} f(x_i) \qquad \lim_{n\to\infty} \hat{I} = I$$

$$\boxed{\hat{I} = (b-a)\,\bar{f}} \qquad \bar{f} \equiv \frac{1}{n}\sum_i f(x_i)$$

$$Var(\hat{I}) = (b-a)^2\,\sigma_{\bar{f}}^2 = \frac{(b-a)^2}{n}\,\sigma_f^2$$

$$\sigma_f^2 = \langle f^2 \rangle - \langle f \rangle^2 \qquad \hat{\sigma}_f^2 = \frac{1}{n-1}\sum_i \left(f(x_i) - \bar{f}\right)^2$$

$$\hat{\sigma}_f^2 = \frac{n}{n-1}\left[\frac{1}{n}\sum_i \left(f(x_i)-\bar{f}\right)^2\right] = \frac{n}{n-1}\left[\left(\frac{1}{n}\sum_i f^2(x_i)\right) - \bar{f}^2\right]$$

$$\boxed{\delta\hat{I} = \frac{b-a}{\sqrt{n}}\,\hat{\sigma}_f}$$

χρειαζομαστε τα

$\sum_{i=1}^{n} f^2(x_i)$ και $\sum_{i=1}^{n} f(x_i)$

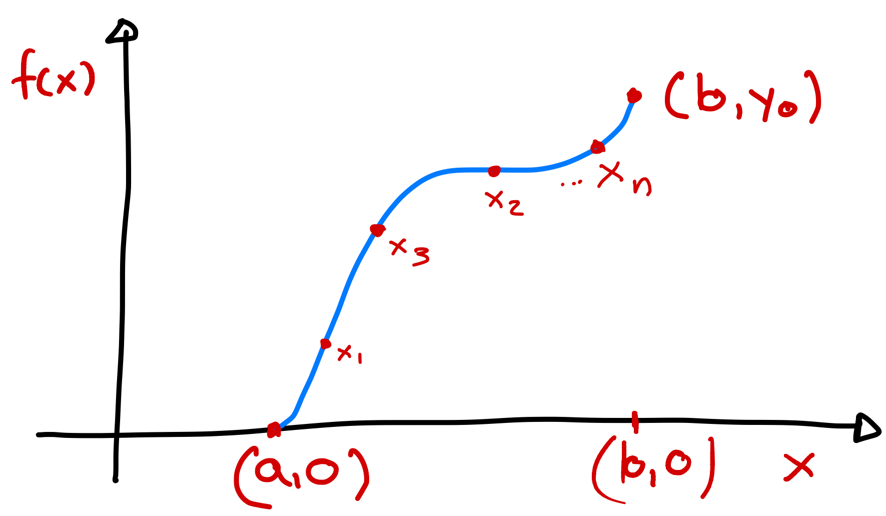
εντος του for loop

Πολυδιάστατη ολοκλήρωση (N-dim)

$$\int_a^b f(x)\,dx \;\longrightarrow\; \underbrace{\iint\cdots\int}_{\text{περιοχη}} dx_1\,dx_2\ldots dx_N\, f(x_1,x_2\ldots x_N)$$

$$b-a \;\longrightarrow\; V = \underbrace{\iint\cdots\int}_{\text{περιοχη}} 1\; dx_1\,dx_2\ldots dx_N$$

$$\boxed{\hat{I} = V\,\bar{f}} \qquad \boxed{\delta\hat{I} = \frac{V}{\sqrt{n}}\,\hat{\sigma}_f}$$



Εαν αντι για ομοιομορφη p(x) εχουμε εξυπνη pdf
η μεθοδος παυει να εχει το προσωνυμιο
crude - απλοϊκη → σημαντικη "εξυπνη" δειγματοληψια
importance sampling

$$\hat{I} = \frac{1}{n}\sum_i^N \frac{f(x_i)}{p(x_i)} = \frac{1}{n}\sum_i^N g(x_i)$$

$$\delta\hat{I} = \hat{\sigma}_g/\sqrt{n} \qquad \text{με} \qquad \hat{\sigma}_g^2 = \frac{1}{n-1}\sum_i\left[(g(x_i))^2-(\bar{g})^2\right]$$

εξυπνη → απλοϊκη αν $p(x) = \frac{1}{V}$

$$I = \int_0^{10} e^x \, dx = \underline{e^{10} - e^0} = I_{exact}$$

$\delta I_{exact} = ? \quad \sim 440$

$$I = V \langle f \rangle \qquad \langle f \rangle = \int_0^{10} f(x) \cdot \overset{\frac{1}{10}}{\cancel{p(x)}} \, dx$$

$$\delta I^2 = V \, \sigma_f^2 / n = \qquad \langle f^2 \rangle = \int_0^{10} \left(f(x)\right)^2 \cdot \underset{1/10}{p(x)} \, dx$$

$$\sigma_f^2 = \langle f^2 \rangle - \langle f \rangle^2$$

Μεθοδος απορριψης (accept – reject hit – or – miss)

$$I = \int_a^b f(x)\,dx = \int_a^b \int_0^{f(x)} 1 \, du\, dx = \int_a^b \int_0^{f_{max}} g(x,u)\, du\, dx$$

$$g(x,u) = \begin{cases} 1 , & u \leq f(x) \\ 0 & u > f(x) \end{cases} \quad \text{με} \quad u \sim U(0, f_{max})$$

$$\hat{I} = V \cdot \bar{g}$$ παρόμοιο MC σε 2-d (u,x) με $V = (b-a)\cdot f_{max}$

$$\bar{g}(x,u) = \frac{1}{n} \sum_i g(x_i, u_i) = \frac{m}{n} \rightarrow \left(\frac{\#\text{ hits}}{\#\text{ total}} = \hat{p} \right)$$

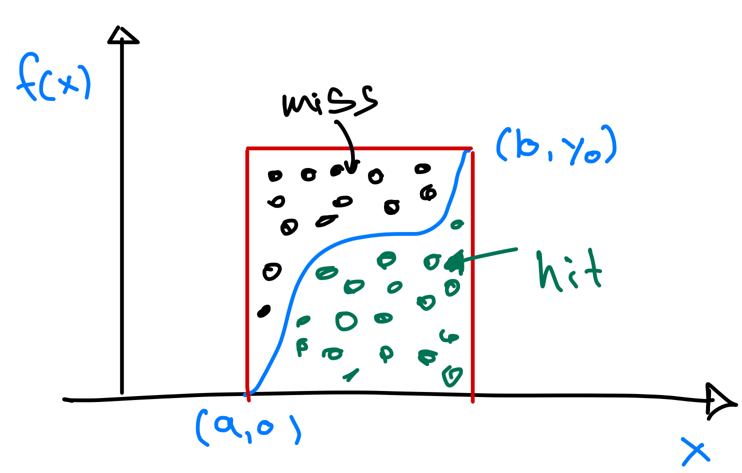
$$\boxed{\hat{I} = V\hat{p}}$$

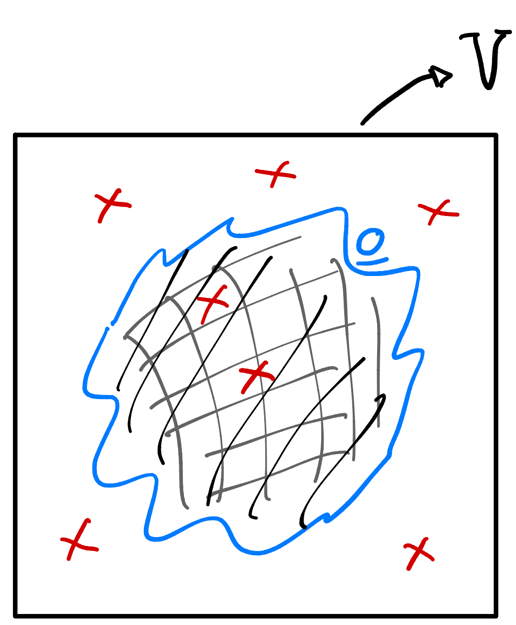
$$Var(\hat{I}) = V^2 \frac{Var(m)}{n^2} = \sigma^2_{\hat{I}}$$

$$m \sim \binom{n}{m} (1-p)^{n-m} p^m$$ (διωνυμική)

$$E[m] = p \cdot n \qquad Var(m) = np(1-p)$$

$$\boxed{\delta\hat{I} = \frac{V}{\sqrt{n}} \sqrt{\hat{p} - \hat{p}^2}}$$





$$I = \iint_{\Omega} 1\, dx\, dy = ?$$

$$I = pV = \left(\frac{m}{n}\right) V$$

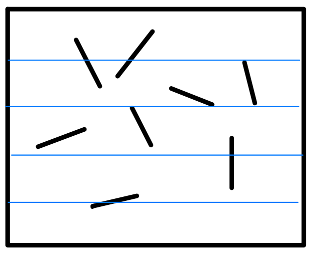
$$\sqrt{Var(m)}$$

$$\delta I = V \delta p = \frac{V}{n} \sqrt{npq}$$

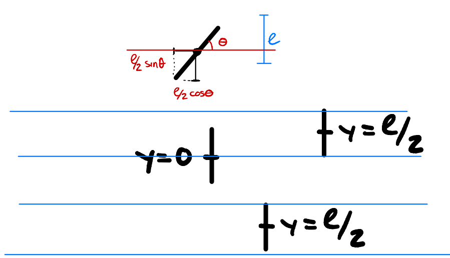
$$\frac{\delta I}{I} = \frac{V}{\sqrt{n}} \sqrt{pq} \Big/ pV = \frac{1}{\sqrt{n}} \sqrt{\frac{q}{p}}$$

$$= \frac{1}{\sqrt{n}} \sqrt{\frac{1-p}{p}}$$

Βελονα Buffon (Buffon's needle)



μηκος βελονας = αποσταση μεταξυ γραμμων = ℓ



$$dp = \frac{2}{\ell} dy \, \frac{2}{\pi} d\theta = \frac{4}{\ell\pi} dy \, d\theta$$

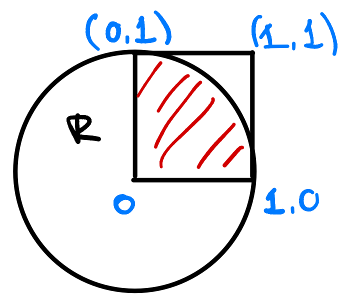
hit $\Rightarrow$ $\boxed{y \le \frac{\ell}{2} \sin\theta}$

$$P = \int_0^{\pi/2} \int_0^{\frac{\ell}{2}\sin\theta} \frac{4}{\ell\pi} \, dy \, d\theta = \int_0^{\pi/2} d\theta \, \frac{4}{\ell\pi} \frac{\ell}{2} \sin\theta = \frac{2}{\pi}$$

$$\hat{p} = \frac{m}{n}$$

ποσο μεγαλο πρεπει να ειναι το n ετσι ωστε $\frac{\Delta\pi}{\pi} \le 1‰$; *

* Απ: $n \geq 570796$



$\boxed{R=1}$

$\pi R^2/4$

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \, (1) = \ldots = \frac{\pi}{4}$$

$$= \int_0^1 \int_0^1 g(x,y)\, dx\, dy$$

$$g(x,y) = \begin{cases} 1 & , x^2+y^2 \leq 1 \\ 0 & , x^2+y^2 > 1 \end{cases}$$

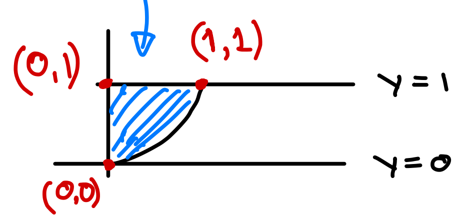
Άσκηση

github

$$M = \iint_D \rho(x,y)\,dx\,dy \quad , \quad D\{(x,y)\in\mathbb{R}^2 : x \geq 0,\ y \leq 1,\ y \geq x^2\}$$

εφαρμογή

πυκνότητα 2-d υλικού



και $\rho(x,y) = \frac{20}{13}(x+y)$ [kg]

$M = 1$ kg αναλυτική λύση

Άσκηση

github

$$M = \int_1^2 \int_1^2 \int_0^1 \rho(x,y,z)\,dx\,dy\,dz = 31/12 \text{ [kg]}$$

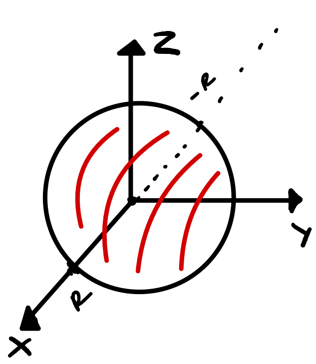
με $\rho(x,y,z) = \frac{12}{31}\left(x^2 + y\cdot z\right)$ [$kg \cdot m^{-3}$]

Παράδειγμα

$$\int_0^{10} e^x\,dx = e^{10} - 1$$

Παράδειγμα

$$M = \iiint_D \rho(x,y,z)\,dx\,dy\,dz$$



$\rho(x,y,z) = \frac{5}{648\pi}\left(x^2 + y^2\right)$ $\left[\frac{kg}{m^3}\right]$

$R = 3$ [m]

$D = \{(x,y,z)\in\mathbb{R}^3 : x^2 + y^2 + z^2 \leq 3^2\}$

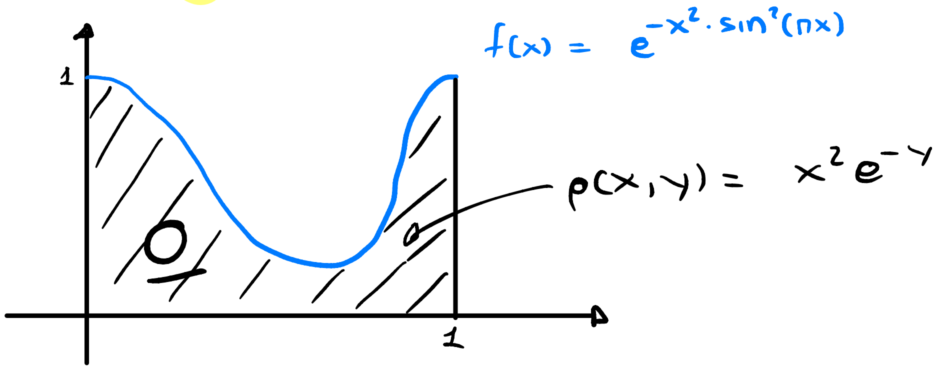
καρτεσιανές συντεταγμένες
πολικές συντεταγμένες
σφαιρικές συντεταγμένες

crude vs h-o-m

Απλοϊκό MC σε "δύσκολο" όγκο ολοκλήρωσης

$$I = \iint\limits_{\underline{\Omega}} \cdots \int f(\bar{x})\, d\bar{x} \qquad \text{όταν} \qquad V = \iint\limits_{\underline{\Omega}} \cdots \int 1\, d\bar{x} = \;;$$

πχ σε 2D



$$V = \int_0^1 dx \int_0^{e^{-x^2\sin^2(\pi x)}} dy = \int_0^1 e^{-x^2\sin^2(\pi x)}\, dx = \;;$$

$$I = \iint\limits_{\underline{\Omega}} x^2 e^{y}\, dx\, dy = \int_0^1 dx \int_0^{e^{-x^2\sin^2(\pi x)}} dy\; x^2 e^{y}$$

- παράγουμε ομοιόμορφα n (x_i, y_i) σημεία στην περιοχή $\overset{V_{hm}}{[0,1]\times[0,1]}$
- έστω m τα σημεία με $y_i \leq f(x_i)$ $[h\text{-}o\text{-}m]$

$\to$ απλοϊκό MC με m δείγματα στο $\underline{\Omega}$

$$\hat{V} = \frac{m}{n} \times V_{h\text{-}o\text{-}m} = \frac{m}{n}(1-0)(1-0)$$

$$\hat{I} = \frac{\hat{V}}{m} \sum_i^m p(x_i, y_i) = \hat{V} \cdot \bar{p}$$

$$\delta \hat{I}^2 \overset{*}{\simeq} \left(\frac{\partial \hat{I}}{\partial \bar{p}} \delta \bar{p}\right)^2 + \left(\frac{\partial \hat{I}}{\partial \hat{V}} \cdot \delta \hat{V}\right)^2 = \underbrace{\left(\frac{\hat{V}}{\sqrt{m}}\, \hat{\sigma}_p\right)^2}_{\delta \hat{I}^2_{cr}} + \underbrace{\left(\bar{p}\, \frac{V_{hm}}{\sqrt{n}} \sqrt{\frac{m}{n}\left(1-\frac{m}{n}\right)}\right)^2}_{\delta \hat{V}^2}$$

$$\delta \hat{I}^2 \overset{*}{\simeq} \delta \hat{I}^2_{cr} + \delta \hat{V}^2$$

*για $\mathrm{cov}(\hat{p}, \hat{V}) \simeq 0$

Μη-γραμμικα αλγεβρικα συστηματα

$$\begin{cases} x^2 + y^2 = 4 \\ x^2 - y^2 = 1 \end{cases} \qquad \begin{cases} f_1(x,y) = x^2 + y^2 - 4 = 0 \\ f_2(x,y) = x^2 - y^2 - 1 = 0 \end{cases} \Bigg| \rightarrow \vec{f}(\vec{x}) = 0$$

$$\vec{f} = \begin{bmatrix} f_1(x+\varepsilon_1, \ y+\varepsilon_2) \\ f_2(x_2+\varepsilon, \ y+\varepsilon_2) \end{bmatrix} = \begin{bmatrix} f_1(x,y) + \varepsilon_1 \frac{\partial f_1}{\partial x} + \varepsilon_2 \frac{\partial f_2}{\partial y} \\ f_2(x,y) + \varepsilon_1 \frac{\partial f_2}{\partial x} + \varepsilon_2 \frac{\partial f_2}{\partial y} \end{bmatrix} \simeq 0$$

$$\vec{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = - \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

$$\boxed{J(f)\, \varepsilon = - f}$$

επιλυση γραμμικου συστηματος με γνωστα τα $J(f)$, f και αγνωστο το $\vec{\varepsilon}$

$$\varepsilon = - J^{-1} f$$

$$x^{n+1} = x^n - J^{-1} f \qquad \text{n-dim}$$

$$x^{n+1} = x^n - (f')^{-1} f \qquad \text{1-dim}$$

Newton–Raphson

```
if ( 0.1 + 0.2 - 0.3 == 0) :
        print ('A')            44/73     60%
else :
        print ('B')            31/71     40%
```

$(101)_2 =$

5	:	A	40 %
3	:	B	23 %
9	:	C	20 %
6	:	D	18 %

$(101)_{10} = 1 \cdot 10^2 + 0 \cdot 10^1 + 1 \cdot 10^0$

$(101)_2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 5$

$(101.01)_2 =$

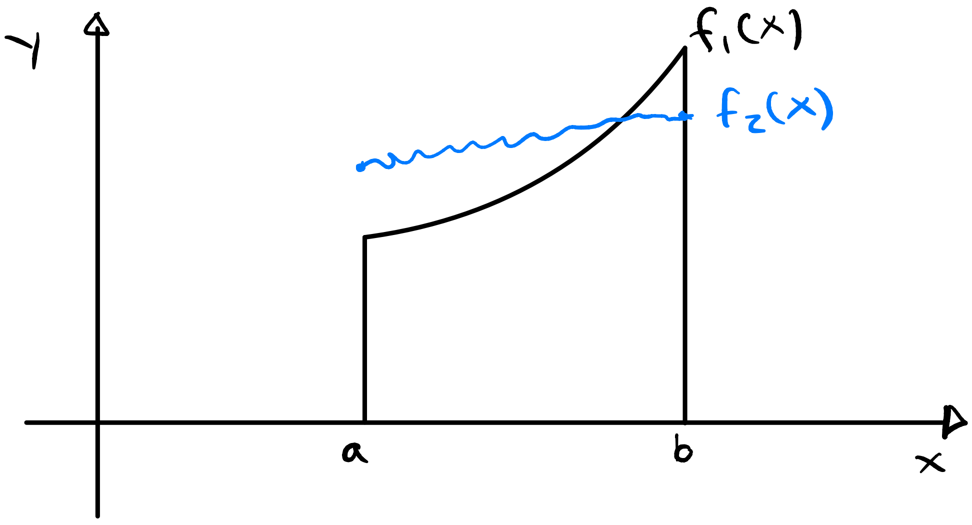
5.01	:	A	31 %
5.5	:	B	8 %
~~5.25~~	~~:~~	~~C~~	~~51 %~~
21	:	D	10 %

$(101.01)_2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2}$

| 5 bits |

$= 5 + 1/4 = 5.25$

MC - crude



$$I_1 = \int_a^b f_1(x)\,dx$$

$$I_2 = \int_a^b f_2(x)\,dx$$

$$\boxed{\begin{array}{ll} A: & \delta I_1 > \delta I_2 \\ B: & \delta I_1 < \delta I_2 \end{array}}$$

$$\hat{I} = V \frac{1}{n} \sum_{i=1}^{n} f(x_i) = V \bar{f}$$

$$\delta \hat{I} = V \cdot \hat{\sigma}_{\bar{f}} = V \cdot \frac{\hat{\sigma}_f}{\sqrt{n}}$$

$$\langle f \rangle = \int_{-\infty}^{+\infty} f(x)\,p(x)\,dx = \int_a^b f(x) \frac{1}{b-a}\,dx = \frac{1}{V} \int f(x)\,dV = \frac{1}{b-a} \int f(x)\,dx$$

$$\sigma_f^2 = \langle f^2 \rangle - \langle f \rangle^2$$

$$I = V \cdot \langle f \rangle$$

$$\delta I = V \cdot \sigma_{\bar{f}} = V \frac{\sigma_f}{\sqrt{n}}$$

$f(x) = e^x$

$a, b = 0, 10$

$$I = \int_0^{10} e^x\,dx = 10 \int_0^{10} e^x \frac{dx}{10} = e^{10} - e^0$$

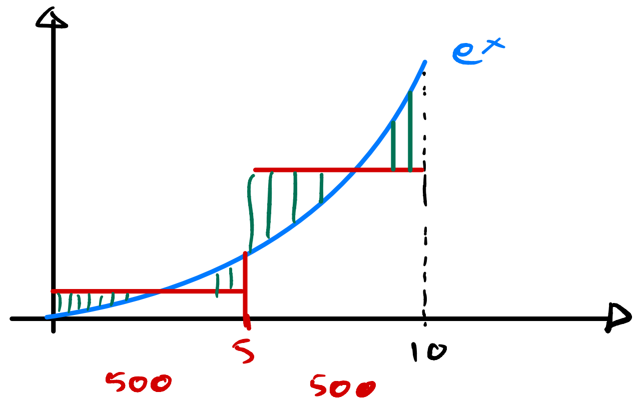
$$\langle f \rangle = \frac{e^{10} - e^0}{10 - 0} = \frac{1}{b-a} \int f(x)\,dx$$

$$\langle f^2 \rangle = \frac{1}{10} \int_0^{10} e^{2x}\,dx = \frac{1}{10} \frac{1}{2} \left[e^{20} - e^0\right]$$

$$\sigma_f^2 = \frac{1}{20} \left[e^{20} - e^0\right] - \left(\frac{e^{10} - e^0}{10}\right)^2 \qquad n = 1$$

$$\delta I = V \frac{\sigma_f}{\sqrt{n}} = 10 \frac{\sqrt{\frac{1}{20}\left[e^{20} - e^0\right] - \left(\frac{e^{10} - e^0}{10}\right)^2}}{\sqrt{1000}} = 1393$$

δ)



(39%) A: $\delta I < \delta I_1 \oplus \delta I_2$

7% B: $\delta I = \delta I_1 \oplus \delta I_2$

(54%) Γ: $\delta I > \delta I_1 \oplus \delta I_2$ ✓

$$I = \int_0^{10} e^x\,dx = I_1 + I_2 = \int_0^5 e^x dx + \int_5^{10} e^x dx$$

$n = 1000$ $\quad n_1 = 500 \quad n_2 = 500$

$e^{2x} - e^{2x}$

$$\delta I_1 = 5\,\frac{\sqrt{\frac{1}{10}\left[e^{10} - e^{0}\right] - \left(\frac{e^5 - e^0}{5}\right)^2}}{\sqrt{500}} = 8.2$$

$$\delta I_2 = 5\,\frac{\sqrt{\frac{1}{10}\left[e^{20} - e^{10}\right] - \left(\frac{e^{10} - e^5}{5}\right)^2}}{\sqrt{500}} = 1211.8$$

$$\boxed{\delta I = 1393.1 \qquad \delta I_1 = 8.2 \qquad \delta I_2 = 1211.8}$$

$$\boxed{\delta I_{o\lambda} \neq 8.2 + 1211.8 \;!}$$

$\delta I_{o\lambda} = \sqrt{8.2^2 + 1211.8^2}$

$\simeq 1211.8$

$z_1 = f(x, y)$

$$\delta z \simeq \sqrt{\left(\frac{\partial f}{\partial x}\delta x\right)^2 + \left(\frac{\partial f}{\partial y}\delta y\right)^2}$$

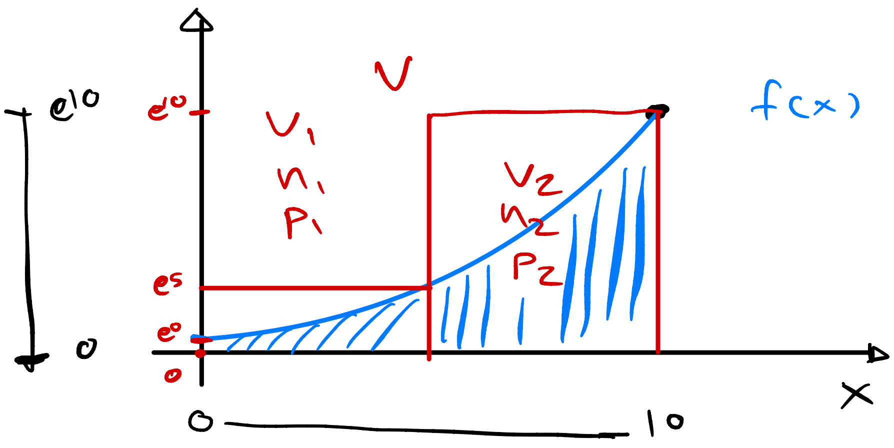
αν $\quad \mathrm{cov}(x, y) = 0$

και $\quad \delta x, \delta y$ μικρα

$$I = I_1 + I_2 = I(I_1, I_2)$$

$$\frac{\partial I}{\partial I_1} = 1 \qquad \frac{\partial I}{\partial I_2} = 1 \qquad \delta I = \sqrt{(1 \cdot \delta I_1)^2 + (1 \cdot \delta I_2)^2}$$

$$P_1 = \frac{e^5 - e^0}{e^{10} \cdot 10}$$



m = hits

n = συνολικα

$$p = \frac{m}{n} = \frac{I}{V}$$

$$V = (b-a) \cdot e^{10}$$

$$\hat{I} = \hat{p} \cdot V$$

$$\delta \hat{I} = \delta \hat{p}\, V = \delta\left(\frac{m}{n}\right) V = \frac{V}{\sqrt{n}} \sqrt{p(1-p)}$$

$$\langle m \rangle = n \cdot p$$

$$P(m,n) = \binom{n}{m} \cdot p^m (1-p)^{n-m}$$

$$V(m) = n \cdot p \cdot (1-p)$$

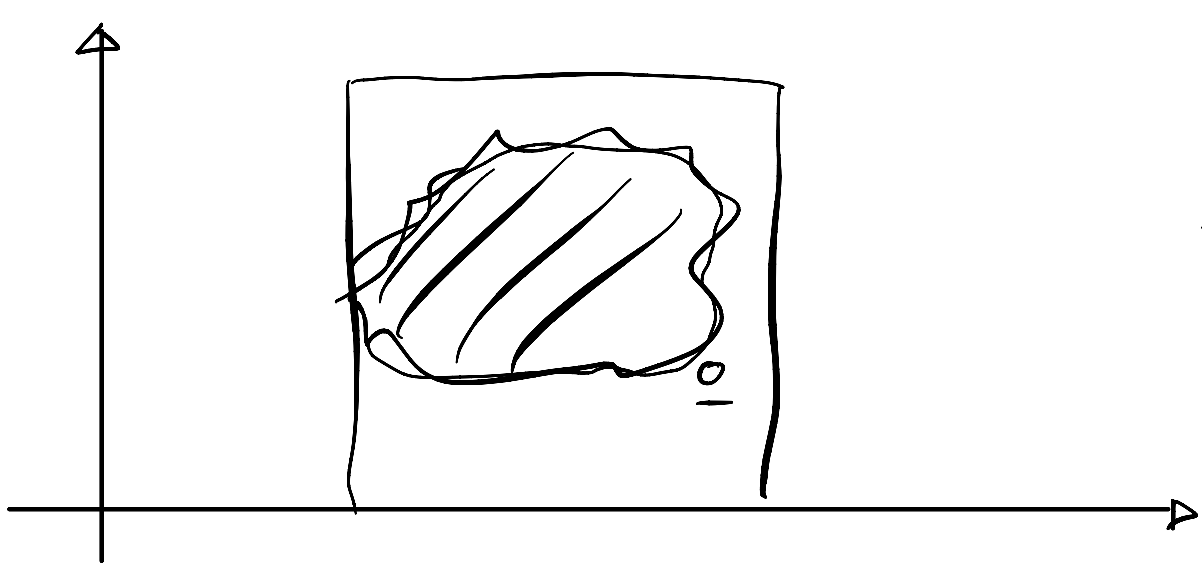
$$\delta(m) = \sqrt{n\, p(1-p)}$$

$$p = \frac{e^{10} - e^0}{e^{10} \cdot (10-0)} = \frac{I}{V}$$

$n = 1000$ (x, y)

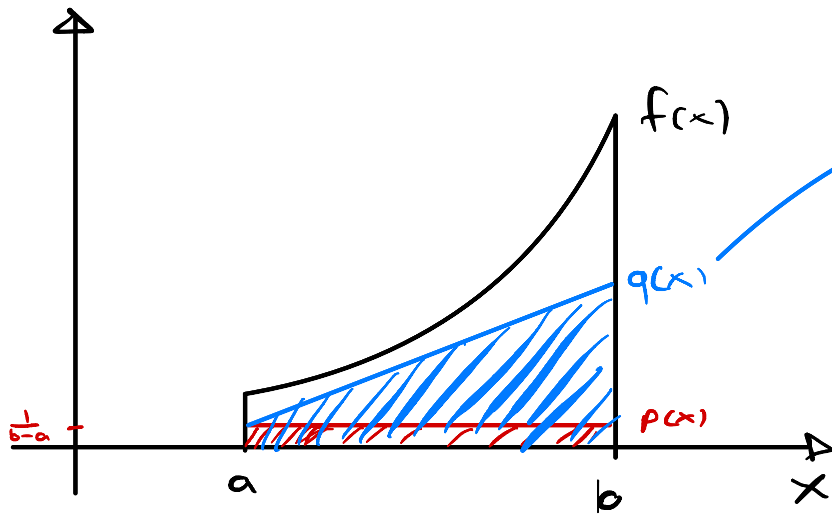
$$\delta I = \frac{10\, e^{10}}{\sqrt{1000}} \sqrt{\frac{e^{10} - e^0}{e^{10} \cdot (10-0)} \left(1 - \frac{e^{10} - e^0}{e^{10} \cdot (10-0)}\right)} = 2089.6$$

$$I = 22025.5$$



$$\iint_{O} f(x,y)\, dx\, dy$$

$$f(\vec{x}) = f(x_1, x_2 \ldots x_n)$$



όχι σε κλίμακα, πρέπει $\int_a^b q(x)\,dx = 1$

$p(x) = \frac{1}{b-a} \quad a \leq x \leq b$

$q(x) \overset{\text{π.χ.}}{=} cx$ με $\left.\frac{cx^2}{2}\right|_a^b = 1$

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} f(x)\left[p(x)^{-1}\right] = (b-a)\,\frac{1}{n}\sum_{i=1}^{m} f(x)$$

$$\hat{I}' = \frac{1}{n}\sum_{i=1}^{n} f(z)\left[q(z)^{-1}\right] \quad \text{με} \quad z \sim cx$$

$g(z) = f(z)/q(z)$

$$\hat{I}' = \bar{g} = \frac{1}{n}\sum_{i=1}^{n}\frac{f(z)}{q(z)}$$

$$\delta\hat{I}' = \hat{\sigma}_g/\sqrt{n}$$