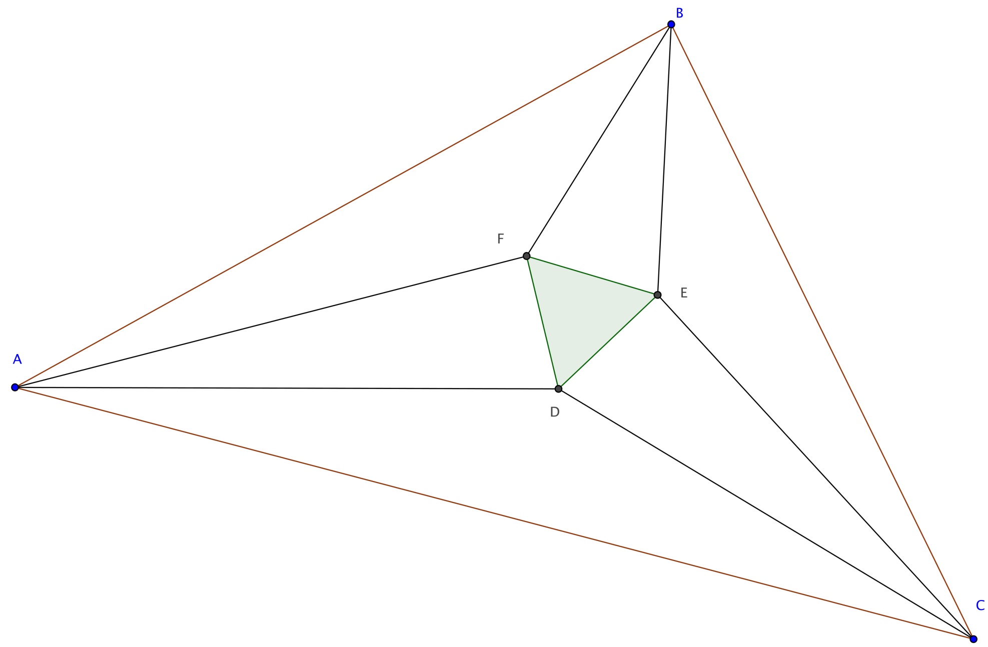
Transactions
in
Euclidean Geometry



Volume 2018F

Issue # 6

Table of Contents

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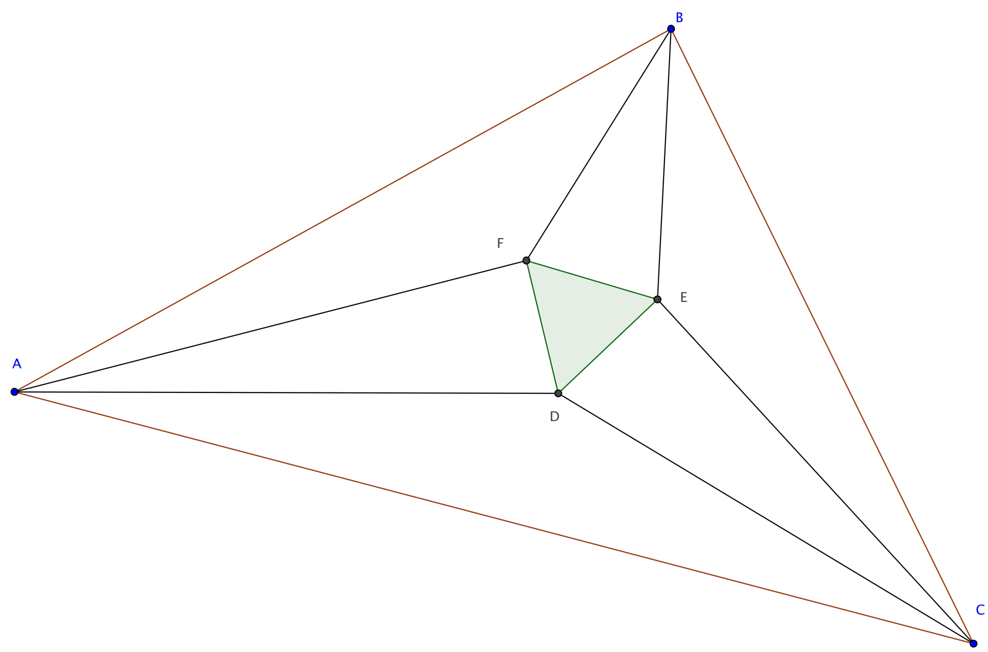

Volume 2018F

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Table of Contents

Pairs of Opposite Sides of a Rectangle are Congruent

Brad Warner

December 7, 2018

Communicated by: Ms. Miller

Definition 1. A rectangle is a quadrilateral which has all four interior angles that are right angles.

Theorem 3.2. Let R be a rectangle. Then each pair of opposite sides of R is a pair of congruent segments.

Proof. Let ADCB be a rectangle. Thus, ADCB has four interior angles that are right angles by definition. Then angle ADC is congruent to angle ABC. By Euclid Postulate I.1, draw line segment AC. By Theorem 3.1, the rectangle is a parallelogram. Thus, AB is parallel to DC by definition. Since AC falls on side AB and side DC then alternate interior angles DCA and BAC are congruent by Euclid Proposition I.29. Since angles ADC and ABC are congruent, angles DCA and BAC are congruent, and line segment AC is common, then triangle ABC is congruent to triangle ADC by Euclid Proposition I.26. Thus corresponding parts of triangles ABC and ADC are congruent, so line segment AD is congruent to line segment BC and line segments AB and DC are congruent. Hence, opposite sides of a rectangle are congruent.

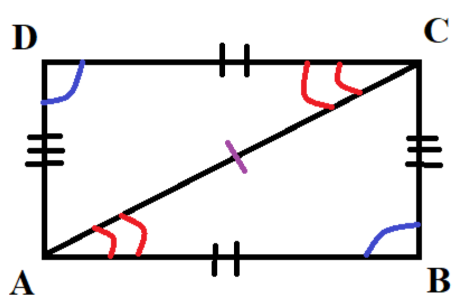

Figure 1: Rectangle ADCB

□

Diagonals of a Rectangle

Payson VandeLune

December 4, 2018

Communicated by: Lexis Wiegmann.

We will begin with the definition of a rectangle and some important characteristics from Theorems 3.1 and 3.2, to be used in the proof of Thereom 3.3.

Definition 1. A rectangle is a quadrilateral which has all four interior angles that are right angles.

Theorem 3.1. Let R be a rectangle. Then R is a parallelogram.

Theorem 3.2. Let R be a rectangle. Then each pair of opposite sides of R is a pair of congruent segments.

Theorem 3.3. The two diagonals of a rectangle are congruent and bisect each other.

Proof. Let ABCD be a rectangle. By Euclid Postulate I.1, draw segments AC and BD, the diagonals of ABCD. Recall that since a rectangle is a convex quadrilateral, the diagonals cross inside the figure. Let the point where AC and BD cross be denoted as E. Notice that AB and CD are parallel by Theorem 3.1. Then by Euclid Proposition I.29, angle BAE is congruent to angle DCE, and angle ABE is congruent to angle CDE. By Theorem 3.2, opposite sides AB and CD are congruent, therefore by Euclid Proposition I.26, angle-side-angle, triangle ABE is congruent to triangle CDE.

Without loss of generality, notice triangle BCE is congruent to triangle ADE. By this congruence, we conclude segment BE is congruent to segment ED and segment AE is congruent to segment EC. Therefore, point E bisects segments AC and BD, so these two segments, the diagonals, bisect each other.

Now notice that angle ABC and angle DAB are both congruent to right angles, by Definition 1. Therefore triangle ABD and triangle BAC are congruent by Euclid Proposition I.4, side-angle-side. Thus, segments AC and BD, the diagonals of ABCD, are congruent. $\square$

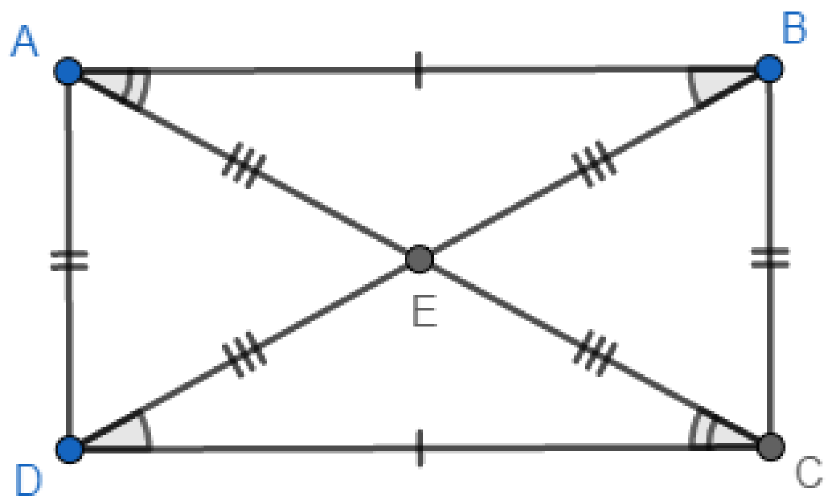

Figure 1: This is a diagram of rectangle ABCD

Neat, It's a Rectangle

Jason Stine

December 7, 2018

Communicated By: Ms. Wiegmann

Theorem 3.4. Let ABCD be a quadrilateral such that angles ABC and ADC are right angles. If side AB is congruent to CD, then ABCD is a rectangle.

Proof. Let ABCD be a quadrilateral such that angles ABC and ADC are right angles and side AB is congruent to CD. Let AC be a diagonal of quadrilateral ABCD. Notice that triangles ABC and CDA are right triangles in which the hypotenuse and leg are congruent. Because of Ms. Wiegmann's Hypotenuse Leg Theorem, triangles ABC and CDA are congruent. Because corresponding parts of congruent triangles are congruent, angle CAD is congruent to ACB, angle ACD congruent to CAB, and side AD congruent to BC. Notice that angles BCA and CAD are congruent, by Euclid Proposition 1.28 lines CD and BA are parallel. Since CD is parallel to BA, by Euclid Proposition 1.33 BC is parallel to AD. Thus quadrilateral ABCD is a parallelogram. By Euclid Proposition 1.34, angles BAD and BCD are congruent. Recall that the four interior angles of a quadrilateral taken together is equal to four right angles. Since ABC and CDA are right angles, the sum of BAD and BCD are two right angles. However, since these two angles are congruent, they must both be right angles. Notice we have a quadrilateral with four right angles, thus ABCD is a rectangle.

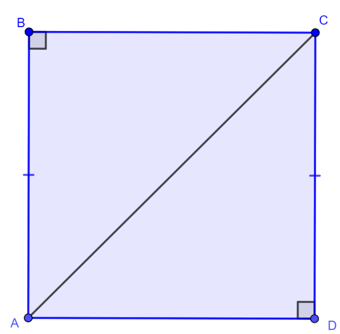

Figure 1: Quadrilateral ABCD

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Equilateral Triangles are Equiangular

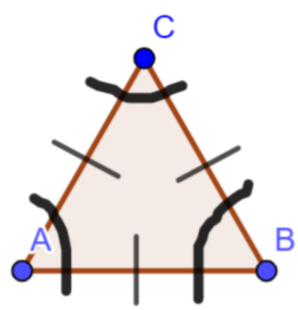
Lauren Falck

December 7, 2018

Communicated by: Mr. Warner

Conjecture 6.1. An equilateral triangle is equiangular, hence regular.

Proof. Since triangle ABC is an equilateral triangle, all of the sides are congruent. By using Euclid I.5, we can look at the congruent sides AB and BC and call triangle ABC an isosceles triangle. Since triangle ABC is an isosceles triangle, base angles A and C are congruent. Similarly, since side AB is congruent to AC, base angles B and C are congruent. Since angles A and C are congruent and angle C is also congruent to angle B, angles A, B, and C are all congruent. Thus, triangle ABC is both equilateral and equiangular, hence regular. □



A Kite Is Not a Parallelogram

Lexis Wiegmann

December 6, 2018

Communicated by: The Editor.

A kite is never a parallelogram. The next part will be solved by way of contradiction.

Theorem F. Let ABCD be a kite. If ABCD is not a rhombus then it is not a parallelogram.

Proof. Let ABCD be a kite, and assume that it is not a rhombus. From the definition of a kite, we know that there are two pairs of adjacent and congruent sides. Without loss of generality, line segments AB and BC are congruent to each other and line segments CD and DA are congruent to each other. Angle A and angle C are congruent from Mr. Warner's Theorem 2.1.

From Ms. Miller's Theorem 2.2, we know that BD bisects angle B and angle D. By Euclid Proposition I.18, we know that line CD is bigger then line BC, so angle CBD has to be larger then CDB, as well as the other side of the kite. Line AD is bigger then line AB, so angle ABD has to be larger then ADB. Since line BD bisects angle B and angle D, angle ABD and CBD are congruent, as well as angle ADB is congruent to angle CDB.

Euclid Proposition I.27 says if a straight line falls on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another. We are going to negate that and say if those angles are not congruent, the lines are not parallel. We know that angle ABD is congruent to CBD. By Euclid Proposition I.18, we know that angle CBD and angle BDA are not congruent. Because of the negation of Euclid Proposition I.27, line AB and line DC are not parallel. Therefore, kite ABCD is not a parallelogram.

□

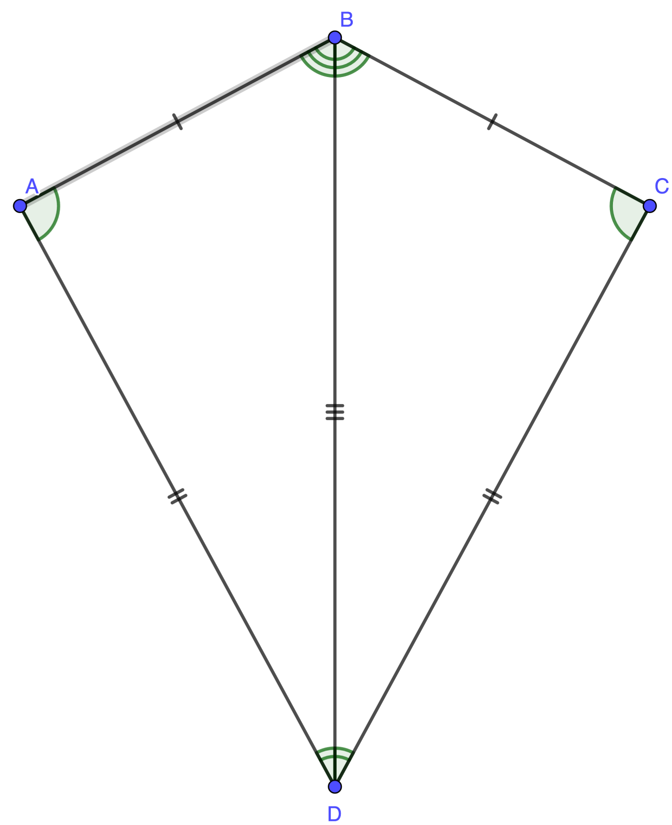


Figure 1: Kite ABCD.

Convex vs. Non-Convex Quadrilaterals

Lexis Wiegmann

December 5, 2018

Communicated by: The Editor.

Definition 1. Let ABCD be a quadrilateral. The *diagonals* of ABCD are the segments AC and BD.

We will formulate a workable definition of the terms convex and non-convex for quadrilaterals.

Definition H.a. A shape would be *convex* if the diagonals of the shape cross.

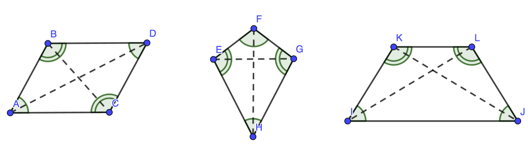


Figure 1: These are all examples of convex quadrilaterals.

Definition H.b. A shape would be *non-convex* if the diagonals of the shape do not cross.

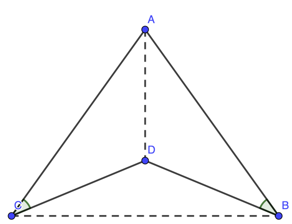


Figure 2: This is a non-convex quadrilateral.

Parallel and Perpendicular Lines

Alexa DeVore

December 7, 2018

Communicated by: Ms. Wiegmann.

Conjecture I. Let line AB be parallel to CD. If BC is perpendicular to AB and line AD is perpendicular to line CD, then BC is parallel to AD.

Proof. Let AB, BC, CD, and AD be lines such that the noted conditions are met. Let point E exist on line AB and point F exist on DA, both past point A. Then from Euclid 1.29 we know that angle DAE is congruent to angle CDA since they are alternate interior angles of parallel lines. We also know from Euclid 1.15 that the vertical angle BAF is congruent to DAE and therefore a right angle. Since angle BAF is congruent to angle CBA, from Euclid 1.27 we know that lines BC and AD must be parallel since those angles are congruent alternate interior angles □

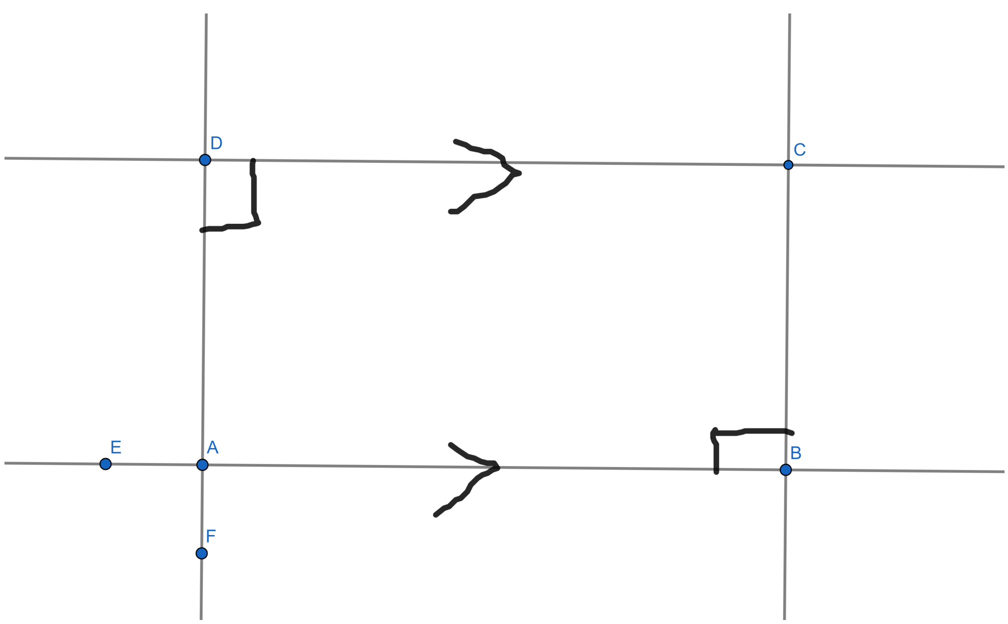


Figure 1: This is a picture of the lines.

Angle Bisector Concurrence

Payson VandeLune

December 5, 2018

Communicated by: Jaclyn Miller.

This paper includes two theorems and a lemma. Theorems 8.1 and 8.2 go hand-in-hand, as 8.1, in conjunction with Lemma 8.2, prove Theorem 8.2.

Theorem 8.1. Let ABC be a triangle, with rays r and s the angle bisectors at A and B, respectively. Suppose that r and s meet at the point I which lies inside the triangle. Draw lines l and m through I that are perpendicular to AC and BC respectively. If l meets AC at point X and m meets BC at Y, then triangle IXC is congruent to triangle IYC.

Proof. Let triangle ABC, rays r and s, lines l and m, and points I, X and Y be as stated in the theorem. We will show triangle IXC is congruent to triangle IYC. Construct a segment perpendicular to side AB that falls on point I. Let this segment be called n. Let the point where n falls on side AB be called Z.

Consider triangles ZBI and YBI. Since shared side BI is the bisector of angle ABC, angles ZBI and YBI are congruent. Since angles IZB and IYB are right angles, triangles ZBI and YBI are congruent by Theorem 7.2 (Hypotenuse Leg Theorem - Wiegmann). Therefore segments ZI and YI are congruent. Similarly, triangles ZAI and XAI are congruent, and segment XI and ZI are congruent. Thus, segments XI and YI are congruent, as they are commonly congruent to segment ZI.

Now consider triangles IXC and IYC. Since angles IYC and IXC are right angles, side IC is shared, and we have concluded that segments XI and YI are congruent, triangle IXC is congruent to triangle IYC, again by Theorem 7.2.

□

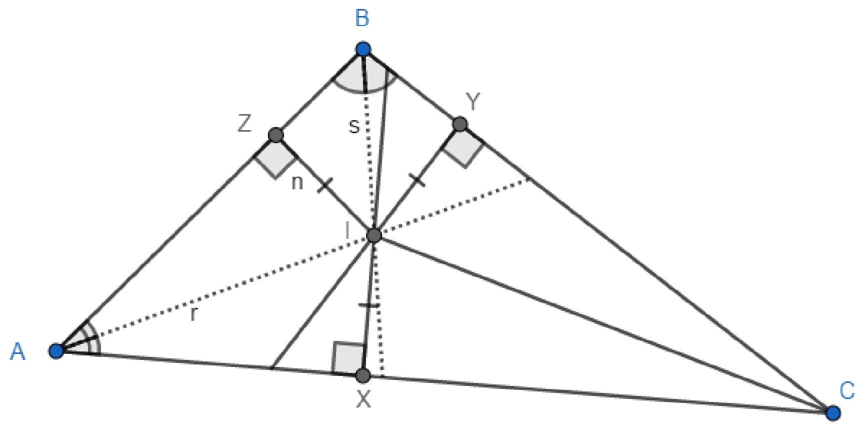


Figure 1: Triangle ABC from Theorem 8.1

Lemma 8.2. For any two angle bisectors in a triangle, those bisectors meet. (That is, they are not parallel.)

Proof. Let ABC be a triangle. Let rays l and m bisect angles ABC and BCA respectively. We will show l is not parallel to m. Extend side BC to some arbitrary points W, past B, and Z, past C. Let point X be somewhere on l not at B and let point Y be somewhere on m not at C. By way of contradiction, suppose l is parallel to m.

Since l and m are parallel, then by Euclid I.29, angle XBC is congruent to angle YCZ. Also notice since l is the bisector of angle ABC, angle XBC is congruent to angle ABX. We will consider the measure of these angles to be α.
Similarly, by Euclid I.29, angle WBX is congruent to angle BCY and since ray m bisects angle BCA, angle BCY is congruent to angle YCA. We will consider the measure of these angles to be β. So we have:

$$\angle XBC \cong \angle YCZ \cong \angle ABX \cong \alpha$$
$$\angle BCY \cong \angle YCA \cong \angle WBX \cong \beta$$

Notice that angles WBX and XBC fall on a straight line, and thus by Euclid I.13, taken together they equal two right angles. Therefore, α and β taken together equal two right angles. Now notice that angles ABC and BCA taken together are equal to twice α and twice β taken together. So we have:

$$\angle ABC + \angle BCA = 2\alpha + 2\beta$$
$$\angle ABC + \angle BCA = \alpha + \alpha + \beta + \beta$$
$$\angle ABC + \angle BCA = (\alpha + \beta) + (\alpha + \beta)$$
$$\angle ABC + \angle BCA = 2 \text{ right angles } +2 \text{ right angles}$$
$$\angle ABC + \angle BCA = 4 \text{ right angles}$$

Here we have a contradiction, as the sum of two of the interior angles of triangle ABC is equal to 4 right angles. Thus the sum of all interior angles of triangle ABC will be greater than 4 right angles. But we know that the sum of the interior angles of a triangle must equal 2 right angles. By this contradiction, we know our supposition cannot be true, and l and m cannot be parallel. Thus for any two angle bisectors in a triangle, they will meet. □

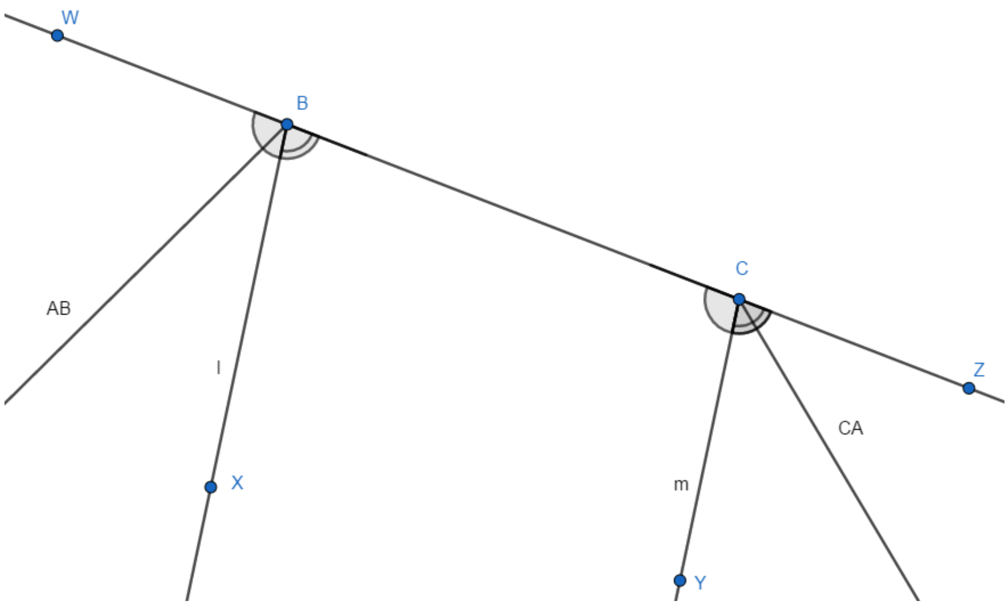


Figure 2: "Triangle" ABC from Lemma 8.2. Note that rays AB and CA "meet" at vertex A. They do not actually meet, as Lemma 8.2 concludes that ABC is not a triangle.

Theorem 8.2. The three angle bisectors of a triangle are concurrent.

Proof. By Theorem 8.1, we know that if we are given two angle bisectors of a triangle that meet at a point, namely, x, the third angle bisector will also pass through x, making the three bisectors concurrent. Given Lemma 8.2, we know that for every triangle, two angle bisectors of that triangle will meet. Thus, we can conclude that the three angle bisectors of any triangle are concurrent.

□

Stine's ASS Theorem

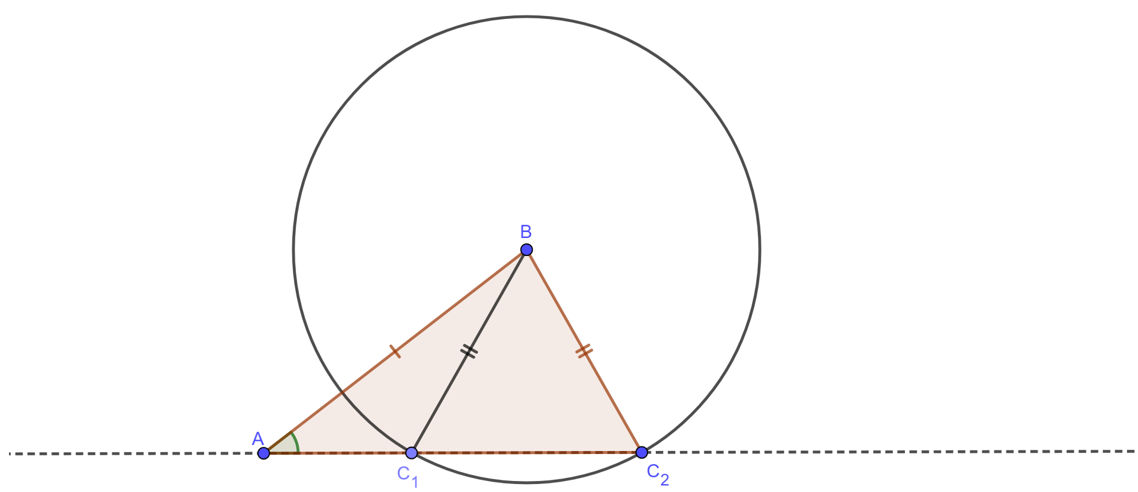
Mr. Stine

December 7, 2018

Communicated by: Ms. Miller

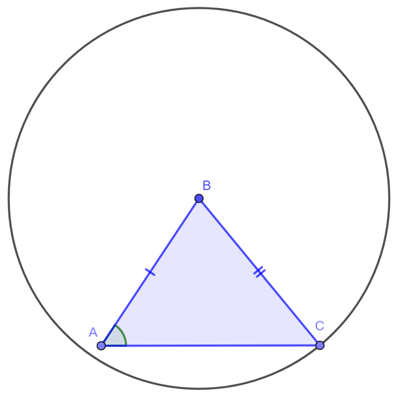
Theorem Stine's ASS A. Given an angle α and two segments s_1, s_2 there are at most two triangles ABC such that angle CAB is congruent to α, AB is congruent to s_1, and BC is congruent to s_2. Moreover, if there are two such triangles, denoted as ABC_1 and ABC_2, then angle AC_1B and angle AC_2B taken together make 2 right angles.

Proof. Let ABC be a triangle, such that angle BAC is α, side B is s_1 and side C is s_2. Let circle B be centered at point B with radius s_2. Let side AB be extended into a line. Because of the circle line intersection theorem, we know that line AB will intersect circle B at only 0, 1, or 2 points. This means that our radius AB only intersects line AC at most at two points, creating two possible angles BCA. Thus the triangle ABC can only be constructed at most in two ways. Let triangle construction 1 be denoted by ABC_1 and construction 2 be denoted by ABC_2.



Since C_1BC_2 is an isosceles triangle, by Euclid Proposition 1.5 we know that angle BC_2C_1 is congruent to angle BC_1C_2. By Euclid Proposition 1.14, taken together, angle AC_1 and angle BC_1C_2 are equal to two right angles. Thus, angle AC_1B and BC_2C_1 taken together sum to two right angles.

Thus, we know that when taken together, angles BC_1A and BC_2A, are equal to the sum of two right angles. □



Theorem Stine's ASS B. Let ABC and DEF be triangles. If angle BAC and EDF are congruent, side AB is congruent to DE, side BC is congruent to EF, and side AB is shorter than side BC, then ABC and DEF are congruent.

Proof. Let ABC and DEF be triangles such that angle BAC and EDF are congruent, side AB is congruent to DE, side BC is congruent to EF, and side AB is shorter than side BC. Let circle B be the circle centered at B with radius BC. Since segment AB is shorter than radius BC, point A lies within the circle. Since point A lies within the circle, segment AC only falls on circle B at one point. Thus, side BC only has one possible position that will construct triangle ABC. Thus, there is only one possible way to construct a triangle with these traits. Thus, any two triangles that share these traits must be congruent. Thus, triangles ABC and DEF are congruent. Notice that in triangles where angle BAC is right or obtuse, side BC will be longer than AB. We know this due to Euclid Proposition 1.18, the longer side will subtend the larger angle. Thus, Theorem Stine's ASS B will always apply to right and obtuse triangles. □

In conclusion, we know that ASS can prove triangle congruence in the following situations: When angle α is a right or obtuse angle. When angle α is acute and s_2 is greater than s_1. In triangles where ASS can not prove triangle congruence, we know that the angles subtending s_1 taken together will be congruent to two right angles.

Construction of an Equilateral Pentagon

Lauren Falck

December 5, 2018

Communicated by: Mr. Vandelune.

In order to prove a few of the conjectures in Task Sequence 6, we need to know that it is possible to construct an equilateral pentagon. I will now prove such a construction is possible.

Falck Theorem Pentagon A. Show how to construct an equilateral pentagon.

Proof. Let there be a line, called L. Place points A and B on line L.

1. By Euclid I.1, draw a circle AB with center A and radius AB.
2. Also by Euclid I.1, draw circle BA with center B and radius AB. The intersections of circles AB and BA create points X and Y.
3. By Postulate 1, create a line through points X and Y. By Euclid I.12, the intersection of line L and line WY creates a perpendicular bisector of line L at a point Z.
4. By Euclid I.31, construct a parallel line to line XY that goes through point A. The intersection of Circle AB and the parallel line creates point W.
5. Construct a circle ZW centered at Z with radius ZW. The intersection of line L and circle ZW creates point V.
6. Construct a circle BV with center B and radius BV. The intersection of circle BV and line XY creates point D.
7. Construct a circle D with radius AB. The intersection of circle D and circle BA creates point C. The intersection of circle D and circle AB creates point E.
8. Connect point A to B, A to C, C to D, D to E, and E to A.

Pentagon Construction.png

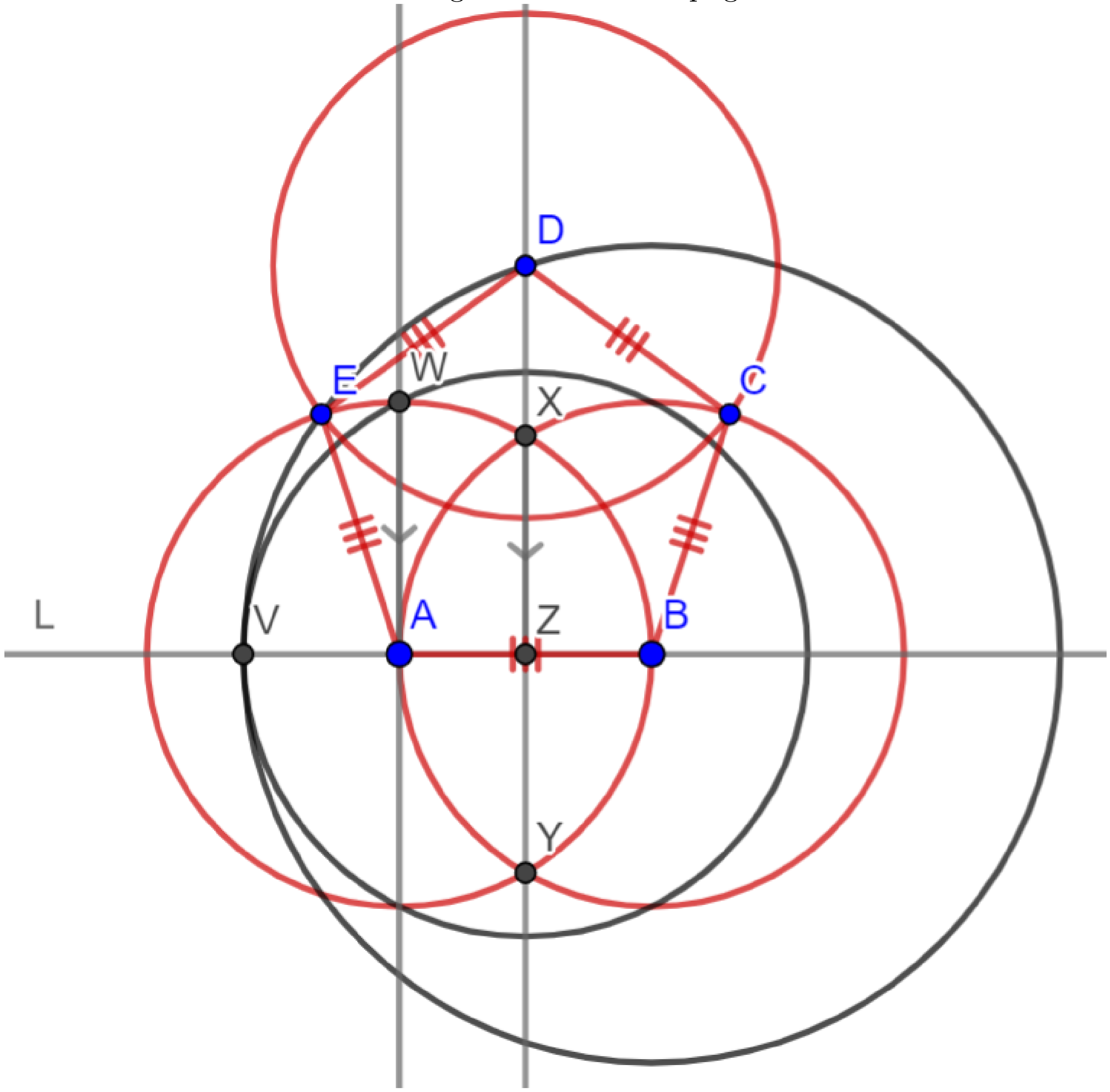


Figure 1: Construction of an equilateral pentagon

Notice the sides are all congruent because they are all radii of circles with a radius AB. Points A and B are chosen as the first side. Then point D is found on the perpendicular bisector of line L. Point E is the intersection of the circle centered at D with radius AB and the circle centered at A with radius AB. Point C is the intersection of the circle centered at D with radius AB and of the circle centered at B with radius AB. Therefore, all of the sides are radii of congruent circles. Thus, this is a construction of an equilateral pentagon.

$\square$

Construction of a Regular Pentagon

Lauren Falck

December 5, 2018

Communicated by: Ms. Miller.

In order to prove a few of the conjectures in Task Sequence 6, we need to know that it is possible to construct a regular pentagon.

Falck Theorem Pentagon B. Recall our construction of an equilateral pentagon from Theorem Falck Pentagon A. It is a unique construction of not only an equilateral pentagon, but it is also the construction for a regular Pentagon.

Pentagon Construction.png

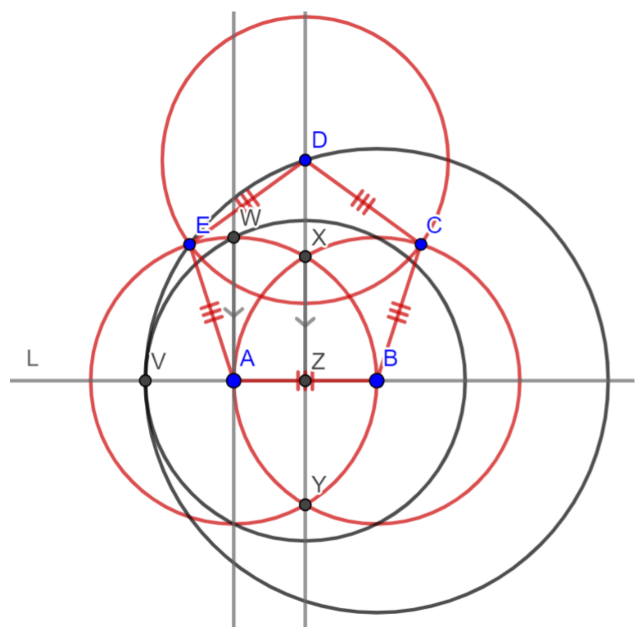


Figure 1: This is a construction of an equilateral pentagon.

Proof. Construct an equilateral pentagon using Falck Theorem Pentagon A. We will show that this pentagon is also an equiangular pentagon, also known as a regular pentagon. By Euclid Postulate I.1, connect all of the nonadjacent vertices of the pentagon.

By Euclid Proposition I.11, a line through the intersections of the circle centered at A with radius AB and the circle centered at B with radius AB is a perpendicular bisector of segment AB. Notice this perpendicular bisector also goes through point D. Call the intersection of the perpendicular bisector and segment AB point Z. Notice segment BE and segment AC both intersect the perpendicular bisector DZ at the same point. Call this intersection point U. Since DZ is the perpendicular bisector of AB, segment AZ is congruent to segment BZ and angle AZD is congruent to angle BZD. By Euclid Proposition I.4, since segment AZ is congruent to segment BZ, angle ADZ is congruent to angle BZD, and segment DZ is shared, triangle ADZ is congruent to triangle BDZ. Since triangle ADZ is congruent to triangle BDZ, segment AD is congruent to BD and angle DAZ is congruent to angle DBZ. By Euclid Proposition I.4, since segment AZ is congruent to segment BZ, and angle AUZ is congruent to angle BZU, and segment UZ is shared, triangle AUZ is congruent to triangle BUZ. Since triangle AUZ is congruent to triangle BUZ, angle UAZ is congruent to angle UBZ. By Euclid's definition of an isosceles triangle, since side AB is congruent to side BC, triangle ABC is an isosceles triangle. By Euclid Proposition I.5, since triangle ABC is an isosceles triangle, angle BAC is congruent to angle ACB. By Euclid's definition of an isosceles triangle, since side AB is congruent to side AE, triangle ABE is an isosceles triangle. By Euclid Proposition I.5, since triangle ABE is an isosceles triangle, angle ABE is congruent to angle AEB.

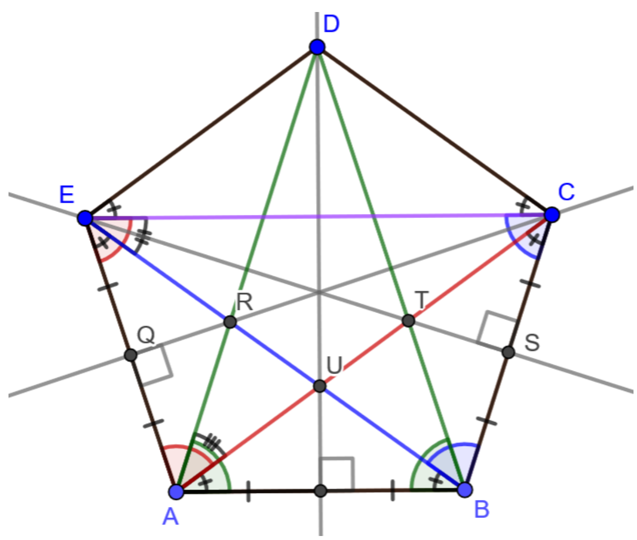
By Euclid Proposition I.11, a line through the intersections of the circle centered at A with radius AB and the circle centered at D with radius AB is a perpendicular bisector of segment BC. Notice this perpendicular bisector also goes through point E. Call the intersection of the perpendicular bisector and segment BC point S. Notice segment BD and segment AC both intersect the perpendicular bisector EU at the same point. Call this intersection point T. Since ES is the perpendicular bisector of BC, segment BS is congruent to segment CS and angle BSE is congruent to angle CSE. By Euclid Proposition I.4, since segment BS is congruent to segment CS, angle BSE is congruent to angle CSE, and segment ES is shared, triangle BES is congruent to triangle CES. Since triangle BES is congruent to triangle CES, segment BE is congruent to CE and angle EBS is congruent to angle ECS. By Euclid Proposition I.4, since segment BS is congruent to segment CS, angle BSE is congruent to angle CSE, and segment ST is shared, triangle BST is congruent to triangle CST. Since triangle BST is congruent to triangle CST, angle SBT is congruent to angle SCT.

By Euclid Proposition I.11, a line through the intersections of the circle centered at B with radius AB and the circle centered at D with radius AB is a perpendicular bisector of segement AE. Notice this perpendicular bisector also goes through point C. Call the intersection of the perpendicular bisector and segment AE point Q. Notice segment CE and segment AC both intersect the perpendicular bisector CQ at the same point. Call this intersection point R. Since CQ is the perpendicular bisector of AE, segment EQ is congruent to segment AQ and angle EQC is congruent to angle AQC. By Euclid Proposition I.4, segment EQ is congruent to segment AQ, angle EQC is congruent to angle AQC, and segment CQ is shared, triangle ECQ is congruent to triangle ACQ. Since triangle ECQ is congruent to triangle ACQ, segment CE is congruent to AC and angle CEQ is congruent to angle CAQ.

By Euclid Common Notion I.1, Since segment BE is congruent to segment CE and segment CE is congruent to segment AC, segments BE, CE, and AC are all congruent. By Euclid Proposition I.8, since segments AE, AB, BC, CD, and DA are congruent and segments BE, CE, and AC are congruent, triangles ABE, ABC, and CDE are congruent. Since triangles ABE, ABC, and CDE are congruent, angles BAE, ABC, and CDE are congruent. Since triangle CDE is congruent to triangle ABC, their base angles must be congruent, so angle DCE and angle CED are both congruent to angle BAC.

Since angle CEA is congruent to angle CAE and angle BEA is congruent to angle DAE, angle BEC is congruent to angle CAD. Since angles BAC, DAE, AEB, and CED are congruent and angles CAD and BEC are congruent, angle BAE is congruent to angle AED. By Euclid Proposition I.4, since segments AB, AE, and DE are congruent and angles BAE and AED are congruent, triangle ABE is congruent to triangle ADE. Since triangle ABE is congruent to triangle ADE, Segment BE is congruent to segment AD.

By Euclid Proposition I.8, Since segment AD is congruent to segment BD and segments AE, DE, BC, and CD are congruent, triangle AED is congruent to triangle BCD. Since triangle AED is congruent to triangle BCD, angle AED is congruent to angle BCD. Therefore, all of the angles of this pentagon are congruent. Thus this construction of an equilateral pentagon is actually the construction of a regular pentagon.



Pentagon.jpg

Figure 2: This is a picture of the equilateral pentagon proving that it is also a regular pentagon

□

Angles Within a Circle

Jaclyn Miller

December 5, 2018

Communicated by: Mr. VandeLune.

Theorem 10.1. Let τ be a circle with center O. Let X be a point in the interior of the circle, and suppose that two lines l and m intersect at X so that l meets τ at points A and A' and m meets at B and B'. Then twice angle AXB is congruent to angle AOB and angle A'OB' taken together.

Proof. Let τ be a circle with center O. Let X be a point in the interior of the circle, and suppose that two lines l and m intersect at X so that l meets τ at points A and A' and m meets at B and B'.

Angle A'AB' and angle A'OB' both stand on arc A'B'. Since angle A'OB' sits at the center of the circle, and angle A'AB' sits at the circumference, angle A'OB' is the same as twice the angle A'AB' by Euclid Proposition III.20.

Similarly, angle AB'B and angle AOB both stand on arc AB. Since angle AB'B sits at the circumference of the circle, and angle AOB sits at the center of the circle, angle AOB is the same as twice the angle AB'B by Euclid Proposition III.20.

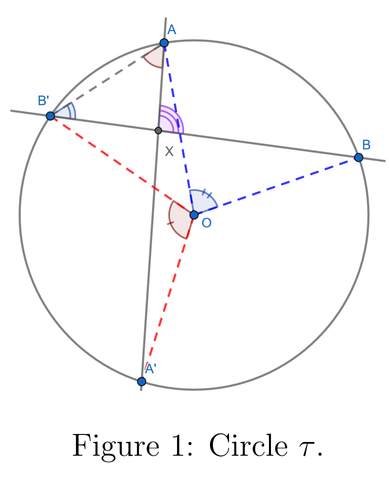


Figure 1: Circle τ.

Looking at triangle B'AX, angles A'AB' and AB'B taken together are congruent to exterior angle AXB by Euclid Proposition I.32. Then, twice angle AXB is congruent to twice the angle AB'B and twice the angle A'AB' taken together. Recall that twice angle AB'B is congruent to angle AOB, and twice angle A'AB' is congruent to angle A'OB'. Then, twice the angle AXB is congruent to angle AOB and angle A'OB' taken together.

$\square$

Proving HL using Pythagoras

Mr. Stine

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Communicated by: Ms. DeVore

Conjecture T. Let STUV and WXYZ be squares. If STUV and WXYZ have equal content, then sides ST and WX are congruent.

Theorem 13.4. Let ABC and DEF be triangles. If side AC is congruent to DF, side BC congruent to EF, angle BAC and EDF be right, and conjecture T is true, then ABC and DEF are congruent.

Proof. Let ABC and DEF be triangles such that angle BAC and angle EDF are right angles, side BC and EF congruent, AC and DF congruent. Construct square a such that it has side AB as it's side length. Similarly construct square b with side BC, square c with side CA, square d with side DE, square e with side EF, and square f with side FD.
By Euclid Proposition 1.47 we know that the sum of the content a and c is equal to the content of b. Similarly we know that the sum of the content of d and f is equal to the content of e. Squares c and f have equal content and squares b and e have equal content. We know this because of Euclid Proposition 1.36, we have parallelograms which are on equal bases and in the same parallels. Since the content of b and e are equal and the content of c and f are equal, by Euclid Common Notion 3, the content of squares a and d are equal. Since squares a and d are equal, by Theorem T, the side lengths of squares a and d are equal. Since squares a and d have equal side lengths, side AB and DE are congruent. Thus triangles ABC and DEF have three corresponding congruent sides. By Euclid Proposition 1.8, triangles ABC and DEF are congruent.

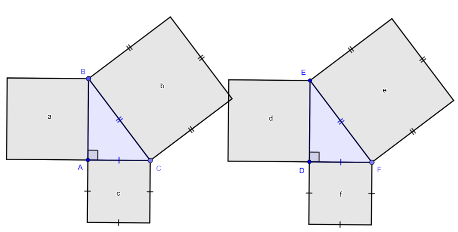

Figure 1: Triangles ABC and DEF

□

Squaring a Rectangle

Payson VandeLune

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Communicated by: Jason Stine.

This is a 7-step construction followed by the proof that it solves the problem.

Theorem 13.8. Given a rectangle, construct a square of equal content.

Consider rectangle ABCD.

Steps 1-2) Extend segments AB, CB each through point B to be rays.

Step 3) Construct a circle centered at B with radius BC. The intersection of this circle and ray AB will be called point E.

Steps 4-6) Using construction 11.2, find the midpoint of segment AE, named M.

Step 7) Construct a circle centered at M with radius ME. The intersection of this circle and ray CB will be called point F.

The square on segment BF has equal content to rectangle ABCD.

Proof. Let ABCD, E, M, and F be as constructed above. We will show the square on segment BF has equal content to rectangle ABCD.

Notice triangle MFB is a right triangle, by construction. Then by Euclid I.47, we know the square on MF has equal content to the squares on FB and BM taken together. This can be written:

$$MF^2 = FB^2 + BM^2$$
$$MF^2 - BM^2 = FB^2$$
$$MF^2 - MB^2 = BF^2$$

But notice also that segments MF and ME are congruent, as they are both radii of circle ME. Thus:

$$ME^2 - MB^2 = BF^2$$

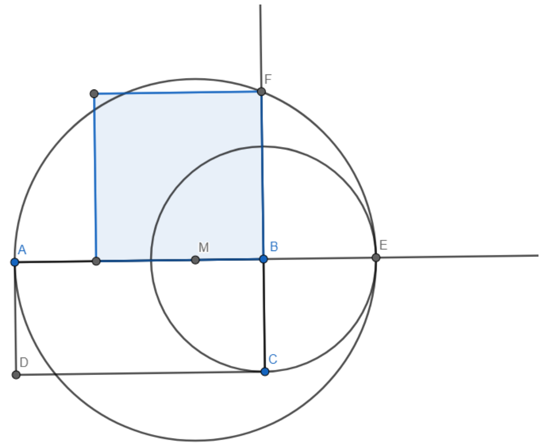

Figure 1: The construction of the square on BF, given rectangle ABCD.

Recall by Euclid 2.5 that rectangle ABCD, together with the square on MB, is equal to the square on ME. Thus:

$$ABCD = ME^2 - MB^2$$

But now we know that:

$$BF^2 = ME^2 - MB^2$$

So:

$$ABCD = ME^2 - MB^2 = BF^2$$

And therefore, the square on BF has equal content to rectangle ABCD.

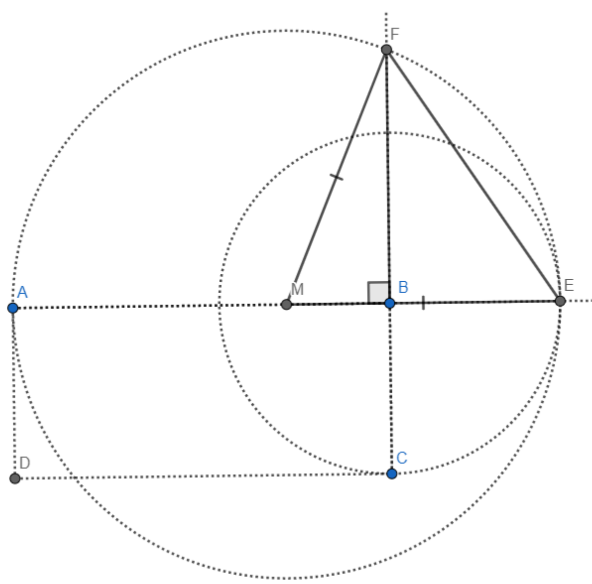

Figure 2: Triangle MFE emphasized within the construction.

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