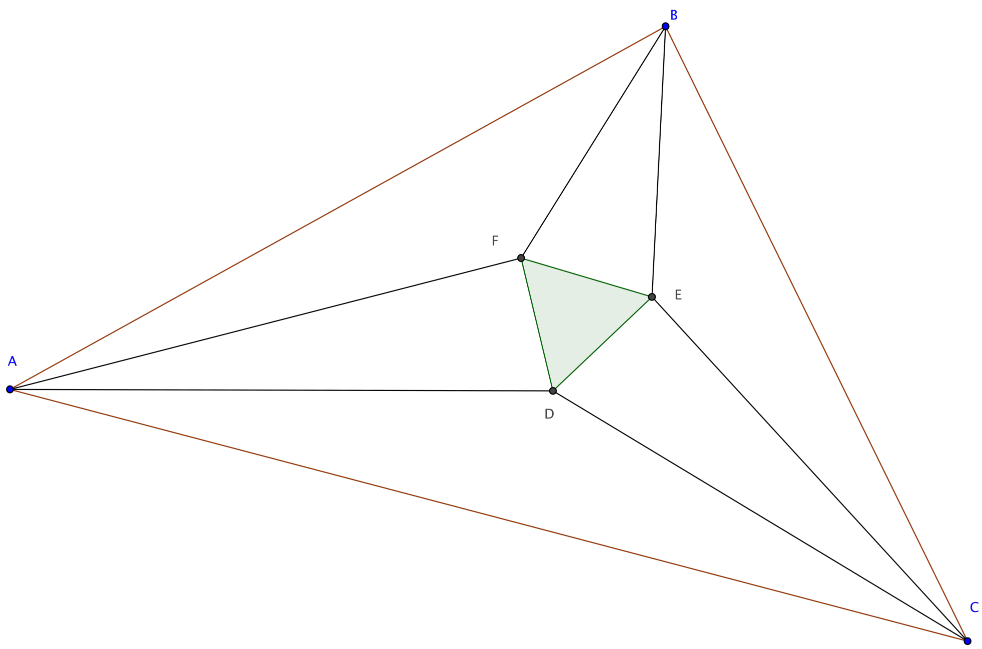
Transactions
in
Euclidean Geometry



Issue # 11

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Internal Common Tangents

Juliana Herran

December 8, 2016

This is a continuation of the study on the common tangent lines through two circles presented in journal issue 8. The construction proof presented below shows the internal common tangents to two circles. The task is as follows: Given two circles Γ and Γ' with centers O,O', respectively, construct a line tangent to both circles.

Lemma 1. If OCP'P is a quadrilateral such that: [i] OP is congruent to CP' [ii] OP is parallel to CP' [iii] Angle POC and angle P'CO are right

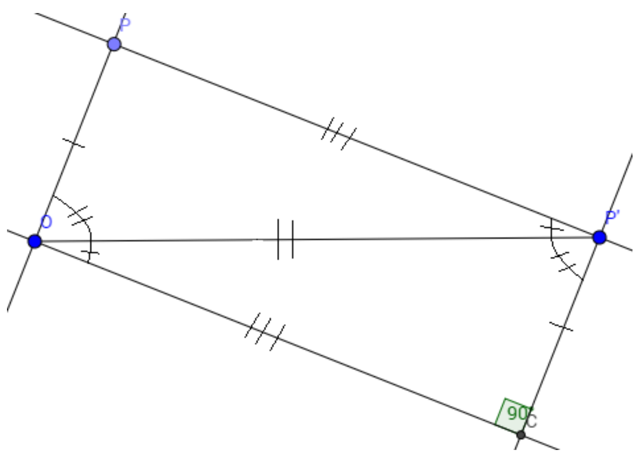

Figure 1: The angle CP'P and the angle OP'P are both right angles

Proof. Let a diagonal from O to P' exist in the quadrilateral OCP'P. Given that segments OP and CP' are parallel and congruent, the diagonal OP' joins congruent parallel lines at their extremities (Euclid 1.33). Then, by Euclid 1.33, segment PP' is parallel and congruent to segment OC. Since PO is parallel to CP', and the diagonal OP' crosses both lines, by Euclid 1.29 angles P'OC and OP'P are congruent. Similarly, angles CP'O and POP' are congruent. Since angle POC is a right angle, angle POP' plus angle P'OC is a right angle. Then angle CP'P is also a right angle. Also, since triangle POP' is congruent to triangle P'OC, their angles must also be congruent by Euclid 1.8. A similar argument can be used to prove that angle OPP' is also a right angle. ☐

Construction

First we join the centers of the two circles Γ and Γ' with different radii to create the line segment OO'. Note that circle Γ is the circle centered at O and circle Γ' is the circle centered at O'. Then, find the perpendicular bisector of OO' using the standard construction of a perpendicular bisector as shown in case one and label the midpoint of OO' as point X. Label the intersection point of circle Γ with segment OO' as point B and the intersection point of the circle Γ' with OO' as point A. Then, draw a circle with center X and radius OX. Construct a circle of center O' and radius AO'+BO (this will be referred to as the outer circle). Label the points of intersection of the circle centered at X and the outer circle as P and Q. Draw a ray passing through O' and point P. The points where ray O'P intersects with the original circle Γ' centered at O' is point C. Create a parallel line to PC going through point O and label the intersection point of this line with the original circle centered at O as P'. The line that goes through points C and P' can be constructed by Euclid's postulate 1. This line through C and P' is an internal common tangent of circles Γ and Γ'. A similar construction method can be used to create the second internal common tangent to these circles shown in the figure below as the line through DQ'.

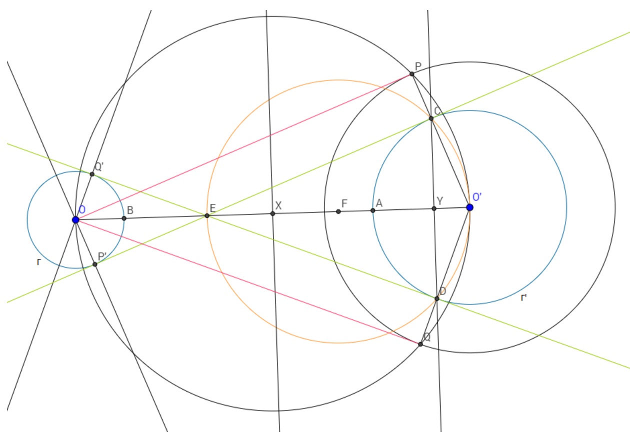

Figure 2: The purpose of the color coded graph is a better visualization of the important segments and triangle relations relevant to the proof below. The original circles Γ and Γ' are in blue and the internal tangent lines are in green.

Proof. Since line segments OX, XO' and XC are all radii of the circle centered at X, they are all congruent to each other. By Miss. King's theorem 7.4, angle OPO' is a right angle. Also, by construction segments OP' is parallel and congruent to segment PC. Then, we have a quadrilateral with two parallel congruent sides and one right angle. By Euclid 3.3, the line segment PQ is perpendicular to the segment OO' since PQ is a cord of circle Γ' and the diameter of the circle that creates a right angle with the given cord is OO'. Then, through a similar construction as the one provided by Miss. Goedken in 3.6 we can create a line segment joining points C and D that is parallel to line segment PQ. Then, we have a quadrilateral ECO'D that looks suspiciously like a kite. Since segment CD is parallel to PQ by Euclid 3.3, the angle CYO' is a right angle. By Euclid 1.13, DYO' is also a right angle. Moreover, since in triangles CYO' and DYO' the side YO' is congruent to itself and by Euclid 3.3 line segment CD is bisected by segment OO' and angles CYO' and DYO' are congruent, by Euclid 1.26 (so called side-angle-side SAS), triangles CYO' and DYO' are congruent. Then, segment CO’ is congruent to segment DO’. Similarly, segment EY is congruent to itself and segments CY and YD are congruent by Euclid 3.3. Since angles CYE and EYD are right, triangles ECY and EDY are congruent to each other. Then, the quadrilateral CEDO’ must be a kite as we have two pairs segments that are congruent to each other and the diagonals of the quadrilateral meet at a right angle (by Miss. Worsfold theorem 2.5). Then, the midpoint of segment EO is constructed and found to be point F. A circle with radius EF centered at F is constructed (shown as the orange circle in the figure above). Since the points of the quadrilateral fall in the circumference of the circle with radius EF and triangles ECO’ and EDO’ are congruent, angles ECO’ and EDO’ are congruent. Finally, by Miss. King’s theorem 7.4, angles ECO’ and EDO’ are also right angles. Then, the line through points C and P’ is an internal common tangent of the circles Γ and Γ'. A similar argument can be given to show that the line through points Q’ and D is also an internal common tangent of circles Γ and Γ'. □

Inscribed Squares in an Isosceles Right Triangle

Duece K Phaly

December 8, 2016

Communicated by Ms. Van Ryswyk.

There is an assumption that the construction of the inscribed squares in an isosceles right triangle occur in such a way introduced below.

Conjecture 13.5 There are two ways to inscribe a square in an isosceles right triangle. Which one has greater content?

Theorem 13.5. There are two ways a square can be inscribed in an isosceles right triangle. In which the case when two vertices are the midpoint of the the legs of the triangle has a greater content than the second case when the vertices are not the midpoint of the triangle.

Proof. Let triangle ABC be an isosceles right triangle with segments AB and AC as well as angle ABC and angle ACB be congruent to one another respectively. We know this by proposition I.6

We will inscribe a square ADEF in the assumed way, such that point D is the midpoint of AB and point F is the midpoint of AC. By definition of a square, we know that segments AD,DE,EF,AF are all congruent to one another and we also know that angle DAC,AFE,FED,EDA are all right angles. Since, point D is the midpoint of segment AB we know that segments AD and DB are congruent. Similarly, segments AF and FC are congruent.

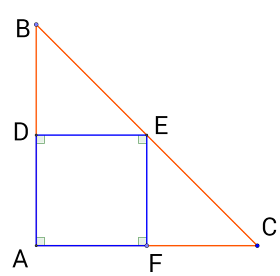

Figure 1: Triangle ABC with inscribed square ADEF (1st way to inscribe a square).

Since, there is a straight line ED which lies on another straight line AB. By Euclid's proposition I.13 we know that angle ADE and angle EDB must equal the sum of two right angles. Since, ADE is a right angle then EDB must also be a right angle. Similarly, angle

EFC is a right angle. Construct a diagonal DF by postulate 1 from Book 1 of Euclid. Since segment BD is congruent to segment EF,angle BDE is congruent to angle EFC and segment DE is congruent to segment FC, then triangle BDE is congruent to triangle EFC. By proposition I.4 of Euclid. By a similar approach we can also prove the congruence of all the triangles within the original triangle ABC. Thus, triangles ADF,EFD,FEC,DBE are all congruent to one another. By axiom EC1 from the theory of content page, we know that these congruent figures all have equal content. Since there are four figures with equal content that make up the larger triangle ABC, we can say that one of these triangles is equal to 1/4 the content of the entire triangle. Thus, the two triangles that make up the inscribed square (triangle ADF and EFD) make up 1/2 the content of triangle ABC.

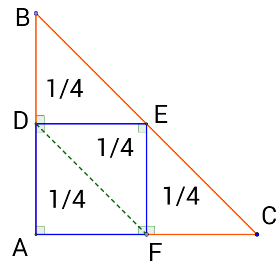

Figure 2: Triangle ABC broken up into four congruent triangles

Let the same triangle ABC have an inscribed square DEFG in such a way that exists below. By definition of a square we know that segments DE,EF,FG and GD as well as angles DEF,EFG,FGD,GDE are all right angles congruent to one another respectively.

We will construct the diagonal DF by postulate 1.By Euclid proposition I.32 we know that the angles ABC,ACB and CAB must add up to two right angles. Since angle CAB is a right angle, angles ABC and ACB must add up to one right angle. Since angles ABC and ACB are congruent, they must each equal 1/2 of a right angle.Since there is a straight line DE which lies on another straight line BC, we know that angle BED and angle DEC must sum up to two right angles.By Euclid's proposition I.13. Since angle DEC is a right angle then, angle BED must also be a right angle. Similarly, angle CFG is a right angle. We know that angles ABC, ACB and BED,CFG are congruent respectively, and side DE is congruent to segment FG. Thus by proposition I.26 triangles BED and CFG are congruent. Then, the angles BDE and CGF are congruent.

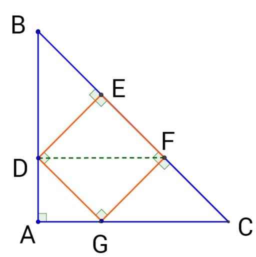

Figure 3: Triangle ABC with inscribed square DEFG (2nd way to inscribe a square)

We can make a similar case to prove that triangles BED,DEF,DFG,GFC are all congruent to one another.By axiom EC1 from the theory of content page, we know that these congruent figures all have equal content.

We know that the individual triangles in this case are smaller than the ones in the previous case because if we remove the content of these triangles from the larger triangle ABC we are left with additional content triangle ADG. Which, differs from the case before where the 4 individual triangles made up of the entire figure. Since the smaller triangles in the first construction were equal to the entire triangle ABC and the smaller triangles in the second construction consist of 4 triangles with an addition triangle ADJ, the 4 congruent triangles in the second construction do not make up the original triangle, thus the second construction will have triangles which are less than 1/4 of the content of the triangle ABC. Thus, when we add two of these triangles respectively in each construction to make up the inscribed squares, the square in the first case has greater content than the square in the second case.

$\square$

To Square a Rectangle

Rebecca Shere

December 8, 2016

Communicated by Mr. Conger.

Problem 13.8 Given a rectangle ABCD, construct a square of equal content.

Note: Let line segment AB be greater than BC.

Construction:

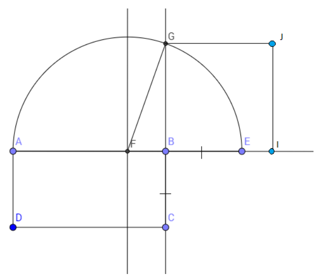

Figure 1: The construction

1. With a straightedge, create a ray from A through B.

2. With a compass, place point E on ray AB, such that, line segment BC is congruent to line segment BE.

3-5. Using Ms. Cohen's perpendicular bisector lemma, construct the perpendicular bisector of AE. We will label the intersection of the bisector with the ray AB point F.

6. With a compass, construct a semicircle with radius FE that lies above line AE.

7. With a straightedge, extend line BC until it intersects with the semicircle centered at F. Label the intersection G.

8. With a straightedge, connect the points G and F.

9. With a compass, place point I on ray AB, such that, line segment BI is congruent to BG.

10-14. With Ms. Schwan's 11.6, we will construct a line parallel to BI through point G.

15. With a compass, place point J on the parallel line through G, such that, GJ is congruent to line segments BG and BI.

16. With a straightedge, connect point J to point I. Thus, we have constructed the square on BG which has equal content to the rectangle ABCD.

Proof. This proof will have 2 parts. Part 1, we will prove BGJI is a square. Part 2, we will prove that the square BGJI has equal content to the given rectangle ABCD.

Part 1: Since ABCD is a rectangle, we know angle ABC is right. By Euclid.I.13, angle IBC is also right. Then, again by Euclid.I.13, GBI is a right angle. We constructed line segment GJ to be parallel and congruent to line segment BI. Thus by Euclid.I.29, angle JGB is also right. Then by Euclid.I.33, we have 2 straight lines, BG and JI, joining congruent and parallel lines(GJ and BI) making line segments BG and JI equal and parallel. Since line BG has right angles JGB and GBI(when meeting lines GJ and BI), we know it is perpendicular to GJ and BI. Since BG is parallel to JI, JI is also perpendicular to both line segments GJ and BI. Therefore, the angles GJI and JIB are right angles. Thus, BGJI has all four right angles along with all four sides congruent proving BGJI is a square.

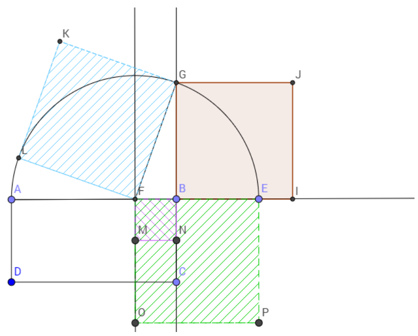

Figure 2: Square on BG is shown in solid. Other squares used in part 2 are shown with hatching.

Part 2: By Euclid.II.5, since the straight line AE has been cut into congruent segments at F(line segments AF and FE), and into non-congruent segments at B(line segments AB BE), the rectangle contained by AB,BE together with the square on BF has equal content to the square on FE. Also, FE is congruent to FG as they are both radii of semicircle centered at F. Then, by EC_1, the square on FE has equal content to the square on FG. So by EC_2 , we can say, the rectangle AB,BE together with the square on FB has equal content to the square on FG.

By Euclid.I.47, taken together the squares on GB and BF have equal content to the square on FG.

Since the rectangle AB,BE together with the square on FB has equal content to the square on FG, and the squares BG and BF taken together have equal content to the square at FG, we can say, by EC_2, rectangle AB,BE together with the square on FB has equal content to the squares on BG and BF taken together. Let the square on BF be subtracted from each side. Then, the rectangle contained by AB,BE, has equal content to the square on BG. By EC_1, we know rectangle AB,BE has equal content to the rectangle contained by AB,BC. Thus, by EC_2, the rectangle contained by AB,BC(rectangle ABCD) has equal content to the square on BG. $\square$

Special Case of Creating a Square to Make Two Rectangles of Equal Content

Perry Kessel

December 8, 2016

Theorem 14.1. Given two rectangles, P and R, construct a square S so that, when taken together the content of R and S is equal to the content of P.

Proof. For this special case we have rectangle ABCD acting as rectangle P and rectangle R being represented by ECGF. We are letting AD be congruent to GF. Also we are letting AD - AB equal to CG. Now I am going suppose one can rearrange the to rectangles such that P is horizontal and R is perpendicular to P.The way I used II.4 is making a square off of AB. Then make a the diagonal BH. We know that E lay on this line because we constructed rectangle P and R specially. We also know that the figure BCEI is a square by proposition 4 of book 2 in Euclid's Elements to construct the square S is constructed by square IBCE in figure 2. Thus when we put the square S and rectangle R together it has the same content as rectangle P. $\square$

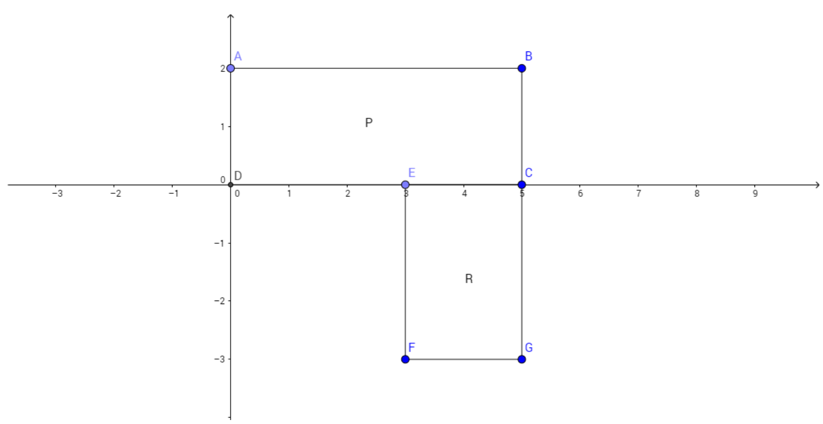

Figure 1: Rearranging the Figures

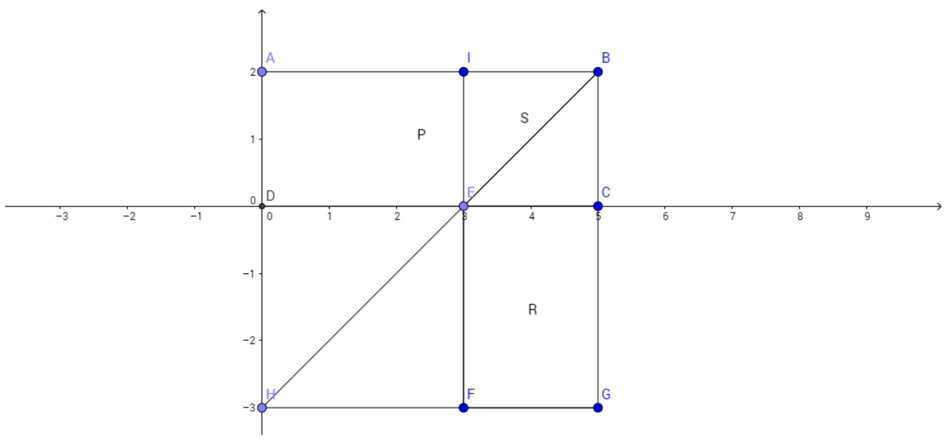

Figure 2: Creating Square S

Equal Content with Right Triangles

Amanda Worsfold

December 8, 2016

Communicated by Ms. Schwan.

Theorem 14.7. Let ABC be a right triangle with the right angle at A. Let AD be the altitude from A to side BC. The square on side AD has equal content with the rectangle on BD and DC.

Proof. We will show the square on side AD has equal content with the rectangle on BD and DC, shown in Figure 1.

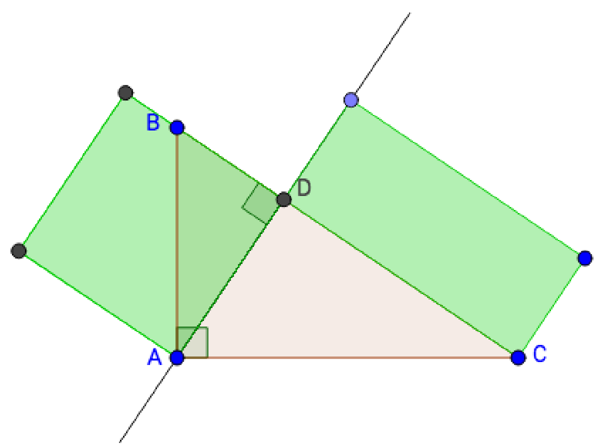

Figure 1: The square on side AD and the rectangle on BD and DC

Notice we have three distinct right triangles formed: Triangle ADB, Triangle ADC, and Triangle ABC. By Euclid's I.47, we can say some things about the squares within these three triangles.

1. Triangle ADB: the square on side BD taken together with the square on side AD has equal content with the square on side AB.

2. Triangle ADC: the square on side CD taken together with the square on side AD has equal content with the square on side AC.

3. Triangle ABC: the square on side AB taken together with the square on side AC has equal content with the square on side BC.

I will references these three ideas later in the proof.

Next, we will turn our attention to the square on side BC. Because the rectangle of interest is constructed with sides BD and DC, the square on side BC can be broken into two squares, namely the square on side BD and the square on side DC, and two rectangles. This is shown in Figure 2. By Euclid's II.4, we can say the square on side BC is of equal content with the sum of 2 rectangles containing sides BD and DC, the square on side BD, and the square on side DC.

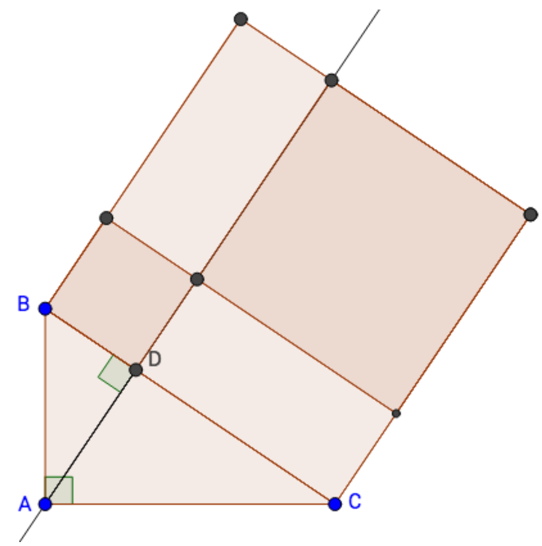

Figure 2: The square on side BC, displaying Euclid's II.4

Using Triangle 3 from above, we can substitute the square on side AB taken together with the square on side AC in for the square on side BC. Using Triangle 1 from above, we can substitute the square on side AB minus the square on side AD in for the square on side BD. Similarly, using Triangle 2 from above, we can substitute the square on side AC minus the square on side AD in for the square on side CD.

Now we have an equation that looks reads the square on side AC taken together with the square on side AB is of equal content to the rectangle on sides BD and DC taken together with the square on side AB minus the square on side AD taken together with the square on side AC minus the square on side AD.

On both sides of the equation, we have the square on side AB and the square on side AC. These cancel with one another. We can also combine the two squares on AD. Now we have an equation that read 0 is of equal content to the 2 rectangles on sides BD and DC minus 2 squares on side AD. We can move the 2 squares on side AD over to the other side of the equation. Then, on both sides of the equation we have a 2. These cancel with one another leaving the square on AD having equal content with the rectangle on sides BD and DC.

□

Constructing a Square

Megan King

December 8, 2016

Communicated by Ms. Schwan.

Theorem 15.3. Construct a square in as few steps as possible. (par 9)

Construction

(1) Draw a circle O with diameter AC. (2) Draw diameter AC. (3)-(5) Construct the perpendicular bisector of segment AC by drawing two circles and the perpendicular line to segment AC. Call the intersection of two circles, point B above segment AC and point D below segment AC. (6) Connect points A and B to create segment AB. (7) Similarly, connect points B and C to create segment BC. (8) Again, connect points C and D to create segment CD. (9) Lastly, connect points D and A to create segment AD. Thus, ABCD is a square.

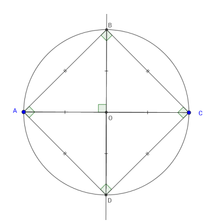

Figure 1: Square ABCD within Circle O

Proof. By construction, segment AC is the diameter of circle O. Similarly, we constructed the perpendicular bisector, so segment BD is also a diameter of circle O. By Theorem 7.4, angle ABC is a right angle, because AC is the diameter and B lies on the circle O. Similarly, angle ADC is a right angle. Now use Theorem 7.4 again; however, looking at diameter BD. Thus, angle BAD and angle BCD will be right angles.

Segments AO, BO, CO, DO are all congruent to each other, because they are the radii of the circle O. Since we constructed the perpendicular bisector BD of segment AC, angles AOB, BOC, COD, DOA are all right angles. Then, triangles AOB, BOC, COD, and DOA are all congruent to each other by Euclid I.4. Therefore, segments AB, BC, CD, and DA are congruent.

Since there are four congruent sides and all interior angles are right angles, ABCD is a square by the definition of a square.

□

Construction of a Regular Hexagon

Mackenzie Mitchell

December 8, 2016

Communicated by Ms. Maus.

Theorem 15.7. Construct a regular hexagon with as few steps possible. What should the par value be?

To construct a regular hexagon using a compass and straightedge (par 11):
1. Create a circle with center A and radius AB (Euclid's Postulate 3).

2. Create a circle with center B and radius BA (Euclid's Postulate 3), with C and D being the intersection points. (Circle-circle intersection property).

3. Create a circle with center D and radius DA (Euclid's Postulate 3), with B and E being the intersection points. (Circle-circle intersection property).

4. Create a circle with center E and radius EA (Euclid's Postulate 3), with D and F being the intersection points. (Circle-circle intersection property).

5. Create a circle with center F and radius FA (Euclid's Postulate 3), with E and G being the intersection points. (Circle-circle intersection property).

6-11. Draw segments GC, CB, BD, DE, EF, and FG by Euclid's Postulate 1.

Proof. We know segments AB, AD, AE, AF, AG, and AC are all congruent since they are all radii of circle A. Circle B is congruent to circle A since they share the same radius. Similarly, circle A is congruent to circles D, E, and F since they all share the same radii by construction. Therefore, all the radii of these circles are congruent to each other. Thus, line segments BC, BD, DE, EF, and FG are all congruent to one other.

It still remains to show line segment GC is congruent to all other segments of the hexagon. Consider triangles ABD, AED, and AEF (refer to figure 2). We know these three triangles are equilateral since they are all composed of radii from congruent circles A, B, D, and E. By Euclid Proposition I.8, these triangles are also all congruent to each other since they have three congruent sides. Since these triangles are equilateral, they are also regular by Theorem 6.1. Thus, all the angles in triangles ABD, AED, and AEF are congruent to each other.

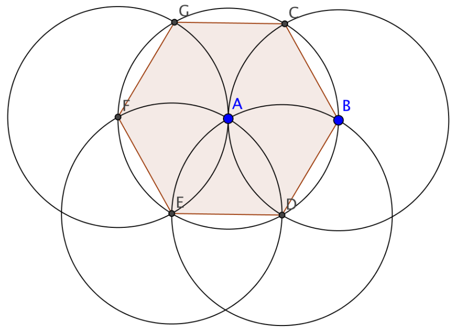

Figure 1: Construction of a Regular Hexagon

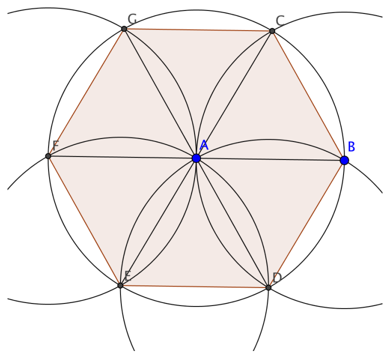

Figure 2: Six Triangles in the Hexagon

We know by Euclid Proposition I.32, a triangle has all three interior angles summing to two right angles. Triangles ABD, AED, and AEF would then add up to a total of six right angles. Since these triangles are regular, all the angles are congruent. Thus, each angle will be 2/3 of a right angle. Consider angles FAE, EAD, and DAB. Summing each angle would result in two right angles. Since angles FAE, EAD, and DAB add up to two right angles, by Euclid Proposition I.14, these angles will lie on the straight line FAB. Again, by Euclid Proposition I.14, angles GAF, GAC, and CAB must add up to two right angles since they lie on line segment FAB. We know angles GAF and CAB are each 2/3 of a right angle since they are regular triangles. Therefore when we take them together they will be 4/3 of a right angle. In order to add up to two right angles to satisfy Euclid Proposition I.14, angle GAC must be 2/3 of a right angle.

By Euclid Proposition I.4, since GA, AC, EA, and DA are all congruent to each other and angle GAC is congruent to angle EAD, triangles AED and GAC are congruent to each other. Thus GC is congruent to ED. Since ED is congruent to all the other sides of the hexagon and congruent to GC, by common notion 1, all of the sides of the hexagon are congruent to one another. Therefore, the hexagon is equilateral.

To prove this equilateral hexagon is regular, we must show the hexagon has all of its angles congruent. In our equilateral hexagon, we have six regular triangles. Each angle of the hexagon is summed by two angles from the regular triangles. For example, angle FGC composes of angles FCA and GCA. Angles FCA and GCA taken together would be 4/3 of a right angle. This argument continues at each of the angles: GCB, CBD, BDE, DEF, and EFG. Hence, all of the angles in the equilateral hexagon are congruent to each other. Thus, since the hexagon is equilateral and has all angles congruent to one another, the hexagon is regular.

$\square$