

Edited: 11:05am, February 15, 2017

JOURNAL OF INQUIRY-BASED LEARNING IN MATHEMATICS

Euclidean Geometry: An Introduction to Mathematical Work

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To the Student

We have two goals this semester: First and foremost, we shall learn to do mathematics independently. Also, we shall study planar Euclidean geometry. Euclidean geometry has been used throughout recorded history as a way to learn to do mathematics and as a way to sharpen thinking skills like problem solving and making logical arguments. We shall join the tradition.¹

About the Mathematics

The general plan of mathematical work has a recurring structure. It starts when somehow, some way, a person has had an idea about a bit of mathematics and has asked a *Question*. Perhaps, this person even believes that he or she knows the correct answer and is brave enough to share it publicly, and then we have a *Conjecture*. Now we have a job to do in two parts. First, we must decide what we believe the answer is. This is a lot of hard work where you look at examples, draw pictures, use your imagination and any other tools at your disposal to make up your mind. Second, you must find an argument supporting your answer. This argument is called a *proof*. We will spend almost all of our time on proofs this term. We will think them up, share them with others and argue over their correctness. When we as a community agree that a proof is correct, then the statement we started with will be called a *Theorem*. To restart the cycle, we collect observations and ask new questions based on what we have just accomplished.²

The Elements is full of theorems and proofs.³ You can use these as examples and guides for how a proof should look, but keep in mind that Euclid lived a long time ago. What we have now is translated from ancient Greek to English, and the translator did his job about a century ago, so the language is a bit odd. I want you to write your proofs in standard

¹Euclid's *Elements* is the world's oldest surviving textbook, and still one of the best, so we shall study it.

²... Or *failed* to accomplish.

³In fact, it has little else in it.

English prose. So, look at Euclid's first theorem and its proof as given by Heath's translation of Euclid's original, and compare it to this update that I'll write to show how one might do it today. Note that the descriptions given are pretty clear and match the figure. Generally, one should write as if there is no diagram, and then draw one for the reader anyway.

Theorem (Proposition One). *If AB is a line segment, then there exists an equilateral triangle ABC having AB as one of its sides.*

Proof. What we really need is a very special point C . The idea is to find the required point as the intersection of two circles.

By Postulate 3, there exists a circle $\odot AB$ with center A which passes through B , and also a circle $\odot BA$ with center B which passes through A . These two circles intersect twice. Choose one of the points of intersection and call it C . By Postulate 1 we may draw the segments AC and BC . This forms a triangle ABC , by definition of a triangle. What remains is to show that

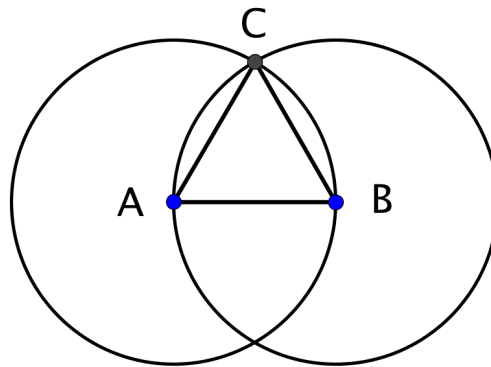


Figure 1: Euclid's First Figure

the three sides of the triangle are mutually congruent.

Since C and B both lie on circle $\odot AB$, by the definition of a circle, segments AC and AB are congruent. Similarly, since C and A both lie on circle $\odot BA$, by the definition of a circle, segments BC and AB are congruent. Now, by Common Notion 1, since AC is congruent to AB and AB is congruent to BC , we see that AC is congruent to BC . This means that all three of the sides of triangle ABC are mutually congruent, so ABC is equilateral, by definition. This concludes the proof. \square

Note that Euclid phrases his result as a construction problem and the “proof” has two parts: the first is a routine for doing the construction and the second is an argument for why it works. The more modern view is to see this as an *existence* result with a *constructive proof*.

Also, note how the proof works. Each statement is justified by reference to something we have already agreed upon. Since this is our first result, we can only refer to assumptions that we have agreed upon in advance. These assumptions are called *Postulates* or *Axioms* or *Common Notions*. Also, we have *definitions*, which are shorthands telling us that when we use a certain word, it has exactly some specific meaning.⁴ Later proofs can (and will) also rely on previously proved theorems to justify some of their steps.

This way of working is called the *Axiomatic Method* and is characteristic of mathematics. “Axiomatic Proof” is what distinguishes mathematics from practically everything else. It can feel a bit unnatural at first, but you will get used to it.

Finally, notice that one step really isn’t justified. Did you see it when reading? It is easier to spot in my writing than in Euclid’s version. If you didn’t notice it, go back and try to find it before reading further.

What is the gap in the proof? Euclid just assumes that the two circles $\odot AB$ and $\odot BA$ intersect twice. This seems obvious from the picture he draws, but nowhere in his list of Postulates and Common Notions does he say anything about how two circles should intersect! What is needed here is another assumption, which I hope you will be willing to believe, called the “circle-circle intersection property.”

Postulate (Circle-Circle Intersection Property). *If circle C contains a point in the interior of a circle D and also contains a point in the exterior of circle D , then circle C and circle D meet at two points.*

There are lots of little things like this in Euclid’s work. It is hard to know how picky one wants to be. My feeling is that we should just note what our assumptions are as we go, and to try to keep the list of assumptions relatively short and simple. But mathematicians are habitually picky, and you should get in the habit of reading with a very critical eye.

⁴We will spend a lot of time thinking about the way mathematical definitions work.

Advice About Doing Mathematics

It is impossible to summarize all one needs to know to do mathematics successfully, and I don't even pretend to know how to write an exhaustive list.⁵ But I want to give you some basic advice anyway.

Getting Stuck

If you are really working at a level that will help you improve, you will get stuck and confused a lot. Being confused is the first step to learning.⁶ When you feel stuck, come talk to me. I like teaching, I like geometry, and I want to help you get through this. I am sure that I can nudge you in a productive direction when you get stuck.

Good Work Habits

Work every day, at the same time each day if you can. Mathematics doesn't happen fast, so you must give yourself time to get to your goal. Also, when you have an idea, take notes. Most ideas don't work, but some that don't work now might work for another problem later. It is a terrible waste of time to "reinvent the wheel" for each problem. Keep a geometry research notebook.

Collaboration

Most mathematicians and students of mathematics find great value in talking to others about mathematics. I encourage you to do this, too. Getting the most out of a collaboration requires two things:

- Work hard on your own first.⁷ You can't have a conversation if you don't know what is going on and have no ideas to share.
- Give credit where it is due. If you talked to someone about a problem and then you solved it, you should mention that conversation when you give presentations of your work (orally or in writing).⁸ If you worked together through the whole process, then you should consider *joint authorship* on any written work.

⁵I am sure that I have much to learn, still.

⁶"Suckin' at something is just the first step to being sorta good at something." —Jake the Dog, *Adventure Time*

⁷No one wants a dead weight collaborator.

⁸Selfish people shed collaborators, while generous ones keep them.

Expectations

The expectations for this class are simple. You should come to class every day prepared to discuss the mathematics. First and foremost, this means:

You should try to solve every problem.⁹

You must solve several problems by the end of the semester. When you solve a problem, write up your arguments before class so that you are ready to present. During class when someone else is presenting, pay attention. Try to catch mistakes, both in the mathematics and in the writing.

Finally, doing mathematics is hard work, and presenting to the class is psychologically difficult (especially the first few times). I demand that we treat each other with the utmost respect at all times. Keep in mind that our differences will be over the content. Try to phrase your comments as questions as often as possible. The ideal question here starts with the phrase “Excuse me. I don’t understand...” and then ends with something specific like “...how do you use the definition of circle in line five?”

What are we up to?

It is likely that the way this course is conducted is new to you.¹⁰ Let me share some perspective about what will happen, and why I’ve arranged things this way.

I have three goals for this semester. In order, they are:

1. To help you gain power and strength as a mathematician.
2. To get you engaged in mathematical work, and to help you learn how mathematics is done successfully.
3. To talk about geometry.

These goals drive all of the choices I make when conducting this class, and they are the main reasons for the structure we use. The best way to learn to do mathematics is to engage with the material and *do mathematics*. So, everything about our structure is predicated on you and your classmates working in the same way that mathematicians do. You will

⁹TRY TO SOLVE EVERY PROBLEM.

¹⁰This course is a particular flavor of Inquiry Based Learning known as a Modified Moore Method. Our principal “modifications” from R.L. Moore’s method are (1) the use of a textbook as reference, and (2) allowing collaboration.

answer questions, prove theorems, make conjectures, and ask questions. Generally, you will work to *make sense of things on your own terms*.

Believe it or not, mathematics is a social activity. Progress is really made when you have reached a new level of understanding, and can clearly explain your ideas to another who is engaged in the same type of work. We will together be a little mathematics research community. We will take Euclid's *Elements* as our entire collection of literature, and I will be the grand guru who provides questions to begin our investigations.

So, how will we do things?

- You will work hard to prove theorems. You can do this on your own, or in small teams of collaborators. Because it is important to develop your own talents, I ask that you use only the *Elements* and no other references.
- Every class day will be devoted to “communication.” You and your classmates will take turns presenting your arguments, sharing your observations, and helping others to refine their ideas by asking questions.

Mathematics is difficult to do.¹¹ Everyone makes mistakes, great and small, in large numbers. This will undoubtedly happen to each of you “in public” at least once (probably repeatedly). But making mistakes is an important part of learning, and I don't want you to be afraid of them. Accept them and move on.

More about grades

NOTE: This section and the next addressed to the author's students. If you are not the author's student, the details in this section may not apply to you directly. Check with your instructor about how these details will work.

The most important thing you can do to achieve a good grade in this course is to get actively involved.¹² Prove things, share your work in presentations and in writing. At the end of the semester, I will assign grades in a holistic way. We will have a midterm and a final, but these are just another chance for you to make a good impression.

To be clear: In a typical mathematics class, the exams are worth just about everything, and class participation is worth just a little. In this class,

¹¹This is an understatement. But you knew that, didn't you?

¹²I have put a significant amount of material on the course web page about this semester's experiment in *Specifications Grading*.

things are the opposite. Your grade depends largely on the quality and quantity of presentations you give and papers you write. The midterm and final exam are just extra opportunities for me to check up on your progress.

A student who does not present at all is guaranteed to fail this course. In the past, students who earn passing grades have solved, presented and published several solutions. Earning an A requires more, both in quality and quantity.

About Sharing Your Work

When you have successfully completed a task from our list, you will get a chance to share it with the class. The first stage is to present your arguments to the class for evaluation by your peers. You should prepare carefully for this, as it is impolite to waste everyone's time with a sloppy and poorly given presentation. I advise you to write up your ideas and arguments carefully *before the class where you want to present*. Also, try to anticipate some questions that your classmates might ask so that you can handle them.

In the class meeting immediately following your successful presentation, you are required to turn in your work in the form of a paper to be published in our class journal. I'm sure you see that mathematical writing is very different from any writing you have been asked to do. One of our goals this semester is to get a start on developing a mathematical writing style. What should a paper look like? I have typed out a paper (with an old chestnut of an argument) that models the generally accepted style and included it as an appendix. You can use this paper as a guide for writing your first paper. Also, there is a template file you can use to properly format your work in \LaTeX .¹³

About drawing figures

It is very important when working to draw your own figures. There is no substitute for it. Even when reading Euclid, I suggest that you frequently draw your own diagrams.

How should you do it? Well, you could just go freehand. This is sufficient for poking around in a general way. But sometimes, you really want an accurate representation of what is going on, and then you need tools.

¹³The course website has links to both the example and a template. For this paper version of the notes, I have included the example at the end.

There are two basic tool sets available.

A physical compass and straightedge set These are available in many places, like the bookstore.

A digital compass and straightedge There are several software packages available that allow you to do geometric constructions. My favorite is *GeoGebra*¹⁴. This is a free, cross-platform, open-source software package for playing with lots of interesting mathematics, including planar geometry. It enables you to make constructions and then drag around the inputs to obtain other versions of the same construction. This is like making hundreds of example pictures all at once!

On the Different Types of Arguments

As the semester progresses, we will see several different types of arguments. Of course, the *Elements* is full of arguments, so you have lots of examples to absorb. It is good to have a mental list of argument styles, because one often works to prove a statement without knowing in advance if it really is true. After all, that is the point of finding a proof—to be really sure that the statement is true. If I am stuck in one attempt, I find it useful to switch styles and see what else I might learn.

With this in mind, here is a short guide to common types of arguments you might try.

Direct Proof Also called a *positive* argument. Use a logical chain of deduction to get from the hypothesis to the conclusion.

Indirect Proof There are two types of proof here, and some mathematicians are picky about the distinction, but they start the same way. One assumes that the conclusion is false, and then tries to derive some contradiction, either to the given hypothesis¹⁵ or to some other known truth.¹⁶ I am often sloppy about the distinction and call both of these “proof by contradiction.”

Counterexample A given statement about “all” objects of a certain type may be false. To disprove the statement, it is enough to give one explicit example of when the statement fails to hold.

There are other bits of mathematician’s jargon I could list here, but this is enough to get going with. So, on to the problems!

¹⁴<http://www.geogebra.org/cms/en/>

¹⁵proof of the *contrapositive*

¹⁶*reductio ad absurdum*

Chapter 1

Beginnings: The Rhombus

Read Euclid's *Elements* Book I Propositions 1-34. Pay particular attention to the triangle congruence theorems in I.4, I.8 and I.26. This material is all of your allowed references.

Now we take up our mathematical work.

Definition 1. *We say that some set of points is collinear when there exists a line passing through all of the given points.*¹

Definition 2. *A quadrilateral is a figure consisting of four points, no three of which are collinear, in a given order and the four line segments joining points next to each other in the list. Usually, we specify only the four points, so quadrilateral ABCD consists of the points A, B, C and D, called vertices of the quadrilateral and the line segments AB, BC, CD and DA, called the sides.*

Definition 3. *A rhombus is a quadrilateral having all four sides mutually congruent.*

Conjecture 4. *Let ABCD be a rhombus. Then angle ABC is congruent to angle ADC. Similarly, angle BAC is congruent to angle BDC.*

It is possible your proof uses something about rhombi that is just plain obvious from a picture.² But “obvious from the picture” is exactly the sort of thing we are trying to avoid when writing an axiomatic argument. So, let's formulate this “obvious fact” and try to prove it.

Definition 5. *Let ABCD be a quadrilateral. The diagonals of ABCD are the segments AC and BD.*

¹Here is a block of new terms given by definitions. The best way to approach a new definition is to try to make some examples and some non-examples.

²Each and every step must be justified by something we have already agreed is true.

Conjecture 6. *The diagonals of a rhombus must cross.*

Do not worry if your argument for Conjecture 4 did not use Conjecture 6, as there is more than one way to approach Conjecture 4. In fact, there is a lot of value in having multiple arguments, as each variant might shed light on some different aspect of the question.

Challenge 7. *If your argument for Conjecture 4 did not use Conjecture 6, find a new argument that does. Conversely, if your argument did use Conjecture 6, find an argument that does not.*

We have been talking about rhombi, but how are we sure that they even exist? If this seems a silly question, keep in mind that a rhombus is just described by a funny definition that someone made up. It is not necessarily clear that this definition means anything at all. So, what is to be done about this awkward situation? Well, we could try to construct a rhombus. This is a lot like the problem Euclid faced at the beginning of the *Elements*. He really needed an equilateral triangle, so he began by proving that one could be found in the first place.

Challenge 8. *Given a segment AB , find a compass and straightedge construction of a rhombus $ABCD$. Enumerate your steps and give a proof that the construction works.³*

Question 9. *How flexible or rigid is the construction solving the last problem? Can one use the construction to create many non-congruent rhombi, or are there only a few options?*

Sometimes, a proof really has more information in it than you think. Then we can get a *corollary* by digging a little deeper into the understanding we have gained. Can you use these hints to make headway on the next few problems?

Conjecture 10. *If $ABCD$ is a rhombus, then $ABCD$ is a parallelogram.*

Conjecture 11. *Let $ABCD$ be a rhombus. Suppose that the diagonals AC and BD meet at a point X . The angle AXB is a right angle.*

Note that we added the hypothesis that the diagonals cross. This is because we recognize that Conjecture 6 may not be a theorem by the time we have an argument for Conjecture 11. This kind of hypothesis making allows us to go forward in several directions at once. Feel free to add hypotheses when the need arises.

³A step only counts if you draw something, like a segment or an arc of a circle.

Chapter 2

The Geometry of Kites

Now we turn our attention to a different special class of quadrilaterals: *kites*.

Definition 12. *Two sides of a quadrilateral are called adjacent when they share a vertex and opposite if they do not.*¹

Definition 13. *A kite is a quadrilateral with two pairs of adjacent and congruent sides.*

Notice that a rhombus is always a kite.² The reason is that if all four sides are mutually congruent, then we can pick any way we like to divide the sides into adjacent pairs (there are only two ways), and these pairs will consist of congruent sides.

Sometimes a proof has features that can be carried over to other situations. In this assignment, you should try to build on the ideas we have developed in studying rhombi in this new situation. Each of the below is an adaptation of a statement we have for rhombi.

Conjecture 14. *Pairs of opposite angles in a kite are congruent.*

Conjecture 15. *The diagonals of a kite must cross.*

Problem 16. *Give a construction (with proof) of a kite. How general is your construction?*

Conjecture 17. *If $ABCD$ is a kite, then it is a parallelogram.*

Conjecture 18. *If the diagonals of a kite meet, then they meet at a right angle.*

¹Again, here are some new definitions. To make sense of them, try creating some examples and some “non-examples.”

²This is a theorem (and its proof). Can you write it out so it looks like a formal theorem statement?

Chapter 3

The Geometry of Rectangles

Rectangles are probably familiar to you, but to be clear we give a precise definition.

Definition 19. *A rectangle is a quadrilateral which has all four interior angles that are right angles.*

Notice that the definition only speaks about angles. There is nothing at all said about the sides. But that doesn't mean that the sides have no special properties—it is just that those properties are really theorems.¹

Conjecture 20. *Let R be a rectangle. Then R is a parallelogram.*

Conjecture 21. *Let R be a rectangle. Then each pair of opposite sides of R is a pair of congruent segments.*

Conjecture 22. *The two diagonals of a rectangle are congruent and bisect each other.*

These conjectures describe some qualities that are common to all rectangles. Mathematicians say that these are *necessary conditions* for having a rectangle. If a figure is a rectangle, then these properties are “necessarily” also true. Now we shall try to go in the opposite direction. The following conjectures are possible answers to the question, “What kind of information does one need to claim that that our figure is a rectangle?” Of course, we can use the definition above—that is one of the important roles of a definition, to test when we are allowed to use a word. But what we want now are *sufficient conditions*, these are conditions that allow us to conclude that our figure is a rectangle by checking something other than the definition.²

¹This is something many people have to get used to. It is a way in which mathematical definitions differ from common definitions in English.

²We will later see an example of conditions which are both *necessary* and *sufficient* for something, and thus provide an example of an *equivalent statement*.

Conjecture 23. *Let $ABCD$ be a quadrilateral such that angles $\angle ABC$ and $\angle ADC$ are right angles. If segments AB and CD are congruent, then $ABCD$ is a rectangle.*

Conjecture 24. *Let $ABCD$ be a quadrilateral such that angles $\angle ABC$ and $\angle ADC$ are right angles. If segments AB and CD are parallel, then $ABCD$ is a rectangle.*

Notice the importance that pairs of parallel lines play in these statements.³

Conjecture 25 (Midline Theorem). *Let ABC be a triangle, D the midpoint of AB and E the midpoint of AC . Then the line through E and D , called a midline, is parallel to the line through B and C .*

Conjecture 26 (Varignon's Theorem). *Let $ABCD$ be a quadrilateral. The midpoints of the four sides are the vertices of a parallelogram.*

³These results don't *look* to be about rectangles.

Chapter 4

Developing an Attitude of Skepticism

I hope you have taken the time to look over the example paper by Professor Ball. One of our goals this semester is to develop a mathematician's healthy skepticism. This can be a bit unsettling at first, but the basic idea is like a tee-shirt quotation: "Question Authority." Since this might be new to you, let me lead you a bit with the following items.

Problem 27. *If you have not, yet, read Professor Ball's argument in the appendix and figure out what went wrong. Pinpoint his error and present it to the class.*¹

Let's pretend that we overheard a tea-time conversation where a famous mathematician said, "Sure Euclid's proof of I.7 is bad, but I can fix that. It's proposition I.4 I'm worried about!"

Problem 28. *What might this person's mysterious objection to the argument for Euclid's Proposition I.7 be? Do you agree? If so, can you fix the argument?*

Problem 29. *What might be this person's mysterious objection to the argument for Euclid's Proposition I.4 be? Do you agree? If so, can you fix the argument?*

So far, I have strongly hinted that the arguments for Propositions I.1, I.4 and I.7 have sometimes been judged imperfect in some way. Unfortunately, they are not alone.

Problem 30 (Standing Problem). *As the semester progresses, we will have occasion to read over a hundred different arguments by Euclid, if any of*

¹Prof. Ball uses a result he calls the hypotenuse-leg theorem, which states that if two right triangles have two pairs of corresponding sides congruent, and one of the pairs is the hypotenuses, then the triangles are congruent. For now, we grant that this theorem is true, and is not the source of Prof. Ball's error. We'll come back to address this theorem later when we study triangles in detail.

the others we read have gaps, please consider giving a short presentation about it. (Hint: there are several of these to be found.)

The big lesson here is that *The Elements* is not perfect, from a modern point of view. This worried mathematicians for a long time, and it took two millennia until all of the issues were finally sorted out to the community's general satisfaction. Several re-axiomatizations of planar geometry have been proposed, the most famous versions are due to David Hilbert, George Birkhoff, and The School Mathematics Study Group.

Chapter 5

Polygons

Now it is time to extend our venue to *polygons* with an arbitrary number of sides.

Definition 31. *Let n be a natural number. An n -gon is a figure consisting of n points A_1, A_2, \dots, A_n , prescribed in order and called vertices, and the n line segments, called sides, $A_1 A_2, A_2 A_3, \dots, A_{n-1} A_n, A_n A_1$. A polygon is an n -gon where n has not been specified.¹*

Problem 32. *Suppose that A, B, C are three consecutive vertices of a polygon. If at the vertex B we extend one of the two sides through B to a ray, then we create a new angle, called an exterior angle to the polygon at B . This construction has a choice in it. In principle, this could be a problem. Describe the problem, then state and prove a theorem that resolves the issue.*

Conjecture 33. *The exterior angles of a pentagon, one choice made at each vertex, add up to four right angles.*

Question 34. *What is the sum of the exterior angles of a hexagon? What about a general n -gon? Can you find a way to build on our understanding from small values of n , to general values of n ?*

¹Note: Commonly used terminology includes the following: 3-gon = triangle, 4-gon = quadrilateral, 5-gon = pentagon, 6-gon = hexagon.

Chapter 6

Regular Figures, A Warm-up

A great part of the allure of geometry is figures with symmetry. Inspired by this, let us study some polygons that have a lot of symmetry.

Definition 35. *A polygon is said to be equilateral if all of its sides are congruent, equiangular if all of its angles are congruent, and regular if it is both equilateral and equiangular.*

Conjecture 36. *An equilateral triangle is equiangular, hence regular.*

Conjecture 37. *Let $ABCD$ be a rhombus. If angle A is congruent to B , then $ABCD$ is regular.*

Definition 38 (reminder). *A regular quadrilateral is called a square.*

Problem 39. *Does Conjecture 37 hold if we replace “angle B ” by “angle C ”? State a result and prove it.*

Conjecture 40. *Let $ABCDE$ be an equilateral pentagon. If angle A is congruent to angle B , then $ABCDE$ is regular.*

Conjecture 41. *Let $ABCDE$ be a regular pentagon. The triangle ACD is isosceles.*

Problem 42. *Let $ABCDE$ be a regular pentagon. State the relationship between the angles CAD and ACD that shows how special the triangle is. Prove your observation.*

Problem 43. *Find experimental evidence for the number of regular pentagons with a given side. (Try using five toothpicks!)*

Chapter 7

Deeper Into Triangles

So far, our most useful tools have been facts about triangles. Euclid makes a pretty thorough study of triangles, but he doesn't cover everything he could. Here we shall plug gaps in Book I with some very useful understanding.

Problem 44. *Show how to construct three segments which are not congruent to the sides of any triangle.*

Definition 45. *A triangle is said to be right if one of its angles is a right angle. The side opposite the right angle is called the hypotenuse. The other sides are called legs.*

Conjecture 46. *Let ABC and DEF be two right triangles, with the angles at A and D right angles. Suppose that BC is congruent to EF and AB is congruent to DE . Then the triangles are congruent.¹*

Conjecture 47. *If we weaken the hypothesis of the previous conjecture so that the angles A and D are still congruent but no longer assumed to be right angles, and leave the other hypotheses intact, the conclusion still holds.*

Right triangles are very useful objects. But we need to know when we have a right angle! Here is a wonderful characterization of right angles.

Conjecture 48. *If AB is the diameter of a circle and C lies on the circle, then angle $\angle ACB$ is a right angle.*

Conjecture 49. *If $\angle ACB$ is a right angle, then C lies on the circle with diameter AB .*

Note that, together, Conjectures 48 and 49 yield this result:

¹This is sometimes called the *hypotenuse-leg* theorem.

Theorem (Thales' Theorem). *Let A, B, C be three points. The angle $\angle ACB$ is a right angle if and only if C lies on the circle with diameter AB .²*

This is a case of a set of *equivalent conditions*. We have stated it here with “if and only if” language, but we also could have said “for angle $\angle ACB$ to be a right angle it is necessary and sufficient that C lies on the circle with diameter AB .” The necessity is in Conjecture 49. The sufficiency is in Conjecture 48.

²It is mathematical folklore that Thales' theorem is the oldest theorem. That is, this theorem was the first one that was given what would today be called a proof. Thales is thought to be the originator of our axiomatic method of working from assumed statements (axioms or postulates) to prove true statements (theorems) which followed only from logical reasoning. It is also part of the folklore that Thales was so overjoyed at the discovery of this theorem and proof that he sacrificed an ox as thanksgiving. I don't know if this is true—I've also heard such a story associated with the Pythagoreans and their famous triangle theorem, though it was 100 oxen. Euclid's *Elements* is really an ancient Greek textbook. It seems clear that the method of axiomatic work typical of mathematics originated somehow with the Greeks before Euclid—say with Thales or with the Pythagoreans. But all of the literature has been lost to the ages.

Chapter 8

The Center of a Triangle

What might be called the center of a triangle? There have been many proposed answers to this question over the centuries. In this assignment, we study two of them.

Conjecture 50. *Let ABC be a triangle, with rays r and s the angle bisectors at A and B , respectively. Suppose that r and s meet at the point I which lies inside the triangle. Draw lines l and m through I that are perpendicular to AC and BC respectively. If l meets AC at point X and m meets BC at Y , then triangle IXC is congruent to triangle IYC .*

Definition 51. *Three segments (or lines or rays) are called concurrent if they all pass through a common point.*

Conjecture 52. *The three angle bisectors of a triangle are concurrent.*

Definition 53. *The point just discovered is called the incenter of the triangle.*

Conjecture 54. *Let T be a triangle. For any pair of sides of T , the perpendicular bisectors of those sides meet. (That is, they are not parallel.)*

Conjecture 55. *The three perpendicular bisectors of any triangle are concurrent.*

Definition 56. *The point where the three perpendicular bisectors of a triangle meet is called the circumcenter of the triangle.*

Chapter 9

Circles

We have learned quite a bit about basic polygonal shapes, especially triangles, and various species of quadrilaterals. Now we turn our attention to circles. This is the subject of Book III in Euclid's *Elements*. We already have one beautiful theorem about circles, that of Thales, but we'd like to have more.

Read *The Elements* Book III Propositions 1-34.¹ For the following propositions you should work in the axiomatic style of Euclid using I.1-34, III.1-34 and any previously proved results.

Conjecture 57. *Let AB and AC be two tangent lines from a point A outside a circle. Then AB is congruent to AC .*

Definition 58. *We say that two circles meet at right angles if the radii of the two circles to a point of intersection make a right angle.*

Conjecture 59. *Let Γ and Ω be two circles with centers G and O , respectively. Suppose that these circles meet at two points A and B . If GAO is a right angle, then GBO is a right angle.*

Definition 60. *A quadrilateral $ABCD$ is said to be a cyclic quadrilateral if there is a circle Γ such that the four vertices A, B, C and D lie on Γ .*

Conjecture 61. *A rectangle is always a cyclic quadrilateral.*

Conjecture 62 (Cyclic Quadrilateral Theorem). *Let A, B, C and D be four points. The quadrilateral $ABCD$ is cyclic if and only if angle DAC is congruent to DBC .*

Conjecture 63. *Let two circles be tangent at a point A . If two lines are drawn through A meeting one circle at further points B and C and meeting the other circle at points D and E , then BC is parallel to DE .*

¹Pay special attention to III.16, III.18, III.20, III.21, III.31 and III.32.

Chapter 10

Circles, Coming 'Round Again

One of the most useful results about circles is Proposition III.20 which relates an *inscribed* angle in a circle to a *central* angle in that circle. Let us try to see what happens when the angle does not sit on the circumference of the circle.

Conjecture 64. *Let Γ be a circle with center O . Let X be a point in the interior of the circle, and suppose that two lines ℓ and m intersect at X so that ℓ meets Γ at points A and A' and m meets Γ at B and B' . Then twice angle AXB is congruent to angle AOB and angle $A'OB'$ taken together.*

Question 65. *Consider the situation from the last conjecture, but instead assume that X lies outside Γ . What happens here? Formulate a conjecture.*

Conjecture 66. *If two chords of a circle subtend different acute angles at points of a circle, then the smaller angle belongs to the shorter chord.*

Conjecture 67. *If a triangle has two different angles, then the smaller angle has the longer angle bisector (measured from the vertex to the opposite side).*

Conjecture 68 (Steiner-Lehmus). *If a triangle has two angle bisectors which are congruent (measured from the vertex to the opposite side), then the triangle is isosceles.*

Conjecture 69. *Let BC be a chord of circle \mathcal{C} , let \widehat{BC} be the arc of \mathcal{C} which is bounded by B and C and does not contain the center of \mathcal{C} . Let M be the midpoint of \widehat{BC} . For a point A on the arc \widehat{BC} , show that as A moves along the arc from B to M , the sums $AB + AC$ increase.*

The next theorem is very pretty, and is commonly attributed to Archimedes.

Conjecture 70 (Archimedes' Theorem of the Broken Chord). *Let AB and BC be two chords of a circle \mathcal{C} , where BC is greater than AB . (Such a configuration is sometimes called a "broken chord.") Let M be the midpoint of*

arc ABC and F the foot of the perpendicular from M to chord BC. Then F is the midpoint of the broken chord, that is, AB and BF taken together are congruent to FC.

Chapter 11

Construction Problems

The following problems are construction challenges. In each case, you are to

1. find a compass and straightedge construction,
2. enumerate your steps,¹ and
3. prove that your construction works.

You may use any parts of the *Elements* Book I.1-34 or Book III.1-34 as part of your reasoning.

For most challenges, I have included a “par value” indicating the number of steps an experienced constructor would need. Can you meet this or do better?

Challenge 71. *Given an angle, construct the angle bisector. (par 4)*

Challenge 72. *Given a segment, find the midpoint. (par 3)*

Challenge 73. *Given a line ℓ and a point A not lying on ℓ , construct a line perpendicular to ℓ through A . (par 4, possible in 3)*

Challenge 74. *Given a line ℓ and a point A lying on ℓ , construct a line perpendicular to ℓ through A . (par 4, possible in 3)*

Challenge 75. *Given an angle at a point A and given a ray emanating from a point B , construct an angle at B congruent to the angle at A having the given ray as a side. (par 4)*

¹A “step” for our counting purposes happens exactly every time you draw something with either the compass or straightedge. We count steps using the modern “fixable compass.” This means that you may copy some segment length for the radius of a circle, but place it at a new point for a center all in one step. If you like, this collapses the content of Euclid Proposition I.2 in to a single step.

Challenge 76. *Given a line ℓ and a point A not lying on ℓ , construct a line parallel to ℓ which passes through A . (par 3)*

Challenge 77. *Given the circumference of a circle, find the center of the circle. (par 5)*

Challenge 78. *Given a circle with center O , and given a point A outside the circle, construct a line ℓ through A which is tangent to the circle. (par 6)*

Chapter 12

More Advanced Constructions

Definition 79. *A circle is said to be circumscribed about a figure if the figure lies in the interior of the circle, except for the vertices which lie on the circle.*

Definition 80. *A circle is said to be inscribed in a figure if the circle lies in the interior of the figure and is tangent to each of the sides of the figure.*

Challenge 81. *Construct a circle inscribed in a given triangle ABC . (par 13)*

Challenge 82. *Construct a circle circumscribed about a given triangle ABC . (par 7)*

Challenge 83. *Given a line ℓ , a line segment d and a point O , construct a circle with center O that cuts off a segment from line ℓ which is congruent to d .*

Challenge 84. *Construct three circles such that each pair meets at right angles. (par 10)*

Challenge 85. *Given a segment d , a circle with center O and a point P inside the circle, construct a line through P on which the circle cuts off a segment congruent to d . When exactly is this construction possible?*

Challenge 86. *Given a segment AB and an angle α and given another segment d , construct a triangle ABC with base equal to AB , angle α at C and such that $AC + CB = d$. Exactly how often is this construction possible? How many ways can the conditions be met?*

Challenge 87. *Given two circles Γ and Γ' with centers O, O' , respectively, construct a line tangent to both circles. How many such lines are there?*

Chapter 13

The Theory of Content

Read the parts of the *Elements* that have to do with the basic theory of area. These are Book I, Propositions 35-48, Book II, and Book III, Propositions 35-37.

Euclid usually calls two planar figures “equal” without explanation. From the work that he does, it seems that he means equality of area, or something like it, but he never defines an “area function” which assigns a (non-negative, real) number to each planar figure. Looking more closely at the text, it appears that Euclid is using a new undefined term and assuming some “common sense” axioms to control use of the word. For clarity, we try to make things more explicit here.¹ We introduce a new undefined term *equal content* which satisfies the following axioms:

- (EC1) Congruent figures have equal content.
- (EC2) If each of two figures each have equal content with a third figure, then the two figures have equal content.
- (EC3) If pairs of figures with equal content are “added,” in the sense of being joined without overlap to make bigger figures, then these added figures have equal content.
- (EC4) The same holds for “subtraction,” noting that it is immaterial where the equal figures are removed.
- (EC5) Halves of figures with equal content have equal content, and doubles of figures with equal content have equal content.
- (EC6) The whole is greater than the part, so if one figure is properly contained in another, then the two figures cannot have equal content.

¹This follows the basic structure of axiomatic work. We have a new undefined term, and the axioms are in place to make sure that we have concrete and specific rules for the allowable use of that new term. In this case, the axioms are basically restatements of Euclid's common notions.

Attention! Attention!

Note that Euclid never uses algebra. (It hadn't been invented, yet.) So there are no equations in his work. We shall not use them either! Writing a "Euclid-style proof" demands no equations.

For the following tasks, work in the style of Euclid using any result from the first three books.

Problem 88. *Prepare a presentation of Euclid's beautiful proof of Proposition I.47.*

Conjecture 89. *Let ABC and DEF be triangles. Let X be the midpoint of DE , Y the midpoint of BC . If AB is congruent to DX and EF is congruent to BY , then ABC and DEF have equal content.*

Problem 90. *Expand on the statement of the last conjecture—find some theorems and prove them.*

Problem 91. *The hypotenuse-leg Theorem 7.2 can be proved using the theory of equal content. Find such a proof.*

Question 92. *There are two ways to inscribe a square in an isosceles right triangle. Which one has the greater content?*

Conjecture 93 (The Parallelogram Law). *Let $ABCD$ be a parallelogram. Then the squares on the diagonals taken together have equal content with the squares on the four sides taken together.*

Problem 94. *Given a triangle ABC and a segment DE , construct a rectangle with equal content to ABC and with DE as one side.*

Problem 95. *Given a rectangle, construct a square of equal content.*

Chapter 14

More Content

The following are some challenging tasks having to do with the theory of equal content. The first two are important to us. The rest are opportunities to show off.

Problem 96. *Given two rectangles, P and R , construct a square S so that, when taken together the content of R and S is equal to the content of P .*

Problem 97. *Use the theory of content to give a new proof of the midline theorem, Theorem 25.*

Challenge 98. *Given a line ℓ and given two points A and B not lying on ℓ , construct a circle passing through A and B and tangent to ℓ .*

Challenge 99. *Given two lines ℓ and m and a point P not on either line, construct a circle through P and tangent to both ℓ and m .*

Conjecture 100. *Let ABC be a triangle, DE a line parallel to the base BC , and F the midpoint of segment DE , where D lies on ray AB and E lies on ray AC . Let AF meet BC at G . Then G is the midpoint of BC .*

Conjecture 101. *Let Γ be a circle with center O . Let A be a point outside the circle, and AB and AC tangents to Γ from A , with B, C lying on Γ . Let BC meet OA at D . Then the rectangle on OA and OD has equal content with the square on OB .*

Conjecture 102. *Let ABC be a right triangle with right angle at A . Let AD be the altitude from A to side BC . The square on side AD has equal content with the rectangle on BD and DC .*

Conjecture 103. *Let ABC be a triangle and D a point on side BC . Let E be the midpoint of BC and draw the parallel to AD through E . Let this new line meet the union of AB and AC at a point F . Then the segment DF cuts the triangle into two polygons of equal content.*

Chapter 15

Regular Figures, especially the Pentagon

Read Book IV of the *Elements*. Pay particular attention to propositions 10-12.

Problem 104. *Prepare a presentation of Euclid's construction of a regular pentagon.*

Problem 105. *Given a circle, but not its center, construct an inscribed equilateral triangle in as few steps as possible. (par 7)*

Problem 106. *Construct a square in as few steps as possible. (par 9)*

Problem 107. *Given a line segment AB , construct a regular pentagon having AB as a side. (par 11)*

Problem 108. *Given a circle Γ and its center O , construct inside Γ three equal circles, each one tangent to Γ and to the other two. (par 13)*

Problem 109. *Let ABC be an equilateral triangle inscribed in a circle. Let D and E be the midpoints of two sides, and extend segment DE to meet the circle at F so that E lies between D and F . Show that the rectangle on EF and DF has the same content as the square on DE .*

Challenge 110. *Construct a regular hexagon in as few steps as possible. What should the par value be?*

Chapter 16

The Regular Pentagon Rides Again

The next three tasks concern the following construction template for inscribing a regular pentagon in a circle.

Given a circle with center O ,

1. Draw any line through O ; get A and B .
2. Draw $\odot AB$.¹
3. Draw $\odot BA$, get C as intersection of last two circles.
4. Draw line OC , get D between O , C as intersection with given circle.
5. Draw $\odot DO$, get E, F as intersections with given circle.
6. Draw EF , get G as intersection with OC .
7. Draw circle $\odot GA$, get point H as intersection with ray DO .
8. Draw circle $\odot A(OH)$, get points I, J as intersections with given circle.
9. Draw circle $\odot B(IJ)$, get points K, L as intersections with the given circle.
10. (five steps) Draw segments BK, KJ, JI, IL, LB .

Conjecture 111. *Triangle OAI is isosceles and its base angles are twice the angle at O .*

Conjecture 112. *Triangle BJI is isosceles and its base angles are twice the angle at B .*

Conjecture 113. *Show that $BKJIL$ is a regular pentagon.*

¹The notation $\odot A(BC)$ means a circle with center A and radius congruent to segment BC .

Appendices

1. The example paper containing Rouse Ball's fallacious argument that all triangles are isosceles.

THE SYMMETRIC NATURE OF TRIANGLES

W. W. ROUSE BALL

In the process of some advanced geometrical investigations, we stumbled upon the following simple result about triangles. The result seems to be of independent interest, so rather than wait to include this as a nugget in a longer manuscript, we make it available here in a short note. As we found it a pleasant part of this investigation, we invite the reader to draw his or her own diagrams.

Theorem. All triangles are isosceles.

Proof. Let ABC be a triangle. Let D be the midpoint of segment BC . Let the perpendicular to BC at D meet the angle bisector of A at the point E .

Suppose first that E is inside the triangle.

Drop perpendiculars EF and EG from E to the sides of the triangle. Draw segments BE and CE . The triangles AEF and AEG have the side AE common and two angles congruent, so they are congruent by Euclid I.26 (AAS). Hence AF is congruent to AG and EF is congruent to EG . The triangles BDE and CDE have DE common, two other sides congruent, and the included right angles equal. Hence they are congruent by Euclid I.4 (SAS). In particular, BE is congruent to CE .

Now, the triangles BEF and CEG are right triangles with hypotenuses and a pair of legs congruent, so by Theorem on Hypotenuse-Leg for Right Triangles, they are congruent. Hence BF is congruent to CG . Adding equals to equals, we see that AB is congruent to AC , so that triangle ABC is isosceles.

There are other cases to consider. If the point E lies outside the triangle, one can use the same proof to conclude that AB and AC are the differences of equals, hence equal.

If E coincides with the point D , or if the angle bisector at A is parallel to the perpendicular to BC at D , the proof is even easier, and we leave it to the reader to complete these cases. \square