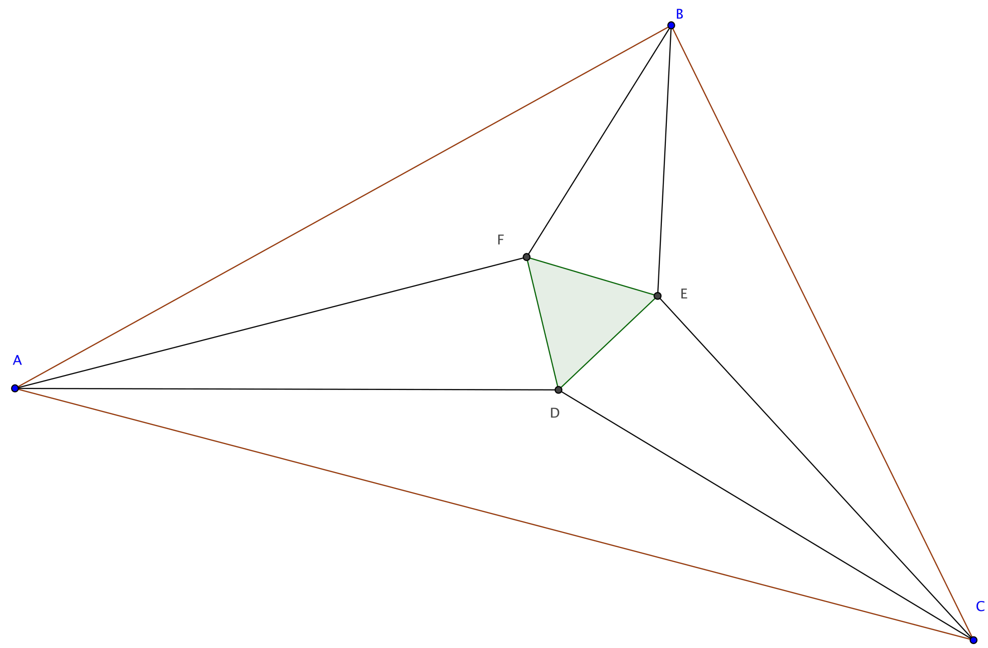
Transactions in **Euclidean Geometry**



Issue # 4

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Euclid's Proposition I.4: Controversy?

Duece K Phaly

October 12, 2016

The following will discuss the problem with Euclid's Proposition I.4 and does not follow the traditional format of a proof.

Challenge 4.3: Figure out what the problem is with Euclid's Proposition I.4.

Explanation:

In Euclid's Proposition I.4, it states that if two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-lines equal then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. In other words, Euclid proves that if there exist two triangles in which two sides are congruent respectively and the angle shared between those two sides are congruent then the both triangles will be congruent to one another as well as well as the remaining corresponding components of the triangles.

To prove this Euclid uses an "application" method of one triangle upon another. He uses the "application" to prove that lines and angles coincide with one another proving congruence of the respective lines and angles. However, Euclid fails to explain what "applying" is in his previous definitions and postulates. This idea of motion or moving objects is unclear and thus this proposition is invalid.

Exterior Angles of a Polygon

Tessa Cohen and Erica Schultz

October 16, 2016

At the start of our polygon studies, we are introduced to a different type of angle than that of the angles which we are used to working with. This new angle is called an exterior angle. Constructing the exterior angle at a vertex comes with a choice which could prove to be a problem for us. Let us first look at how this construction is performed: Suppose that A, B, C are three consecutive vertices of a polygon. If at the vertex B we extend one of the two sides through B to a ray, then we create a new angle, called an exterior angle to the polygon at B.

As we go through the process of constructing an exterior angle, we see that there is a choice we must make. We can choose to use side AB to construct the ray which gives us the exterior angle, or we can choose to use side CB. In principle, this could be a problem because we are unsure if one choice will give us a different exterior angle than the other choice. We will now see a theorem that resolves this problem.

Theorem 5.1. If A, B, C are three consecutive vertices of a polygon, the exterior angles of the polygon created at B will be congruent to one other.

Proof. Let A, B, C be three consecutive vertices of a polygon. We will show that either choice made when constructing an exterior angle at B will give us an angle which is congruent to the angle created by the opposite choice. Using the construction we are given, let us extend sides AB and CB as rays through vertex B. These rays create both possible exterior angles of the polygon at B. These angles can be seen in Figure 1, given below. By extending the sides AB and CB, we have created two straight lines which cut one another and create exterior angles which are vertical angles. Therefore, the exterior angles are congruent to one another by Euclid 1.15.

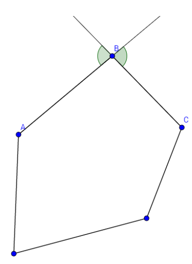

Figure 1: Polygon with highlighted exterior angles at vertex B

□

Addition of Exterior Angles of a Pentagon

Duece K Phaly

October 12, 2016

Theorem 5.2. Suppose that ABCDE is a pentagon such that inserting the diagonals BD and BE splits it into three non-overlapping triangles. Then the exterior angles of ABCDE taken together make four right angles.

Proof. Let ABCDE be a pentagon such that inserting the diagonals BD and BE splits it into three non-overlapping triangles (figure 1). By Postulate 2, we will extend line segments to create rays extending from each vertex. So, extending side AE through vertex A creates a ray to some other point on the line we will call A1. Lines will be extended the same way in a similar format creating rays BB1, CC1,DD1,EE1 (figure 2). These rays will create exterior angles by our current definition of exterior angles. Let us label the exterior angles at each vertex a,b,c,d,e to correspond with the vertices A,B,C,D,E respectively (figure 2).Let us also label the angles adjacent to each exterior angle a1,b1,c1,d1,e1 respective to a,b,c,d,e.

Diagonals BD and BE splits the pentagon into three non-overlapping triangles.It is known by Proposition 32 that the three angles in a triangle combine to make two right angles(2R). Thus since we have three triangles in our figure we can say that the angles in our figure make up six right angles (6R) (a1 + b1 + c1 + d1 + e1 = 6R).

Proposition 13 allows us to conclude that the any exterior angle of our figure when combined with its adjacent angle combine to make two right angles, because each side (a straight line) falls on another straight line. Thus we can say a + a1 = b + b1 = c + c1 = d + d1 = e + e1 = 2R. If we were to combine these equations we would have a + b + c + d + e + a1 + b1 + c1 + d1 + e1 = 10R. We know that a1 + b1 + c1 + d1 + e1 = 6R, so we can substitute and we have a + b + c + d + e + 6R = 10R. By subtracting 6R from both sides we are left with a + b + c + d + e = 4R. Since a + b + c + d + e is the exterior angles of polygon ABCDE taken together and it equals four right angles, it is proven what was intended to show. □

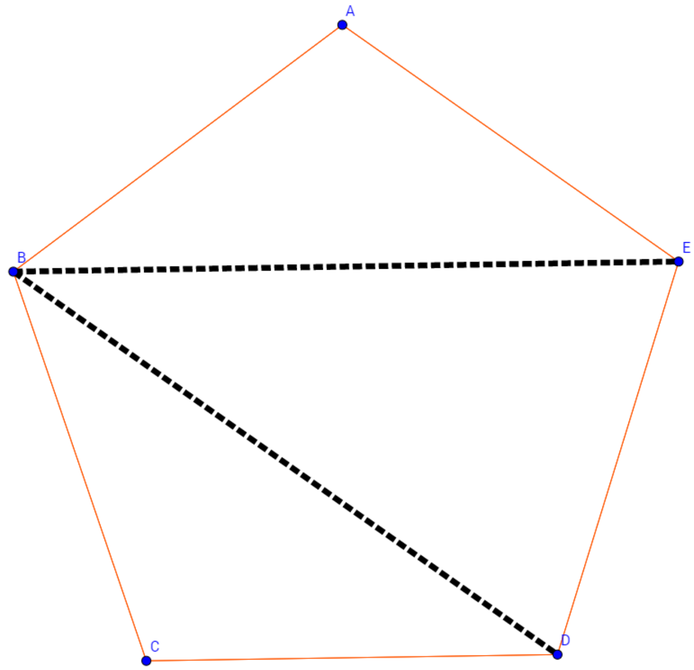

Figure 1: Pentagon ABCDE with diagonals BE and BD

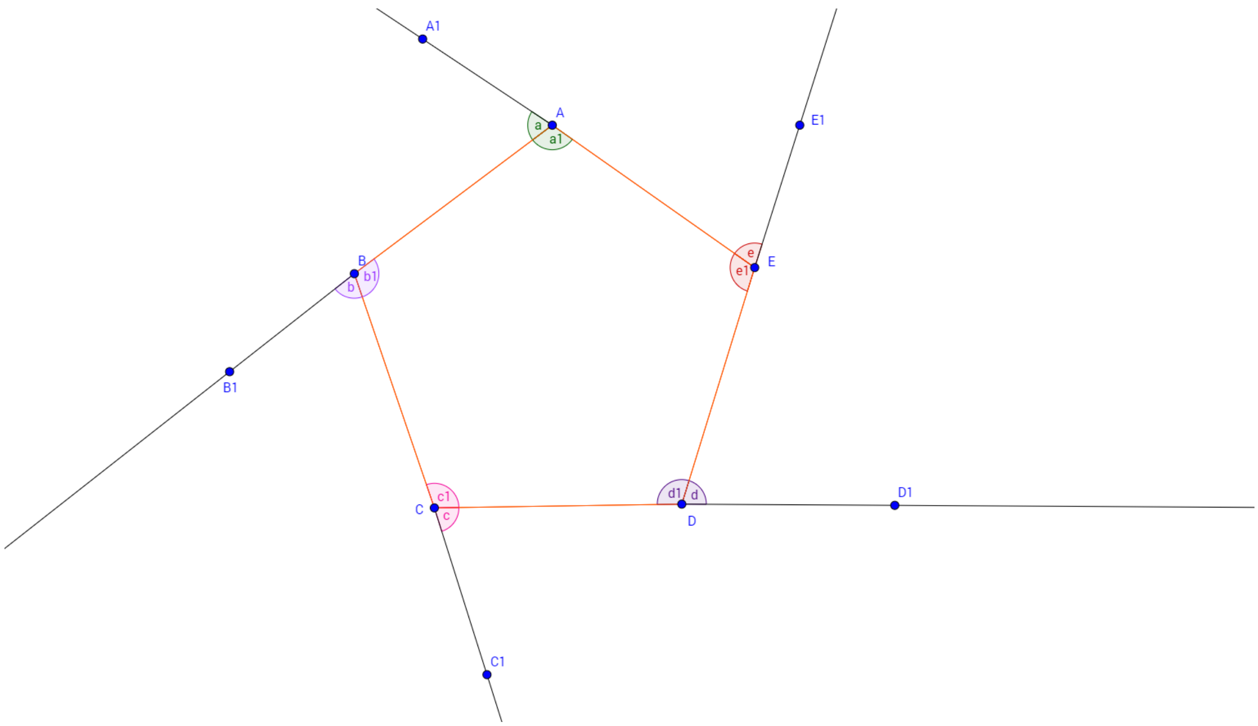

Figure 2: Pentagon ABCDE with exterior and their respective adjacent angles

Simple Polygons

Rebecca Shere

November 29, 2016

Communicated by Ms. Maus.

We have discussed the definition of a polygon to be as follows:

Definition A. Let n be a natural number. An *n-gon* is a figure consisting of n points A_1, A_2,..., A_n, prescribed in order and called *vertices*, and n line segments, called *sides*, A_1A_2, A_2A_3,..., $A_{n-1}A_n$, A_nA_1. A *polygon* is an *n-gon* where n has not been specified.

In many cases, it is helpful to talk about a certain type of polygon; therefore, we will create two categories of polygons. We will call them simple and non-simple.

Definition B. A polygon is *simple*, when the following three statements hold:
1. The vertices A_1, A_2,...,A_n are all distinct.
2. No two adjacent sides, A_1A_2, A_2A_3,..., $A_{n-1}A_n$, A_nA_1, are collinear.
3. If two sides, A_1A_2, A_2A_3, ..., $A_{n-1}A_n$, A_nA_1, meet they meet at a vertex, A_1,A_2, ..., A_n, which is an endpoint of both of the sides.

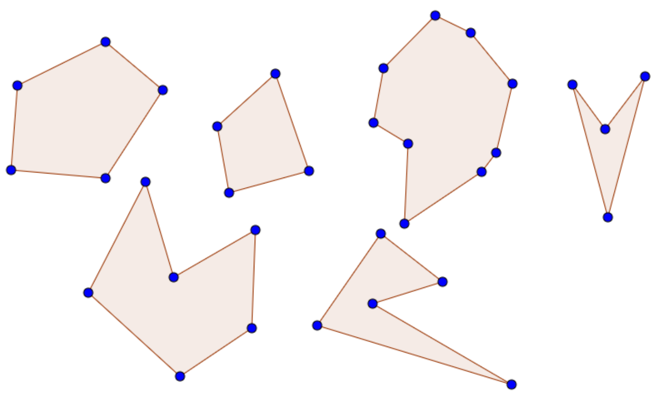

Figure 1: Simple polygons

If a polygon does not meet all three requirements used to define simple, then it will be known as a non-simple polygon.

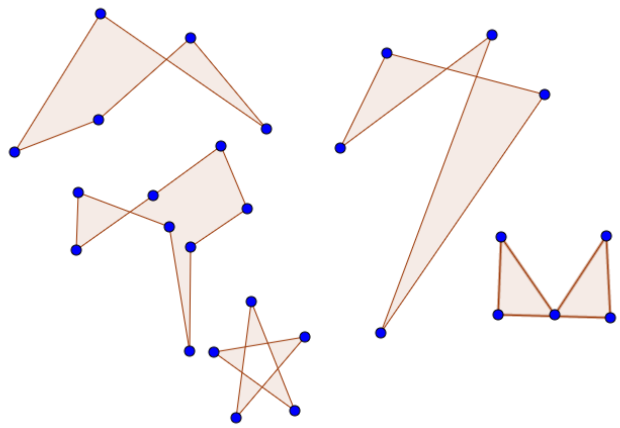

Figure 2: Non-simple polygons

Definition of an Interior Angle of a Quadrilateral

Duece K Phaly

November 29, 2016

Communicated by Ms. Worsfold.

Question J. Find a reasonable way to define the term ”interior angle” for quadrilaterals.

Definition J: Let ABCD be a quadrilateral. We say that the angle ABC is an interior angle of ABCD when the angle bisector of ABC meets the quadrilateral in a point different from B on the quadrilateral. (The definition is being applied in figure 1. Figure 2 is an example of an angle that is not interior by definition.)

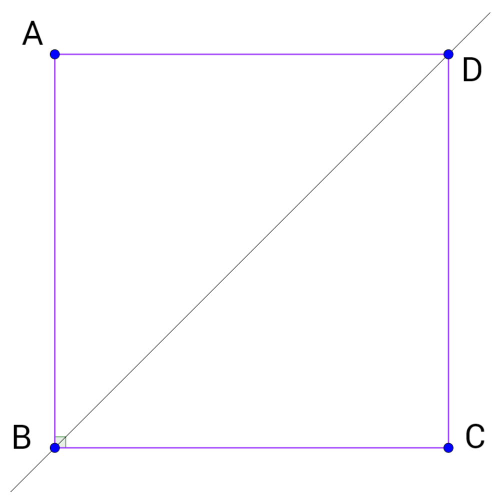

Figure 1: The angle bisector created at B, intersects the point D on the figure. Thus angle ABC is an interior angle by definition.

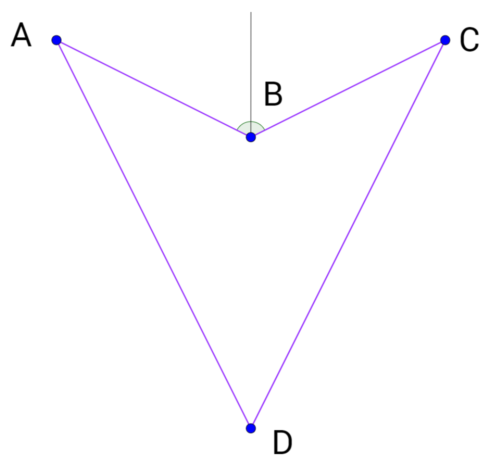

Figure 2: The angle bisector created at B, does not intersect another point on the figure. Thus angle ABC is not interior by definition.

The Converse of Euclid's Postulate 5

Taryn Van Ryswyk

October 13, 2016

This proof will show that the converse of Euclid's Postulate 5 is indeed true.The converse of Euclid's Postulate 5 is stated that if a straight line falling on two straight lines make the interior angles on the same side greater than or equal to two right angles, the two straight lines, if produced indefinitely, will not meet on that side on which are the angles great than or equal to two rights. This proof will also be split into three cases.

Theorem I. Draw a diagram of the kind where ABCD is like a rectangle, if angles ABC and BCD taken together are greater than or equal to two right angles, then ray BA does not meet ray CD.

Proof. CASE 1: Let angle ABC and angle BCD taken together be equal two the sum of right angles. Prove that if the lines are parallel, then ray BA and ray CD do not meet. Then by Euclid's Postulate 2, extend line segment AB, BC, and CD. By Euclid's Proposition I.13, since the angles made by two straight lines will make either two right angles or angles equal to the sum of two right angles, then angle ABE and angle ABC equal the sum of two right angles. Similarly, angle BCD and DCF will equal the sum of two right angles. By Euclid's Proposition I.28, if a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another. Thus since straight line BC falls on straight line AB and straight line CD and exterior angle ABE is equal to interior and opposite angle BCD, then lines are parallel. By Euclid's definition of parallel lines, since lines are parallel they will never cross, thus ray BA does not meet ray CD.

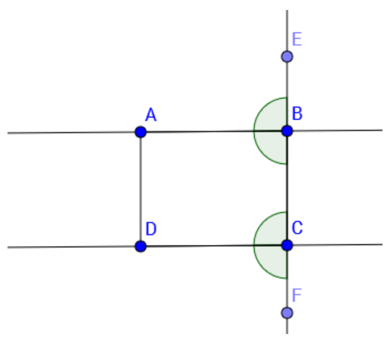

Figure 1: This image shows rectangle ABCD where angles are all right angles, thus angle ABC and angle BCD equal two rights.

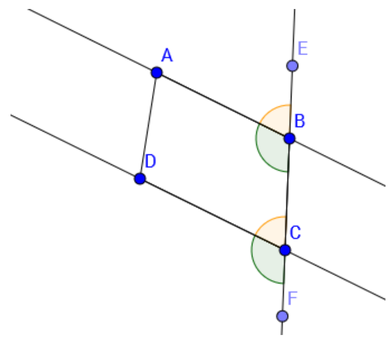

Figure 2: This image shows a figure like a rectangle where angle ABE and angle ABD equal two right angles. Similarly for angle BCD and angle DCF. Also, angle ABE is congruent to angle CDB. Similarly for angle ABD and angle CDF.

CASE 2: Let angle ABC and angle BCD taken together be greater than the sum of two right angles, where both angles are obtuse. Then by Euclid's Postulate 2, extend line segment AB, BC, and CD. By Euclid's Proposition I.13, since the angles made by two straight lines will make either two right angles or angles equal to the sum of two right angles, then angle ABC and angle EBC equal the sum of two right angles. Similarly, angle BCD and angle ECB equal the sum of two right angles. Since angle ABC is greater than a right angle, then angle EBC is less than a right angle since angle ABC and angle ECB equal the sum of two right angles. Similarly, since angle BCD is and obtuse angle, by Euclid's definition of obtuse the angle is greater than a right angle, then angle ECB is an acute angle, by Euclid's definition of acute the angle less than a right angle. Since interior angle EBC and interior angle ECB are both acute angles, the angles together will be less than the sum of two right angles. Then by Euclid's Postulate 5, lines will meet on the side where the angles are less than two rights. Since the lines cross at point E on the side where angles are less than two right angles the lines will not cross on the side where angles are greater than two right angles, thus ray BA does not meet ray CD.

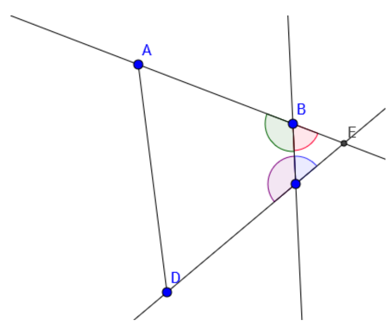

Figure 3: This image shows two interior obtuse angles which are greater than two right angles and two interior acute angles which are less than two right angles Thus, the side which has interior angles greater than two right angles will not meet, and the side which has interior angles less than two right angles will meet.

CASE 3: Let angle ABC and angle BCD taken together be greater than the sum of two right angles, where angle BCD is acute and angle ABC is obtuse. Then by Euclid's Postulate 2, extend line segment AB, BC, and CD. By Euclid's Proposition I.13, since the angles made by two straight lines will make either two right angles or angles equal to the sum of two right angles, then angle ABC and angle EBC equal two right angles. Similarly, angle BCD and angle ECB equal two right angles. Since angle ABC is obtuse, then angle EBC is acute. Similarly, since angle BCD is acute, then angle ECB is obtuse. Since angle ABC and BCD must be greater than two right angles and the angles made by two straight lines will make either two right angles or the sum of two right angles by Euclid's Proposition I.13, then angle ECB and EBC will be less than two right angles. Thus, by Euclid's Postulate 5 lines will meet on the side where the angles are less than two right angles. Since the lines cross at point E on the side where angles are less than two right angles the lines will not cross on the side where angles are greater than two right angles, thus ray BA does not meet ray CD.

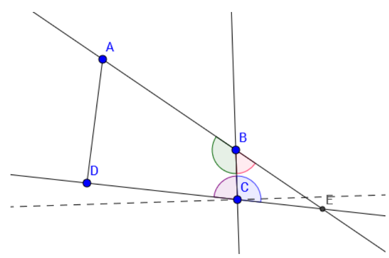

Figure 4: This image shows acute angle BCD and obtuse angle ABC that are greater than two right angles, and acute angle EBC and obtuse angle ECB that are less than two right angles. Thus the lines cross on the side of AC which angles are less than two right angles. The dotted line in the image represents a line perpendicular to AC going through point C.

Since all cases conclude that ray BA does not meet CD, then if a straight line falling on two straight lines make the interior angles on the same side greater than or equal to two right angles, the two straight lines, if produced indefinitely, will not meet on that side on which are the angles great than or equal to two rights. ∎

Special Diagonals of a Regular Pentagon Form an Isosceles Triangle

Maria Ahrens

November 29, 2016

When two distinct diagonals are constructed in a regular pentagon, those diagonals along with one of the sides of the regular pentagon form an isosceles triangle.

Theorem 6.5. Triangle ACD is isosceles when ABCDE is a regular pentagon.

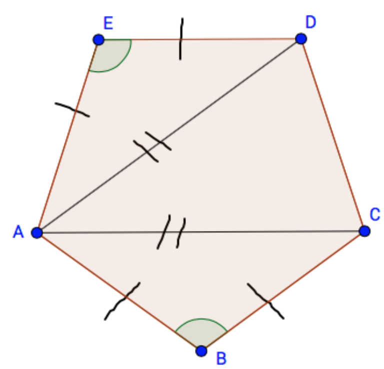

Figure 1: Triangle ACD is an isosceles triangle.

Proof. Let ABCDE be a regular pentagon.

From any vertex on ABCDE, draw diagonals from that point such that each diagonal extends to each non-consecutive vertex from that chosen point. We can do this by Postulate 1 of Euclid's Elements. In Figure 1, two diagonals were extended from Vertex A to the vertices, C and D.

Now we are going to focus on the two formed from the diagonals which do not share a side. In Figure 1,the two triangles are ADE and ACB. Since Pentagon ABCDE is regular, it is both equilateral and equiangular. Thus, the side, AE, is congruent to ED, which are also congruent to the sides, AB and BC, and Angle ABC is congruent to Angle AED. Triangle AED is then congruent to Triangle ABC by Euclid I.4. We can say that the diagonals, AC and AD, are congruent by Euclid's Common Notion 4, forming two congruent sides of Triangle ACD. Therefore, ACD is an isosceles triangle. □

Special Angles in a Regular Pentagon

Mackenzie Mitchell

November 29, 2016

Communicated by Ms. Cohen.

Theorem 6.6. Let ABCDE be a regular pentagon. Then angle DAC is congruent to half of angle ADC.

Proof. Let ABCDE be a regular pentagon, meaning all sides are congruent and all interior angles are congruent. Referring to figure 1, we can construct three triangles within the regular pentagon by constructing line segments AC and AD by postulate 1. Euclid Proposition I.32 states the sum of the interior angles of a triangle is congruent to 2 right angles. Since there are three triangles within the regular pentagon, the sum of the interior angles in the regular pentagon is congruent to 6 right triangles.

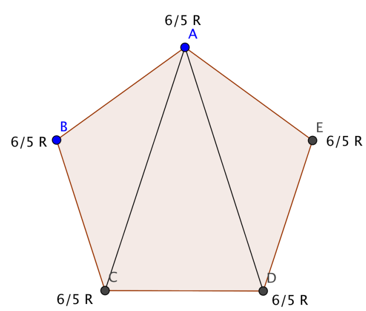

Figure 1: Pentagon with 3 Interior Triangles

Since there are 5 congruent interior angles in the regular pentagon and the sum of the interior angles are congruent to 6 right angles, each angle is congruent to 6/5 right angles. This means angles A, B, C, D, and E are congruent to 6/5 right angles. Referring to figure 1, we will look at triangle ABC. Sides AB and BC are congruent by the definition of a regular pentagon, thus making ABC an isosceles triangle by the definition of an isosceles triangle. By Euclid Proposition I.5, since triangle ABC is an isosceles triangle, the base angles are congruent. Therefore angles BAC and BCA are congruent. Since AB and BC are congruent to AE and ED by the definition of a regular pentagon and have congruent angles ABC and AED, the triangles ABC and AED are congruent by Euclid Proposition I.4. Thus line segments AB, BC, AE, and ED are congruent to each other, line segments AC and AD are congruent to each other, angles ABC and AED are congruent to each other, and angles BAC, BCA, EAD, and EDA are congruent to each other as well since they are isosceles triangles and congruent triangles.

As mentioned before, angles ABC and angles AED are congruent to 6/5 right angles. Using Euclid Proposition I.32, we know triangle ABC's interior angles sum is congruent to 2 right angles. Since angle ABC is congruent to 6/5 right angles, the sum of angles BAC and BCA are congruent to 4/5 right angles by Euclid Proposition 1.32. Since angle BAC and BCA are two congruent angles, each angle is congruent to 2/5 right angles. Since each angle (BAC and BCA) is congruent to 2/5 right angles and those angles are congruent to EAD and EDA, by Euclid's common notion 1, angles EAD and EDA are also congruent to 2/5 right angles (Refer to figure 2).

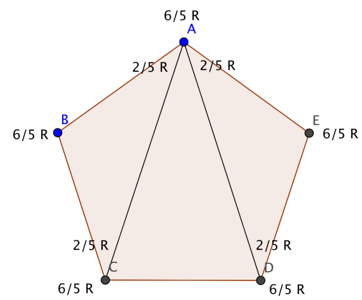

Figure 2: Angles in triangles ABC and AED

Consider angle EDC. Angle EDC consists of angles EDA and ADC. Being that angle EDA is congruent to 2/5 right angles and angle EDC is congruent to 6/5 right angles, then angle ADC is congruent to 4/5 right angles.

Now consider angle BAE. Angle BAE consists of angles BAC, CAD, and EAD. Angles BAC and EAD are each congruent to 2/5 right angles, thus the sum of angles BAC and EAD are congruent to 4/5 right angles. Being that angle BAE is congruent to 6/5 right angles, then angle CAD is congruent to 2/5 right angles (Refer to figure 3).

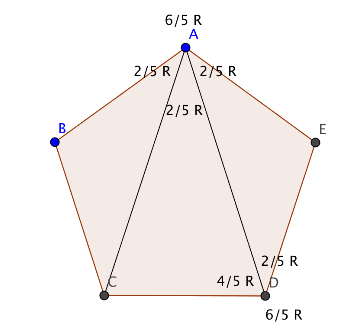

Figure 3: Looking at angles DAC and ADC

Thus we have angle DAC congruent to 2/5 right angles and angle ADC congruent to 4/5 right angle. Therefore, in a regular pentagon, angle DAC is congruent to half of angle ADC. □

An Impossible Triangle

Rebecca Shere

November 29, 2016

Communicated by Ms. Mitchell.

Problem 7.1 Show how to construct three segments which are not congruent to the sides of any triangle.

By Euclid's postulate 1 create line segment AB. Similarly, create a ray from A through B and construct point C such that line segment BC is congruent to line segment AB. Thus, line segment AC is congruent to line segment AB + line segment AB. Next, we create point D on ray A through B such that line segment AD is greater than line segment AC.

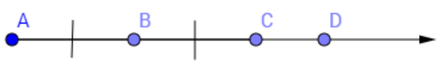

Figure 1: Ray from A through B with points C and D.

If we attempt to create triangle EDF with line segment ED congruent to line segment AB, line segment DF congruent to line segment AB, and line segment EF congruent to line segment AD, we will find it to be impossible. We created line segments ED and DF to each be congruent to line segment AB; therefore, the two line segments ED and DF taken together will be congruent to line segment AC. Line segment AD, which is congruent to EF, was constructed to be greater than line segment AC. It follows that line segment EF is greater than line segments ED and DF taken together. By Euclid's proposition I.20, we know that any two sides of a triangle taken together must be greater than the remaining side; however, the sides ED and DF taken together are less than side EF. Therefore, this triangle EDF cannot exist.

Regularity in Equilateral Triangles

Christopher Merck

November 30, 2016

Communicated by Ms. Mitchell.

Theorem 6.1. An equilateral triangle is equiangular, hence regular.

Proof. Let ABC be an equilateral triangle. Because triangle ABC is equilateral, line segments AB, BC, and AC are congruent. Because an equilateral triangle is also an isosceles triangle, then Euclid Proposition I.5 can be applied. In it, the angles at the base are congruent to each other. Hence, angles at the base of AB and AC, angles ABC and ACB, are congruent. Since all line segments are congruent, the Euclid Proposition I.5 can be applied again. Then the angles at the base of line segments BC and CA are also congruent. Hence, angles BAC and ACB are congruent. By Common Notion 1, in which things that are equal to the same thing are equal to one another, then angle ACB, which is congruent to ABC, is also congruent to BAC. Hence, the triangle ABC is equiangular, and therefore, regular.

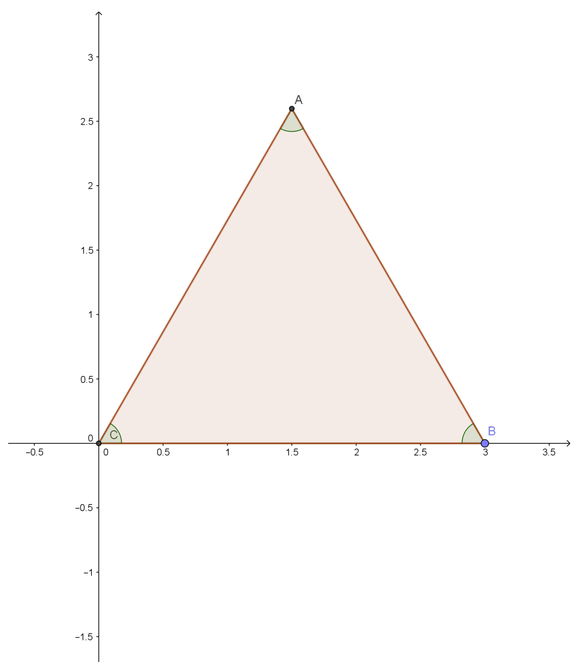

Figure 1: Triangle ABC

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