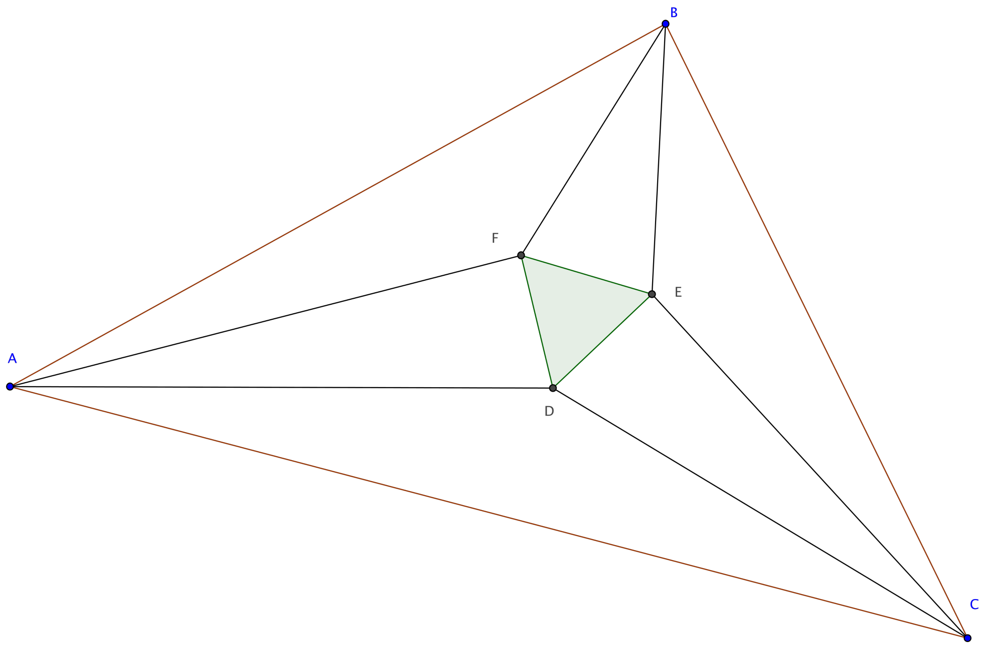
Transactions

in

Euclidean Geometry



Issue # 6

APPLYING A SPECIFIC AGRUMENT TO TRY AND PROVE DIAGONALS OF A RECTANGLE CROSS

EMILY HERBST

Communicated by Katy Goodmundson

Question K asks: Will Herbst's arguement for the diagonals of a rhombus work in this context? If not, where does it fall? To answer this question, we will attempt to prove Theorem 3.3 using Herbst's argument.

Theorem 3.3. Let ABCD be a rectangle. The diagonals, AC and BD, must cross.

Proof. Let ABCD be a rectangle such that line AB is congruent to line CD which is greater than line BC which is congruent to line AD. Let there be the line AC.

By Euclid Proposition I.9, choose a point X1 on the line AC, where angle ABX1 is congruent to the angle CBX1.

Notice that the line BX1 is congruent to the line BX1 and angle ABX1 is congruent to the angle CBX1. But the line AB is not congruent to the line BC. Thus by by Euclid Proposition I.4, triangle ABX1 is not congruent to the triangle CBX1. Since the line AB is greaeter than the line BC, then angle AX1B is greater than the angle CX1B by Euclid Propostion I.18. Since triangle ABX1 is not congruent to triangle CBX1, line AX1 is not congruent to line CX1 and thus X1 is not the midpoint of line AC.

By Euclid Proposition I.9, choose a point X2 on the line AC, where angle ADX2 is congruent to the angle CDX2. Notice that the line DX2 is congruent to the line DX2 and angle ADX2 is congruent to the angle CDX2. But the line AD is not congruent to the line CD. Thus by Euclid Proposition I.4, triangle ADX2 is not congruent to the triangle CDX2. Since the line CD is greaeter than the line AD, then angle CX2D is greater than the angle AX2D by Euclid Propostion I.18. Since triangle ADX2 is not congruent to triangle CDX2, line AX2 is not congruent to line CX2 and thus X2 is not the midpoint of line AC.

Since X1 is located on the line AC such that the line AX1 is not congruent to the line CX1 and X2 is located on the line AC such that the line AX2 is not congruent to the line CX2, X1 and X2 are not the same point thus not creating the collinear line BD.

This proves that Theorem 3.3 cannot be proven using Herbst's arguement.

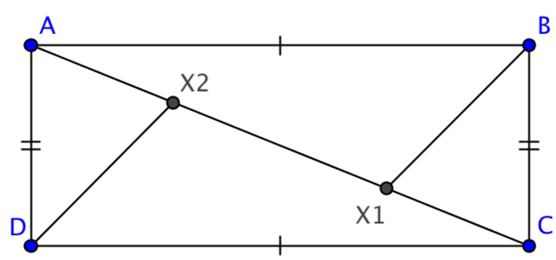


FIGURE 1. Rectangle ABCD

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INTERIOR AND EXTERIOR ANGLES

KEVIN WALTERS

Communicated by John Fisher

The following is a definition of the interior and exterior angle and also a proof that goes with the definition.

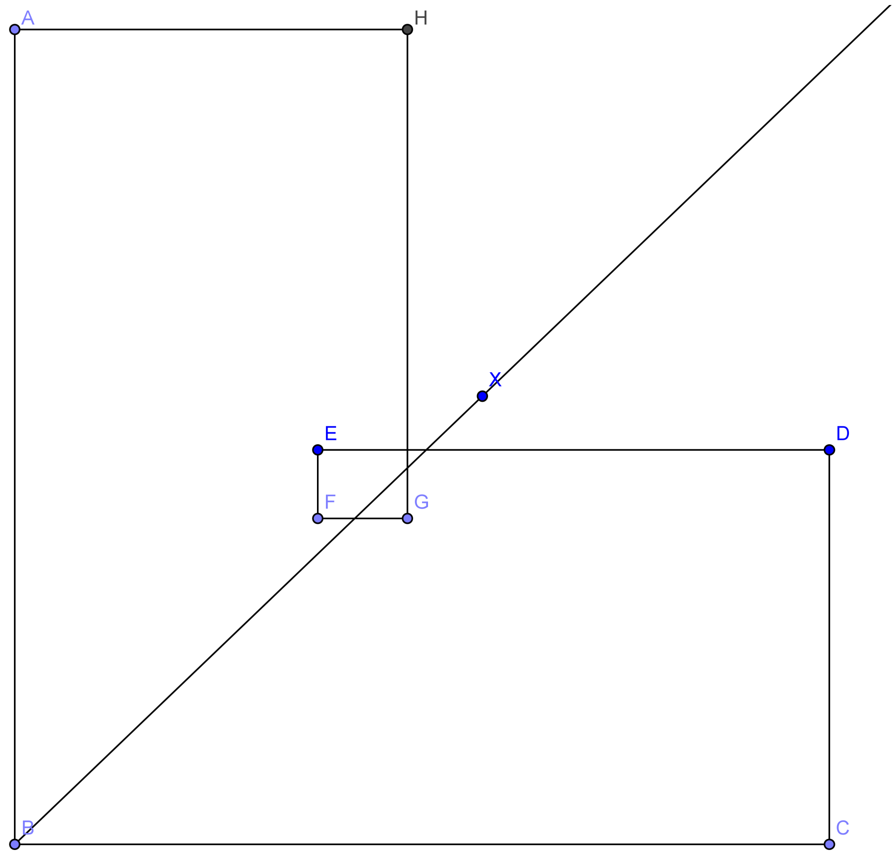
Question M. Find a workable definition for interior and exterior angles.

Definition M. Given a polygon P with three consecutive vertices A,B, and C. The three vertices form two angles. We will call the angle that is less than two right angles angle ABC, and the angle that is greater than two right angles angle ABC'. We can say that angle ABC lies inside the polygon if we draw ray x that divides angle ABC such that it does not pass through a vertex and passes through an odd number of sides. We can say angle ABC lies outside the polygon if we draw a ray x that divides angle ABC such that it does not pass through a vertex and passes through an even number of sides. Similarly we can find whether angle ABC' lies inside or outside the polygon.

Theorem M. Three consecutive, non-colinear vertices form one angle greater than two right angles and one angle less than two right angles.

Proof. Pick three non-colinear points and call them A, B, and C. Draw segment AB and segment BC. This will form two angles which we will call angle ABC and angle ABC'. Let the smaller angle be angle ABC and the larger angle be angle ABC'. Extend segment BC and choose a point X on the line opposite of B compared to C. By proposition 13 we know that the sum of angle XBA and angle ABC equals two right angles. Since the sum of angle ABC and angle XBA is equal to two right angles we know angle ABC is less that two right angles. We also know that angle ABC' is made up of line CB and angle ABX. We know that a line is equal to two right angles so the sum of a line and angle ABX is greater than two right angles. Therefor angle ABC' is greater than two right angles. Therefor three consecutive non-colinear vertices will form one angle less than two right angles and one angle greater than two right angles.

In the image below, ray AX passes through an odd number of sides so angle ABC lies inside the polygon. □


A
H
X
E
D
F
G
B
C

Interaction of Vertex Angles with Diagonals in a Regular Pentagon

Matt Griffen

Communicated by Emily Herbst

Theorem 6.6. The diagonals of a vertex in a regular pentagon trisect that vertex angle

Proof. Let ABCDE be a regular pentagon with the vertex angles labeled as X. Let the diagonals be drawn.

Each of the triangles formed by a diagonal and two adjacent sides of the pentagon (i.e. triangle ABC) are congruent by Euclid's proposition 1.4, because they all have an X angle contained by two congruent sides. Therefore the diagonals are congruent to one another. Also, by definition these triangles are isosceles and, by Euclid's proposition 1.5, the angles of the base are congruent. Let them be labeled as Y.

Let the intersection of segment EC and AD be labeled as point F. By Euclid's proposition 1.32 the sum of the angles of a triangle are congruent to 2 right angles (2RA's). So it can be shown that angle EFD is congruent to 2RA's minus 2Y and angle ABC is congruent to 2RA's minus 2Y. Therefore angle EFD is congruent to angle ABC. Since segment EC and AD are straight lines that cut one another at point F, Euclid's proposition 1.15 states that angle EFD and AFC must be congruent.

Each of the triangles formed by two diagonals and a side of the pentagon (i.e. triangle EBD) are congruent by Euclid's proposition 1.8. Therefore the angles contained by adjacent diagonals are congruent (i.e. angle EBD). Let them be labeled as angle Z.

By Euclid's proposition 1.32, the interior angles of triangle AFC sum up to 2RA's. Therefore 2RA's are congruent to X plus 2Z. It is also known that 2RA's are congruent to X plus 2Y. Therefore 2Z must be congruent to 2Y. So Z must be congruent to Y.

Thus each angle X is formed by 3 angle Y's created by a vertex angle and its diagonals.

□

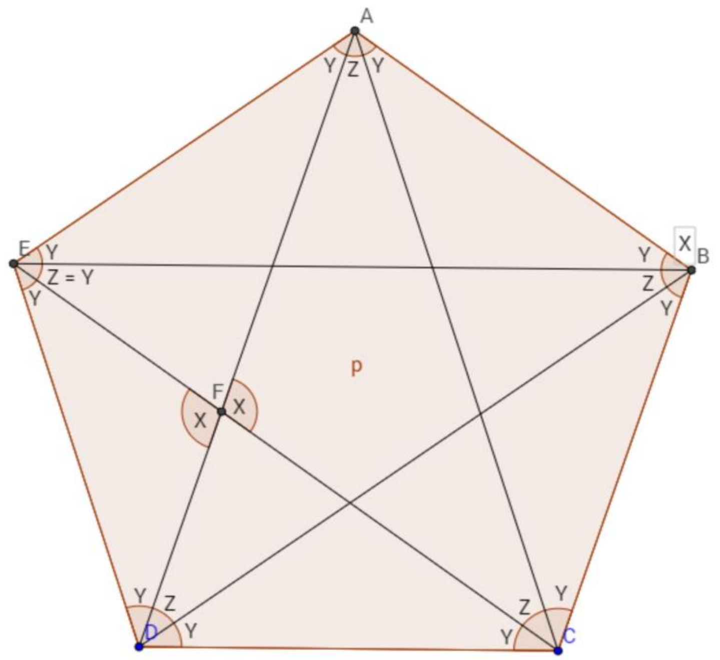


Figure p

Furthering Theorem 6.6

Matt Griffen

Communicated by Emily Herbst

Theorem 6.4. Assuming it is possible to construct a regular pentagon, then it is also possible to construct an equilateral pentagon with two congruent and adjacent angles that is not equiangular.

Proof. Let the end figure of Theorem 6.6 be constructed. Since segment AC is congruent to itself and angle AFC is congruent to angle ABC, triangle AFC is congruent to triangle ABC by Euclid's proposition 1.26. So segments AF and FC are congruent to segments AB and BC.

Therefore Pentagon AFCDE is an equilateral pentagon with two congruent and adjacent angles (E and D), but it is not equiangular (angle DCF is not congruent to angle D).

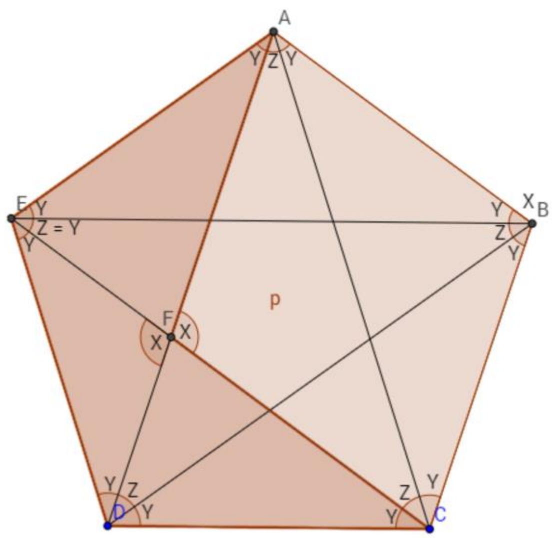


Figure 1: Pentagon AFCDE is highlighted

□

RIGHT ANGLE INSCRIBED IN A CIRCLE

AARON WENDT

Communicated by Emily Bachmeier

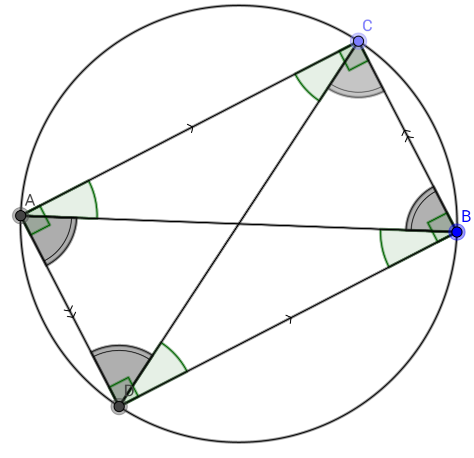
Theorem 7.4. If AB is the diameter of a circle and C lies on the circle, then angle ACB is a right angle.

Proof. Let AB be the diameter of a circle. Let C be a point on the circumference such that C does not coincide with A or B. Now choose point D on the circumference such that C is on the opposite side of AB, AD is congruent to CB and AC is congruent to DB. ACBD forms a cyclic quadrilateral.

Since AC is congruent to DB, CB is congruent to AD, and AB is congruent to AB, by Euclid I.8, triangle ABC is congruent to triangle BAD. Furthermore, the corresponding angles BAC and DBA are congruent, and angle ABC is congruent to angle BAD. By Euclid I.27 we can conclude that AC is parallel to DB and AD is parallel to BC, making the quadrilateral ACBD a parallelogram.

ACBD is a cyclic parallelogram so by Euclid III.21 we know that with the common subtending side AD, angle ACD is congruent to angle DBA. Furthermore, through congruency of alternate interior angles we confirm that angles ACD, BDA, BAC, CDB are congruent. Similarly angles DAB, DCB, ADC, CBA are congruent.

By adding equal angles to equal angles we finally find that the interior angles of ACBD are mutually congruent. The corollary by Herington in issue 2 shows that "the interior angles of a parallelogram taken together make four right angles." Since the sum of the interior angles in ACBD equals four right angles, we conclude that each mutually congruent interior angle of ACBD is a right angle. Most notably of these is ACB, constructed from the diameter AB and point C on its circumference. □



Parallelism of Perpendicular Bisectors in Triangles

Matt Griffen

Communicated by Megan Westervelt

Theorem 8.3. For any pairs of sides of triangle T, the perpendicular bisectors of those sides meet.

Proof. Assume that the pairs of perpendicular bisectors are parallel. Let the perpendicular bisectors be labeled L and M and let the midpoints from which they begin be labeled W and Z respectively. Also let the intersections of lines L and M with T be labeled X and Y respectively.

T1, T2, and T3 contain two parallel straight lines L and M cut by a straight line segment AC. By Euclid I.29, since angle YZX is a right angle then angle WXZ must be a right angle. By Euclid I.13 the angle WXA must also be a RA. According to Euclid I.28 WA and XA are parallel. But WA and XA intersect at point A. This is a contradiction; therefore lines L and M cannot be parallel.

T4 also contains two parallel straight lines L and M cut by a straight line segment AC. By Euclid I.29, angle WXY plus angle ZYX must be congruent to two right angles. Bieghler's theorem 5.2 shows that the interior angles of a pentagon must sum up to six right angles. Pentagon BWXYZ contains four known right angles (BWX, WXY plus XYZ, YZB), therefore angle B must be congruent to two right angles. Then the interior angles of T4 sum up greater than two right angles which is impossible according to Euclid I.32. Therefore lines L and M cannot be parallel.

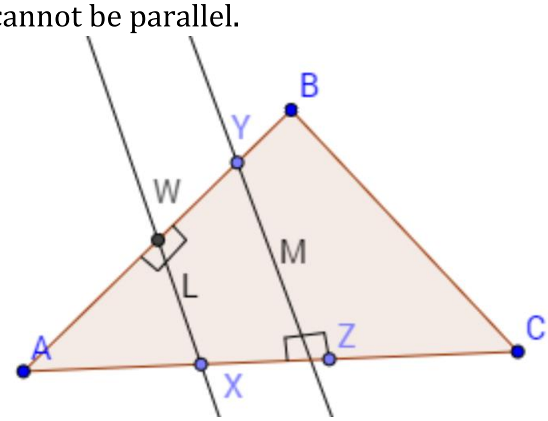


Figure T1

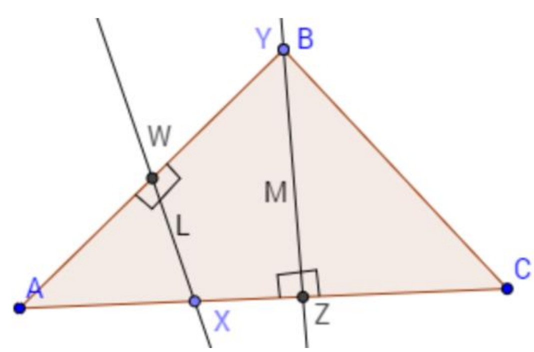


Figure T2

T1, T2, T3, and T4, cover all possible arrangements for how the perpendicular bisectors can interact with the sides of a triangle.

□

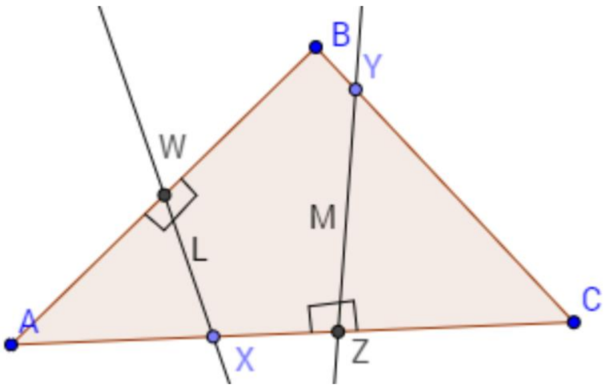


Figure T3

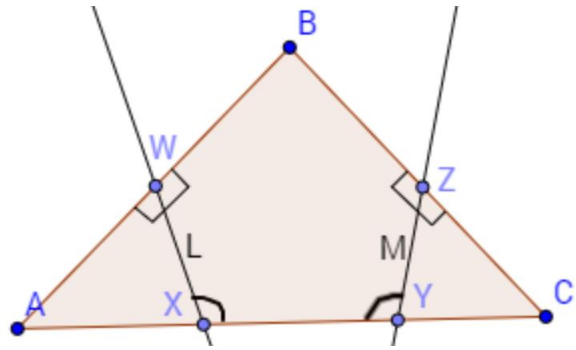


Figure T4

IF QUADRILATERAL ABCD IS CYCLIC, THEN ANGLE DAC IS CONGRUENT TO ANGLE DBC

JALEN RAYMOND

John Fisher

Theorem 9.4. Let A, B, C and D be four points. If Quadrilateral ABCD is cyclic, then angle DAC is congruent to angle DBC.

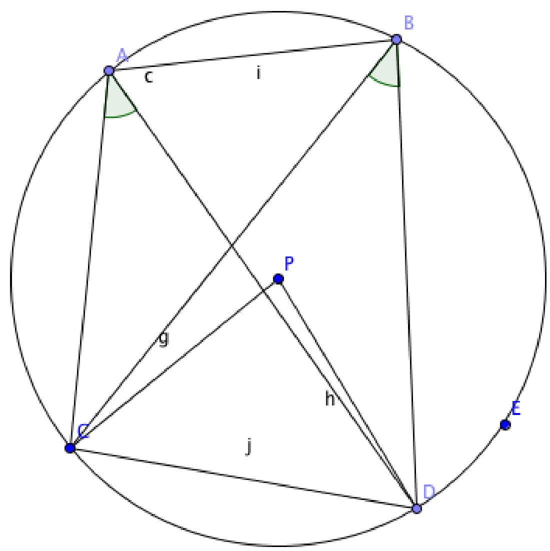


FIGURE 1. Quadrilateral ABCD

Proof. Let ABCD be a quadrilateral. I will show that in order for ABCD to be cyclic, angle DAC will be congruent to angle DBC. First, I construct a point P, which is the midpoint of the cirlce. Then connect point P to point D. Then connect point P to point C. Based on proposition 20, angle DBC is double angle DPC, because the share the same base on the circumference, and angle DPC is at the midpoint while angle DBC is at the circumference (on the circle). Similarly, angle DAC is double angle DPC, based on proposition 20. Since angle DBC and angle DAC are both double angle DPC, they are congruent. Thus, Since quadrilateral ABCD is cyclic, angle DAC is congruent to angle DBC.

□

PARALLEL SEGMENTS OF TRIANGLES INSCRIBED IN TANGENT CIRCLES

MEGAN WESTERVELT

Communicated by Joellen Hatchett

Theorem 9.5. Let two circles be tangent at a point A. If two lines are drawn through A meeting one circle at further points B and C and meeting the other circle at points D and E, then BC is parallel to DE.

Proof. By Euclid III.17, create the line tangent to both circles at point A. Let F and G be points on the tangent line such that A is between F and G. These points will strictly be used to reference specific angles. By Euclid III.32, angle FAB is congruent to angle ACB. Euclid I.15 states that vertical angles are congruent. Therefore, angle FAB is congruent to angle GAE. Again, by Euclid III.32, angle GAE is congruent to angle ADE. Since angle ACB is congruent to angle FAB which is congruent to angle GAE that is congruent to angle ADE, angle ACB is congruent to angle ADE. By Euclid I.29, line segment BC is parallel to line segement DE.

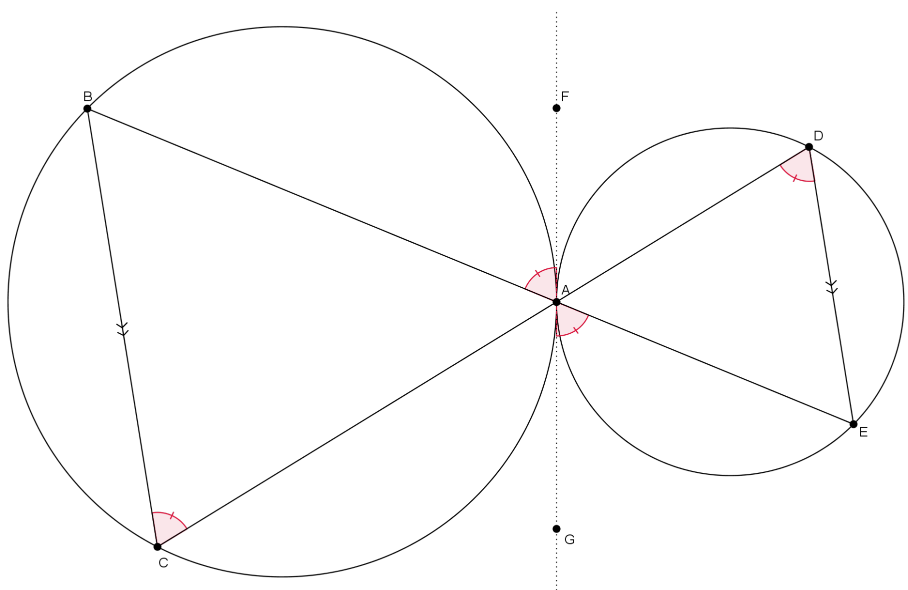


Figure 1

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LOCATING THE MIDPOINT OF A LINE SEGMENT

JOELLEN HATCHETT

Communicated by Katy Goodmundson

Theorem 11.2. Given a segment AB, it is possible to find the midpoint using a compass and straight edge.

Proof. Given segment AB.

1. Construct circle AB.
2. Construct circle BA. Label the interections of circle AB and circle BA, C and D.
3. Construct segment CD. The midpoint is the intersection of segment AB and segment CD.

Since segments AC, AD, BC, and BD are all radii of congruent circle they are congruent. Segment CD is congruent to CD. By Euclid I.8 triangle ACD is congruent to triangle BCD. By Euclid I.15 angle AEC is congruent to angle BED. By Euclid I.27 triangle ACE is congruent to triangle BDE. By correspponding parts of congruent triangles, AE is congruent to DE, thus E is the midpoint of AB.

□

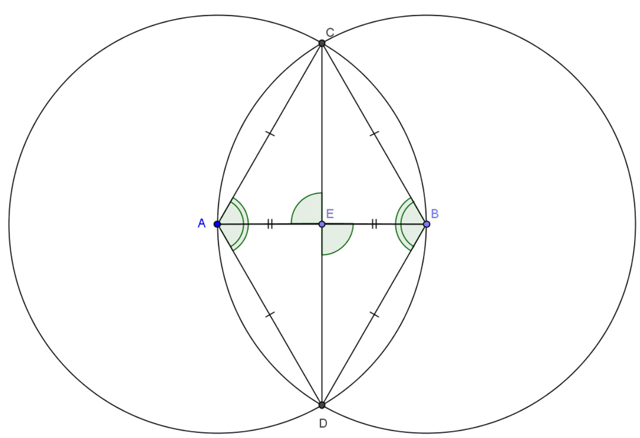

FIGURE 1. Proving E is the midpoint on segment AB.

LOCATING THE MIDPOINT OF A LINE SEGMENT

JOELLEN HATCHETT

Communicated by Katy Goodmundson

Theorem 11.2. Given a segment AB, it is possible to find the midpoint using a compass and straight edge.

Proof. Given segment AB.

(1) Construct circle AB.
(2) Construct circle BA. Label the interections of circle AB and circle BA, C and D.
(3) Construct segment CD. The midpoint is the intersection of segment AB and segment CD.

Since segments AC, AD, BC, and BD are all radii of congruent circle they are congruent. Segment CD is congruent to CD. By Euclid I.8 triangle ACD is congruent to triangle BCD. By Euclid I.15 angle AEC is congruent to angle BED. By Euclid I.27 triangle ACE is congruent to triangle BDE. By correspponding parts of congruent triangles, AE is congruent to DE, thus E is the midpoint of AB.

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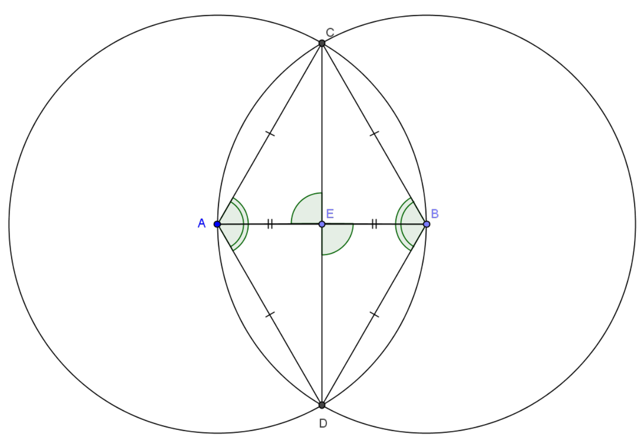

FIGURE 1. Proving E is the midpoint on segment AB.

CONSTRUCTING A TANGENT LINE, L, TO A CIRCLE WITH CENTER O, FROM A POINT A, OUTSIDE OF SAID CIRCLE

JOHN FISHER

Communicated by Katy Goodmundson

Theorem 11.8. Let there be a circle with center, O and point A outside of the circle. Thus it is required to draw a tangent line, l, to the circle from point A.

Proof. First, draw OA. Next, draw two circles: one with center A, radius AO and the latter with center O, radius OA. Label the intersections of the two circles X and Y. Draw XY. Let K be the intersection of OA and XY. By Euclid Proposition I.12, K is the perpendicular bisector. Next, draw circle with center K, radius O. Label the intersection of circle K and circle O, point B. Draw a line l, through AB.

By Theorem 7.4 (Bieber,Wendt) angle ABO is right, since B lies on the circle with diameter OA. Thus, by Euclid Proposition III.19, line l is tangent to the circle with center O. □

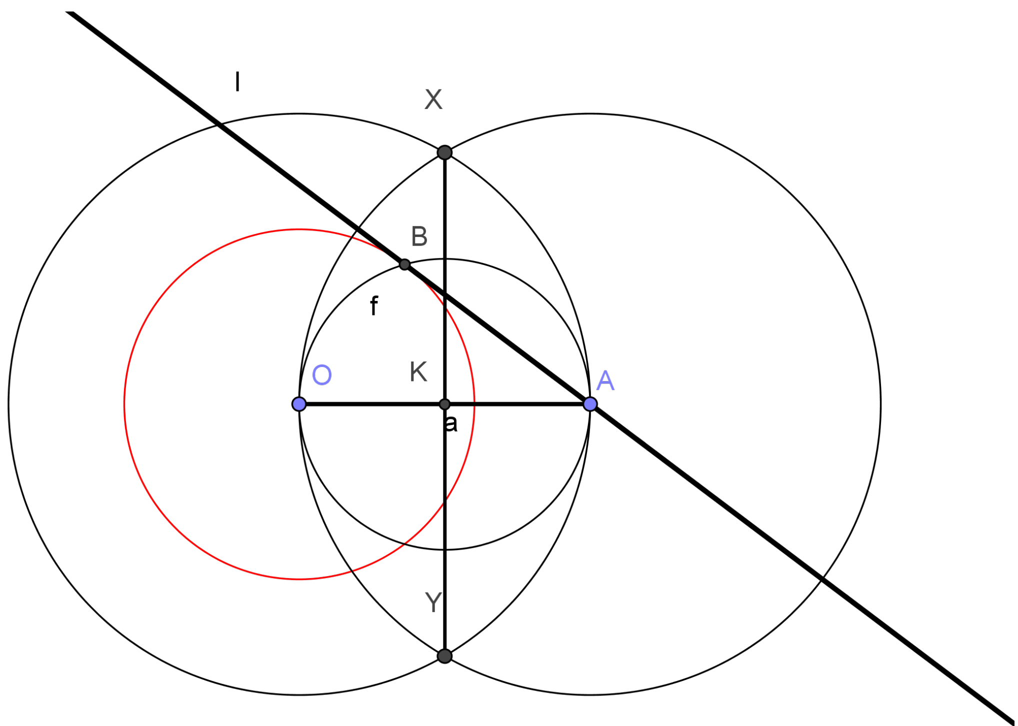

Figure 1. Notice circle with center O, highlighted in red. Angle ABO is perpendicular, thus line l which passes through the circle at point B, is tangent to the circle with center O.

CONSTRUCTION AND PROOF OF AN INSCRIBED CIRLCE

KAYLEE BENSON

Communicated by Emily Bachmeier

Theorem 12.1. Construct a circle inscribed in a given triangle ABC.

Proof.
(1) Draw circle AD. Label point E where line segment AB and circle AD intersect.
(2) Draw circle ED.
(3) Draw circle DE. Label point F where circles ED and DE meet.
(4) Draw line AF.
(5) Draw circle CG. Label point H where line segment AC and circle CG meet.
(6) Draw circle HG.
(7) Draw circle GH. Label point I where circles GH and HG meet.
(8) Draw line CI. Label point J where lines AF and CI meet.
(9) Draw circle CJ. Label point K where circle CJ meets line segment AC.
(10) Draw circle KJ. Label point M where circles CJ and KJ meet on the opposite side of AC as J.
(11) Draw line JM. Label point N where line JM and AC meet.
(12) Draw circle JN.

Since lines AF and CI are angle bisectors by Theorem 11.1 it is known that they intersect at a given point, in this case point J, known as the incenter by Theorem 8.2. JM is perpendicular to AC by Theorem 11.3. Then circle JN is tangent to line AC by Theorem 9.1. Also using Theorem 8.2 it is known that from point J, all perpendicular lines to each side are all congruent. Therefore, circle JN is inscribed within triangle ABC.

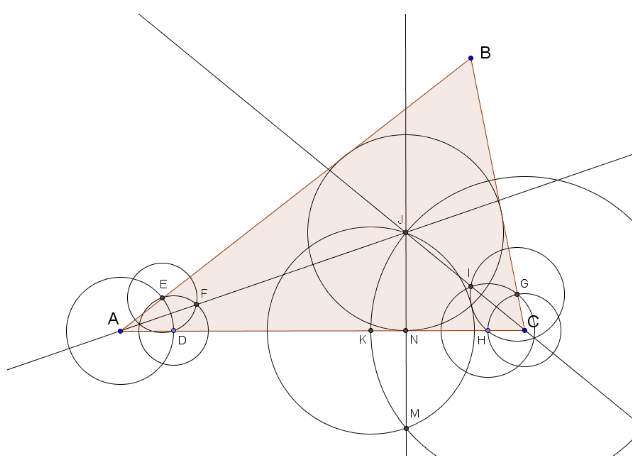

FIGURE 1. Triangle ABC with inscribed circle JN

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