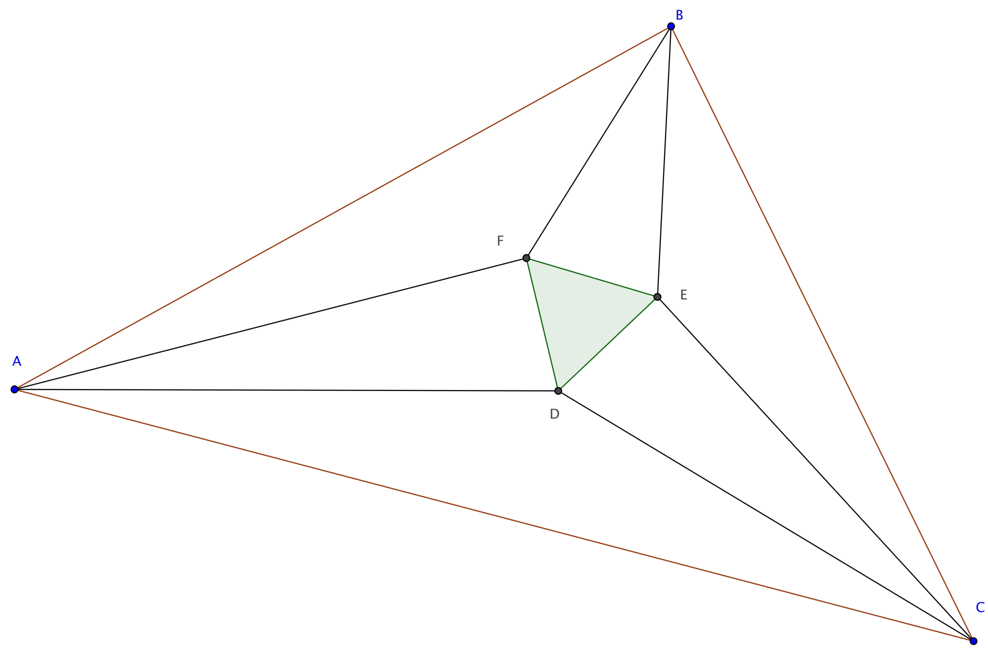
Transactions

in

# Euclidean Geometry




Issue # 6

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# Given a Line Segment and Any Angle a Rhombus Can be Constructed

Abigail Goedken

December 12, 2016

*Communicated by Ms. Mitchell.*

For the following proof the use of the construction of a rhombus given in theorem 1.4 is necessary. We will show that any rhombus can be constructed using this construction of a rhombus when given a random angle.

**Theorem B.** Let AB be a given line segment and let RST be a given angle. Then it is possible to chose a point D in the construction so that angle BAD is congruent to RST.

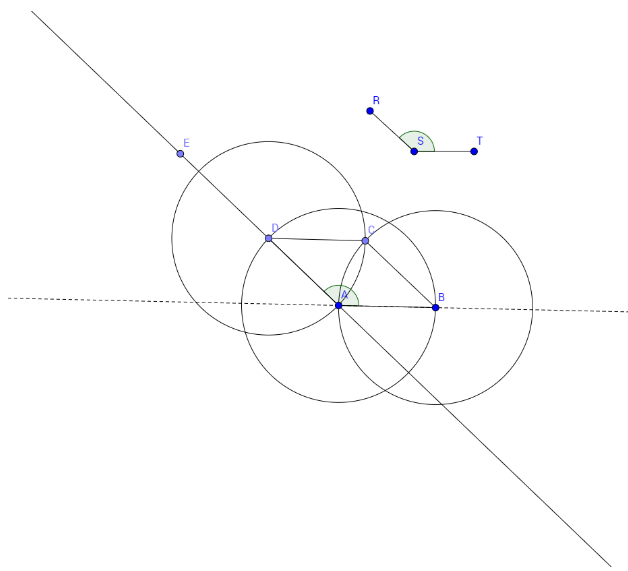



Figure 1: Copying any angle into Theorem 1.4: construction of a rhombus

*Proof.* Let AB be a given line segment on the construction of a rhombus. Let RST be a given angle.By Euclid I:23 we can create an angle congruent to RST at point A since we were given a line segment, a point on it and an angle. Let AE be the line that intersects line segment AB in order to create the angle congruent to RST. By Euclid 1:2 we can chose a point D on line AE such that Point D lies on the circle A with a radius AB. Since point D lies on the circle then AD is a radius. Since AD is a radius of circle A and AB is a radius of circle A then AD is congruent to AB. Using the construction in theorem 1.4 the rest of the rhombus can be created. Since line AE was the line that created the angle congruent to RST at point A, and AD is a segment of AE, then the angle at BAD is also congruent to angle RST. In conclusion given the line segment AB and a given angle RST a point D can be chosen in the construction such that angle BAD is congruent to angle RST.

$\square$

# How to Construct a Perfect Rhombus

Perry Kessel

October 23, 2016

A perfect rhombus is a rhombus ABCD that has segments AB congruent to segment AC.

**Question C.** Is it possible to construct a "perfect rhombus?" That is it possible to construct a rhombus ABCD so that AB is congruent AC.

*Proof.* Draw a circle A of radius segment AB. Draw an another circle at point B with radius of BA. Then, find the points of intersection of the two circles. We know there are points of intersection by Circle-Circle Intersection Property. According to postulate 1, make segments: AC,CB,BD,AD, connecting all vertices. (see Figure 1) We know that AC is congruent to AB because that points C and B land on circle A. Therefore, AC and AB are radii of circle A. □

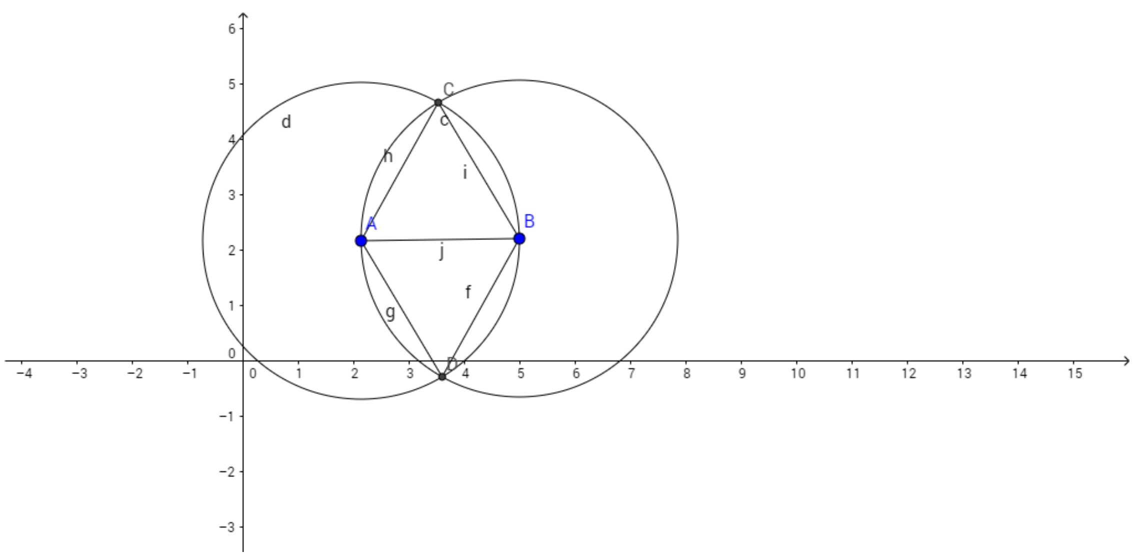



Figure 1: AB is congruent to AC

# A Kite is Not a Parallelogram

Kandy Schwan

December 5, 2016

I am going to show that a kite is not a parallelogram, unless it is a special type called a rhombus.

**Theorem 2.4.** If ABCD is a kite, it is not a parallelogram.

*Proof.* By the definition of a kite, we know the line segment AB is congruent to the line segment BC. We also know the line segment AD is congruent to the line segment CD, by the definition of a kite. Then, we can use postulate 1 and draw line segments AC and BD. The line segment BD is congruent to itself, and the line segment AC is congruent to itself. By proposition 5, the angle BCA is congruent to the angle BAC, because triangle ABC is an isosceles triangle. Also, by proposition 5, the angle DAC is congruent to the angle DCA, because triangle ACD is an isosceles triangle. Then, we can conclude the triangle ABC is not congruent to the triangle ADC by the parts of the triangles not being congruent of the previous statements.

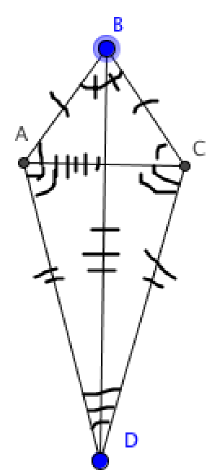



Figure 1: ABCD is a kite.

By the proposition 32, all angles of a triangle will add up to the sum of two right angles. Since the angles BAC and BCA are congruent to each other, but different than the congruent angles of DAC and DCA, the angles ADC and ABC are not congruent to each other. So, by this statement and previous statements, the triangles ABC and ADC are not congruent.

By postulate 2, we can extend the lines AB, BC, CD, and DA infinitely in both directions. Then, by proposition 27, since the angles ABD and BDC are not congruent, the line AB and CD are not parallel. Similarly, since the angles DBC and BDA are not congruent, the lines BC and AD are not parallel. So, by the definition of a parallelogram, the kite ABCD is not a parallelogram.

If line segments we chosen such that ABCD fit the definition of a rhombus, ABCD would be a parallelogram. A kite has two pairs of adjacent congruent sides, while a rhombus has four mutually congruent sides. $\square$

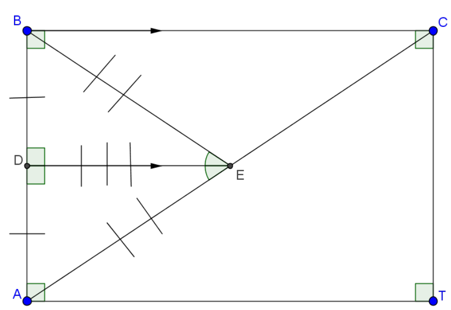
# Midline Theorem: A Special Case

Samantha Ancona

October 17, 2016

**Theorem 3.6.** Let ABCT be a rectangle. Let line segment AC exist by postulate 1. Thus we have the right triangle ABC. Let D be the midpoint of AB, and E be the midpoint of AC. Then the line segment ED, called a midline also constructed by postulate 1, is parallel to the line segment BC.

*Proof.* By Theorem 3.3 we will draw line segment BE that will be congruent to segment AE. Now we have an isosceles triangle BEA. With the Folklore Theorem, we can say that ED bisects the angle BEA, meets AB at its midpoint D, and BD is congruent to AD. Also, we know that the midline, ED, is congruent to itself. Since all sides are congruent, their adjacent angles are congruent. Since the angles are congruent, they must both be right angles by Proposition 11. Postulate 4 states that all right angles are congruent to other right angles; therefore, right angle ABE is congruent to right angle BDE. Thus by Proposition 28, the midline ED is parallel to segment BC. □



# The Midline of a Triangle is Parallel to the Base

Abigail Goedken

December 5, 2016

*Communicated by Ms. Van Ryswyk.*

**Theorem 3.6.** Let ABC be a triangle, D the midpoint of AB and E the midpoint of AC. Then the line through E and D, called the midline, is parallel to the line through BC.

*Proof.* Let ABC be a triangle. Let triangle ABC have point D and point E on it such that D is the midpoint of segment AB and E is the midpoint of segment AC. By Euclid postulate 1 we can draw the line segment between the points D and E. By Euclid postulate 2, we can extend the line segment DE. On line DE we can create a point F such that the line segment FD is congruent to the line segment DE by Euclid I:2. Using Euclid I:23, we can create an angle on the line ED at point F that is congruent to the angle ADE. By Euclid I:23 we can construct triangle GFD at point F such that GF is congruent to AD, GD is congruent to AE, and FD is congruent to DE. Therefore the angle GFD is congruent to angle ADE. Similarly, the angle at GDF is congruent to the angle AED by Euclid I:23. Angle GFD is congruent to angle ADE. Segment FD is congruent to DE. Angle GDF is congruent to angle AED. Thus, triangle FGD is congruent to triangle DAE by Euclid I:26.

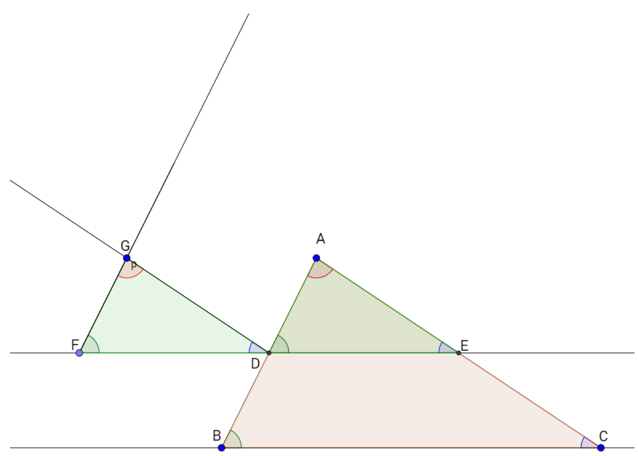



Figure 1: Triangle GFD is Congruent to Triangle ADE

Euclid I:32 tells us that the interior angles of a triangle equal the sum of two right angles. Since ADE is a triangle, then the sum of angles ADE, AED, and DAE equal two right angles. In triangle GFD, angle GFD is congruent to angle ADE and angle GDF is congruent to angle AED. Since the angles of a triangle must equal the sum of two right angles, angle FGD must be congruent to angle DAE. Since angle GFD is congruent to angle ADE, angle GDF is congruent to angle AED, angle FGD is congruent to angle DAE, and segment FD is congruent to DE, then triangle GFD is congruent to triangle ADE since all angles between the triangles are congruent and they have a side congruent.

Since points F, D, and E are co-linear the angles at point D must be equal to the sum of two right angles by Euclid I:13. Since angle GDF is congruent to angle AED and angle ADE is congruent to itself the angle GDA must be congruent to angle EAD. Thus all of the angles that fall on line FE will equal the sum of right angles. By Euclid postulate 1 we can draw the line segment from point G to point A. Since segment AD is congruent to DA, angle ADG is congruent to angle DAE, and segment DG is congruent to AE, then triangle DAG is congruent to triangle ADE.

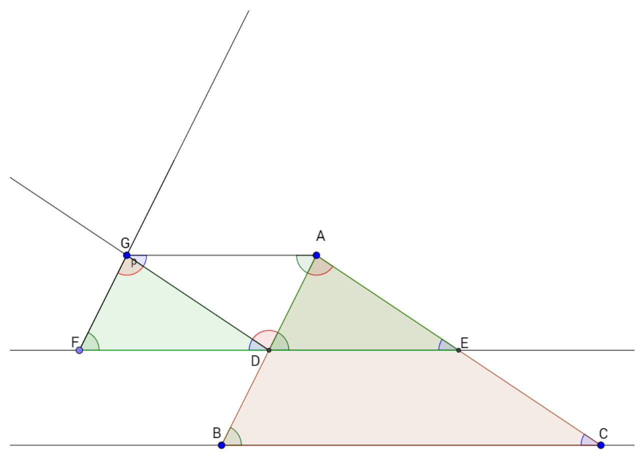



Figure 2: Triangle DAG is Congruent to Triangle ADE

By Euclid postulate 2 we can extend the line segment FG into a line. By Euclid I:2 we can create a point H on line FG so that segment GH is congruent to segment FG. Three angles fall on line FG: angle HGA, AGD, and DGF. By Euclid I:13 we know that angles that fall on a line must equal the sum of two right angles. Since DGA is congruent to angle AED and angle FGD is congruent to angle DAE, then angle HGA is congruent to angle ADE since the interior angles of triangle ADE add up to two rights by Euclid I:32. Since segment GH is congruent to segment FG and segment FG is congruent to segment DA, then segment GH is congruent to segment DA. Since segment GH is congruent to segment DA, angle HGA is congruent to angle ADE and segment GA is congruent to segment DE, then triangle HGA is congruent to triangle ADE by Euclid I:4.

Since point D is the midpoint of segment AB, segments AD and DB are congruent. Therefore, the sum of segments AD and DB is equal to 2 segments AD. 2 segments AD is

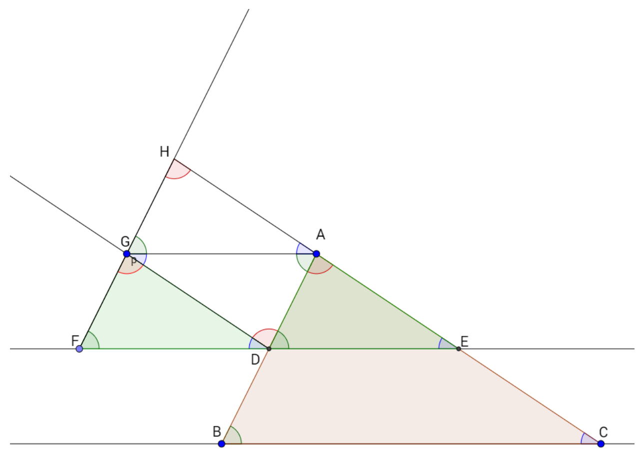



Figure 3: Triangle HGA is Congruent to Triangle ADE

congruent to segment AB. The sum of segments HG and GF is equal to segment HF. Since segment HG is congruent to segment AD and segment GF is congruent to AD, then the sum of segments HG and GF is equal to 2 segments AD. Thus segment HF is congruent to 2 segments AD. Therefore, segment HF is congruent to segment AB. Similarly since point E is the midpoint of segment AC, segments AE and EC are congruent. Therefore, the sum of segments AE and EC is equal to 2 segments AE. 2 segments AE is congruent to segment AC.

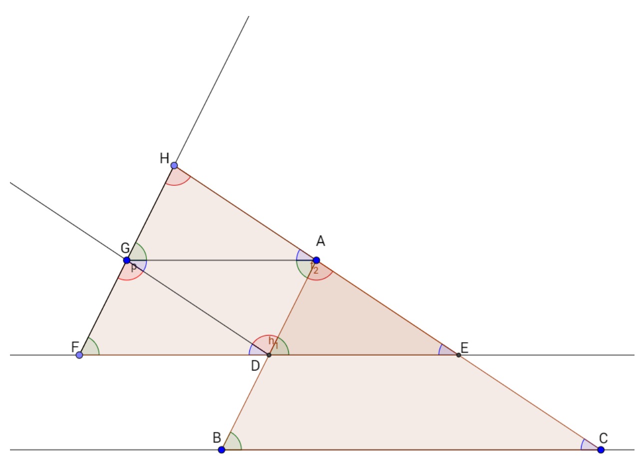



Figure 4: Triangle HFE is Congruent to Triangle ABC

The sum of segments HA and AE is equal to segment HE. Since segment HA is congruent to segment AE and segment AE is congruent to itself, then the sum of segments HA and AE is equal to the sum of segments AE and AE. Thus segment HE is congruent 2AE. Therefore, segment HE is congruent to segment AC. Since segment HF is congruent to segment AB, angle FHA is congruent to angle BAC, and segment HE is congruent to segment AC, then triangle HFE is congruent to triangle ABC by Euclid I:4. Since triangle HFE is congruent to triangle ABC, then angle ABC is congruent to angle HFE. Angle HFE is congruent to angle ADE since angle GFE is congruent to angle ADE. Since angle ABC is congruent to angle HFE and angle ADE is congruent to angle HFE, then angle ABC is congruent to angle ADE by Euclid CN:1. Since angle ADE is congruent to angle ABC, then the line segments DE and BC are parallel by Euclid I:28. Therefore the midline, DE, of triangle ABC is parallel to the base, BC.

$\square$

# Regular Rhombus Must Be a Square

Taryn Van Ryswyk

December 5, 2016

*Communicated by Ms. Mitchell.*

Theorem 6.2 states: Let ABCD be a rhombus. If angle A is congruent to B, then ABCD is regular. The results of this proved that a rhombus must be a square in order for the rhombus to be regular.

**Theorem 6.3.** Let ABCD be a rhombus. If angle A is congruent to angle C, then ABDC is regular.

*Proof.* Theorem 6.2 will not hold the definition for a regular polygon ABCD, if we replace "angle b" by "angle c". By Theorem 1.1, angle A is congruent to angle C and angle B is congruent to angle D. However, when a rhombus is not a square, angles A and C are not congruent to angles B and D, thus it will not be equilangular. We know that a rhombus equals four right angles, thus if angle A and angle C are greater than a right angle, we can conclude that angle B and angle D are less than a right angle. Similarly if angle A and angle C were less than a right angle, and angle B and angle D were greater than a right angle. Therefore, when angle A and angle C are congruent in a rhombus, angle B and angle D will not be congruent to angle A and angle C unless the rhombus is a square. This then concludes that a rhombus is not a regular polygon unless it is a square where all angles are equal. □

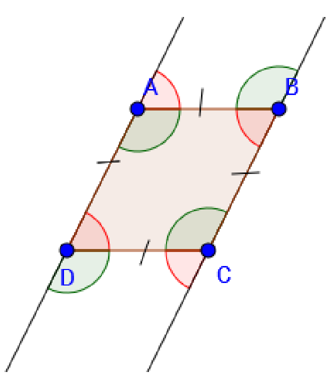



Figure 1: Rhombus where angle A and angle C are obtuse and angle B and angle D are acute.

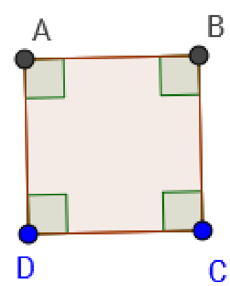



Figure 2: Rhombus where all angles are equal, otherwise known as a square.

# Construction of Line Segments Not Congruent to the Sides of any Triangle

Abigail Goedken

October 27, 2016

**Theorem 7.1.** We will be showing how to construct line segments not congruent to the sides of a triangle.

*Proof.* First, we must construct a circle AB with center A and radius of AB lets call this circle D. By Euclid postulate two we can extend segment AB into a line. By the circle line relationship line AB will cross D at a second point. Let's call this point C. Line segment BC is the diameter of circle D. Lastly, by Euclid postulate 1 we can draw line segment CA. CA is a radius of circle D. If we take the three segments CA, AB, and BC we have three segments that are not congruent to the sides of any triangle. Since CA and AB are both radii of circle D, CA plus AB will be congruent to the diameter of circle D. Since CA plus AB is congruent to the diameter of circle D and BC is a diameter of circle D then, CA plus AB is congruent to CB. Euclid I:20 says that in any triangle two sides taken together in any manner are greater than the remaining one. Since CA taken with AB is congruent to BC then two sides taken together are not greater than the remaining one, therefore these three segments are not congruent to the sides of any triangle.

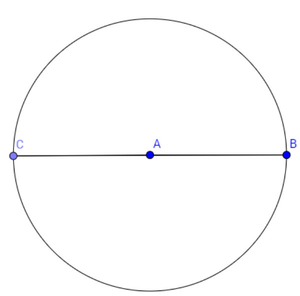



Figure 1: Three Line Segments Not Congruent to the Sides of Any Triangle

□

# Acute-Angled Triangles and Congruence

Mackenzie Mitchell

December 5, 2016

*Communicated by Ms. Shere.*

Theorem 7.2 states: Let ABC and DEF be two right triangles, with the angles at A and D right angles. Suppose that BC is congruent to EF and AB is congruent to DE. Then the triangles are congruent. This leads into Conjecture 7.3 which states: If we weaken the hypothesis of the previous conjecture so that the angles A and D are still congruent but no longer assumed to be right angles, and leave the other hypotheses intact, the conclusion still holds. After further exploration, we have discovered this is not true for all cases. We see that this fails with obtuse-angled triangles, but holds for acute-angled triangles. In order to prove conjecture 7.3, we must first show the altitude of the acute-angled triangle falls within the triangle first.

**Lemma 7.3.** In an acute-angled triangle, the altitude falls within the triangle.

*Proof.* Let ABC be an acute-angled triangle. We will proceed by contradiction.

Assume the altitude falls outside of the triangle at point X (refer to figure 1). This creates a right triangle BXC with BXC being a right angle, by definition of an altitude. By Euclid Proposition I.32, since X is a right angle and a triangle's interior angles must add up to two right angles, angles XBC and XCB add up to one right angle. Thus angles XBC and XCB are acute angles. Consider angles XCB and BCA. By Euclid Proposition I.13, since XCB and BCA lie on a straight line, the angles must add up to two right angles. Since angle XCB is acute, angle BCA must be obtuse. Here lies a contradiction. Since BCA lies in the triangle and the triangle is an acute-angled triangle, BCA must be an acute angle, not obtuse. Thus, the altitude of an acute-angled triangle cannot lie outside the triangle.

Now assume the altitude falls on the side of the triangle. Let this side be BC (refer to figure 2). By the definition of an altitude, angle BCA will be a right angle. Here lies a contradiction. Since angle BCA lies in the triangle and the triangle is an acute-angled triangle, angle BCA is acute, not right. Thus, the altitude of an acute-angled triangle cannot lie on the side of the triangle.

Since the altitude of an acute-angled triangle cannot lay outside the triangle or on the side of the triangle, the altitude must lay on the inside of an acute-angled triangle. □

**Theorem 7.3.** Let ABC and DEF be two acute-angled triangles. Suppose that BC is congruent to EF, AB is congruent to DE, and angle BAC is congruent to angle EDF. Then the triangles are congruent.

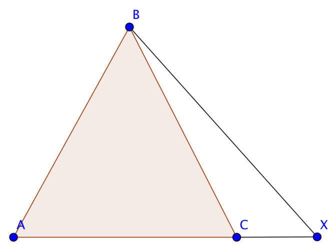



Figure 1: Triangle ABC with the altitude lying outside the triangle

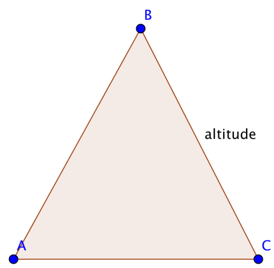



Figure 2: Triangle ABC with the altitude lying on the side of the triangle

*Proof.* Consider figure 3 with triangles ABC and DEF. By the definition of altitude and by Lemma 7.3, create the altitudes of both triangles where triangle ABC's altitude meets at point X (between A and C) and triangle DEF's altitude meets at point Y (between D and F).

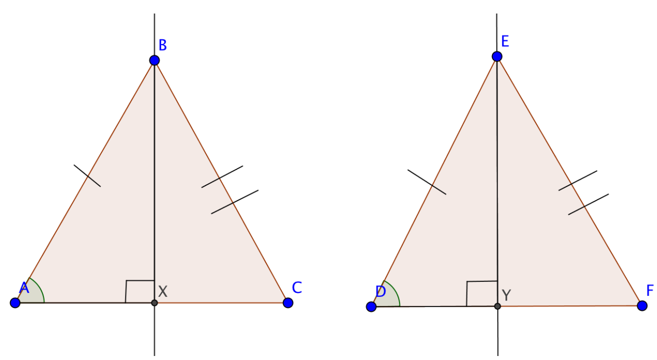



Figure 3: Triangles ABC and DEF with AB congruent to DE, BC congruent to EF, and angles BAC and EDF congruent. Also holding altitudes BX and EY in triangles ABC and DEF respectively

This creates two triangles in each triangle: triangles ABX and BXC in ABC and triangles DEY and EYF in DEF. By Euclid Proposition I.26, these triangles are congruent since they have two congruent corresponding angles and one congruent side. This results in having angles ABX and DEY congruent, BX and EY congruent, and AX and DY congruent.

Referring to figure 4, since the two angles (AXB and BXC, and DYE and EYF) lie

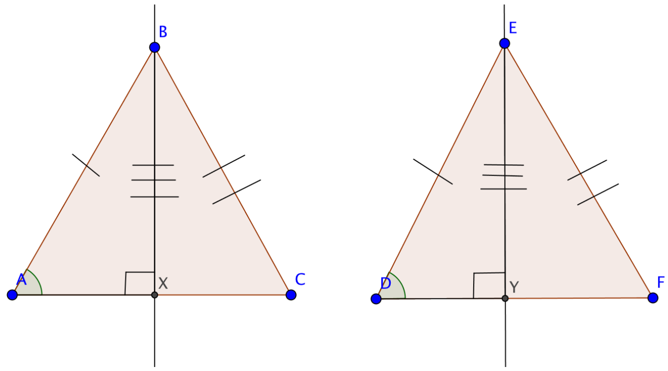



Figure 4: Triangles ABC and DEF with triangles ABX and BEY congruent

on a straight line and angles AXB and DYE are right angles, angles BXC and EYF are right angles by Euclid Proposition I.13 . Since BX is congruent to EY, BC is congruent to EF and angles BXC and EYF are right angles, by Theorem 7.2 (hypotenuse-leg theorem), triangles BXC and EYF are congruent. This results in XC being congruent to YF, angle FEY congruent to angle CBX, and angle XCB congruent.

We know AX and XC taken together is congruent to AC, and DY and YF taken together is congruent to DF. This can be carried out by the definition of the sum of segments. Since AX is congruent to DY, XC is congruent to YF and AC is congruent to AX plus XC, then AC is congruent to DY plus YF by common notion 1. Since AC is congruent to DY plus YF and DF is congruent to DY plus YF, by common notion 1, AC is congruent to DF (refer to figure 5).

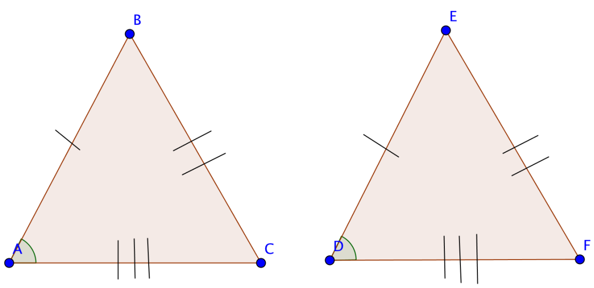



Figure 5: Triangles ABC and DEF with AB congruent to DE, BC congruent to EF and AC congruent to DF.

Since triangles ABC and DEF have three congruent corresponding sides, by Euclid Proposition I.8, triangles ABC and DEF are congruent.

Thus, when triangles ABC and DEF are two acute-angled triangles with angles BAC and EDF congruent, AB congruent to DE, and BC congruent to EF, triangles ABC and DEF are congruent. □

# Right Triangles Within Any Circle

Megan King

December 5, 2016

*Communicated by Ms. Maus.*

**Theorem 7.4.** If AB is the diameter of a circle and C lies on the circle, then angle ACB is a right angle.

*Proof.* Let circle O exist with center O and diameter AB. Picking any point on the circle O, create point C. By postulate 1, we can connect points to create line segments AC, OC, and BC. Segments OA, OB, and OC are all congruent to each other because they are all radii of circle O.

Within the triangle ACB that was created, there are two isosceles triangles, triangle OAC and triangle OBC. By Euclid I.5, angle OAC is congruent to angle OCA. Similarly, angle OBC is congruent to angle OCB. Since ABC is a triangle, by Euclid's I.32, all the interior angles add up to two right angles. Therefore, by taking angle OAC, angle OCA, angle OBC, and angle OCB, together they will equal two right angles.

Since angle OCA is congruent to angle OAC and angle OCB is congruent to angle OBC, we can say that two of angle OCA is congruent to two of angle OCB, which will equal two right angles. Since angle OCA taken together with angle OCB is congruent to angle ACB and we have two of angle OCA congruent to two of angle OCB, we can say two of angle ACB is equal to two right angles. Therefore, by Euclid's Common Notion 4, angle ACB is equal to one right angle.

Thus, angle ACB is a right angle.

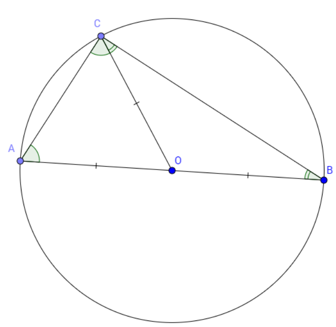



Figure 1: Right Triangle ACB within Circle O

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