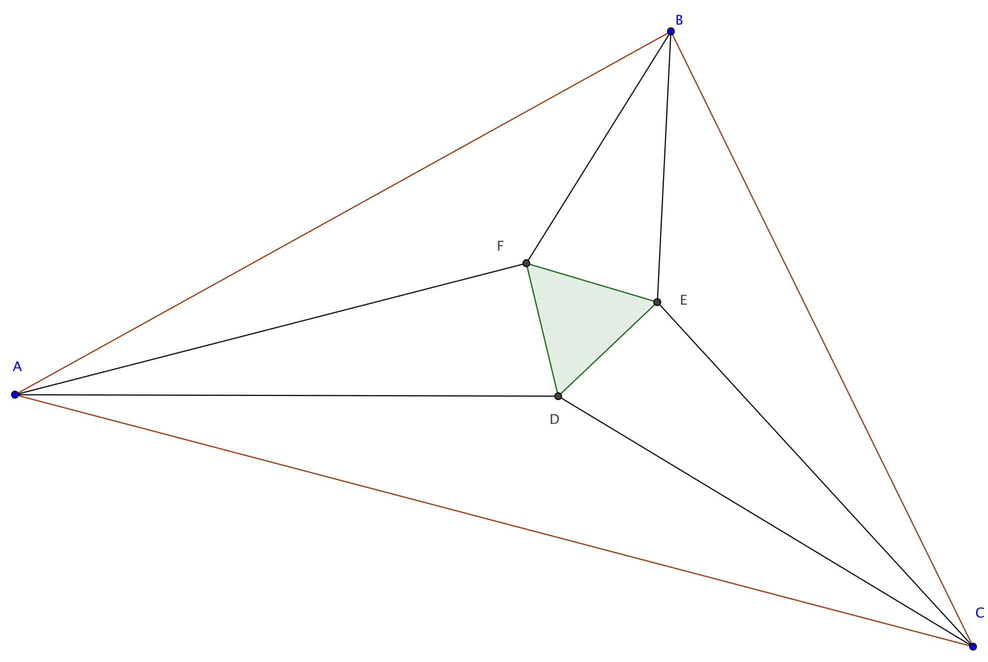
Transactions
in
Euclidean Geometry



Issue # 10

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Parallel Lines in Tangent Circles

Kevin Conger

December 8, 2016

Communicated by Ms. Goedken.

Theorem 9.5. Let two circles be tangent at a point A. If two lines are drawn through A meeting one circle at further points B and C and meeting the other circle at points D and E, then BC is parallel to DE.

Proof. **Case 1:**

Let two circles be tangent at a point A, such that the circles touch internally. Let two lines be drawn through A meeting one circle at further points B and C and meeting the other circle at further points D and E, such that points B and D are on the same line through point A and points C and E are on the same line through point A. By Euclid's Postulate 1, create BC and DE. We will show that BC is parallel to DE.

Let there be a line drawn tangent to the two circles at point A. Let there be a point F on the tangent line such that it is on the same side of the line AB as point C. Then by Euclid III.32, the angle FAE will be congruent to the alternate angle, namely angle ADE, which is on the cut of the circle made by AE. Similarly, by Euclid III.32, the angle FAC will be congruent to angle ABC. Since A, E, and C are collinear, angles FAE and FAC are congruent. It follows, by Euclid's Common Notion 1, angles ADE and ABC are congruent. Then by Euclid I.27, BC is parallel to DE.

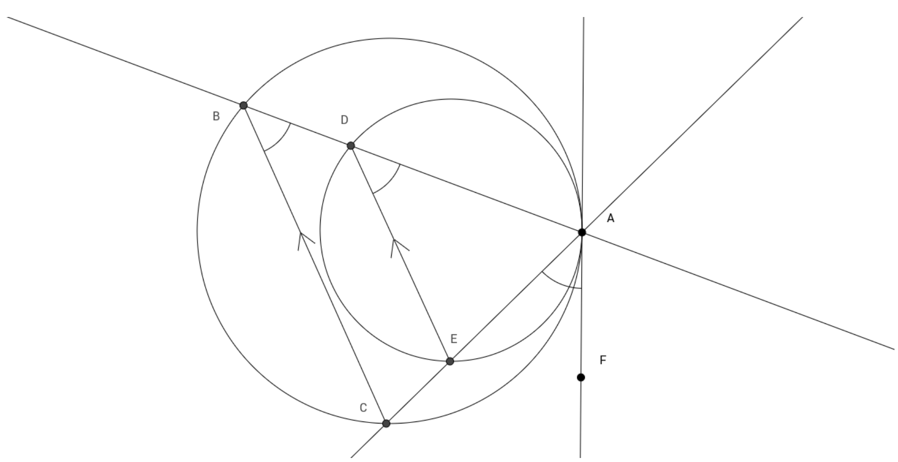

Figure 1: The two circles touch internally and BC is parallel to DE.

Case 2:

Let two circles be tangent at a point A, such that the two circles touch externally. Let two lines be drawn through A meeting one circle at further points B and C and meeting the other circle at further points D and E, such that points B and E are on the same line through point A and points C and D are on the same line through point A. By Euclid's Postulate 1, create BC and DE. We will show that BC is parallel to DE.

Let there be a line drawn tangent to the two circles at point A. Let there be a point F and a point L on the tangent line, such that A is between F and L and F is on the same side of line BE as C. Then by Euclid III.32, the angle FAC will be congruent to the alternate angle, namely angle ABC, which is on the cut of the circle made by AC. Similarly, by Euclid III.32, the angle LAD will be congruent to angle AED. Since FL and CD are straight lines which cut one another, angle FAC is congruent to angle LAD by Euclid I.15. By Euclid's Common Notion 1, angles ABC and AED are congruent. Then by Euclid I.27, BC is parallel to DE.

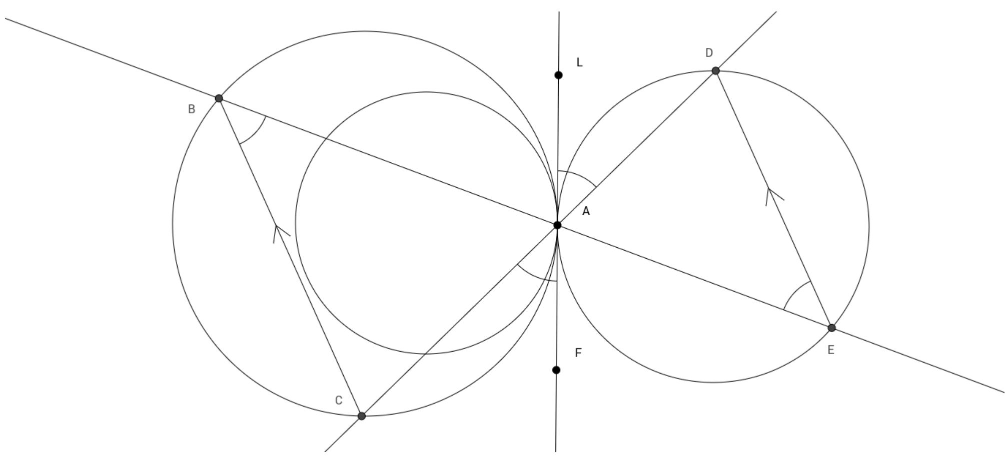

Figure 2: The tangent line is between the two circles and BC is parallel to DE.

□

Circle Inscribed in Triangle Construction

Amanda Worsfold

December 8, 2016

Communicated by Mr. Phaly.

Theorem 12.1. Construct a circle inscribed in a given triangle ABC.

Proof. By Ms. Van Ryswyk's Theorem 8.2, we know the angle bisectors of the triangle are concurrent at the incenter, where the incenter is the center of the inscribed circle. Thus I will use 8 steps to construct an angle bisector of angles BAC and ACB using Ms. Shere/Ms. Schmeling Construction of 11.1. The first 4 steps will be used to bisect angle BAC. This is shown in Figure 1.

Step 1: Draw a circle with center A, intersecting the triangle at Points P and Q.

Step 2: Using the same radius as Circle A, draw a circle with center P.

Step 3: Using the same radius, draw a circle with center Q. This intersects Circle P at Point R.

Step 4: Draw a ray from Vertex A through Point R. This is the angle bisector of BAC.

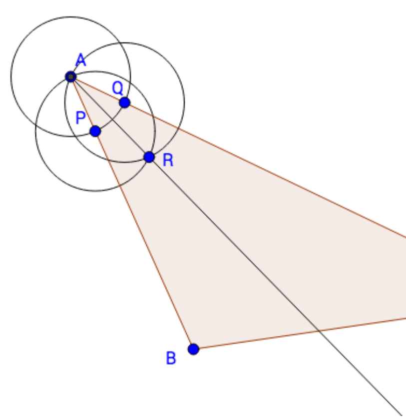

Figure 1: Construction of the Angle Bisector of BAC

Similarly in Steps 5-8, I will construct the angle bisector of angle ACB. This is shown in Figure 2.

Step 5: Draw a circle with center C, intersecting the triangle at Points S and T.

Step 6: Using the same radius as Circle C, draw a circle with center S.

Step 7: Using the same radius, draw a circle with center T. This intersects Circle S at Point U.

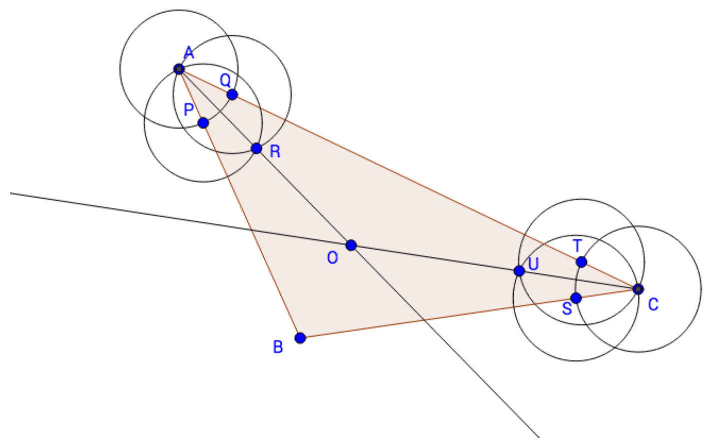

Figure 2: Construction of the Angle Bisector of ACB

Step 8: Draw a ray from Vertex C through Point U. This is the angle bisector of ACB.

Ray CU meets Ray AR at the a Point 0, namely the incenter of our inscribed circle.

Now that we have the center of our circle, we need a radius to complete the construction of the inscribed circle. In steps 9-12, I will construct a perpendicular line to Side AC through the Point O using Ms. Cohen's Construction of 11.4. This is shown in Figure 3.

Step 9: Construct a circle with center O with radius OB. Circle O intersects Side AC twice at Points V and W.

Step 10: Using radius VW, draw a circle with center V.

Step 11: Using the same radius, draw a circle with center W. This intersects Circle V at Point X.

Step 12: Draw the line through Points O and X which is perpendicular to Side AC at Point Y.

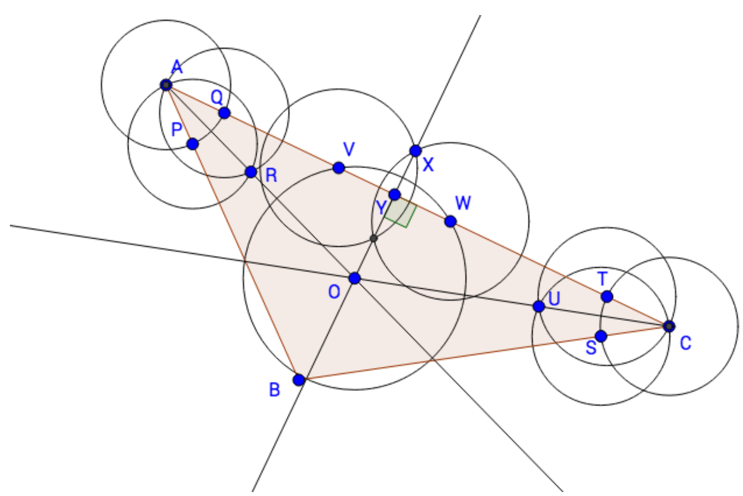

Figure 3: Construction of the Perpendicular Line OX

The final step is constructing the inscribed circle with radius OY. This final circle is shown in Figure 4 as the darker circle.

Step 13: Draw the circle with center O and radius OY. This is the inscribed circle of the triangle ABC.

Since we found the incenter and constructed a circle such that the circle is tangent to all the sides of triangle ABC, then by definition the circle with center O and radius OY is an inscribed circle. □

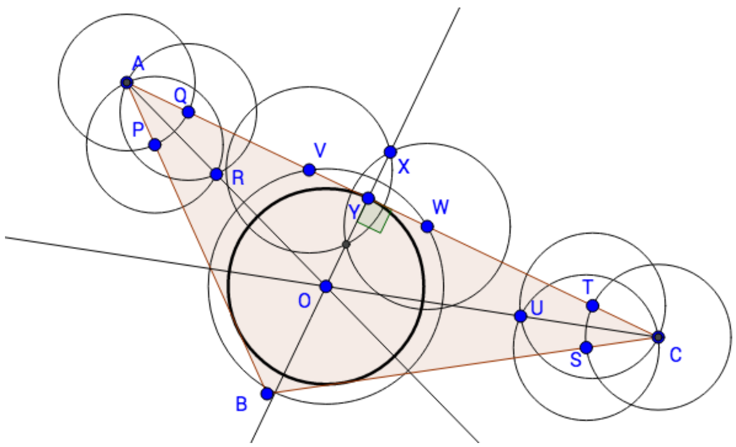

Figure 4: Construction of the Inscribed Circle in Triangle ABC

Circle Circumscribed About a Triangle

Tessa Cohen

December 8, 2016

Communicated by Mr. Conger.

A circle is circumscribed about a figure if the figure lies in the interior of the circle, except for the vertices which lie on the circle. It is possible, using a straight edge and compass, to construct a circumscribed circle about any given triangle in 7 steps.

In order to construct a circumscribed circle about a given triangle, we will need to know how to find the perpendicular bisector of a given segment.

Lemma *Perpendicular Bisector.* It is possible, using a straight edge and compass, to construct a perpendicular bisector of a given any segment AB in 3 steps.

1. Draw a circle with center A through point B.

2. Draw a circle with center B through point A.

3. Draw a line through the two intersection points of circle A and circle B.

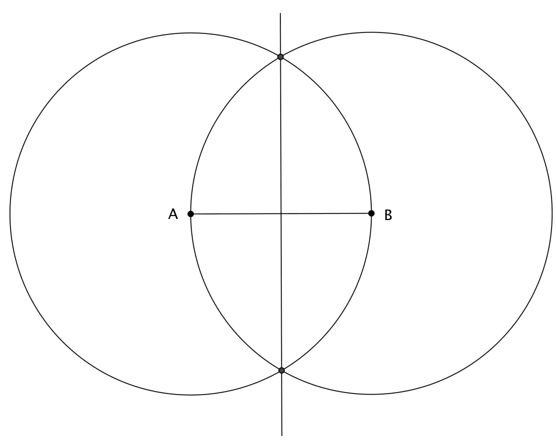

Figure 1: The 3 steps of constructing a perpendicular bisector on segment AB.

Proof. Let the intersection points of circle A and circle B be called point C and point D. By Euclid Postulate 1, create segments AC, BC, BD and AD. Since C and D are the intersection points of circles A and B, AC and AD are radii of circle A, therefore they will be congruent. Similarly, BC and BD are radii of circle B, therefore, BC and BD will be congruent. Circle A and circle B were both constructed with radius AB. Since they have the same radius, circle A will be congruent to circle B. Then, the radii of circle A and circle B will all be congruent. Thus, AC, BC, BD and AD are all congruent. Therefore, ACBD will be a rhombus. By Miss Bavido's Theorem 1.7, the diagonals of a rhombus meet at a right angle. DC and AB are the diagonals of ACBD, so DC will meet AB at a right angle. Let the intersection point of DC and AB be called X. We will consider triangle ACX and triangle BCX. Angles CXA and CXB are right, segment AC is congruent to segment BC and triangles ACX and BCX have shared side XC. By the Hypotenuse-Leg Theorem, triangle ACX is congruent to triangle BCX. Therefore, segment AX is congruent to segment BX.

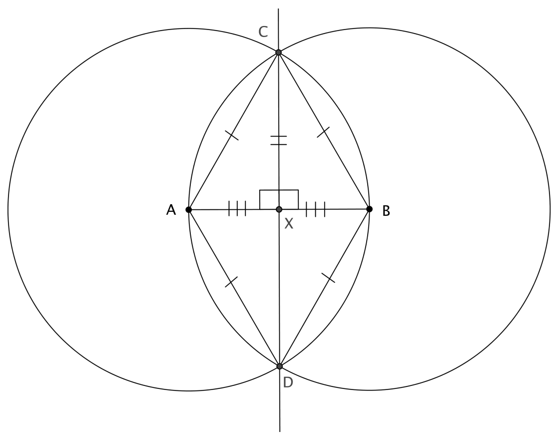

Figure 2: Rhombus ACBD with congruent triangles CXA and CXB.

Since CD meets AB at a right angle and AX is congruent to BX, CD is the perpendicular bisector of AB. □

Construction ***Circumscribed Circle.*** We begin with any triangle, ABC.

1 - 3. Construct the perpendicular bisector of side AB.

4 - 6. Construct the perpendicular bisector of side BC.

7. Create a circle with the center being the intersection of the perpendicular bisectors of AB and BC and through either point A, B, or C.

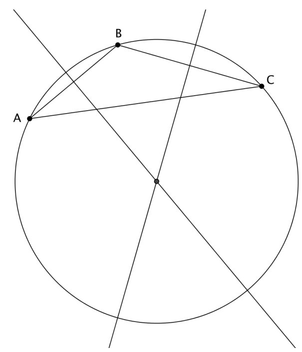

Figure 3: The intersection of the perpendicular bisectors of AB and BC is the center of the circumscribed circle.

Proof. We will let the center of the circumscribed circle be called D. By Euclid's Postulate 1, we will create segments AD, BD, and CD. In order to prove that this circle is circumscribed about our triangle ABC, by Euclid's definition of a circle, we need to prove that segments AD, BD, and CD are congruent. Let the midpoint of AB, also the intersection of the perpendicular bisector and AB, be called E. Let the midpoint of BC, also the intersection of the perpendicular bisector, be called F. Since DF is the perpendicular bisector of BC, we know that segment BF is congruent to segment FC and angles DFB and DFC are right. We also know that segment DF is congruent to itself. Therefore, by Euclid Proposition I.4, triangles BFD and CFD are congruent. Thus we know that segments BD and CD are congruent. Similarly, since ED is the perpendicular bisector of AB, we know that segment AE will be congruent to segment BE and angles AED and BED are right. We also know that segment ED is congruent to itself. Therefore, by Euclid Proposition I.4, triangle BED is congruent to triangle AED. Thus segment AD will be congruent to segment BD.

By Euclid Common Notion 1, we know that since CD is congruent to BD and AD is congruent to BD, then AD is also congruent to CD. Then segments AD, BD, and CD are all congruent to each other, and will lie on the circle with center D. Thus by the definition of a circumscribed circle, circle D is circumscribed about given triangle ABC.

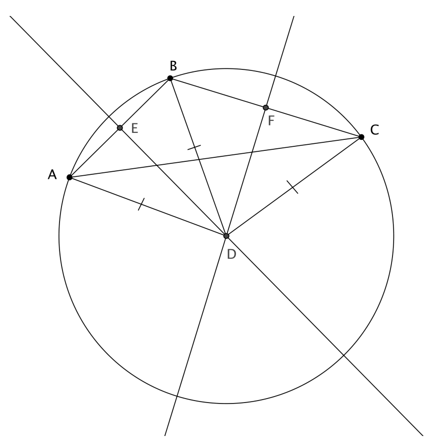

Figure 4: AD, BD, and CD are congruent, therefore are radii of circle D.

□

Constructing Congruent Segments Using a Circle

Taryn Van Ryswyk

December 8, 2016

Communicated by Ms. Bavido.

Theorem 12.3. Given a line l, a line segment d and a point O, construct a circle with center O that cuts off a segment from line l which is congruent to d.

Proof. We are given segment d, and by theorem 11.2 we are able to construct the midpoint of segment d by creating a circle with radius AB, which is equal to segment d. Similarly we will create another circle with radius BA, which is equal to segment d. By the circle-circle intersection theorem, we know that the circles will intersect each other. Next, draw a line m connecting the intersecting points of the two circles. The point where line m intersects segment d is the midpoint of segment d which we will denote as point O. By the definition of midpoint, segment OA is congruent to segment OB. Now create a circle with center O and radius OB.

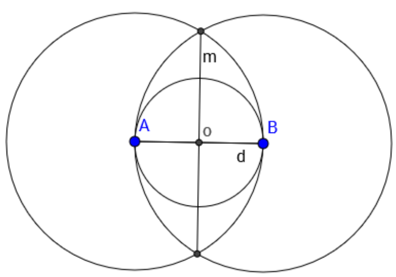

Figure 1: Segment d with midpoint O.

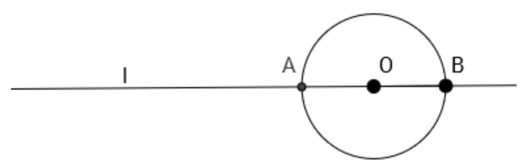

Figure 2: Line l with the circle centered at O.

Now we are going to cut off a segment from line l which is congruent to segment d. We are given line l, and then we will place point O on this line. Then using a compass, create a circle centered at O with radius OB. Since d is the diameter of the circle centered at O with radius OB, then when the same circle is constructed on line l it will cut line l to be the diameter of the circle centered at O with radius OB. Thus d is the diameter of the given circle, and the segment cut from line l is also the diameter of the given circle. Since both are the diameter of the circle centered at O with radius OB, they are congruent.

□

Three Circles Meeting at Right Angles

Kevin Conger and Tessa Cohen

December 8, 2016

Communicated by Ms. Schultz.

Construction 12.4. A pair of intersected circles is said to meet at right angles if at their intersection points, the radii of one circle meets the radii of the other circle at right angles. With a straight edge and compass, it is possible to construct 3 circles such that each pair meets at right angles in 9 steps.

1. Let there be any line segment AB.

2 - 4. By Ms. Cohen's Perpendicular Bisector Lemma, construct the perpendicular bisector of segment AB. Let the perpendicular bisector of segment AB be called l and the midpoint of AB be called X.

5. With a compass, draw circle with center X through point A. Let the intersection of circle X and l be called T.

6. With a compass, draw a circle with center A through point T.

7. With a compass, draw a circle with center B through point T.

8. Fix a compass with radius AT. Draw a circle with center X and with radius AT. Let the point of intersection between this circle and l be called C.

9. Fix a compass with radius XA. Draw a circle with center C and with radius XA. The three circles that meet at right angles are circle A, B and C.

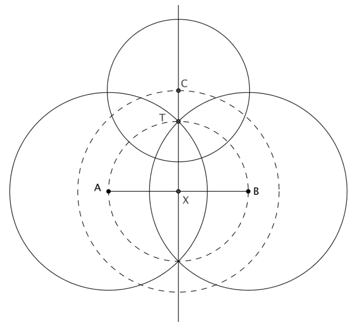

Figure 1: The circles used to construct three circles meeting at right angles.

Proof. By the Circle-Circle Intersection Theorem, a pair of intersecting circles has two points of intersection. Then, between the circles A, B, and C, there will be six points of intersection. In order to prove that these three circles meet each other at right angles we will show that at each of the six points, the radii of the intersecting circles meet at right angles.

Let the two points of intersection between circle A and circle C be called point R and point V. Let the intersection points of circle B and circle C be called point S and point U. We already have one of the intersection points of circle A and circle B called T. Let the other point of intersection of circle A and circle B be called point W.

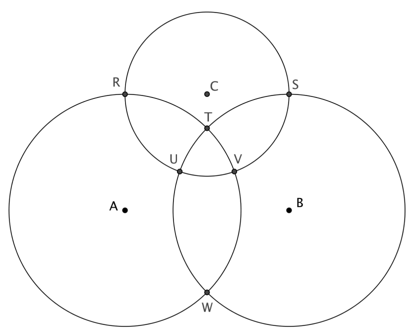

Figure 2: Circle A, B, and C with their labeled intersection points.

By Euclid Postulate 1, create segments AT and BT. We will consider triangle ATX and triangle BTX. Since segment TX is part of l, the perpendicular bisector of segment AB, angles TXA and TXB will be right and segment AX will be congruent to segment BX. Triangles ATX and BTX also have shared side TX. By Euclid Proposition I.4, triangle ATX will be congruent to triangle BTX. Then segment AT is congruent to segment BT. Since point T is the intersection of circle A and circle B, segments AT and BT are radii of circle A and circle B, respectively. Since AT and BT are congruent, circle A is congruent to circle B. Then, the radii of circle A and circle B will be congruent. By Euclid Postulate 1, create segments AR, BS, CR, CS, AC and BC. We will consider triangle ACX and triangle RCA. Circle C was constructed with a radius congruent to segment XA, so XA is congruent to segment CR. Circle XC was constructed with a radius congruent to segment AT. The point of intersection between circle XC with radius AT and the perpendicular bisector of AB is point C. Then, segment XC is the radius of circle XC. Since circle XC was created with radius congruent to segment AT, segment XC is congruent to AT. Point R is the intersection between circle A and circle C, so segment AR is also a radius of circle A. Thus, by Euclid Common Notion 1, since AR is congruent to AT and AT is congruent to XC, then segment AR will be congruent to segment XC. Triangle ACX and triangle RCA have congruent sides AR and XC, XA and CR, and shared side AC. By Euclid Proposition I.8, triangles ACX and RCA are congruent. Since angle CXA is right, and triangles ACX and RCA are congruent, angle ARC is right. Without loss of generality, the same argument can be made considering triangles BCX and CBS, making angle CSB a right angle as well.

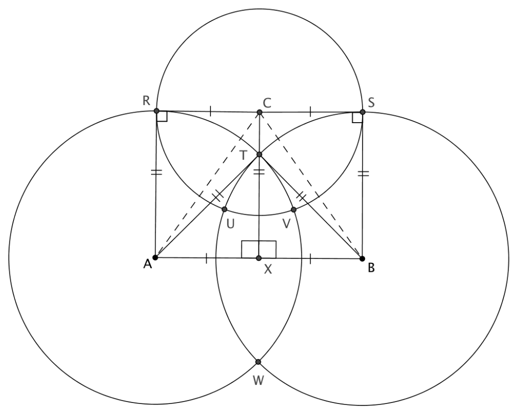

Figure 3: Angle CXA and angle ARC are right.

Therefore, at intersection points R and S, the radii of circle A and circle C, and circle B and circle C, respectively, meet at right angles.

Next we will look at the intersection points of circle A and circle B, namely points T and W. Create segments AT, BT, AW and BW by Euclid Postulate 1. Point T was constructed as the intersection between l and circle X. Then circles A and B were constructed through point T. Therefore, point T is the intersection of circle A, circle B and circle XT. Circle XT has a diameter AB and point T on its circumference. By Thales' Theorem, angle ATB is right. Circles A and B were previously found to be congruent, so the radii are congruent. Segments AT and AW are radii of circle A, and segments BT and BW are radii of circle B. Therefore, segments AT, AW, BT and BW will all be congruent and ATBW will be a rhombus. By Ms. Mitchell's Theorem 1.1, the opposite angles of a rhombus are congruent. Therefore, since angle ATB is right, angle AWB is right.

Thus, at intersection points T and W, the radii of circle A and circle B meet at right angles.

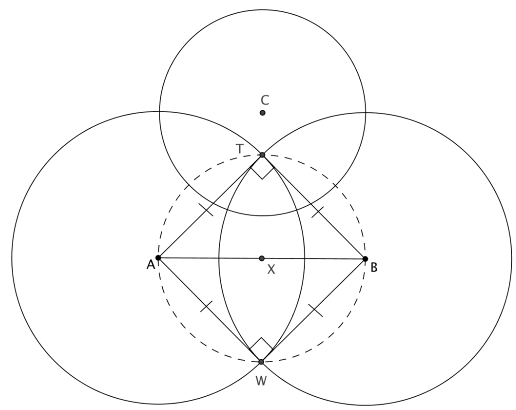

Figure 4: Angle ATW and angle AWB are right.

Lastly, we will consider point U, the intersection of circle B and circle C, and point V, the intersection of circle A and circle C. By Euclid Postulate 1, create segments RC, CV, AV, and AR. Since segments RC and CV are radii of circle C, they are congruent. Similarly, segments AR and AV are radii of circle A and are congruent. With two pairs of congruent sides, RCVA is a kite. By Ms. Ancona's Theorem 2.1, angle CRA is congruent to angle CVA. Angle CRA was previously found to be right, therefore, angle CVA will also be right. Without loss of generality, the same argument can be made for angle CUB when considering kite CUBS.

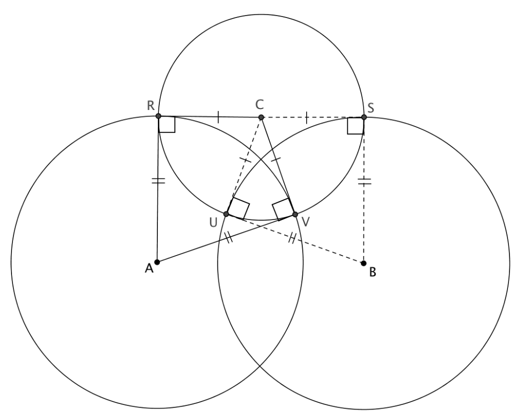

Figure 5: Angle CVA and angle CUB are right.

Therefore, at intersection points U and V, the radii of circle B and circle C and the radii of circle A and circle C, respectively, meet at right angles.

At all six points of intersection, R, S, T, U, V, and W, it was found that the radii of the pairs of the circles meet at right angles. Therefore, all three of the circles meet at right angles.

□

Circles and Tangents

Juliana Herran

December 6, 2016

The following paper presents a set of constructions and proofs of common tangents to two circles. There are three different cases explored here, 1 common tangent, 2 external common tangents and 3 common tangents to two circles. These cases do not include all the possible cases. Further work with internal common tangents will be presented in subsequent papers.

Lemma 1. If OCP'P is a quadrilateral such that: [i] OP is congruent to CP' [ii] OP is parallel to CP' [iii] Angle POC and angle P'CO are right

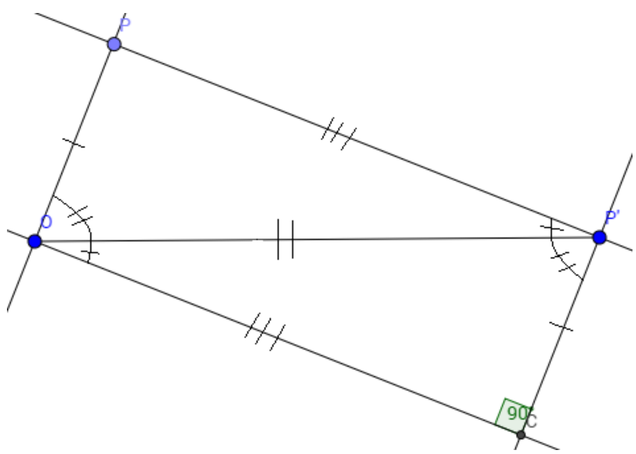

Figure 1: The angle CP'P and the angle OP'P are both right angles

Proof. Let a diagonal from O to P' exist in the quadrilateral OCP'P. Given that segments OP and CP' are parallel and congruent, the diagonal OP' joins congruent parallel lines at their extremities (Euclid 1.33). Then, by Euclid 1.33, segment PP' is parallel and congruent to segment OC. Since PO is parallel to CP', and the diagonal OP' crosses both lines, by Euclid 1.29 angles P'OC and OP'P are congruent. Similarly, angles CP'O and POP' are congruent. Since angle POC is a right angle, angle POP' plus angle P'OC is a right angle. Then angle CP'P is also a right angle. Also, since triangle POP' is congruent to triangle P'OC, their angles must also be congruent by Euclid 1.8. A similar argument can be used to prove that angle OPP' is also a right angle. □

Theorem 12.7. Given two circles Γ and Γ' with centers O, O' respectively there can be one, two or three common tangent lines through both circles.

Proof. This proof will be divided in cases. Case one will show the construction of one common tangent to two circles one inside the other. Case two will show external common tangent lines of two separate circles of with different radii. Case three will show three tangent common lines to the two circles. □

CASE 1: First, construct a circle of center O. Then, find any point in the diameter of the circle at O and call it O'. Make O' the center of the circle inscribed in the circle of center O. The inscribed circle is constructed in such a way that circles O and O' meet at one point P. Then, construct a radius of the circle centered at O which goes from O to P. Then, construct a circle centered at P with radius O'P (The diameter of this circle is O'R). Finally, construct the perpendicular bisector of O'R by setting the compass to the length of the diameter O'R. Then, without changing the setting of the compass, center it at O' first and draw a circle with radius O'R centered at O'. Then, center the compass at R and create circle with radius O'R. The perpendicular bisector is the line that joins the intersection points of the circle with radius O'R centered at O' and the circle centered at R. This perpendicular bisector is tangent to both circles.

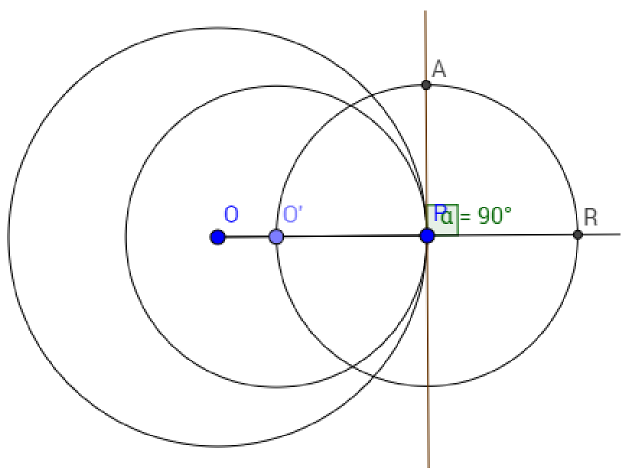

Figure 2: The circles of radius O'R centered at O' and R' mentioned during the construction were not included in the image as these final construction steps follow the standard construction of a perpendicular bisector.

Proof. Since the perpendicular bisector of segment O'R goes through point P, by Euclid 1.13 and 3.18, angle APO' is also a right angle. Similarly, angle APO is also a right angle. Since point P is the only point of intersection of the circles centered at O and O' by construction and the bisector through P is at a right angle with segments O'R and OR, the line through point P is a common tangent of the circles centered at O and O'. □

CASE 2:

First we join the centers of the two circles Γ and Γ' with different radii to create the line segment OO' Note that circle Γ is the circle centered at O and circle Γ' is the circle centered at O'. Then, find the perpendicular bisector of OO' using the standard construction of a perpendicular bisector as shown in case one and label the midpoint of OO' as point X. Label the intersection point of circle Γ with segment OO' as point B and the intersection point of the circle Γ′ with OO' as point A. Then, draw a circle with center X and radius OX. Draw a circle of center O' and radius AO'-BO (this will be referred to as the inner circle). Label the points of intersection of the circle centered at X and the inner circle as C and D. Draw a ray passing through O' and point C and extend the ray all the way to circle Γ′. Create a parallel line to O'C going through point O and label the intersection point as P. The line that goes through points P and P' can be constructed by Euclid's postulate 1. This line through P and P' is an external common tangent of circles Γ and Γ′. A similar construction method can be used to create the second common tangent to these circles shown in the figure below as the line through QQ'.

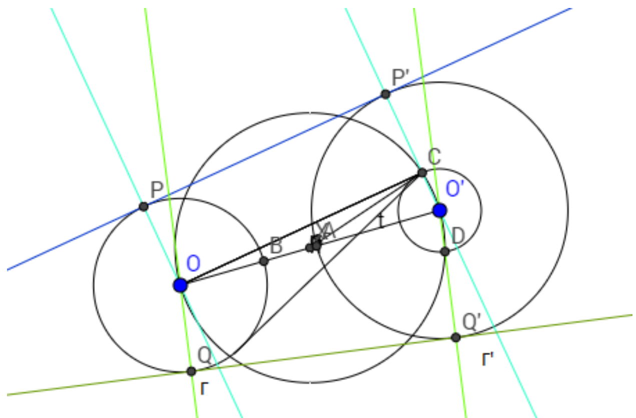

Figure 3: The purpose of the color coded graph is a better visualization of the important segments and triangle relations relevant to the proof below.

Proof. Since line segments OX, XO' and XC are all radii of the circle centered at X, they are all congruent to each other. By Miss. Ahrens theorem 7.4, angle OCO' is a right angle. Then, by Euclid 1.13, angle OCP' is also a right angle. Also, since segment OP is parallel and congruent to segment P'C by construction, the straight line OC falling on the two straight lines OP and P'C is a transversal. Then, since we know that angle OP'C is a right angle it follows that COP is also a right angle as we extend the line OC to find the exterior alternate angle by Euclid 1.29 and then complete the two right angles between two straight lines by Euclid 1.13. Then by the lemma shown at the beginning of the paper, angles OPP' and CP'P are also right angles. Therefore, since the angle between the line PP' and the center of the circles is a right angle and the line touches the circles at one point only (point P for one circle and point P' for the other), the line through PP' is a common tangent between circles Γ and Γ′. □

CASE 3:

First, draw a line segment AB and find its perpendicular bisector following the standard construction of a perpendicular bisector as presented in case 1 . Draw a circle centered at point A with radius AC and a circle centered at B with radius CB. The perpendicular bisector of AB going through C is a common tangent of the circles centered at A and B. Now, draw a circle centered at C with diameter AB. The points at which the perpendicular bisector meets the circle centered at C are H and N. Then, find the perpendicular bisectors of DC and CE. The points where these bisectors meet the circles centered at A and B are K, M, L and O. Create a line through KHL and another line through MNO. These lines are also common tangents to the circles centered at A and B.

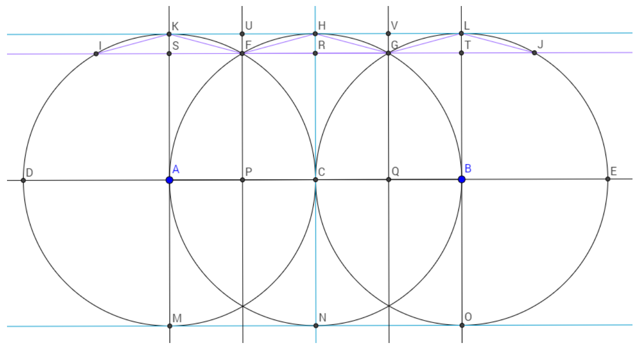

Figure 4: Note the color coded graph indicating the 3 common tangents to these circles in blue and congruent triangles in purple.

Proof. Since the perpendicular bisector of AB goes through C and circles A and B only touch at point C, this perpendicular bisector is tangent to the circles of center A and B. Also, the line through points PF is a perpendicular bisector of AC as it follows the standard construction of a perpendicular bisector. Then, this perpendicular bisector is parallel to the perpendicular bisector through RC by Euclid 1.29 with the line through PC as the transversal. Also, angles FPC, HCQ and GQC are right angles. Since we also know that PC is congruent to CQ, FR and RG as each of these line segments are 1/4 of the initial diameter of the circles centered at A, B and C and PF is congruent and parallel to CR and QG by construction, by my lemma, angles PFR, CRG, QGR are right angles when looking at rectangles PFRC and CRGQ.Then, by Euclid 1.13, angles GRH, FRH are also right angles. By a similar argument, angles ISK, KSF, GTL and LTJ are also right angles. Since SK is parallel and congruent to RH as they fall in the intersection of the perpendicular bisector DC and AB respectively with the line DB which joins the centers of circles A and C, SR is parallel and congruent to KH by my lemma. We can construct a transversal SH and by my lemma, angle RHK is a right angle when looking at rectangle KSRH. This as we have two sets of parallel congruent lines (SR KH and SK RH) and right angles KSF and HRF. Finally, since R lies on the line through H and C, angle CHK is also a right angle. A similar argument can be used to prove that BLH and AKH are also right angles. Then, the line KHL is a common tangent of the circles centered at A and B. A similar argument can be presented to show that AMN and BON are also right angles and therefore that the line through MNO is also a common tangent of the circles with center A and B. Thus, circles centered at A and B have three common tangents.

□

Proof. An Alternative proof for the two common tangents through KHL and MNO of circles centered at A and B goes as follows. Since AC, AK, CB and BL are all radii of the circles centered at A and B respectively, they are congruent to each other by construction. Also, since the line through AK is also a perpendicular bisector of DC and the line through BL is also a perpendicular bisector of CE. By Euclid 1.28, angle KAC and LBE are right angles. Angle LBC is also right by Euclid 1.13. Then, the line through K and L will be parallel to line AB. Since AK and BL are congruent and angle KAP and LBQ is a right angle, line segment KL is also congruent to AB. Since AB and KL are bisected by the perpendicular bisector through C, KH is congruent to HL. Then, we have a pair of congruent parallel lines (KH to AC, HL to CB, AK to CH and CH BL) and two right angles from the construction of the perpendicular bisectors (angles KAC, HCA, HCB and LBC) in each of the quadrilaterals ACHK and CBLH. Thus, by my lemma, angles AKH and BLH are right angles. A similar argument can be built for the common tangent MNO. □

Hypotenuse-Leg Theorem with Equal Content

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Ms. Schultz has previously proven the hypotenuse-leg theorem using triangle congruence and addition of line segments. With Challenge 13.4, we must prove this theorem utilizing equal content properties. To do so, we must first prove a lemma that will be used later in proving Theorem 13.4.

Lemma AF. If two squares, S_1 and S_2, have equal content, then S_1 will be congruent to S_2.

Proof. We will proceed by way of contradiction. Assume that S_1 and S_2 have equal content and S_1 and S_2 are not congruent to one another. Without loss of generality, suppose the side length of S_1 is less than side length of S_2 by definition of "less than" with segments. We will denote a side of S_1 as a and a side of S_2 as b. Since S_1 is less than S_2, by definition of less than regarding line segments, side a is less than side b.

By Euclid's Proposition I.46, we can describe S_1 on the line segment b. This will result in S_1 being inside S_2 since side a is less than side b and S_1 is less than S_2 (refer to figure 1).

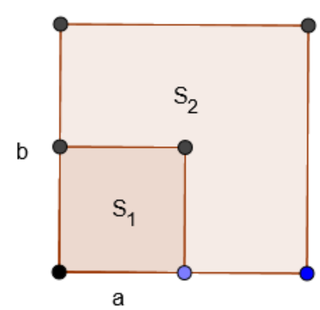

Figure 1: Square S_1 described in square S_2

EC6 says, "The whole is greater than the part, so if one figure is properly contained in another, then the two figures cannot have equal content." Since S_1 is properly contained in S_2, by EC6, the two figures will not have equal content. This is then a contradiction to our assumption. Therefore, when square S_1 and square S_2 have equal content, S_1 will be congruent to S_2.

$\square$

Theorem 13.4. If two right-angled triangles, ABC and DEF, with right angles at angles ABC and DEF, have AB and DE congruent, as well as AC and DF congruent, then the triangles are congruent. This is known as the hypotenuse-leg theorem.

Proof. Let ABC and DEF be right angled triangles with AB congruent to DE and AC congruent to DF. By Euclid Proposition I.47, the square on AC will be of equal content to the sum of the squares on AB and BC. Similarly, the square on DF will be of equal content to the sum of the squares on DE and EF. (refer to figure 2)

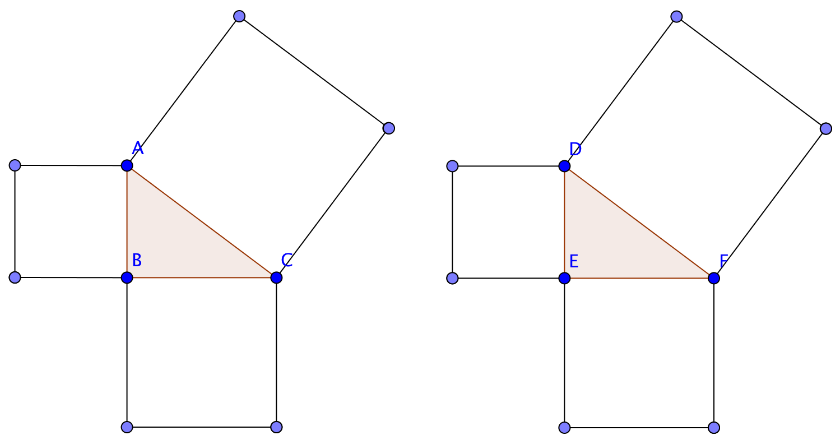

Figure 2: Triangles ABC and DEF with the squares on the sides.

Since the definition of a square is a quadrilateral with four congruent sides and all right angles, and since AB is congruent to DE, the squares on AB and DE are congruent. By EC1, since square AB and square DE are congruent, they will have equal content. Similarly, since AC is congruent to DF, the square on AC is congruent to the square on DF. Again, by EC1, since square AC and square DF are congruent, they will have equal content.

EC 4 states, "subtracting pairs of figures of equal content, will also have equal content." Since square AB and square DE are of equal content and square AC and square DE are of equal content, by subtracting square AB from square AC and square DE from DF, the remaining figures will have equal content by EC4.

By Euclid Proposition I.47, since the content of square AB with square BC is of equal content to square AC, when subtracting the content of square AB from square AC in that statement, the remaining figure will be of equal content to square BC. Similarly, since the content of square DE with square EF is of equal content to square DF, when subtracting the content of square DE from square DF in that statement, the remaining figure will be of equal content to square EF. Thus, as previously stated, since the remaining figures have equal content and are of equal content to square BC and square EF, by EC2, square BC is of equal content to square EF.

By Lemma AF, since square EF and square BC have equal content, square EF is congruent to square BC. Since square BC is congruent to square EF, the sides are all congruent to one another by the definition of a square and the definition of congruence. Thus, segment BC is congruent to segment EF.

By Euclid Proposition I.8, since there are three corresponding congruent sides, triangle ABC is congruent to triangle DEF. Thus when ABC and DEF are right angled triangles with AB congruent to DE and AC congruent to DF, triangles ABC and DEF are congruent to each other.

$\square$