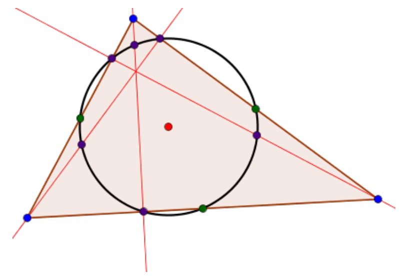
Euclidean Geometry:
An Introduction to Mathematical Work

Math 3600

Fall 2016



Circles, Coming 'Round Again

One of the most useful results about circles is Proposition III.20 which relates an *inscribed* angle in a circle to a *central* angle in that circle. Let us try to see what happens when the angle does not sit on the circumference of the circle.

10.1 Conjecture. Let Γ be a circle with center O. Let X be a point in the interior of the circle, and suppose that two lines ℓ and m intersect at X so that ℓ meets Γ at points A and A' and m meets Γ at B and B'. Then twice angle AXB is congruent to angle AOB and angle $A'OB'$ taken together.

10.2 Question. Consider the situation from the last conjecture, but instead assume that X lies outside Γ. What happens here? Formulate a conjecture.

10.3 Conjecture. If two chords of a circle subtend different acute angles at points of a circle, then the smaller angle belongs to the shorter chord.

10.4 Conjecture. If a triangle has two different angles, then the smaller angle has the longer angle bisector (measured from the vertex to the opposite side).

10.5 Conjecture (Steiner-Lehmus)**.** If a triangle has two angle bisectors which are congruent (measured from the vertex to the opposite side), then the triangle is isosceles.

10.6 Conjecture. Let BC be a chord of circle $\mathscr{C}$, let $\widehat{BC}$ be the arc of $\mathscr{C}$ which is bounded by B and C and does not contain the center of $\mathscr{C}$. Let M be the midpoint of $\widehat{BC}$. For a point A on the arc $\widehat{BC}$, show that as A moves along the arc from B to M, the sums $AB + AC$ increase.

The next theorem is very pretty, and is commonly attributed to Archimedes.

10.7 Conjecture (Archimedes' Theorem of the Broken Chord)**.** Let AB and BC be two chords of a circle $\mathscr{C}$, where BC is greater than AB. (Such a configuration is sometimes called a "broken chord.") Let M be the midpoint of arc ABC and F the foot of the perpendicular from M to chord BC. Then F is the midpoint of the broken chord, that is, AB and BF taken together are congruent to FC.