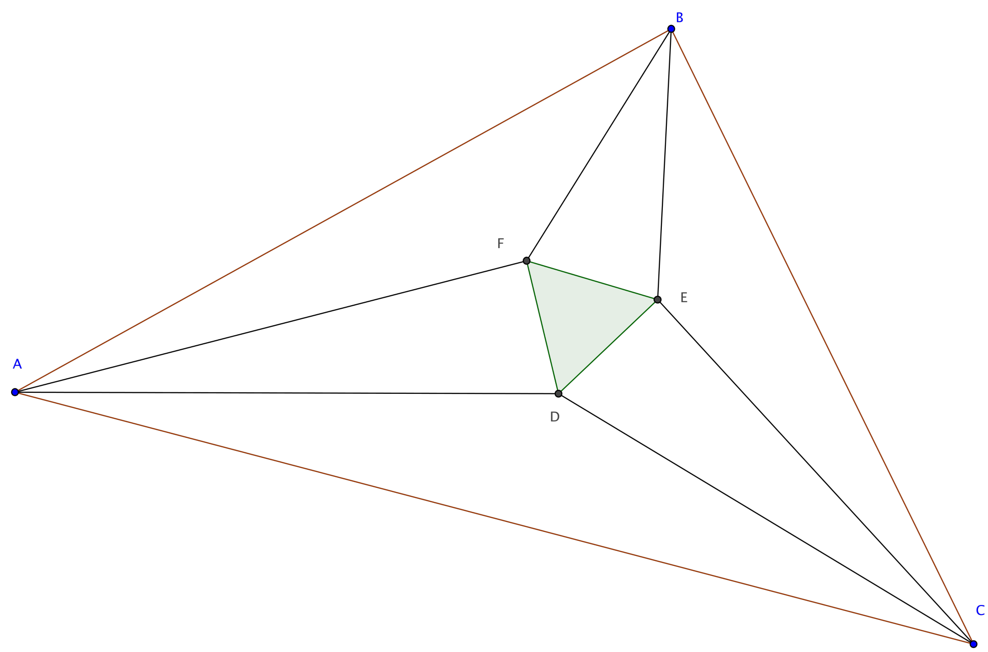
Transactions

in

Euclidean Geometry



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Rhombus Construction with Equilateral Triangles

Alexa DeVore

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Communicated by: Jaclyn Miller.

Theorem A. Given a segment AB it is possible to construct a rhombus ABCD having a diagonal AB and sides which are congruent to AB.

Proof. Let AB be a line segment. Construct a circle with center A such that AB is the radius and similarly, construct a circle with center B with AB as the radius. Due to sharing the same radius AB, circles A and B will intersect at two distinct points. Let one of the points where the circles intersect be called C and let the other be called D. Based off Postulate 1.1, we can connect A to points C and D and also B to C and D. Note that these four segments, AC, BC, BD, and AD, are radii of circles A and B. Since both circles A and B share the same radius of AB, circles A and B must have congruent radii. Since these four line segments are radii of the respective circles, the four segments must be congruent and therefore, the construction, ABCD, is a rhombus.

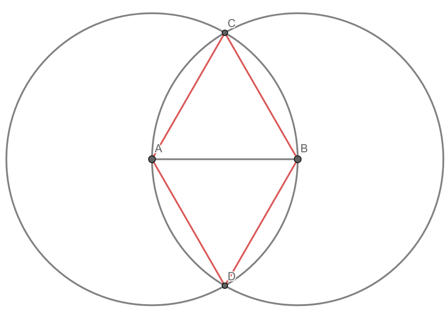

Figure 1: This is a picture of the construction.

□

Diagonals Form Right Angles

Jason Stine and Lexis Wiegmann

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Theorem 1.7. Let ABCD be a rhombus. Suppose that the diagonals AC and BD meet at a point X. The angle AXB is a right angle.

Proof. Let ABCD be a rhombus. By the definition of a rhombus, we have four mutually congruent sides. Let there be diagonals AC and BD, such that they cross at point X. Because of Falck's Theorem of bisectors, we know that AC and BD bisect each other. Since BD bisects AXC at point X, then AX is congruent to XC. We also know BX is congruent to BX.

By Euclid 1.8 we know that the two triangles, AXB and BXC, are congruent because they have three congruent corresponding sides. Since the two triangles are congruent, they also have congruent corresponding angles. Thus, angle AXB is congruent to angle BXC.

Recall, we have a straight line BX set up on a straight line AXC. By Euclid 1.13, we know we have either two right angles or two angles which are equal to two right angles. We know that angle AXB is congruent to angle BXC. Since the angles are congruent and they are equal to two right angles, we know that each of the angles is a right angle. Therefore, angle AXB is a right angle. □

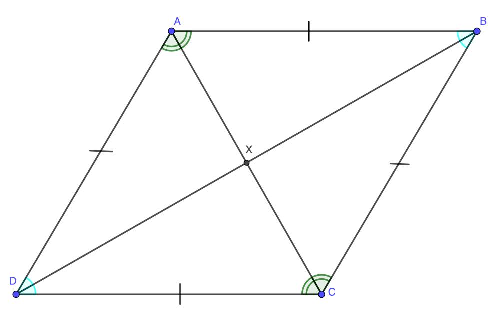

Figure 1: ABCD is a rhombus. X is the point where the diagonals intersect.

Diagonals of a Kite

Jaclyn Miller

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Recall that there is one pair of opposite congruent angles in a kite (Theorem 2.1), and there are two diagonals of a kite.

Lemma 2.2a. The diagonal that passes through the opposite and non-congruent angles in a kite, bisects those angles.

Proof. Let ABCD be a kite. By definition, a kite has two pairs of congruent sides. We assume that the kites vertices are labeled such that line segment AB is congruent to line segment BC and line segment AD is congruent to line segment CD. By Euclid Postulate I.1, draw line segment BD. Since line segments AD and CD are congruent, and line segments AB and BC are congruent, and line segment BD is common, triangle ABD is congruent to triangle BCD by Euclid Proposition I.8. Thus, the corresponding parts of triangles ABD and BCD are congruent as well. So, angle ABD is congruent to angle DBC and angle BDA is congruent to angle BDC. Thus, the diagonal BD bisects angle ABC and angle ADC. □

Theorem 2.2. The extended diagonals of a kite must cross.

Proof. Let ABCD be a kite. By definition, a kite has two pairs of congruent sides. We assume that the kites vertices are labeled such that line segment AB is congruent to line segment BC and line segment AD is congruent to line segment CD. By Euclid Postulate I.1, draw line segment BD and line segment AC. By Euclid Postulate I.1, let diagonals AC and BD be extended to straight lines.

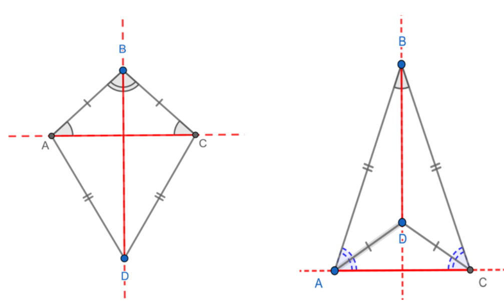

Figure 1: Two different constructions of kite ABCD.

By Euclid Proposition I.32, angles ABC, BAC, and BCA taken together are congruent to two right angles. By Euclid Proposition I.5, angle BAC is congruent to angle BCA. Since angle BAC is congruent to angle BCA and angles ABC, BAC, and BCA are congruent to two right angles, neither angle BAC nor angle BCA is congruent to one right angle.

By Lemma 2.2a (above), diagonal BD bisects angle ABC, so angle ABD is congruent to angle DBC. Thus, angle ABD and angle BAC taken together are less than two right angles. Therefore, by Euclid Postulate I.5, the extended diagonals of a kite must cross.

□

Construction of a Kite

Jaclyn Miller

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Theorem 2.3. It is possible to construct a kite, ACBD, with a compass and straightedge.

Proof. First, construct the circles and line segments needed to make the kites.

1. Let A, B, and C be points, where C is not collinear with A and B.
2. By Euclid Postulate I.3, construct a circle centered at point A, with radius AC, called circle A.
3. By Euclid Postulate I.3, construct a circle centered at point B, with radius BC, called circle B.

Because point C is not collinear with points A or B, and is part of a radius for both circle A and circle B, point C is at an intersection of circle A and circle B. Thus, there exists a point F that is inside both circle A and circle B; and there exists a point E that is inside circle A but outside circle B. Then, by the Circle-Circle Intersection Property, circle A and circle B intersect twice. Let D be the second point of intersection.

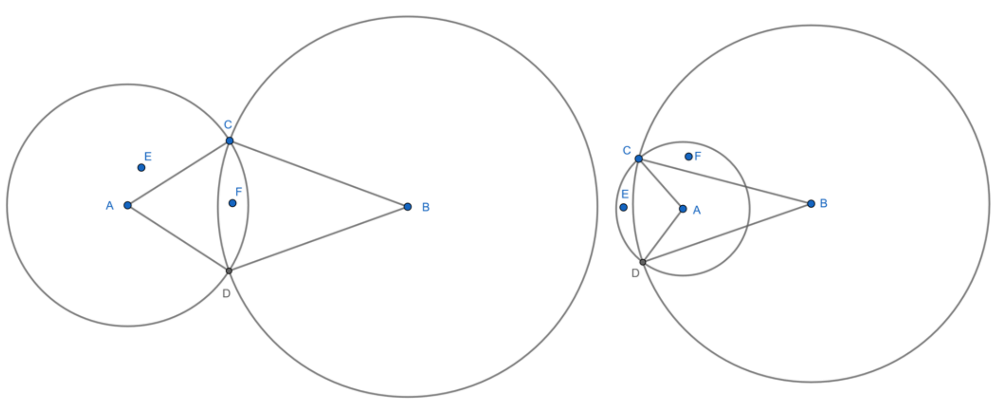

Figure 1: Diagram of Convex Kite ACBD (left) and Non-Convex Kite ACBD (right).

By Euclid Postulate I.1, draw line segments AD, AC, BC, and BD. Since line segments AC and AD are both radii of circle A, line segment AC is congruent to line segment AD. Similarly, since line segments BC and BD are both radii of circle B, line segment BC is congruent to line segment BD. Since line segments AC and AD share angle CAD, they are adjacent by definition. Similarly, since line segments BC and BD share angle CBD, they are adjacent by definition. Thus, each quadrilateral has two sets of adjacent and congruent sides. Therefore, the quadrilateral is a kite by definition.

$\square$

By Ms. DeVore's observation, the kite is convex if point A lies outside of circle B, as seen in the left construction of the kite. The kite is non-convex if point A lies inside of circle B as seen in the right construction.

Midline Theorem

Jaclyn Miller

September 12, 2018

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Theorem 3.6. Let ABC be a triangle, D the midpoint of AB and E the midpoint of AC. Then the line through E and D, called a midline, is parallel to the line through B and C.

Proof. Let ABC be a triangle, with sides AB, BC, and AC. Let D be the midpoint of line segment AB and let E be the midpoint of line segment AC. Because D is the midpoint of line segment AB, line segment AD is congruent to line segment DB. Because point E is the midpoint of line segment AC, line segment AE is congruent to line segment EC. By Euclid Postulate I.3, construct a circle with center E and radius DE. Let F be a point on the circle that is collinear with points D and E. Thus line segment DE is congruent to line segment EF because both are radii of circle E. By Euclid Postulate I.1, construct line segments CF, AF, and DC.

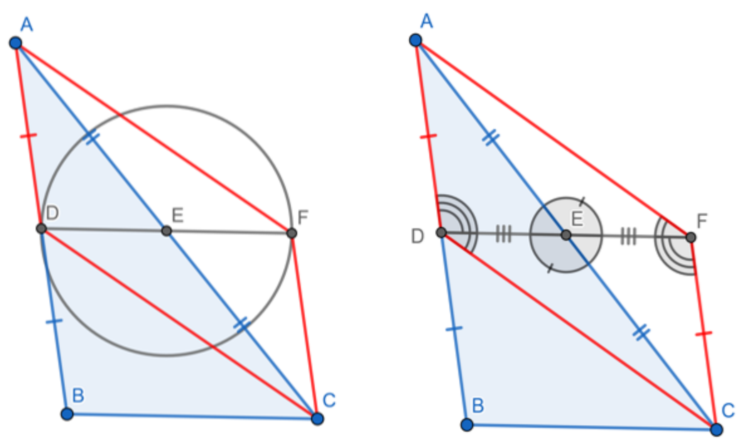

Figure 1: Two Versions of Triangle ABC.

Since line segment DF and line segment AC cut each other, angle AED is congruent to angle FEC by Euclid Proposition I.15. Thus, since line segment DE is congruent to line segment EF, line segment AE is congruent to EC, and angle AED is congruent to angle FEC, triangle AED is congruent to triangle FEC by Euclid Proposition I.4. Since corresponding parts of congruent triangles are congruent, line segments CF and AD are congruent and angles EFC and EDA are congruent. Since alternate interior angles EFC and EDA are congruent, AD is parallel to CF by Euclid Proposition I.27.

Similarly, since line segment AC and line segment DF cut each other, angle AEF is congruent to angle DEC by Euclid Proposition I.15. Thus, since line segments AE and EC are congruent, line segments DE and EF are congruent, and angles AEF and DEC are congruent, triangle AEF is congruent to triangle DEC by Euclid Proposition I.4. Thus, the corresponding parts of triangles AEF and DEC are congruent, so angle AFD is congruent to angle FDC. Since alternate interior angles AFD and FDC are congruent, line segment AF is parallel to line segment DC by Euclid Proposition 27. Since we have two sets of opposite and parallel sides (AD parallel to CF and AF parallel to DC), AFCD is a parallelogram by definition.

Since line segments AD and DB are collinear, and line segment AD is parallel to line segment CF, line segment DB is parallel to line segment CF. Recall that line segments AD and DB are congruent and line segments AD and CF are congruent. Thus, by Euclid Common Notion I.1, line segment DB is congruent to line segment CF. Since line segment DB is parallel to line segment CF and line segments DB and CF are congruent, line segment DF is parallel to line segment BC by Euclid Proposition, I.33. Since line segments DE and DF are collinear, and line segment DF is parallel to line segment BC, line segment DE is parallel to line segment BC. Therefore, in triangle ABC, the midline DE is parallel to line segment BC.

□