Euclidean Geometry:
An Introduction to Mathematical Work

Math 3600

Spring 2015

The Theory of Content

Read the parts of the *Elements* that have to do with the basic theory
of area. These are Book I, Propositions 35-48, Book II, and Book III,
Propositions 35-37.

Euclid usually calls two planar figures "equal" without explana-
tion. From the work that he does, it seems that he means equality
of area, or something like it, but he never defines an "area function"
which assigns a (non-negative, real) number to each planar figure.
Looking more closely at the text, it appears that Euclid is using a
new undefined term and assuming some "common sense" axioms to
control use of the word. For clarity, we try to make things more ex-
plicit here. We introduce a new undefined term *equal content* which
satisfies the following axioms:

(EC1) Congruent figures have equal content.

(EC2) If each of two figures each have equal content with a third
figure, then the two figures have equal content.

(EC3) If pairs of figures with equal content are "added," in the sense
of being joined without overlap to make bigger figures, then these
added figures have equal content.

(EC4) The same holds for "subtraction," noting that it is immaterial
where the equal figures are removed.

(EC5) Halves of figures with equal content have equal content, and
doubles of figures with equal content have equal content.

(EC6) The whole is greater than the part, so if one figure is prop-
erly contained in another, then the two figures cannot have equal
content.

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Attention! Attention!

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Note that Euclid never uses algebra. (It hadn't been invented, yet.)
So there are no equations in his work. We shall not use them either!
"Euclid-style proof" demands no equations.
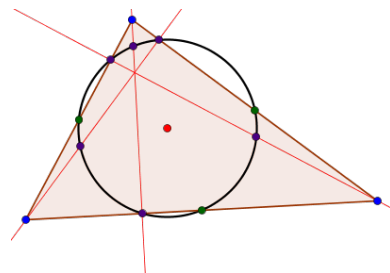
This follows the basic structure of
axiomatic work. We have a new unde-
fined term, and the axioms are in place
to make sure that we have concrete and
specific rules for the allowable use of
that new term. In this case, the axioms
are basically restatements of Euclid's
common notions.

For the following tasks, work in the style of Euclid using any result from the first three books.

13.1 Problem. Prepare a presentation of Euclid's beautiful proof of Proposition I.47.

13.2 Conjecture. Let ABC and DEF be triangles. Let X be the midpoint of DE, Y the midpoint of BC. If AB is congruent to DX and EF is congruent to BY, then ABC and DEF have equal content.

13.3 Problem. Expand on the statement of the last conjecture–find some theorems and prove them.

13.4 Problem. The *hypotenuse-leg* Theorem 7.2 can be proved using the theory of equal content. Find such a proof.

13.5 Question. There are two ways to inscribe a square in an isosceles right triangle. Which one has the greater content?

13.6 Conjecture (The Parallelogram Law). Let $ABCD$ be a parallelogram. Then the squares on the diagonals taken together have equal content with the squares on the four sides taken together.

13.7 Problem. Given a triangle ABC and a segment DE, construct a rectangle with equal content to ABC and with DE as one side.

13.8 Problem. Given a rectangle, construct a square of equal content.