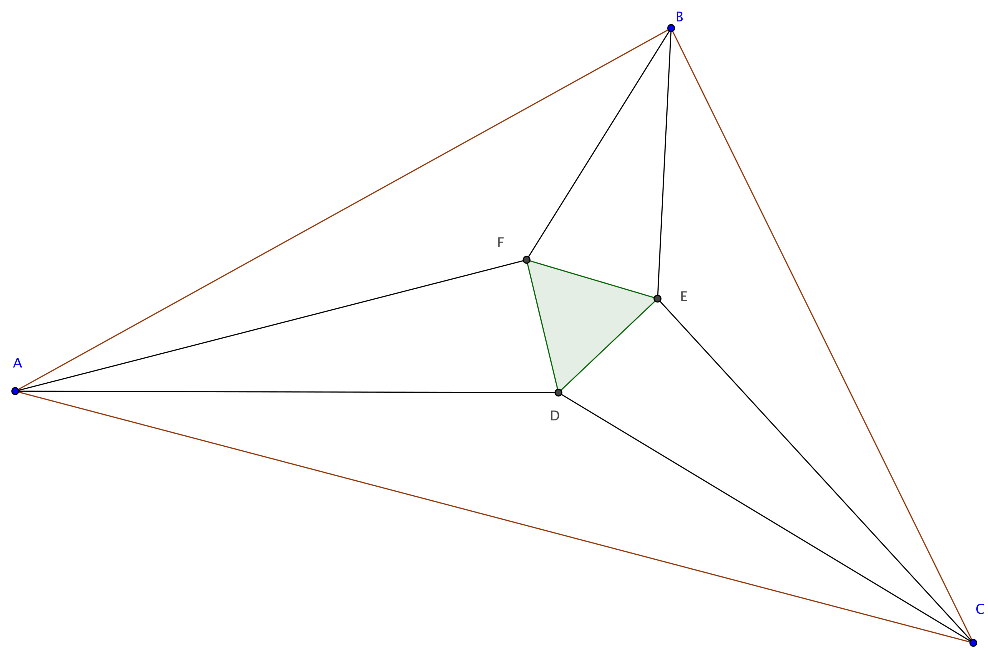
Transactions
in
Euclidean Geometry



Issue # 7

DIAGONALS OF A RECTANGLE ARE CONGRUENT AND BISECT EACH OTHER

JOSHUA HAWKINS

Communicated by Thomas Bieber

Theorem 3.3. If ABCD is a rectangle and the diagonals cross, then diagonals AC and BD are congruent and bisect each other.

Proof. Let ABCD be a rectangle. Draw lines AC and BD. Then by definition of a rectangle, angle DAB and angle CDA are right angles. Since angle DAC is a part of angle DAB, then angle DAC is less than a right angle. Similarly, angle ADB is less than a right angle. So, angle DAC and angle ADB together are less than two right angles. Then by postulate 5 line AC must intersect line DB at apoint. Label their intersection X.

Next, by Theorem 3.2, line AD is coungruent to line BC. Also, by definition of a rectangle angle ADC and angle BCD are right angles. Since line AD is coungruent to line BC, angle ADC is congruent to angle BCD, and line DC is congruent to itself, then by proposition 4 triangle ADC is congruent to triangle BCD. Since triangle ADC is congruent to triangle BCD, then line AC is congruent to line BD.

Also, by Theorem 3.2 line AB is congruent to line DC. Since line AD is congruent to line BC, line AB is congruent to line DC, and angle ADC is congruent to angle ABC since these are both right angles, then triangle ADC is congruent to triangle CBA by proposition 4. Similarly, triangle DAB is congruent to BCD. Since triangle DAB is congruent to BCD, then angle DAX is congruent to angle BCX and angle ADX is congruent to CBX. Since angle DAX is congruent to angle BCX, angle ADX is congruent to angle CBX, and line AD is congruent to line BC, so triangle AXD is congruent to triangle CXB by proposition 26. From this we can see that line DX is congruent to line XB and line AX is congruent to line XC. Since the two parts of line DB, namely line DX and line XB, are congruent, then line DB has has been bisected at point X. Similarly line AC has been bisected at point X.

Therefore line AC and line BD are congruent and have been bisected at X by eacother.

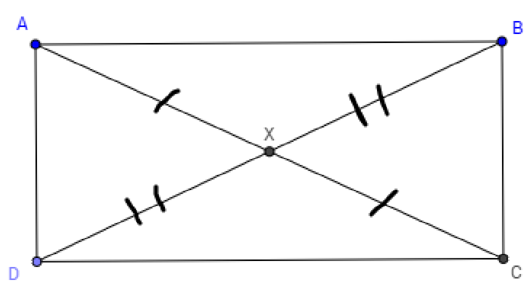


FIGURE 1. A rectangle showing the bisected diagonals

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A COMPLEX QUADRILATERAL WITH OPPOSITE SIDES CONGRUENT AND OPPOSITE ANGLES RIGHT IS NOT A RECTANGLE

JOSHUA HAWKINS

Communicated by Emily Herbst

Theorem 3.4b. Let ABCD be a quadrilateral such that angles ABC and ADC are right angles and segments AB and CD are congruent. If ABCD is complex, then ABCD is not a rectangle.

Proof. Let ABCD be a complex quadrilateral such that angles ABC and ADC are right angles and segments AB and CD are congruent. Since ABCD is complex then a side intersects another side at at least one point. Label that point E. This intersection is either between segments AB and CD or segments AD and CB.

Case 1: Segments AB and CD cross
By theorem 32 the angles of a triangle add up to two right angles. Since EBC is a right angle, then the other two angles are equal to one right angle. Since the whole is greater than the parts, angle ECB is less than one right angle. Since not all the angles of the quadrilateral are right, the figure is not a rectangle.

Case 2: Segments AD and CB cross
Similarly ECD is less than a right angle, so ABCD is not a rectangle.

Since in both cases the quadrilateral is not a rectangle, no complex quadrilateral with angles ABC and ADC being right angles and segments AB and CD being congruent can be a rectangle.

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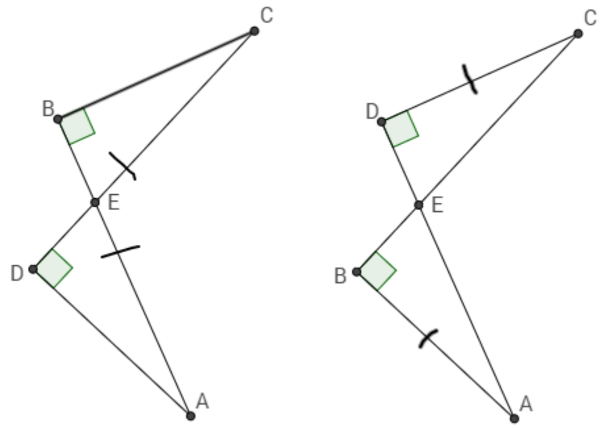

FIGURE 1. Quadrilaterals ABCD with segments AB and CD congruent and angles ABC and ADC as right angles. Two cases with different intersecting sides.

HYPOTENUSE LEG THEOREM

ERIC SCHEIDECKER

Communicated by Emily Herbst

Conjecture 7.2. Let △ABC and △DEF be right triangles, with the angles at A and D right angles. Suppose that $\overline{AB}\cong\overline{DE}$ and $\overline{BC}\cong\overline{EF}$. Then △ABC≅△DEF.

Proof. Let △ABC and △DEF be triangles such that the angles at A and D are right angles, $\overline{AB}\cong\overline{DE}$ and $\overline{BC}\cong\overline{EF}$.

Draw circles ⊙AB, ⊙DE, ⊙CB, and ⊙FE.

By the Circle Intersection Property, ⊙AB intersects ⊙CB at some point that is not B. Let that point be called X. Similarly, ⊙DE intersects ⊙EF at a point we will call Y.

Draw line segments $\overline{AX}$, $\overline{CX}$, $\overline{DY}$, and $\overline{FY}$.

Since X is the intersection of circles centered at A and C, both passing through B, then $\overline{AB}\cong\overline{AX}$, $\overline{BC}\cong\overline{CX}$, and clearly $\overline{AC}\cong\overline{AC}$. Since the sides of △ABC are congruent to the sides of △ACX, △ABC≅△ACX. Similarly, △DEF≅△DFY.

By congruent parts of congruent triangles, ∠CAX≅∠BAC and ∠FDY≅∠EDF. By Theorem D, A, B, and X are collinear and E, D, and Y are collinear. Thus A lies on $\overline{BX}$ and D lies on $\overline{EY}$.

Since $\overline{AB}$ and $\overline{AX}$ are radii of ⊙AB and A, B, and X are collinear, A is the midpoint of $\overline{AX}$. Similarly, D is the midpoint of $\overline{EY}$.

Consider △BCX and △EFY. Since $\overline{AB}\cong\overline{AX}\cong\overline{DE}\cong\overline{DY}$, $\overline{BX}=\overline{AB}+\overline{AX}$, and $\overline{EY}=\overline{DE}+\overline{AX}$, it follows that $\overline{BX}\cong\overline{EY}$.

Since $\overline{BX}\cong\overline{EY}$ and $\overline{CX}\cong\overline{BC}\cong\overline{EF}\cong\overline{FY}$, it follows that △BCX≅△EFY.

Since △BCX≅△EFY, ∠ABC≅∠DEF. Since $\overline{AB}\cong\overline{DE}$ and $\overline{BC}\cong\overline{EF}$ and ∠ABC≅∠DE, by Euclid's Proposition I.4, $\overline{AC}\cong\overline{DF}$.

Therefore △ABC≅△DEF. □

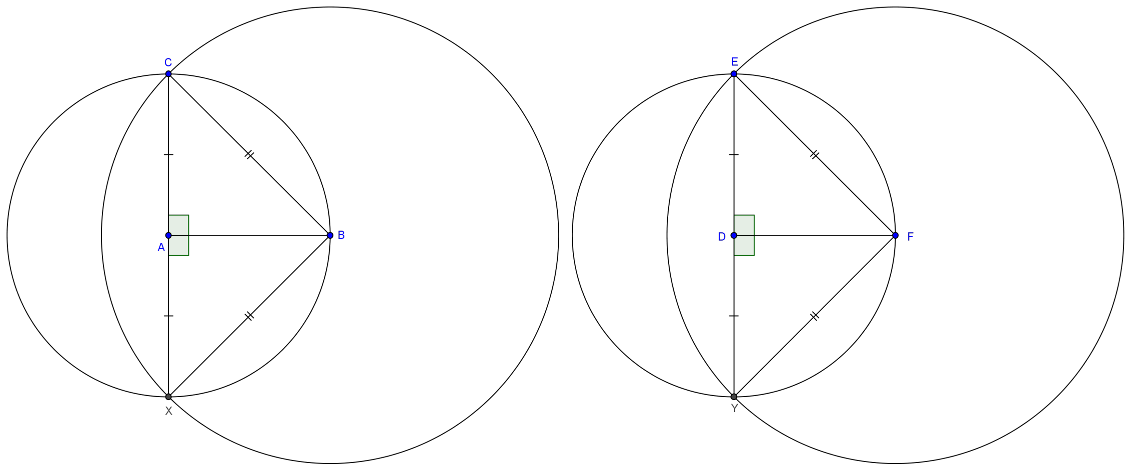

FIGURE 1. Triangles △ABC and △DEF.

CONGRUENCY BETWEEN TANGENT LINES

KEVIN WALTERS

Communicated by Emily Herbst

Theorem 9.1. Let AB and AC be two tangent lines from a point A outside the circle. Then AB is congruent to AC.

Proof. Using proposition 3.1, we will find the center of the circle and call this point D. By proposition 1.18 we can draw a line from D to B which makes angle DBA right. Similarly we can draw DC making angle DCA right. Draw DA. Since DC and DB are both the radius of the circle, we can say DB is congruent to DC and DA is congruent DA. Using hypotenuse leg theorem we can now say triangle ABD is congruent to triangle ACD and therefor AB is congruent to AC. □

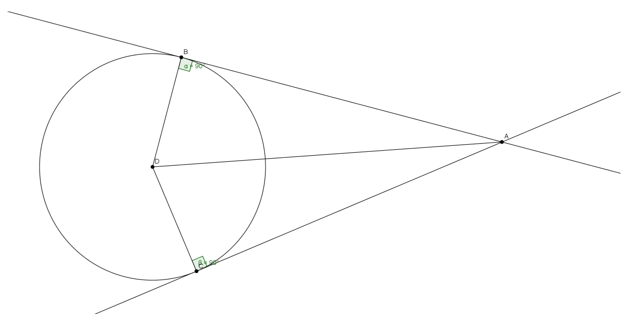

Figure 1

LINES INTERSECTING IN A CIRCLE AND THEIR ANGLES

JOSHUA HAWKINS

Communicated by Eric Scheidecker

Theorem 10.1. Let L be a circle with center O. Let X be a point in the interior of the circle, and suppose that two lines l and m intersect at X so that l meets L at points A and A' and m meets L at B and B'. Then twice angle AXB is congruent to angle AOB and angle A'O'B' taken together.

Proof. Let L be a circle with center O. Let X be a point in the interior of the circle, and suppose that two lines l and m intersect at X so that l meets L at points A and A' and m meets L at B and B'. let segments AB and A'B' be drawn. Then by proposition III.21, angle AOB is equal to twice angle AB'B and angle A'OB' is equal to twice angle B'AA'. Then by proposition I.32 a triangle's angles add up to two right angles, so angles B'AA', AB'B, and B'XA add up to two right angles. So two angles of B'AA', AB'B, and B'XA also add up to four right angles. Earlier is was shown that two angles of AB'B are equal to AOB and two angles of B'AA' are equal to two A'OB', so angle AOB, angle A'OB', and two angles of B'XA equal four right angles. By proposition I.13 angles B'XA and B'XA' together equal two right angles. So two angles of B'XA and two angles of B'XA' together equal four right angles. Which can be rewritten as two angles of AXB' are equal to four right angles minus 2 angles of AXB. Then angles A'OB' and AOB together with four right angles and minus two angles of AXB are equal to four right angles. The four right angles can be subtracted from both and two angles of AXB added to both. Finally, we have angles AOB and A'OB' together equal two angles of AXB.

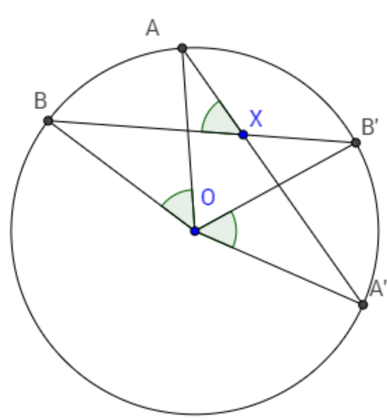

Figure 1. Angles AOB and A'OB' together are equal to twice angle AXB.

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ANGLE AND CHORD RELATIONSHIPS

JOSHUA HAWKINS

Communicated by John Fisher

Theorem 10.3. If two chords of a circle subtend different acute angles at points of a circle, then the smaller angle belongs to the shorter chord.

Proof. Let angles ABC and DEF be different acute angles of the circle where A,B,C,D,E, and F are on the circumference of the circle with center O.

Case 1: Angles are known.
Let angle ABC be less than angle DEF. Let segments AO, CO, DO, and FO be drawn. By propostion III.20 since angles ABC and AOC share the same circumference as a base angle AOC is twice angle ABC. Similarly angle DOF is twice angle DEF. Since each angle, AOC and DOF, is a double of ABC and DEF respectively, then AOC is less than DOF. Since segments AO, CO, DO, and FO are all radii, then they are all congruent. By proposition I.24 since segments AO and CO are congruent to segments DO and EO but angle AOC is less than angle DOF, then segment AC is less than DF. So, the shorter chord belongs to the smaller angle.

Case 2: Chords are known.
Similarly we let segment AC be less than segment DF and conclude ABC is less than DEF. So, the smaller angle belongs to the shorter side.

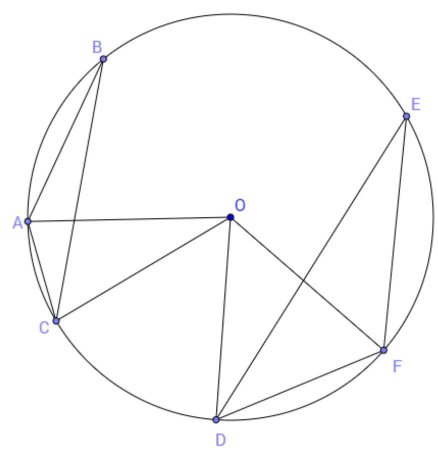


FIGURE 1. Angle ABC is smaller than angle DEF and segment AC is shorter than the segment DF.

□

STRAIGHTEDGE AND COMPASS: PARALLEL LINE

MATT GRIFFEN

Communicated by Emily Herbst

Theorem 10.6. Given a straight line L and a point not on it (A), it is possible to construct a straight line through A that is parallel to L using only a straight edge and compass.

Construction:

1. Draw a circle centered on A such that it cuts line L. Then label an intersection of line L and circle A as point B.

2. Draw a circle with the radius of circle AB centered at a point C on Line L such that it intersects line L at Point B. Label the intersection of circle AB and circle CB as point D.

3. Draw a line M through points A and D.

Line M goes through point A and is parallel to Line L

Proof. Let segments BA and CD be drawn. Since segments AD, AB, BC, and CD are mutually congruent because they are all radii of congruent circles, ABCD is a rhombus. By Goodmundson 1.6 CB is parallel to DA. Therefore L is parallel to M.

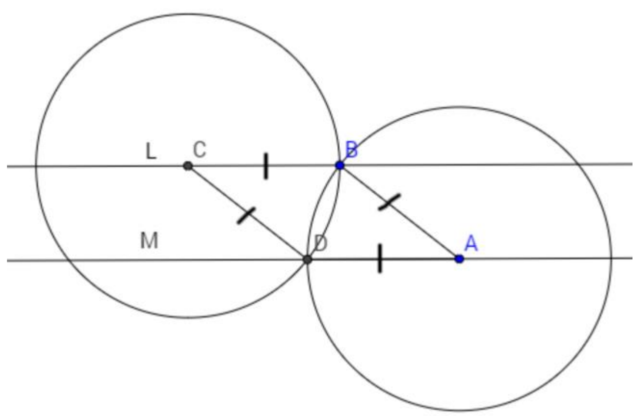


Figure 1 □

CONSTRUCTION OF A PERPENDICULAR LINE THROUGH A POINT

DIANN HERINGTON

Reviewed by Ashley Stuffelbeam

Theorem 11.3. Given a line l and a point A not lying on the line l, a perpendicular line to l and through A can be constructed.

Proof. Given the line l and point A not on the line, construct circle b with radius BA. Construct circle a with radius AB. The point of intersection with line l and circle a is point C. Then, construct cirlcle c with radius CA. Label the intersection of circles b and c point D. Construct the line k through points A and D.
Since AB and BD are radii of circle b, they are congruent. Similarly, since CA and CD are radii of circle c, they are congruent. Because AD is congruent to itself, triangle ABD and triangle ACD are congruent by Euclid's Proposition I.8. So, by congruent parts of congruent triangles, angle BAD is congruent to angle CAD.
Triangle BAX is congruent to triangle CAX by Euclid's Proposition I.4. So, angle BXA is congruent to angle CXA. By Euclid's definition of right angles, angle BXA and angle CXA are right angles, and therefore, line k through A is perpendicular to line l. $\square$

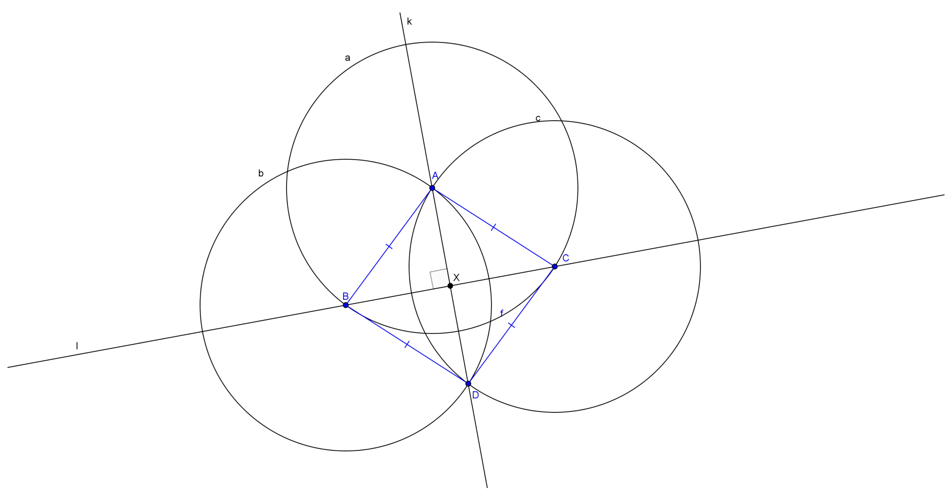


FIGURE 1. Line k perpendicular to line l

CONSTRUCTION FOR A PARALLEL LINE

KEVIN WALTERS

Communicated by Thomas Bieber

Theorem 11.6. Given a line l and a point A not lying on l, construct a line parallel to l which passes through A.

Proof. Given a line l and a point A not lying on l.

Pick points B and C on the line.

(1) Draw a circle centered at B through A.

(2) Draw a circle centered at C through A.

label the intersection of the circles E.

(3) Draw line AE.

Label the intersection of the line l and line AE, D.

(4) Draw a circle centered at A through D.

Label the intersections of circle A and line AE, F.

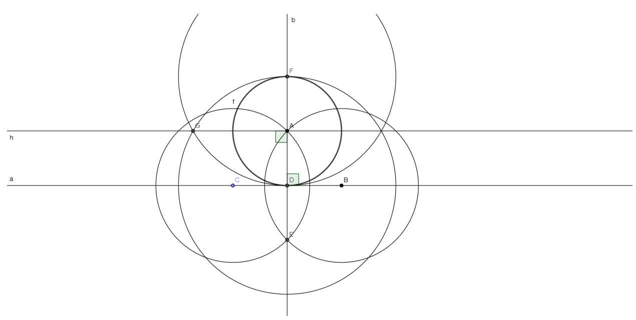
(5) Draw a circle centered at D through F.

(6) Draw a circle centered at F through D.

Label the intersection of the two circles G.

(7) Draw line GA.

We know line AE is perpendicular to line l by construction 11.3 and also that line GA is perpendicular to line AE. This means that angle GAD is a right angle and angle BDA is right. This means that angle GAD and angle BDA are congruent. So by proposition 1.27, line GA is parallel to line l. □



CONSTRUCTION OF AN ANGLE ON A RAY

JOSHUA HAWKINS

Communicated by Thomas Bieber

Challenge 11.5. Given an angle at a point A and given a ray emanating from a point B, construct an angle at B congruent to the angle at A having the given ray as a side. (par 4)

Proof. On angle A pick points C and D one on each leg of the angle.
1. Draw circle BX with radius AC.
Label the intersection of ray B and circle BX E.
2. Draw circle EY with radius CD.
3. Draw circle BZ with radius AD.
Label the intersection of circles EY and BZ F.
4. Draw ray BF.

Since segment BE is a radius of circle BX, segment BE is congruent to segment AC. Similarly segments BF and AD are congruent and segments FE and CD are congruent. Since segments BE and AC are congruent, segments BF and AD are congruent, and segments FE and CD are congruent, then triangles ADC and BFE are congruent by propostion I.8. By congruent parts of congruent triangles, angles DAC and FBE are congruent. So, an angle congruent to the given angle was constructed on the given ray.

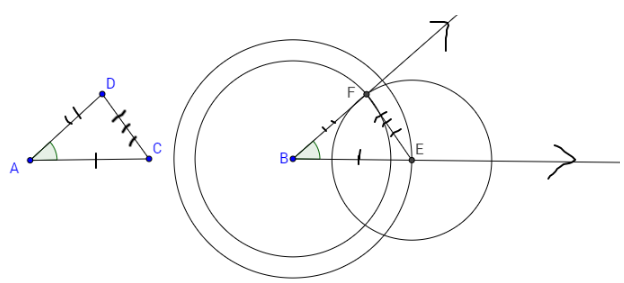


FIGURE 1. Construction based on the above process

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CONSTRUCTING A CIRCUMCIRCLE ABOUT TRIANGLE ABC IN 7 STEPS

JOHN FISHER

Communicated by Jalen Raymond

Theorem 12.2. Construct a circle circumscribed about a given triangle, ABC. (Par 7)

Proof. Given triangle ABC, draw a circle centered at A with radius AB. Next draw a circle centered at B with radius BA. Label the intersections of these two circles points Y and Z. Draw YZ, which will be the perpendicular bisector of side AB of triangle ABC. Next, draw a circle centered at A with radius AC, and then draw a circle centered at C with radius CA. Label the intersections of these two circles W and X. Draw Wx, which will be the perpendicular bisector of side AC of triangle ABC. Label the intersection of YZ and WX, point O. Lastly, draw a circle with center O, and radius C (or any vertice).

By Theorem 8.4, we know that the three perpendicular bisectors of any triangle are concurrent, and meet at the circumcenter of the triangle. Thus O is the circumcenter of the triangle, and it is not required to draw the last perpendicular bisector. By definition, a circle with center O through any vertice will circumscribe triangle ABC.

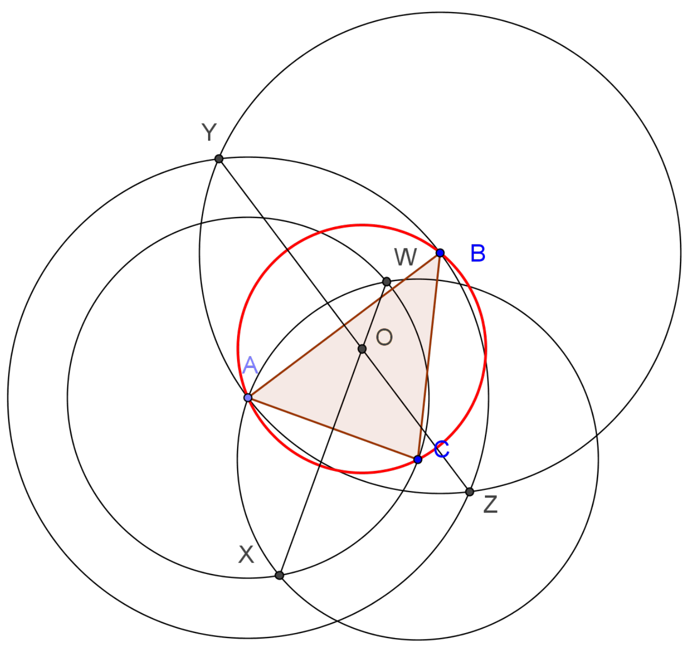

FIGURE 1. Notice circle with center O, radius OC (outlined in red) circumscribes triangle ABC

∎

THREE CIRCLES SUCH THAT EACH PAIR MEETS AT RIGHT ANGLES

KATY GOODMUNDSON

Communicated by Joshua Hawkins

Theorem 12.4. Construct three circles such that each pair meets at right angles.

Proof. (1) Draw line AB

(2) Draw circle AB. Label point C as the intersection of circle AB and line AB.
(3) Draw circle CB
(4) Draw circle BC. Label intersection of circle CB and BC, point D.
(5) Draw ray DA. Label intersection of circle AB and ray DA, point E.
(6) Draw circle BA. Label intersection of line AB and circle BA, point G.
(7) Draw circle GA.
(8) Draw circle AG. Label intersection of circle GA and circle AG, point H.
(9) Draw ray HB. Label intersection of circle BA and ray HB, point J.
(10) Connect segment EJ.
(11) Draw circle JE. Label intersection of circle JE and ray HB, point K.
(12) Draw circle CA.
(13) Draw circle EA. Label intersection of circle EA and circle CA point L.
(14) Draw circle KJ. Label intersection of circle KJ and circle EA, point M.
(15) Draw ray AL. Label intersection of circle AB and ray AL, point N.
(16) Draw ray JM. Label intersection of circle JE and ray JM, point O.
(17) Connect segment NO.
(18) Draw ray BE. Label intersection of ray BE and segment NO, point P.
(19) Draw circle PO.

Since segment AB is congruent to segment JB and the diameters, CB and BK are congruent, circles AB and JB are equal, by Euclid III, Definition 1. By Euclid III.18, line HJ is tangent to the circle centered at A. Similarly, by Euclid III.18, line EJ is tangent to the circle centered at A. Therefore, angles ABJ and AEJ are right angles.

Since the two circles, AB and JB are equal, the angle bisectors at A and J will be congruent. Hence, AN is congruent to JO. Similarly, since circles AB and JB are equal, the angle bisector at B will cut through the line segment NO. By Euclid I.9, P is the midpoint of line segment NO. By Euclid III.18, line NO is tangent to both circles, centered at A and J respectively. Therefore, angles ANP and JOP are right angles.

Hence, the circle AB meets circle JB at right angles. Also, the circle AB meets circle PO at a right angle. Finally, the circle JB meets circle PO at a right angle.

□

Date: December 11, 2014.

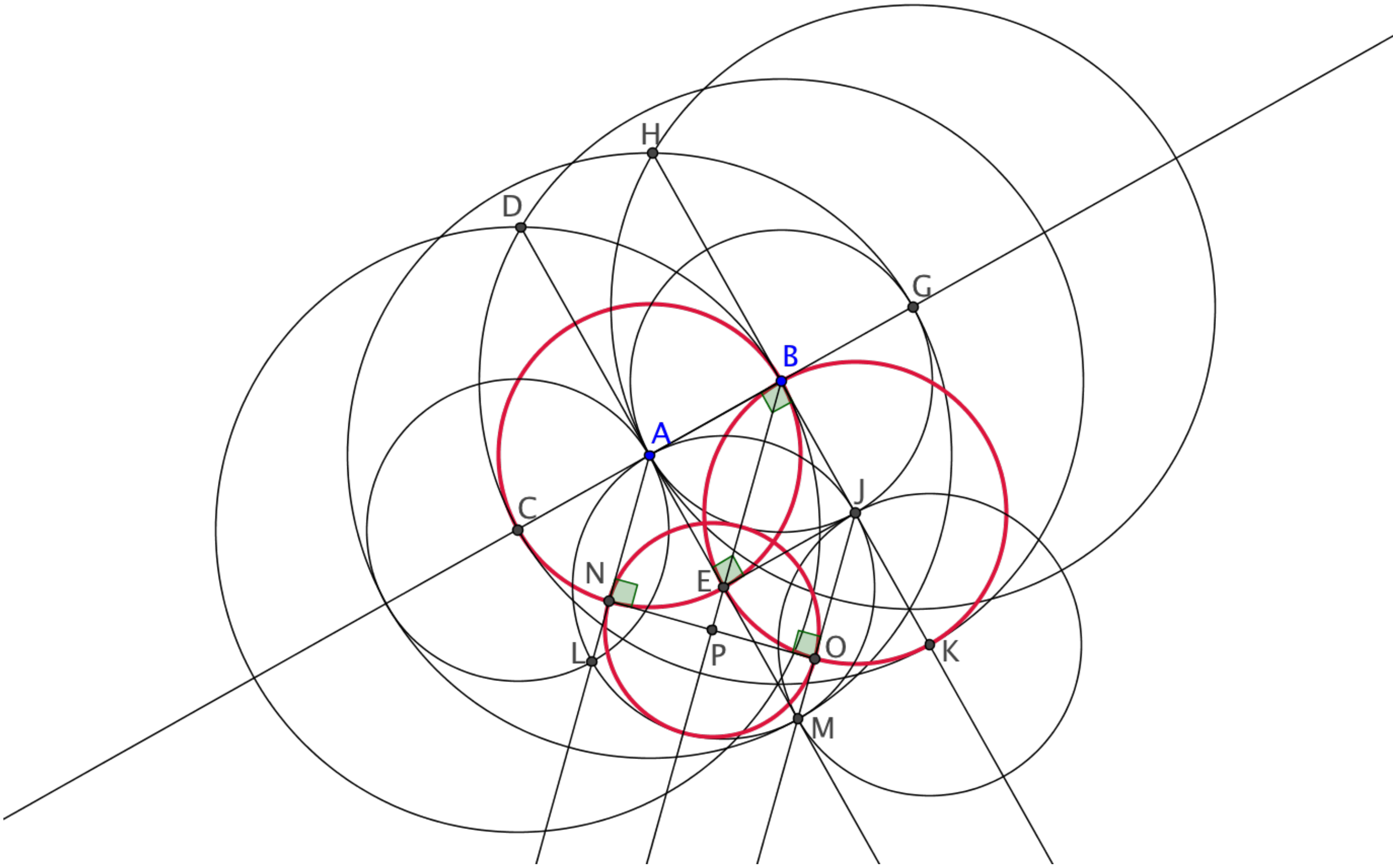

Figure 1. Three Circles That Each Pair Meets at Right Angles

ALGORITHM TO DIVIDE ANY SIMPLE POLYGON INTO A COLLECTION TRIANGLES.

JOSHUA HAWKINS

Communicated by Thomas Bieber

Theorem T.1. Let P be a simple n-gon. Then it is possible to divide P into n-2 non-overlapping triangles by inserting diagonals of P.

Proof. The algorithm has four rules for drawing diagonals.
1: Do not draw a diagonal if it already exists.
2: Do not draw a diagonal if it would cross another diagonal.
3: Do not draw a diagonal if any part of it would fall outside the polygon.
4: Do not draw a diagonal if a point that is not an endpoint would lie on the diagonal.

Algorithm:
1: Pick a starting vertex and a direction around the polygon.
2: As long as no rules will be broken, draw a diagonal connecting your starting vertex to the vertex two vertices in the chosen direction.
3: Go to the next vertex in the chosen direction and repeat step number 2.
4: Repeat step number 4 until you return to the chosen vertex.
5: Repeat steps 2-4 while increasing the number of vertices moved in the chosen direction by one each circuit.
6: End the algorithm when the number of vertices skipped from the first vertex would have you connect to itself. □

Theorem T.2. For any simple polygon with at least four sides, the first diagonal chosen by the algorithm lies inside the polygon and cuts off a triangle.

Proof. Going in order I shall prove that it must be possible for my algorithm to be able to draw a first diagonal.
1: No diagonals exist since we are looking for the first, so this rule is not broken.
2: Similarly to the first since no diagonals exist, then none can be crossed.
3: The only way for a polygon to break rule number 3 is if every diagonal lies outside the polygon. Assume every diagonal is outside the polygon. When we draw a polygon for all the diagonals to be on the outside, we notice that the polygon's sides must meet each other at some point for it to be a polygon. Once this happens we notice that all the diagonals are on the inside of the polygon, but we assumed they were all on the outside of the polygon. Therefore the inside of the polygon is the outside and the outside is the inside. This is impossible, so it is impossible for this rule to be broken.
4: Since the first diagonal will only involve three consecutive vertices, then it is only broken if the three vertices are collinear. This would mean the polygon is no longer the original polygon we started with. So this rule is not breakable.

Since breaking the rules to create a diagonal is not possible, there exists a diagonal on the inside of the polygon that cuts off a triangle.

□

Proof. An Inductive proof of the algorithm.

Base Case: A quadrilateral can be split into two triangles by its diagonals.

Inductive step: Assume a polygon with P vertices can be divided into a collection of triangles. Let there be a triangle that has P+1 vertices. Since the algorithms first move cuts off a triangle, draw the first diagonal. Now we have a polygon with P vertices. This can be divided into a collection of triangles.

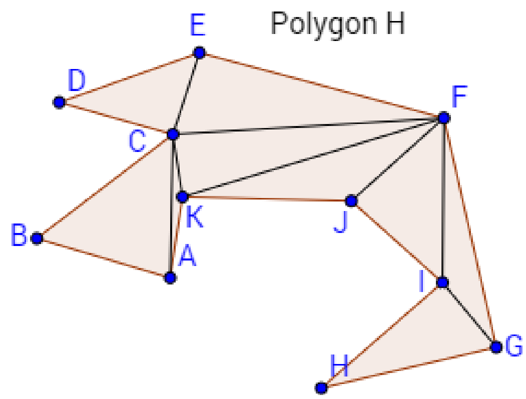

FIGURE 1. Polygon H is divided into triangles by the algorithm.

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