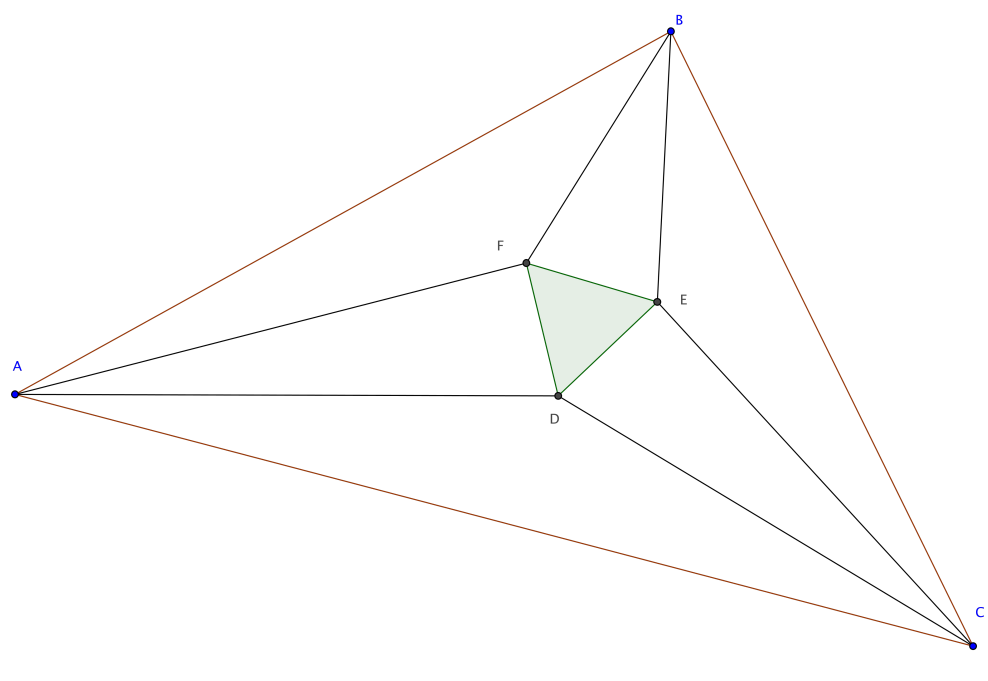
Transactions
in
Euclidean Geometry



Issue # 9

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Given two Segments and an Angle, Create a Kite

Kevin Conger

December 7, 2016

Communicated by Ms. Cohen.

If a man approached me in a trench coat, opened it, and said, "Hey! Want a kite," then I could respond by saying, "No. I can create my own. Thank you, very much." Previously, Mr. Conger provided a general construction for a kite. The following is how to create a kite given two segments and an angle.

Let it be known that the proof which follows is not limited to a specific angle nor specific segments as illustrated by the figures. Since the proof is generalized, this construction works for any given angle and segments, which may result in a non-convex kite. If CD and CB are collinear, the result will not be a kite but will be a triangle.

Theorem E. Given segments GH and TJ, and given an angle QRS, it is possible to make choices in Mr Conger's construction of a kite so that the resulting kite ABCD has AB congruent to GH, BC congruent to TJ and angle ABC congruent angle QRS.

For the purpose of this construction, if the two given segments are of unequal length, we will let TJ be the shorter segment.

`To construct such a kite ABCD:`

1. Let there be a line l, and let there be a point, namely point A, on line l.

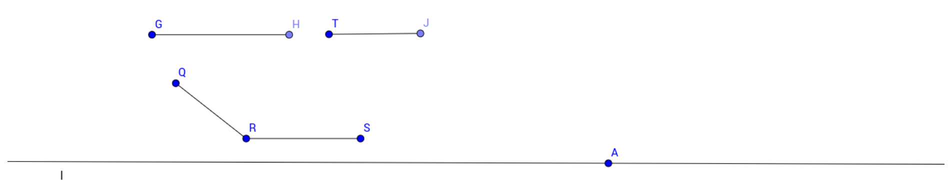

Figure 1: Step 1. The given segments GH and TJ and the given angle QRS with point A on line l.

2. Let point B be placed on l such that AB is congruent to GH.
3. From point B, cut off AB at point X, such that BX is congruent to TJ.
4. With a compass, create a circle with center B and radius BX.
5. Extend RQ and RS of the angle QRS (Euclid Postulate 2).
6. Cut off RS at point Y, such that RY is congruent to TJ.
7. With a compass, create a circle with center R and radius RY.
8. Let the intersection of circle RY and segment RQ be point Z.
9. Create segment YZ (Euclid Postulate 1).
10. On l, at point X, let XW be placed so that XW is equal to YZ.
11. With a compass, create a circle with center X and radius XW.
12. Let the intersection of circle XW and circle RY be point C.

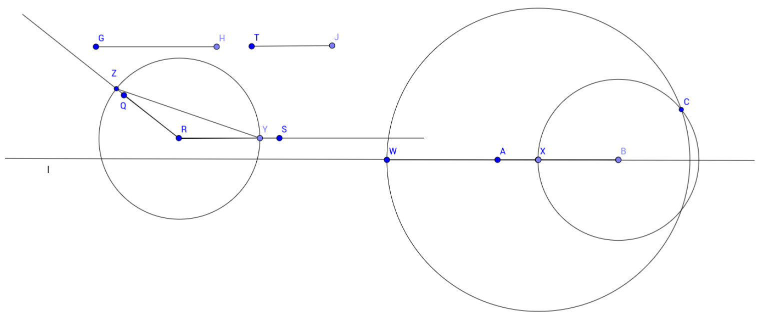

Figure 2: Steps 2-12 of the construction provide points to use with Mr. Conger's construction of a kite, namely points A, B, and C.

13.-17. Use Mr. Conger's construction of a kite by choosing circle A to have radius AB and circle C to have radius CB.

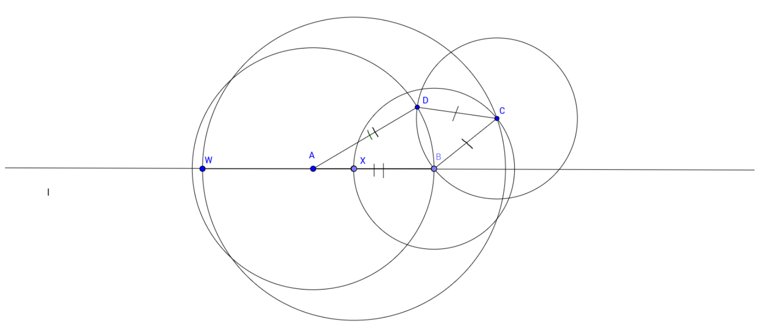

Figure 3: Steps 13-17 result in kite ABCD.

Proof. By Mr. Conger's construction of a kite, quadrilateral ABCD is a kite.

Create segment CX (Euclid Postulate 1). Since CX is a radius of circle XW, then CX is congruent to XW. Since XW is congruent to YZ, then by Euclid's Common Notion 1, CX is congruent to YZ. Since segment RY and segment RZ are radii of circle RY, segment RY is congruent to segment RZ. Since segment BX and segment BC are radii of circle BX, the segment BX is congruent to segment BC. Since BX is congruent to RY, then by Euclid's Common Notion 1, RY, RZ, BX, and BC are congruent segments. Then triangle RZY is congruent to triangle BXC (Euclid I.8). Then angle ZRY is congruent to angle XBC. Since angle ZRY lies on angle QRS, then angle ZRY is congruent to angle QRS. Likewise, since angle XBC lies on angle ABC, then angle XBC is congruent to angle ABC. By Euclid's Common Notion 1, angle QRS is congruent to angle ABC.

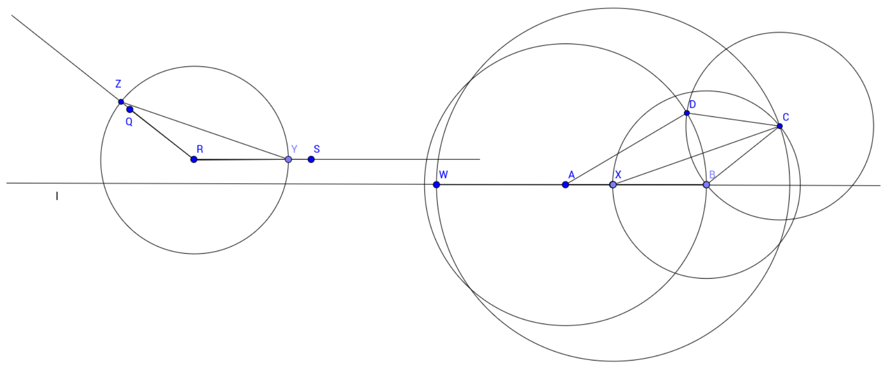

Figure 4: Congruent triangles RZY and BXC result in congruent angles ZRY and XBC, which results in congruent angles ABC and QRS.

By construction AB is congruent to segment GH.

Lastly, since BC is congruent to RY and RY is congruent to TJ, then by Euclid's Common Notion 1, BC is congruent to TJ.

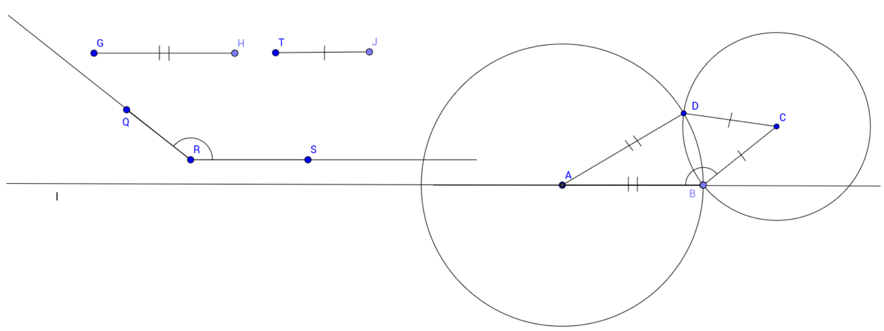

Figure 5: Kite ABCD and its congruence to the given segments GH and TJ and the given angle QRS.

Thus, kite ABCD has angle ABC congruent to angle QRS, AB congruent to GH, and BC congruent to TJ, as was to be proved. □

A Look at Thales' Theorem

Maria Ahrens

December 7, 2016

Communicated by Ms. Cohen.

Recall Miss King's findings in 7.4, which proved that when AB is the diameter of a circle and D lies on the circumference of that circle, then Angle ADB is a right angle. This fact is a part of a bigger theorem known as Thales' Theorem. The purpose of this paper is to further explain Thales' Theorem by showing why Point D must be placed on the circumference of the circle, and not placed outside or inside of the circumference.

Proof. **Case 1. Point D is placed outside the circumference of the circle.**

Let there be a circle CB with a diameter of AB and let D be a point placed on the outside of the circumference. Now let us focus on the triangle formed by connecting the segments AD and BD by Euclid Postulate 1. We want to prove that Angle ADB is not a right angle. Since we already know from Miss King's proof of Theorem 7.4 that a point on the circumference of a circle would create a right angle, let us place a point E where Line Segment AD intersects Circle CB. Then create a line segment BE so that we now have a triangle ABE with a right angle AEB and another triangle DBE. By Euclid 1.13, we know that the angles AEB and BED must add up to two right angles. Since AEB is a right angle, then BED must also be a right angle. Because the interior angles of a triangle must add up to two right angles by Euclid 1.32, we realize that the angles EDB and EBD combined must add up to one right angle since Triangle DBE already has a right angle DEB. Thus, Angle EDB will be acute and not a right angle. □

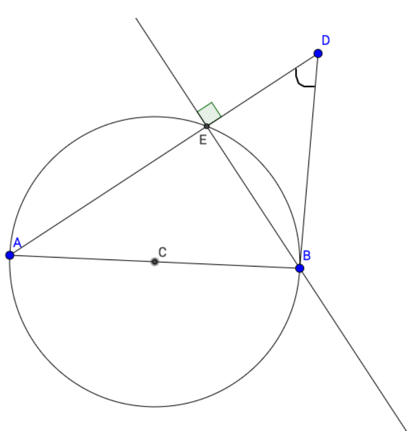

Figure 1: Angle EDB is acute when D is placed outside the circumference of Circle CA.

Proof. **Case 2. Point D is placed inside the circumference of the circle.**

Again, let there be a circle CB with a diameter of AB. But this time, let D be a point placed on the inside of the circumference. Now connect the segments AD and BD by Euclid Postulate 1. Since we want to use Theorem 7.4, we will extend BD and place a point E where Segment BD intersects Circle CB. Then create a line segment AE so that we now have a triangle EAD with a right angle AEB. By Euclid 1.32 we know that the interior angles of a triangle must add up to two right angles. Since Triangle EAD has a right angle AED, the angles DAE and EDA combined must add up to one right angle, making those two angles acute. By Euclid 1.13, we know that a straight line set up on another straight line will add up to two right angles. In order for angles EDA and ADB to be supplementary to make up the line segment BE, then Angle ADB must be obtuse since angle EDA is acute. Therefore, ADB is not a right angle. □

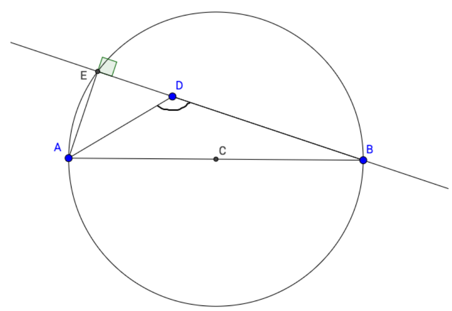

Figure 2: Angle ADB is obtuse when D is placed inside the circumference of Circle CA.

Challenge Construction of a Midpoint

Maria Ahrens

December 8, 2016

Communicated by Ms. Mitchell.

Challenge 11.2. Construct the midpoint of a given line segment using three steps.

The construction is as follows given a line segment AB:

1. Create a circle AB with a radius AB by Euclid Postulate 3.
2. Create a circle BA with a radius BA by Euclid Postulate 3.
3. Circles AB and BA intersect in two locations by the Circle-Circle-Intersection Postulate, which can be denoted as points C and D. Connect those two points with a line segment CD by Euclid Postulate 1. The point of intersection between segments AB and CD can be called E and is the midpoint of AB.

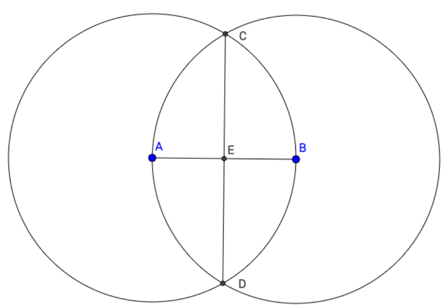

Figure 1: Midpoint E of Segment AB.

Proof. Let AB be a line segment and let AB and BA be circles with the radius AB by Euclid Postulate 3. We know by the Circle-Circle Intersection Postulate that circles AB and BA intersect in two locations, which we will call C and D. Then form a line segment CD by Euclid Postulate 1.

Now the four endpoints of the two line segments AB and CD will be connected to form a quadrilateral. By Euclid Postulate 1, we can draw segments AD and BD. Then AD is congruent to BD since BD and AD are both congruent to the radius AB. When segments AC and BC are drawn using Euclid Postulate 1, we know that AC is congruent to AD and

AB since all are radii of the circle AB. Similarly, BC is congruent to BD and BA since all are radii of the circle BA. Thus, all line segments of the quadrilateral ACBD are all congruent to one another, making ACBD a rhombus by definition.

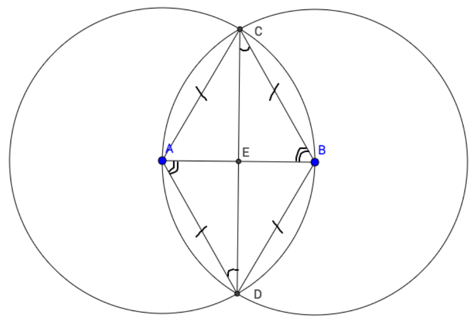

Figure 2: Rhombus ACBD

Using Theorem 1.2, we know that diagonals within a rhombus cross. Thus, the intersection of AB and CD can be denoted as Point E. It is also known that CB is parallel to AD and AC is parallel to DB by Theorem 1.6, which states that all rhombi are parallelograms. Then angle BCD is congruent to angle CDA and angle CBA is congruent to angle BAD by Euclid 1.29. Using these characteristics, triangle AED is congruent to triangle BEC by Euclid 1.26. Then AE is congruent to BE. Thus, E is the midpoint of line segment AB. $\square$

Perpendicular Bisectors of a Triangle

Tessa Cohen

December 8, 2016

Communicated by Ms. Goedken.

Lemma (Cohen). Suppose that line l meets line m and that lines l and r are parallel. Then line r meets line m.

Proof. We will proceed by way of contradiction. Let lines l and r be parallel. Let line m meet line l. We will suppose that m is parallel to r. Since m is parallel to r, m will also be parallel to l, by Euclid Proposition I.30. However, we supposed that m crosses l, therefore, we have a contradiction. Thus, m must cross r. □

Theorem 8.3. Let T be a triangle. For any pair of sides of T, the perpendicular bisectors of those sides meet (they are not parallel).

Proof. We will proceed by way of contradiction. Suppose that ABC is a triangle. Let l be the perpendicular bisector of side AB and r be the perpendicular bisector of side BC. Suppose that l and r are parallel. We will extend side AB into line AD, D being the intersection of AD and r. Let E be the midpoint of AB and F be the midpoint of BC. Let G be the intersection of segment AC and r and H be the intersection of segment AC and l. We know that the angles created by perpendicular bisectors are right, so angle BEH is right. Since l and r are parallel, and BEH is right, by Euclid Proposition I.29, angle BDF is also right. Since r is the perpendicular bisector of BC, angle DFB is right.

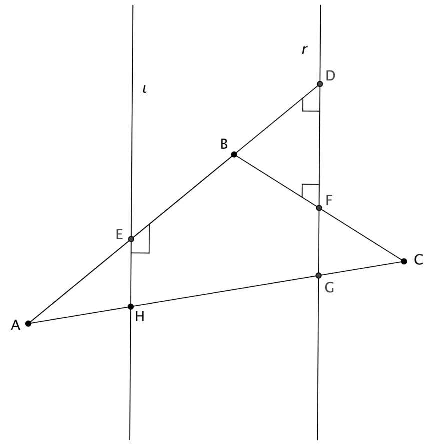

Figure 1: Angles BEH, BDF and DFB are all right.

Now we will consider line AD and segment BC. We have the perpendicular bisector, r, which cuts line AD at point D and line BC at the midpoint, F. Angle BDF and angle DFB taken together are congruent to 2 right angles, therefore, by Euclid Proposition I.28, line AD and segment BC must be parallel. Since, AD and BC will not cross and will not share any point B. Thus, ABC cannot be a triangle and we have a contradiction. Therefore, any 2 perpendicular bisectors of a triangle will cross.

□

The Perpendicular Bisectors of Any Two Sides of a Triangle Will Meet

Maria Ahrens and Abigail Goedken

December 8, 2016

Communicated by Ms. Van Ryswyk.

Theorem 8.3. Let ABC be a triangle. For any pair of sides of ABC, the perpendicular bisectors of those two sides meet.

Proof. Let ABC be a triangle. By Euclid 1.10, we can create perpendicular bisectors for AB and BC. We will denote the midpoint of AB as point D and the midpoint of BC as point E. The place where the perpendicular bisector of AB crosses CA will be called G, and the other point on CA that falls on the perpendicular bisector of BC will be called F. To clarify, segment DG is the perpendicular bisector of AB and segment EF is the perpendicular bisector of BC. Now we will draw the segment DE by Euclid Postulate 1. Since segments DG and EF form right angles with segments AB and BC, then angles BDG and BEF must also be right angles by Euclid 1.13. Drawing the segment DE causes the angles GDE and FED to be acute. By Euclid Postulate 5, the line segments DG and EF will cross at point H. □

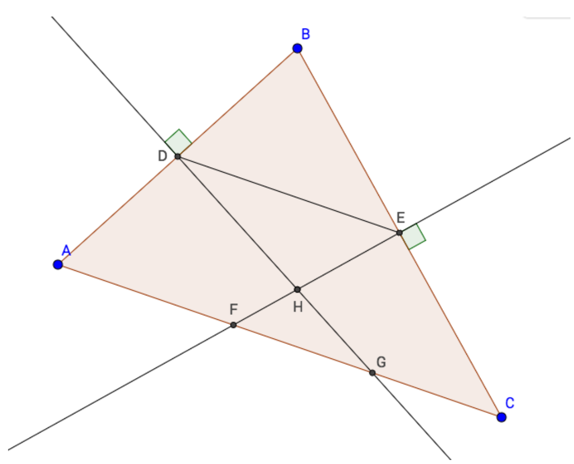

Figure 1: Perpendicular bisectors DG and EF cross at the point H.

Congruence of Two Tangents

Kevin Conger

December 8, 2016

Communicated by Mr. Kessel.

Theorem 9.1. Let AB and AC be two tangent lines from a point A outside a circle, such that B and C are the points at which AB and AC touch the circle, respectively. Then AB is congruent to AC.

Proof. Let AB and AC be two tangent lines from a point A outside a circle, such that B and C are the points at which AB and AC touch the circle, respectively. Let the center of the given circle be point X.

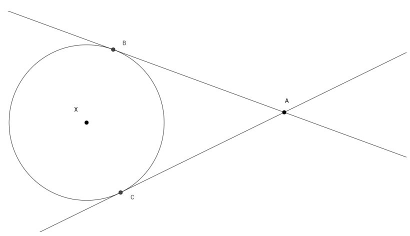

Figure 1: The given tangents and given circle.

By Euclid's Postulate 1, create segments XA, XB, and XC. By the definition of a circle, XB is congruent to XC. Since XB and XC are radii, which touch the intersection points of tangent lines of circle X, angles ABX and ACX are right angles by Euclid III.18. Since triangles ABX and ACX have right angles, a pair of congruent legs, namely XB and XC, and a shared hypotenuse, namely XA, triangles ABX and ACX are congruent by Ms. Cohen's Theorem 11.3. Then segments AB and AC are congruent, as was to be shown.

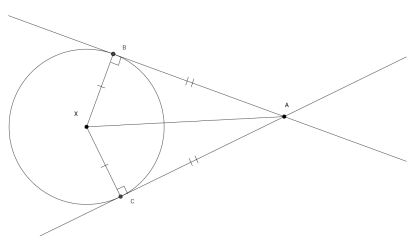

Figure 2: Triangles ABX and ABC are congruent, so segments AB and AC are congruent.

□

Proof of a Rectangles Vertices Lay on a Circle

Perry Kessel

December 8, 2016

Communicated by Mr. Phaly.

Theorem 9.3. Prove that rectangle is always a cyclic quadrilateral such that each vertex around a rectangle will lay on a circle.

Proof. By Miss King's proof of Theorem of 3.3 which states that the two diagonal's of a rectangle will cross and bisect each other. She states by I.4 in *Euclid's Propositions* that triangles created by the diagonals are congruent. Refer to the figure below to see that point X is the intersection point of the rectangle's diagonals, and the segments AX,BX,CX, and DX are congruent. Create a circle from X to point A(in the image below this circle is labeled C). Since we know from point X to each vertex is congruent. We know that each vertex will lay on this circle XA. By definition a rectangle is always a cyclic quadrilateral. □

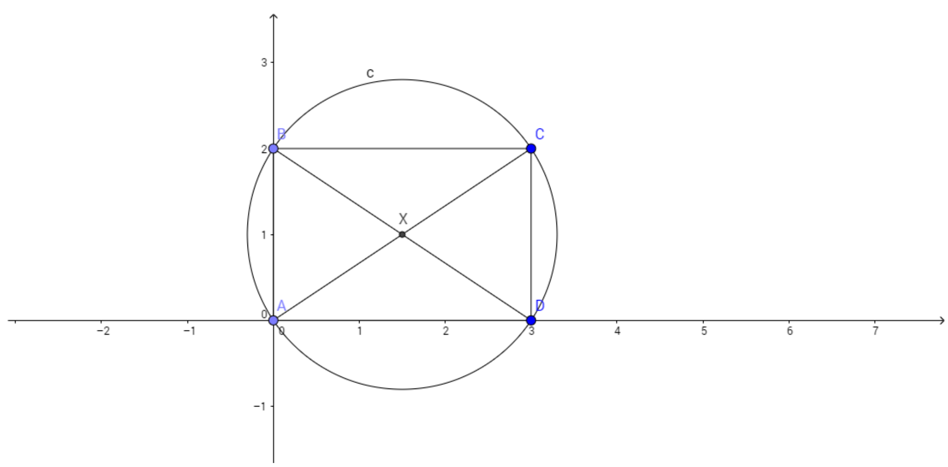

Figure 1: Notice rectangle ABCD's vertices are on circle C

Challenge Perpendicular Line Construction

Maria Ahrens

December 8, 2016

Communicated by Ms. Maus.

Challenge 11.4. Given a line l and a point A lying on l, construct a line perpendicular to l through A.

This construction was completed using four steps as follows given a line l and a point A on l:

1. Pick a point B on Line l and create a circle AB with radius AB. Place a point C where Circle AB intersects Line l.
2. Create circle BC with a radius BC by Euclid Postulate 3.
3. Create Circle CB with a radius CB by Euclid Postulate 3.
4. By the Circle-Circle-Intersection Postulate, circles BC and CB intersect at points D and E. Connect those two points by drawing a line segment DE by Euclid Postulate 1. Line segment DE is perpendicular to line l and goes through point A on l.

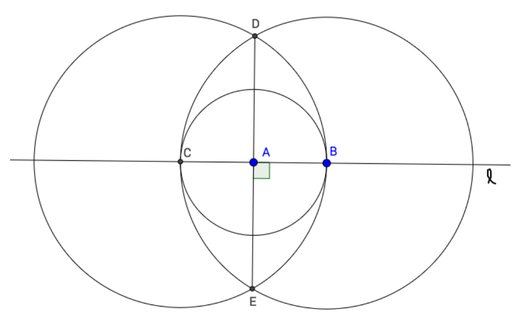

Figure 1: Perpendicular Line DE through Point A on Line l.

Proof. Let l be a line with a point A on it. Place a point B on l such that a circle AB with radius AB can be created by Euclid Postulate 3. Because the circle AB is on l, the circle will cross Line l at two different points, at B and at another point we will denote as C. We now have two points equidistant from each other, since AC and AB are both radii of the circle AB.

The creation of circles BC and CB, by Euclid Postulate 3, results in a common radius BC, which will be important in future steps. By the Circle-Circle-Intersection Postulate, circles BC and CB intersect at points D and E. We can connect these two points by creating a line segment DE and draw segments CD, DB, BE, and EC by Euclid Postulate 1. Since CD, EC, and CB are all radii of circle CB, they are congruent to one another by the definition of a circle. Similarly, since BD, BE, and BC are all radii of circle BC, they are congruent to one another using the same justification.

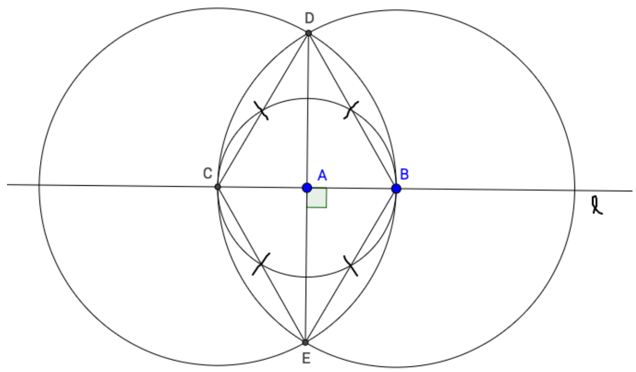

Figure 2: Rhombus CDBE with diagonals meeting at a right angle.

Since CD, EC and CB are congruent to one another and BD, BE, and BC are congruent to one another, then line segments CD, EC, BD, and BE are all congruent to one another by Euclid Common Notion 1. Then the line segments of the quadrilateral CDBE are all congruent to one another, making DCBE a rhombus by definition. We know that the diagonals DE and CB will cross by Theorem 1.2, which states that the diagonals of a rhombus intersect. Since we know that all rhombi are kites, the diagonals DE and CB will cross at right angles by Theorem 2.5. This causes CAD, DAB, BAE, and EAC to be right angles. Then DE is perpendicular to CB with A as their point of intersection. Since points C and B both fall on l, the segment CB exists as a part of Line l. Therefore, since DE is perpendicular to CB, DE is also perpendicular to l. Thus, DE is perpendicular to line l through point A. $\square$

A Parallel Line through a Point

Kevin Conger

December 8, 2016

Communicated by Ms. Schultz.

Theorem E. With a compass and a straightedge, it is possible to construct a line parallel to a given line l which passes through a given point A not lying on l.

`To construct such a line:`

1. Chose any point X on the given line l, and with a compass, construct a circle with center X and with radius XA, using the given point A. Let the two intersection points of circle XA with l be points R and S.
2. Fix a compass to the length of RA. Construct a circle with center S and with a radius congruent to RA. Let the intersection of circle S with circle X, on the same side of line l as point A, be point B.
3. With a straightedge, create line AB (Euclid's Postulates 1 and 2).

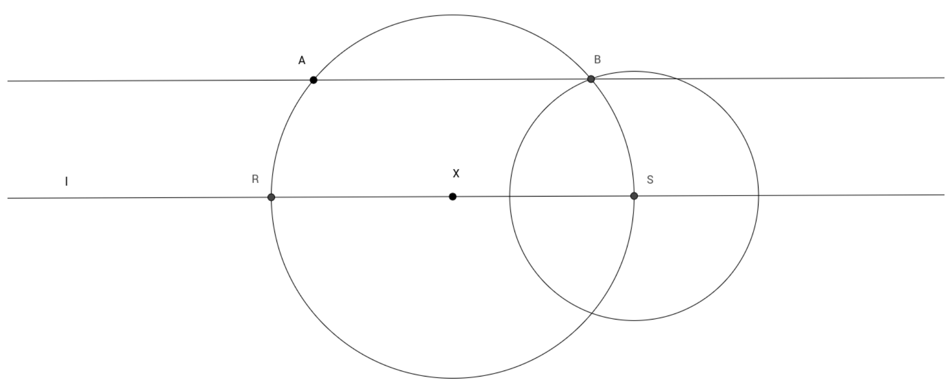

Figure 1: The resulting construction of a line parallel to l and through A.

Proof. By Euclid's Postulate 1, create RA, XA, XB, and SB. By construction, XA, XR, XB, and XS are congruent, since they are the radii of circle X. Also by construction, RA and SB are congruent. Therefore, triangles ARX and BSX are congruent by Euclid I.8. Since triangles ARX and BSX are congruent, angles AXR and BXS are congruent.

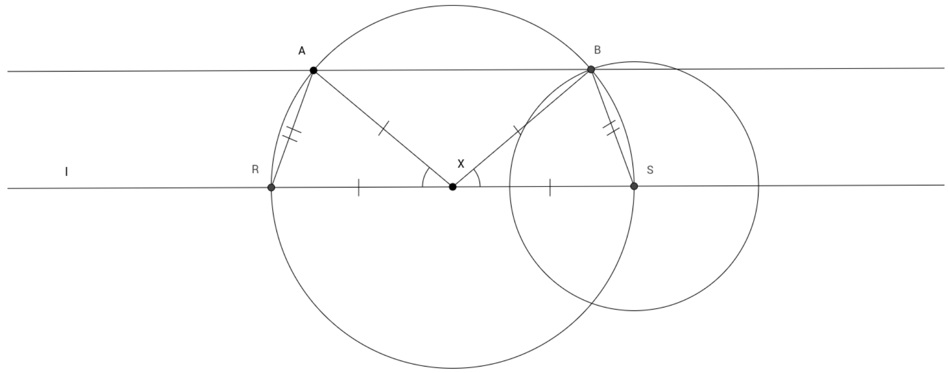

Figure 2: Triangles ARX and BSX are congruent, so angles ARX and BSX are congruent.

Let a perpendicular from segment AB exist such that it intersects with X by Ms. Cohen's Theorem 11.3, and let the point of the perpendicular line on AB be point T. It follows that angles ATX and BTX are right angles, and triangles AXT and BXT are right triangles. Since triangles AXT and BXT are right triangles with congruent hypotenuses, namely XA and XB, and a shared leg, namely XT, triangles AXT and BXT are congruent by Ms. Schultz' Theorem 7.2. It follows that angles AXT and BXT are congruent.

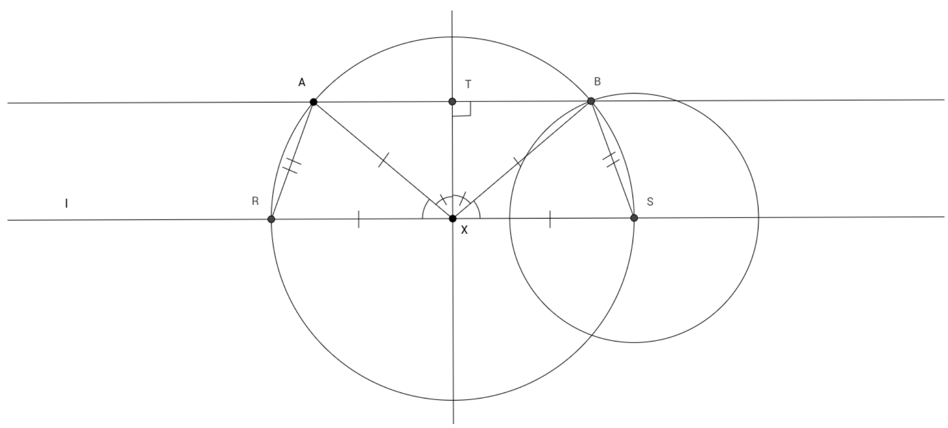

Figure 3: By Ms. Schultz' Theorem 7.2, triangles AXT and BXT are congruent, so angles AXT and BXT are congruent.

Since angles RXA and SXB are congruent, and angles AXT and BXT are congruent, angles RXT and SXT are congruent by Euclid's Common Notion 2. Since XT falls on a straight line, it makes either two right angles or angles equal to two rights by Euclid I.13. Since angles RXT and SXT are congruent, angles RXT and SXT will both be rights. Since angles SXT and BTX are right angles, line AB is parallel to l by Euclid I.28. Thus, AB is a line parallel to l and goes through point A.

□

The Sum of Classes

Rebecca Shere

December 8, 2016

Communicated by Ms. Schwan.

Definition. Let a and b be segment classes. We define the sum of $a+b$ to be the class of the segment AB+CD, where AB is an element of the class a and CD is an element of class b.

Problem A.4 The notion of sum of classes does not depend on the particular choices of segments AB and CD. The defintion holds when adding any line segment in a to any line segment in b.

Proof. Let a be the set of all line segments congruent to AB. Let b be all the line segments congruent to CD. And let c be the segment class of all line segments congruent to AB+CD. First, pick a line segment, let's denote it RS, from class a. Then, pick a line segment, let's denote it TU, from class b. To construct line segment RS+TU($a+b$), create a ray from R through S and place a point V on that ray, such that, line segment SV is congruent to line segment TU. Thus, We have constructed RS+TU, which is congruent to RV.

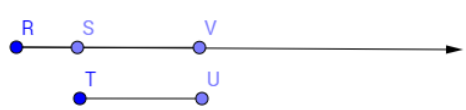

Figure 1: Construction of RS+TU

Now through Euclid's Common Notion 2, we can say RS+TU is congruent to AB+CD. Because the pieces that make up RS+TU, namely line segments RS and TU, are congruent to AB and CD respectively, the sums RS+TU and AB+CD will be congruent. Since the sum RS+TU is congruent to AB+CD, it will belong to the segment class c. This holds no matter what line segments you choose from a and b, when added together $a+b$ will always be in c.

$\square$