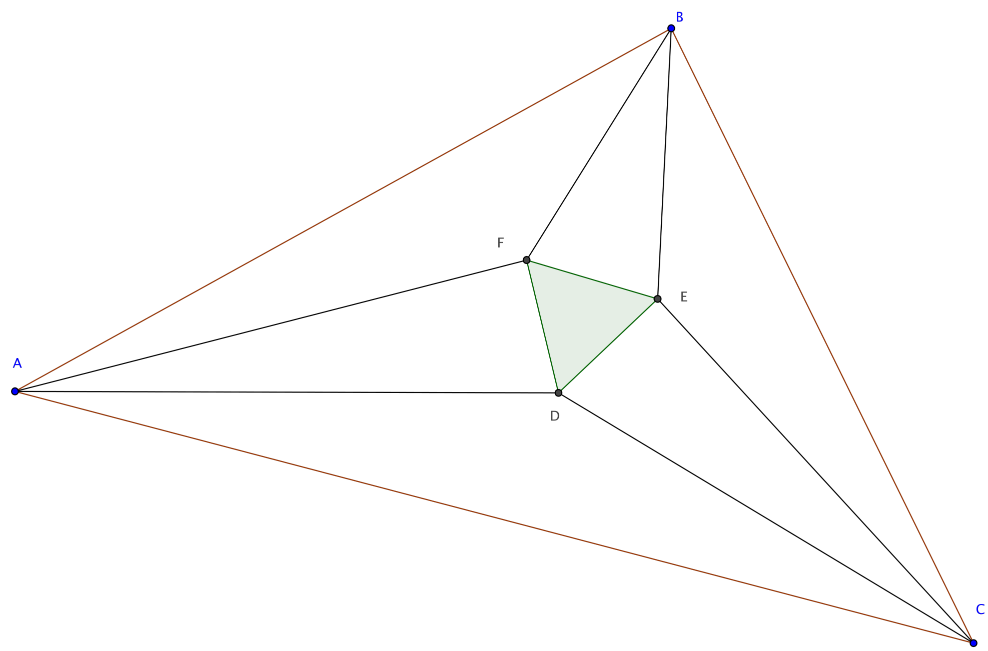
Transactions in **Euclidean Geometry**



Issue # 8

Construction of a Kite

Nathan Opheim
Communicated by Megan Westervelt

December 12, 2014

Theorem 2.3. A kite is constructable

Proof.
1. Construct line AB by Postulate 1
2. Construct circle A through point B by Postulate 3
3. Construct circle B through point A by Postulate 3

Circle AB and circle BA cross by the Circle-Circle-Intersection property.
Label the point where the circle AB and circle BA cross point C.

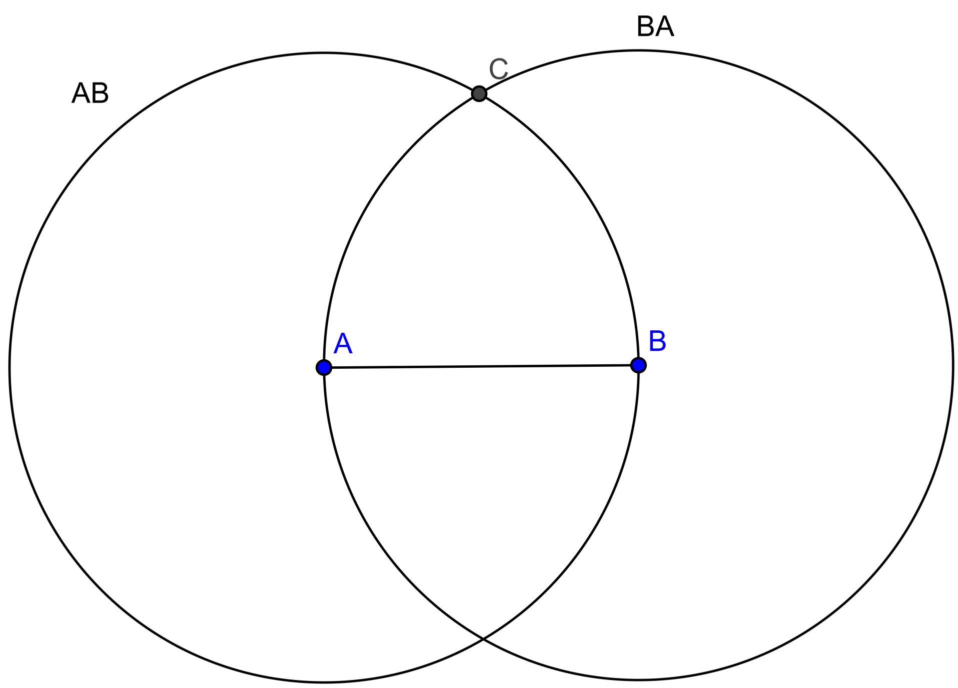


Figure 1: Circle AB intersecting with circle BA at point C

4. Draw line segment AC
5. Draw line segment BC

Then AC is congruent to BC because they have the same radius.

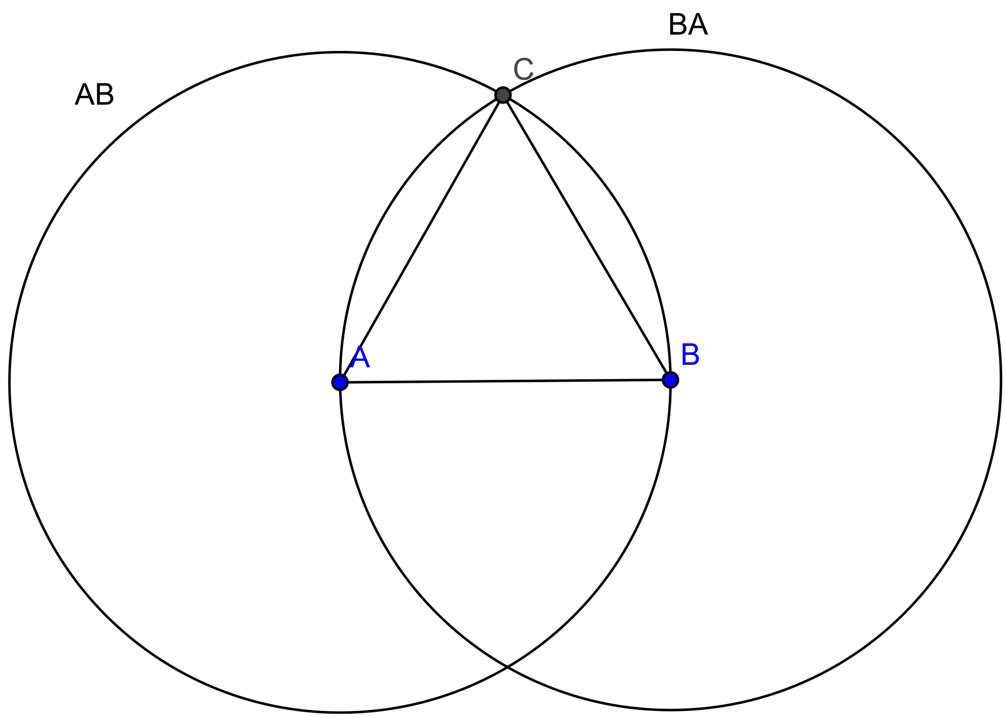


Figure 2: Side AC and BC of kite

6. Construct circle AD with radius greater than AB by Postulate 3
7. Construct circle BD2 with radius equal to AD by Postulate 3

Circle AD and circle BD2 cross by the Circle-Circle-Intersection property.
Label one point where circle AD and circle BD2 cross E.

8. Draw line segment AE by Post 1
9. Draw line segment BE by Post 1

Then AE is congruent to BE because they have the same radius. We have AC is congruent to BC and AE is congruent to BE so ACBE is a kite.

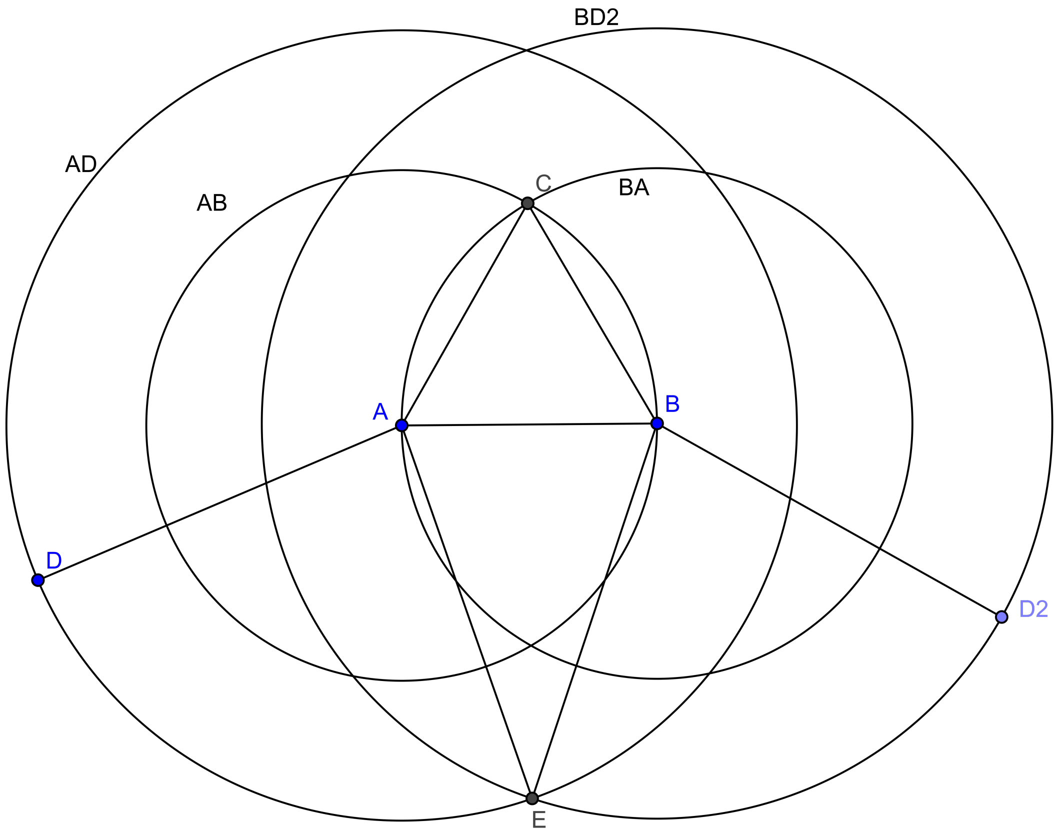


Figure 3: Kite ACBE

How general is this construction?

This construction will allow one to make any type of kite. Any type of kite can be made by adjusting the size of the radius for the circles, making the sides of the kite longer or shorter. Also the point chosen when making the first triangle of the kite (like triangle ABC) can change the kite from concave to convex.

□

MIDLINE OF A TRIANGLE

THOMAS BIEBER

Communicated by Joshua Hawkins

Theorem 3.6. Let ABC be a triangle, D the midpoint of AB and E the midpoint of AC. Then the line through E and D, called a midline, is parallel to the line through B and C.

Proof. Let ABC be a triangle, D the midpoint of AB and E the midpoint of AC. Draw line segment DE. Draw a circle centered at E through D. Extend DE past E until it intersects with circle ED. Label the intersection F. Draw line segment CF. By the definition of midpoint AE is congruent to EC. DE and FE are congruent because they are radii of the same circle. By proposition I.15 angles DEA and FEC are congruent. By proposition I.4 triangle AED is congruent to triangle CEF. By corresponding parts of congruent triangles, angles DAE and FCE are congruent. By proposition I.27 AD is parallel to FC. Since AD and DB are collinear, DB is parallel to FC. Since D is the midpoint of AB, AD is congruent to DB. Since triangles AED and FEC are congruent, AD is congruent to FC. By proposition I.33 we know segments DF and BC are parallel. Since E is collinear with DF, DE is parallel to BC. Thus, the midline of a triangle is parallel to the base. □

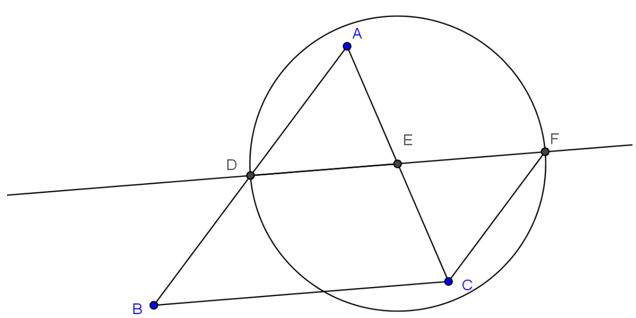


FIGURE 1. Triangle ABC with all constructions needed for the proof

RIGHT ANGLE INSCRIBED IN A CIRCLE

THOMAS BIEBER

Communicated by Joshua Hawkins

Theorem 7.4. If AB is the diamater of a circle and C lies on the circle, then angle ACB is a right angle.

Proof. Let AB be the diameter of a circle. Let C be a point on the circumference such that C does not coincide with A or B. Let X be the midpoint of AB. Draw segment CX. AX is congruent to BX is congruent to XC because they are radii of the circle. Triangle AXC is isosceles, so by proposition I.5 angle CAX is congruent to angle XCA. Similarly, triangle BXC is isosceles, so angles XBC and XCB are congruent. Also notice that angles CAX and CAB are the same angle, and angles XBC and ABC are the same angle. By proposition I.32 We know that the three interior angles of a triangle equal two right angles. Thus, angle ACB plus angle CBA plus angle CAB equal two right angles. Notice that angle ACB equals angle XCA plus angle XCB. From earlier stated congruencies we know ACB equals angle CAB plus angle CBA. So 2 Angle CAB plus angle 2 angle CBA equals two right angles. So angle BAC plus angle ABC equals one right angle. Thus, angle ACB is a right angle. ☐

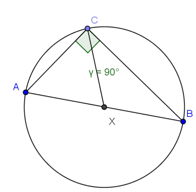


FIGURE 1. This figure shows the triangles used for the proof and that angle ACB is a right angle.

Theorem 8.4

Nathan Opheim
Communicated by Eric Scheidecker

December 12, 2014

Proof completed with help from Kevin Walters, Tim Nieman, and Thomas Bieber

Theorem 8.4. The three perpendicular bisectors of any triangle are concurrent.

Proof.

Let ABC be a triangle. By Proposition 1.10 find the midpoint of AC and label it Z. Also, find the midpoint of BC and label it X. Then by Proposition 1.11 find the perpendicular bisector of AC and BC. By Theorem 8.3 (Let T be a triangle. For any pair of sides T, the perpendicular bisectors of those sides meet.) the perpendicular bisectors of AC and BC cross. Label the point where the perpendicular bisectors cross I.

Then connect A to I, C to I, and B to I. So we have Triangle AIZ, Triangle CIZ, and Triangle BIX. Looking at Triangle AIZ and Triangle CIZ, we know that AZ is congruent to CZ because Z is the midpoint of line segment AC. Also IZ is congruent to itself. Since Z is at a right angle we can say Triangle AIZ is congruent to CIZ by side angle side, Postulate 1.4. So we know that IA is congruent to IC.

We can use a similar argument for Triangles IXC and IXB to show that IC is congruent to IB. So we have that IA is congruent to IC and that IC is congruent to IB. It follows that IA is congruent to IB since IC is congruent to both IA and IB.

Then on line AB construct a perpendicular line to I and label the point where the perpendicular line crosses AB point Y. By the Hypotenuse Leg Theorem Triangle IYB is congruent to Triangle IYA and Y is the midpoint of line AB. Therefore the three perpendicular bisectors of of any triangle are concurrent.

□

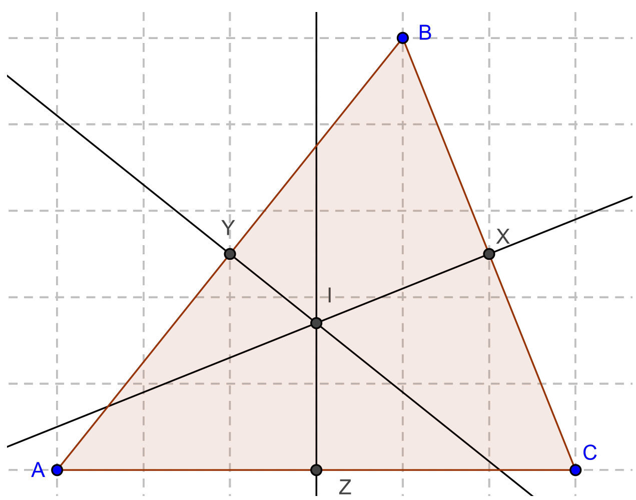


Figure 1: Perpendicular Bisectors of a Triangle are concurrent

Theorem 11.1

Nathan Opheim
Communicated by Tim Nieman

December 12, 2014

Theorem 11.1. Given an angle, construct the angle bisector. (par 4)

Construction:
1.Given an angle A, construct circle A with radius B.
2.Construct circle B with radius AB.
3. Construct circle C with radius AB. Call the point where circle B intersects circle C, point F.
4. Connect A to F

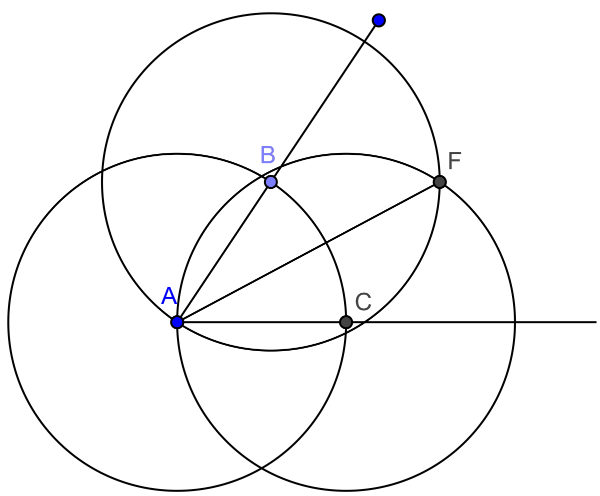


Figure 1: Angle bisector AF

Proof.
Connect B to F and C to F. Then AB is congruent to AC because they have the same radius. AF is congruent to itself. BF is congruent to CF because they have the same radius. So by Proposition 1.8 Triangle ABF is congruent to Triangle ACF and the angle has ben bisected. □

12.2

Nathan Opheim
Communicated by Megan Westervelt

December 12, 2014

Theorem 12.2. Construct a circle circumscribed about a given triangle *ABC*.

Proof.

Construction:

Given triangle ABC
Find the midpoint of AB.
1. Draw circle A with radius B.
2. Draw circle B with radius A.
3. Connect the points where the two circles intersect.

Note this line goes through the midpoint and is a perpendicular bisector by proposition 10 and 11.
Label the midpoint point x.
Find the midpoint of CB
4. Draw circle B with radius C.
5. Draw circle C with radius B.
6. Connect the points where circle C and circle B intersect.

Note this line goes through the midpoint and is a perpendicular bisector by proposition 10 and 11.
Label the midpoint Y.
7. Draw circle O through point C.

Proof:

From theorem 8.4 we know that the three perpendicular bisectors of a triangle are concurrent. By definition we know that the point where the three perpendicular bisectors of a triangle meet is the circumcenter of a triangle. The circumcenter is the center of a circle circumscribed about a triangle.

So by finding the circumcenter, we get the center, O, of the circle. Then by connecting O to C we get a point on the circle O. To show that the other points, A and B, of the triangle lie on the circle we need to show that segments OC, OB, and OA are all congruent.

Given the triangle with all of the perpendicular bisectors drawn in, connect O to C, O to B, and O to A. Then, looking at triangle AOX and triangle BOX, we know that AX is congruent to BX because X is the midpoint. Alos OX is congruent to itself. Since X is at a right angle we can say that triangle AOX is congruent to triangle BOX by side angle side, Postulate 1.4. So we know that OA is congruent to OB.

We can use a similar argument for triangles BOY and COY to show that OB is congruent to OC.

So we have that OA is congruent to OB and OB is congruent to OC. It follows that OA is congurnet to OC since OB is congruent to both OA and OC. Since OA, OB, and OC are all congurent to eachother and C is a point on the circle O, then A and B must lie on circle O since each point is the same distance away from the center of the circle.

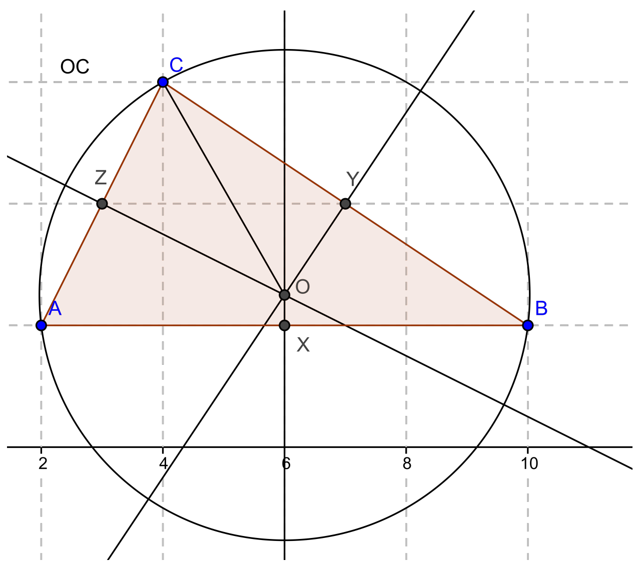


Figure 1: Circle circumscribed about Triangle ABC

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STRAIGHTEDGE AND COMPASS: PAIRWISE ORTHOGONAL CIRCLES

MATT GRIFFEN

Communicated by

Theorem 12.4. It is possible to construct three pairwise orthogonal circles using only a straightedge and compass.

Construction 12.4. 1. Draw a straight line X and pick a point A on it.

2. Draw a circle with center A. Label an intersection of the circle with line X point B.

3. Draw a circle with center B through A. Label the intersections of circle AB and circle BA points C and D

4. Draw a line Y through points C and D. Label the intersection of line Y and X point E

5. Draw a circle with center E through D. Label an intersection of circle ED and X point F

6. Draw a circle with center D through E. Label the intersection of circle DE and line Y point G

7. Draw a circle with center F through E. Label the intersection of circle FE and X point H and the intersection of circle FE and circle DE point I.

8. Draw a circle with center H through E.

9. Draw a circle with center G through E. Label the intersection of circle HE and circle GE point J.

10. Draw a circle with center J through G.

Circles DE, FE, and JG are pairwise orthogonal circles.

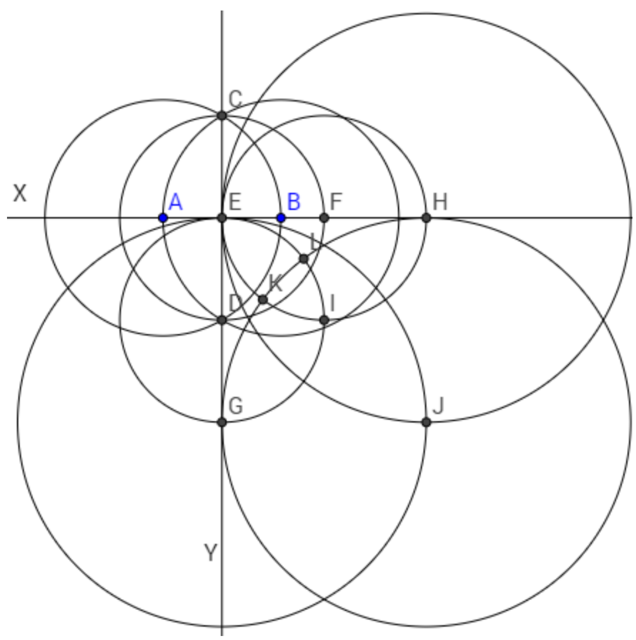

FIGURE 1. Construction of orthogonal circles

Proof. To prove that circles DE, FE, and JG are pairwise orthogonal circles it must be shown that: segment FE is perpendicular to segment DE, segment FI is perpendicular to segment DI, segment EH is perpendicular to segment JH, segment EG is perpendicular to segment JG, segment DL is perpendicular to segment JL, and segment FK is perpendicular to segment JK.

Let the intersection of circle FH and JH be labeled point K and the intersection of circle DG and JG be labeled point L. Let segments FI, DI, JH, JG, JF, JD, JK, JL, FK, and DL be drawn.

Point E was constructed in the manner of Hatchett 11.2, therefore E is the midpoint of segment CD. By Euclid III.3 angle BED is a right angle (RA). Since F is collinear with B, segment FE is perpendicular to segment DE.

Segments DE, DI, FE, and FI are radii of congruent circles. Therefore they are congruent and the quadrilateral DEFI is a rhombus. By the Bieber-Hawkins theorem angle FID is also a RA, so FI is perpendicular to DI.

It can similarly be shown that EHJG is a rhombus and that angle HJG is a RA. By Hawkins 3.4 EHJG is a rectangle, so angle EHJ and angle EGJ are both RA's. Therefore segment EH is perpendicular to JH and segment EG is perpendicular to segment JG.

Segment GJ is congruent to segment JL, segment GD is congruent to DL, and segment JK is congruent to itself. By Euclid I.8 triangle DLJ is congruent to triangle DGJ and angle DLJ is a RA. Therefore segment DL is perpendicular to segment JL.

It can similarly be shown that triangle FHJ is congruent to triangle FKJ and angle FKJ is a RA. Therefore segment FK is perpendicular to segment JK.

Q.E.D.

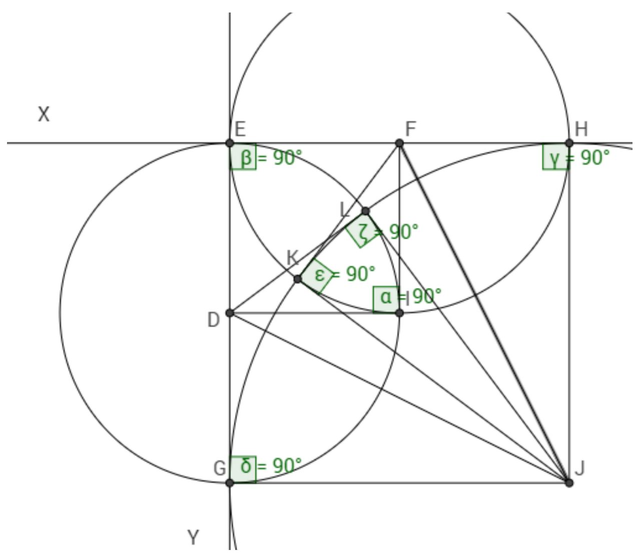

FIGURE 2. The orthogonal circles

□

Corollary 12.4-1. Given a square and the midpoints of two adjacent sides, three pairwise orthogonal circles can be constructed by drawing two circles centered at the midpoints through the adjacent sides' shared point and a third centered on the point diagonal to to the shared point with a radius of a side of the square.

Proof. EHGJ is a quadrilateral with four congruent sides and four congruent right angles, therefore it is a square. Circle DE has a diameter of segment GE. Segment GD is congruent to DE, therefore point D is the midpoint of GE. Similarly point F is the midpoint of EH. Circles DE and FE are centered at the midpoints D and F of adjacent sides of the square EHJG and go through their shared point E. Circle JG is centered at the point diagonal to E (point J) with a radius of a side of the square (JG). By 12.4, circles DE, FE, and JG are pairwise orthogonal circles.

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Theorem 12.4-2. Given a square ABCD and the midpoints of segment AB and segment AD (points E and F resp.), three pairwise orthogonal circles can be constructed by drawing circles FA, EA, and CD.

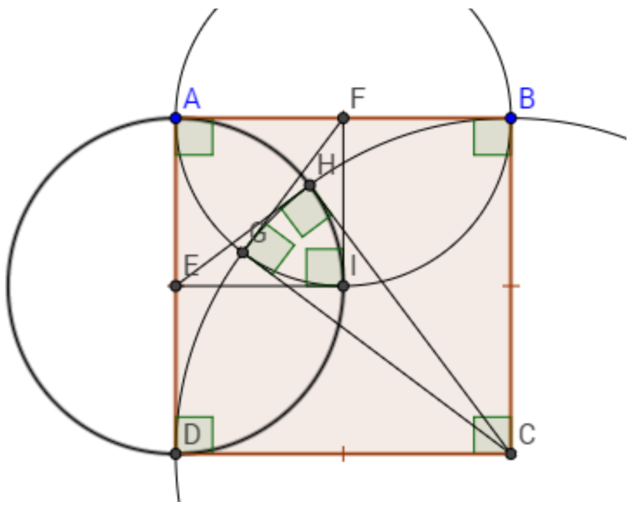


FIGURE 3. Orthogonal circle constructed from square

Theorem 12.4-2 is the general case of corollary 12.4-1 written with more acssible notation

THEOREM 12.7

EMILY BACHMEIER

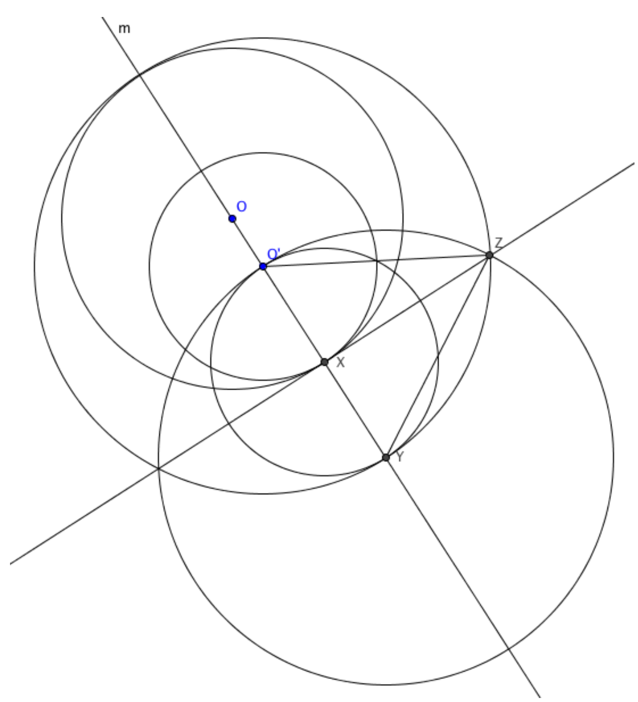
Communicated by Jalen Raymond

Theorem 12.7. Given two circles Γ and Γ' with centers O and O', respectively, construct a line tangent to both circles. How many such lines are there?

Proof. There exist five cases for which it is possible to construct a line tangent to both Γ and Γ'.

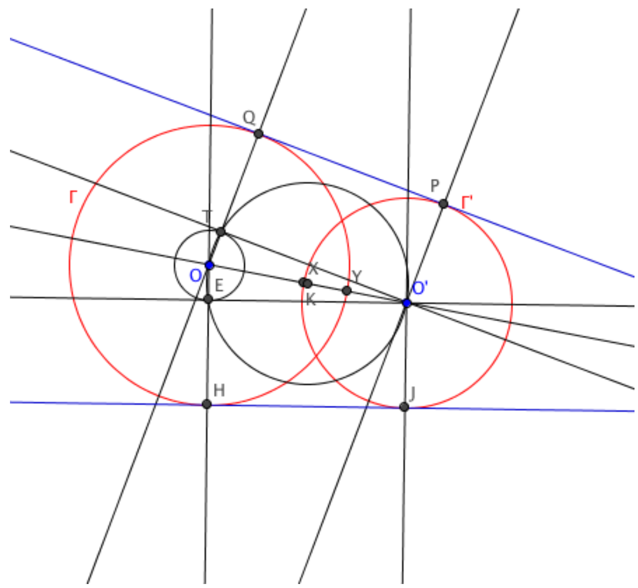
Case 1: Γ' lies inside Γ touching at exactly one point, *one tangent line*

Construct line m through O and O'. Label the intersection of m, Γ, and Γ' as X. By Euclid III.12, this line joining the centers of both circles goes through the point of contact of the two circles, X. By Euclid III.18, the tangent line and the line joining the center and point of contact, m, will be perpendicular. Using Walters' Theorem 11.4, construct a line perpendicular to m and through the point X. Draw circle XO'. Label the intersection of circle XO' and line m as Y. Draw circles YO' and $O'Y$. Label one of the intersection points of these two circles as Z. Draw line segments ZY and $O'Z$ and line XZ. Because these are all congruent radii of circles and by Euclid I.8, triangle $O'XZ$ is congruent to triangle ZXY. By Euclid I.13, two angles set upon a straight line m will add up to two right angles. Therefore, angle ZXO' and angle ZXY are both equal to one right angle. Thus, line XZ is perpendicular to line m and is a tangent line.



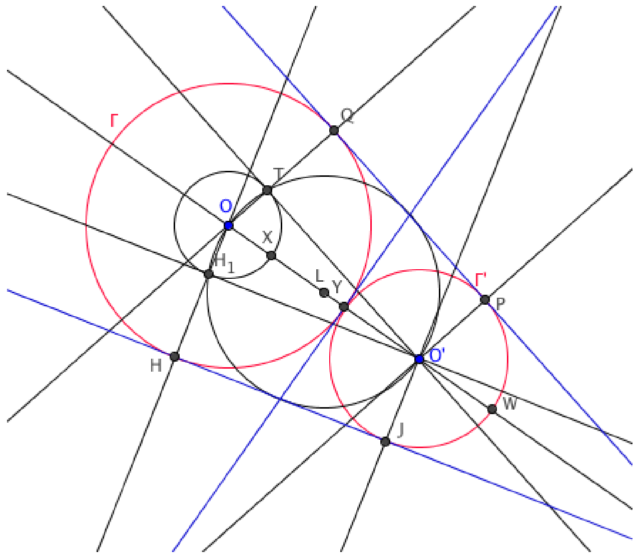
Case 2: Γ and Γ' intersect at exactly two points, *two tangent lines*

Draw line OO' and label interior intersections as X and Y. Construct circle OZ such that OZ is equal to the radius of Γ - radius of Γ'. Using Theorem 11.2, find the midpoint of segment OO', label it K and construct circle KO. Label the intersection of circles KO and OZ as T. Draw line TO' and TO, labeling the intersection of Γ and line TO as Q. By Theorem 7.4, angle $O'TO$ is right. Use Theorem 11.6 to construct a line parallel to TO and through O'. Label the intersection of this line and circle Γ' as P. Construct the line through points P and Q. Segments TQ and $O'P$ are congruent because they were constructed as such, and therefore we have rectangle $TQPO'$ with all interior angles right. Thus, line QP is tangent to both circles because it is perpendicular to both. Similarly, using the additional intersection point of circles $KO$and OZ and the same argument, we can construct line HJ tangent to both Γ and Γ'.



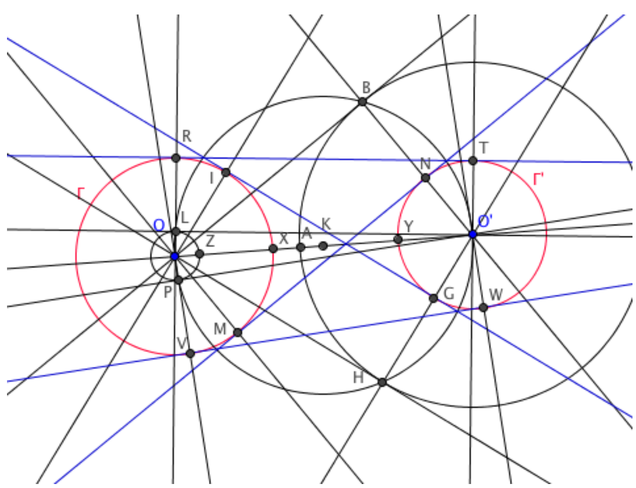
Case 3: Γ' lies outside of Γ and touch at exactly one point, *three tangent lines*

Draw line m through O and O'. Using Euclid III.12, this is the intersection of the two circles. Label the intersection of m and the circles as Y. As in Case 1, using Walters' Theorem 11.4, construct a line perpendicular to m through point Y and this line will be tangent to both circles. Draw circle OX such that the radius OX is equal to segment OY - segment $O'Y$. Next, using Theorem 11.2, find the midpoint of line segment OO' and label as L. Construct circle LO' and label the intersection of circles OX and LO' as T. Draw line TO'. Draw line TO. Label the intersection of line TO and circle Γ as Q. Using Theorem 7.4 which states that two straight lines set upon a diameter of a circle form a right angle, we know that line TO' is tangent to circle OX. Use Theorem 11.6 to construct a line parallel to line TO through O'. Label the intersection of this line and circle $O'Y$ as P. Because it was constructed as such, we know that segment TQ and radius $O'P$ are congruent. Construct line QP. By Euclid I.28, angles QTO' and $PO'T$ each form right angles, making lines TO and $O'P$ parallel. Thus we have the rectangle $TO'PQ$ where all interior angles are equal to one right angle. Then, line *QP* is perpendicular to lines *TO* and *O'P* making it tangent to both circles. Similarly, using the additional intersection point of circles *LO'* and *OX* and the same argument, we can construct line *HJ* tangent to both Γ and Γ'.



Case 4: Γ and Γ' do not touch at any point, *four tangent lines*

Use the exact same construction steps for Case 2 to construct two tangent lines in this case. Draw line *OO'*. Label the intersections of Γ and Γ' with line *OO'* as *X* and *Y* respectively. Include the circle constructed with diameter *OO'* and center *K*. Construct circle *O'A* with radius equal to *O'Y* + *OX*. Label the intersection of circle *KO* and circle *O'A* as *B*. Draw line *OB*. Draw line *O'B*. By Theorem 7.4, angle *O'BO* is a right angle. Label the intersection of Γ' and line *O'B* as *N*. Construct a line parallel to line *O'B* through the point *O*. Label the intersection of Γ and this line as *M*. Draw line *MN*. This line is tangent to both circles Γ and Γ' because, as in previous cases above, a rectangle *BOMN* is formed with two sets of congruent, parallel sides and all interior angles equal to one right angle each. Similarly, using the same method, construct the other interior tangent line to both Γ and Γ'.



Case 5: Γ is equal to Γ', *infinitely many tangent lines*

Since Γ and Γ' are the same circle, any line tangent to Γ will also be tangent to Γ.

Case 6: Γ' lies inside Γ with no points of contact, *zero tangent lines*

Since Γ' lies inside Γ, any line tangent to Γ will never be tangent to or even touch Γ'.

$\square$

GIVEN A RECTANGLE, CONSTRUCT A SQUARE OF EQUAL CONTENT

JOHN FISHER

Communicated by Emily Herbst

Theorem 13.8. Given rectangle WXYZ, construct a square with equal content.

Proof. Begin by extending WX. Then draw a circle centered at W through Z. Call the intersection of WX and the circle WZ, point R. Next, draw a circle centered at R through point X, and a circle centered at X through point R. This is required to find the midpoint of line XR. Label the two points of intersection A and B. Draw AB. The intersection of segment AB and XR shall be labeled S, with S being the midpoint of XR. Now draw a circle centered at S through point X. Lastly, extend WZ. Call the intersection of WZ and the circle centered at S through X, point T.

Now draw TS and label TS as "a", WS as "b", and TW as "c". Since SR is congruent to TS (both radii), we may label SR as "a". By the same reasoning, we may also label WS as "b", WR as "a-b", and WZ as "a-b". This gives rectangle WXYZ's content as WX by WZ, or (a+b) by (a-b) which is equivalent to a squared - b squared. By Euclid's Proposition I.47,in triangle SWT we have b squared + c squared = a squared. So c squared = a squared - b squared. So the content of rectangle WXYZ must also be equal to c squared. Thus it is required to construct a square of equal content.

We will now construct a square of equal content using the side length of c (WT), belonging to trianlge SWT. First draw a circle centered at T through W. Then draw a circle centered at W through T. Call the point where the circle centered at W through point T intersects the extension of WX (from our first step) point W'. From point W', draw W'T. Draw a circle centered at W' through T and a circle centered at T through W'. Join the intersections by drawing a segment. Label the intersection of this segment with W'T as point D (perpendicular bisector). Now draw a circle centered at D through T. Label the intersection of the circle DT and TW as T'. Draw W'T'. This creates a tangent line to circle TW through T' from point W'.

The last paragraph creates a square, TWW'T', with side lengths all equal to "c", because the sides were constructed using circles with equal radii of "c". We know this is a square because the side lengths are congruent, and there are four right angles. We know angle TWW' is right by Euclid Proposition I.13. We know angle WW'T' is also right by corresponding parts of congruent triangles. We know angle W'T'T is right since the segment W'T' is tangent to circle TW at point T', and by Euclid Prop III. 18 the tangent line is perpendicular to the center of the circle. Lastly, we know that angle T'TW is right also by corresponding parts of congruent triangles.Thus, TWW'T' is a square with content of c squared, so TWW'T' and rectangle WXYZ have equal content.

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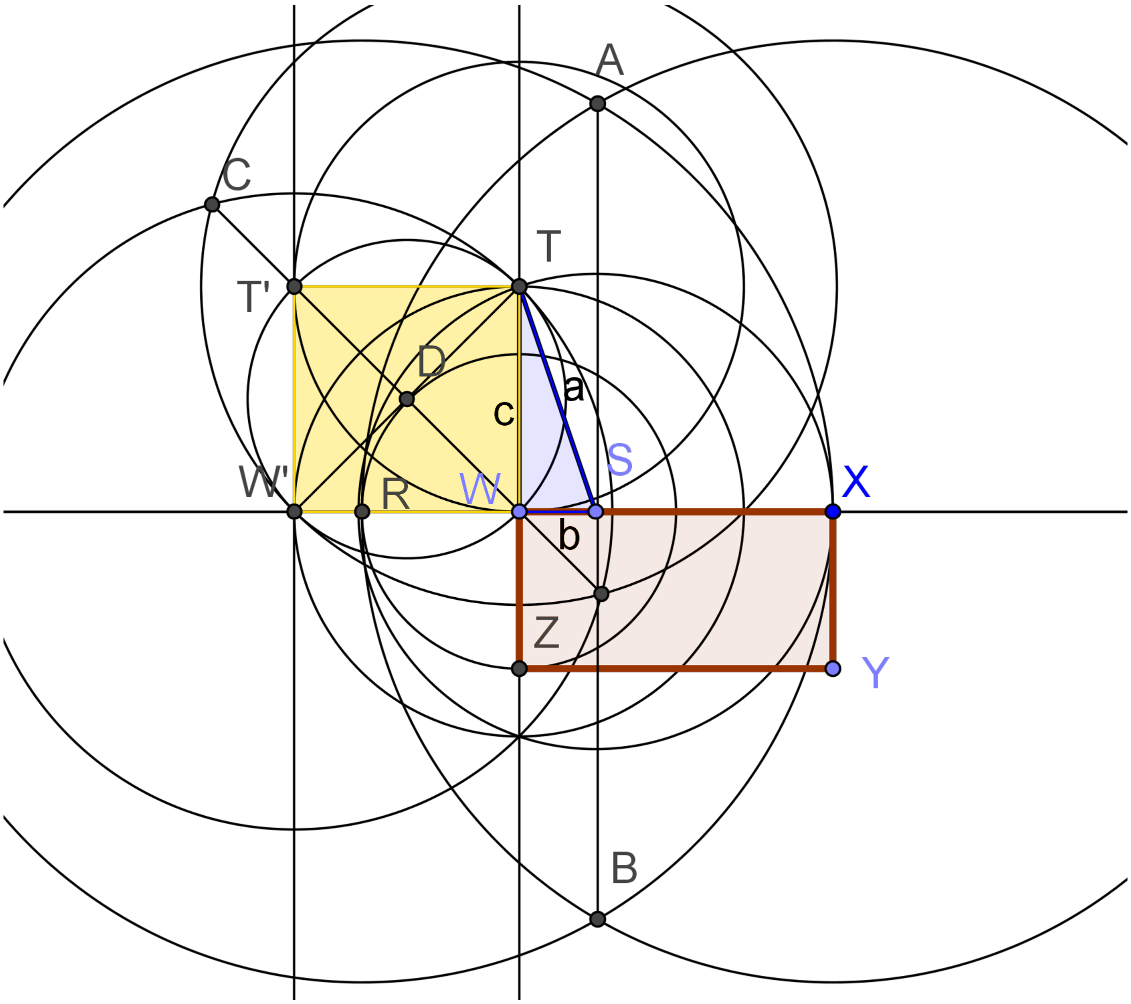

FIGURE 1. Notice the square has content of c squared, and the rectangle has content of a squared-b squared, which is proven to be equal to c squared. The important figures are also shaded for aesthetics.