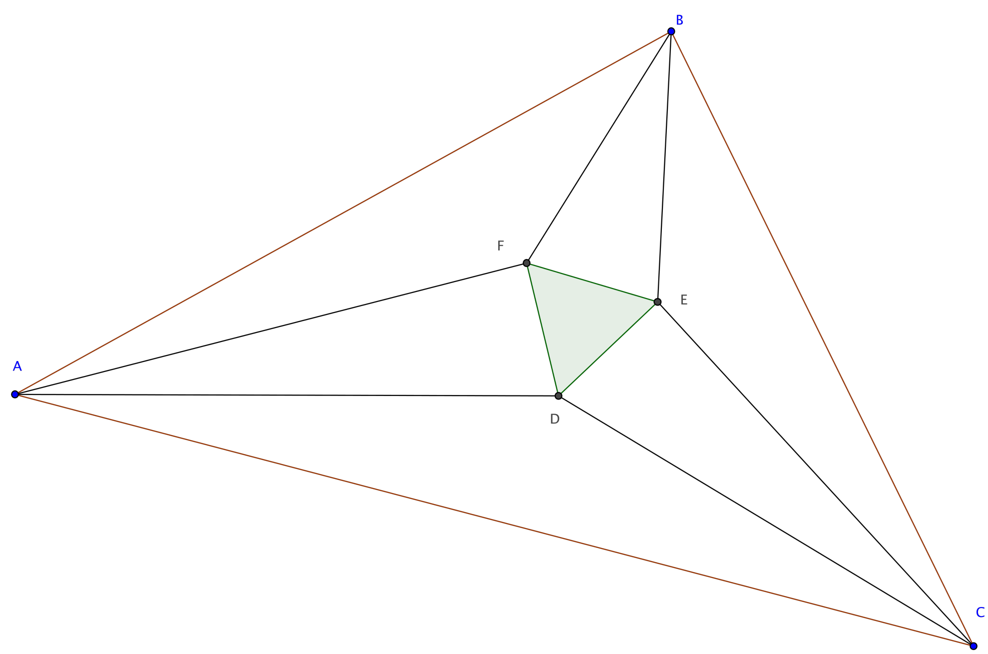
Transactions

in

Euclidean Geometry



Issue # 12

Table of Contents

Sum of Exterior Angles of a Convex N-Gon

Maria Ahrens and Abigail Goedken

December 10, 2016

Communicated by Ms. Shere.

The purpose of this paper is to formulate an equation that will determine the sum of the exterior angles of any number-sided convex polygon. Recall, Mr. Phaly had previously solved that the sum of the exterior angles of a convex pentagon is congruent to four right angles. Building off of his results, we will begin by exploring a hexagon's sum of exterior angles and applying these outcomes to form an equation that will determine the sum of the exterior angles of any convex n-gon.

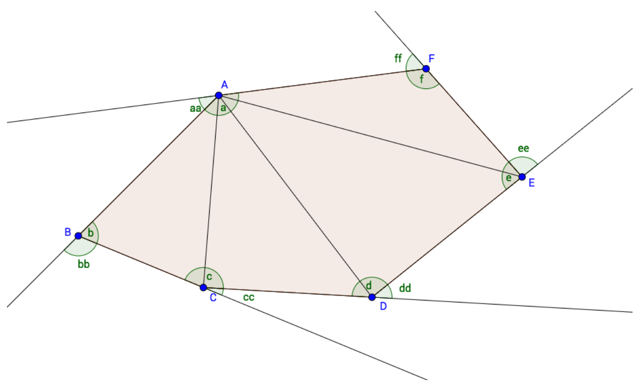


Figure 1: Hexagon ABCDEF

Theorem 5.3.a. The sum of the exterior angles of a hexagon ABCDEF together make four right angles.

Proof. Suppose ABCDEF is a convex hexagon. Create rays AB, BC, CD, DE, EF, FA. Keeping Ahrens' T Theorem in mind, which gave a method to split a pentagon ABCDE into three, non-overlapping triangles by drawing diagonals from a vertex to every non-consecutive vertex from that chosen vertex, we will use a similar method for Hexagon ABCDEF. By drawing diagonals AC, AD, and AE, the hexagon will be split into four, non-overlapping triangles. We now have Triangles ABC, ACD, ADE, and AEF. Since we know, by Euclid 1.32, that the sum of the interior angles of a triangle must add up to two right angles, then the interior angles in a hexagon must add up to eight right angles since four non-overlapping triangles are contained within ABCDEF. Further, by Euclid 1.13, we can say that the sum of each interior angle of ABCDEF along with its associated exterior angle for each of the six rays drawn earlier will add up to two right angles. Thus, the combined sum of both the interior and exterior angles of ABCDEF is congruent to the measure of twelve right angles. Since we are interested in the sum of the exterior angles of the hexagon, we need to take the total measure of angles and subtract the sum of the interior angles. Thus, taking eight right angles from twelve results with the sum of the exterior angles congruent to the measure of four right angles. □

Theorem 5.3.b. The sum of the exterior angles of any n-gon is congruent to the total measure of the interior and exterior angles together minus the sum of the interior angles.

Proof. By extending the method described in the Ahrens' T Theorem to any simple, convex polygons with more than three sides, we can determine the sum of the interior angles. By using Euclid 1.32, the sum of the interior angles of each triangle is two right angles. To find the total sum of both the interior and exterior angles, rays can be extended from each side of the n-gon, and by Euclid 1.13, we know that each ray is made up of one of the n-gon's interior and exterior angles and must add up to two right angles. Thus, we have an equation that looks like this:

2 Right Angles x n Vertices - 2 Right Angles x (n-2) Vertices = Sum of the Exterior Angles

The "2 Right Angles x n Vertices" represents the total sum of both the interior and exterior angles in reference to the rays extended from each vertex of the n-gon. The "2 Right Angles x (n-2) Vertices" represents the total number of non-overlapping triangles contained within the the n-gon, since there are two vertices that will never be used in the creation of a diagonals by the method described in the Ahrens' T Theorem. This is because the two adjacent vertices to the chosen vertex to split the n-gon into triangles already form sides with that vertex. Since diagonals cannot be the sides of the original polygon, there will always be two vertices that will not be included in the action of splitting polygons into triangles. In Figure 2, the vertices B and D are not used in any diagonals. In conclusion, subtracting the sum of the interior angles from the total sum of the interior and exterior angles of the n-gon will result in the sum of the exterior angles.

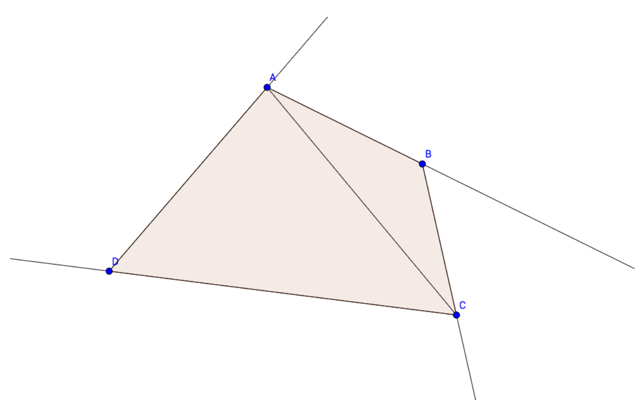


Figure 2: Quadrilateral ABCD

□

Congruent Right Triangles

Erica Schultz

December 9, 2016

Theorem 7.2. Let ABC and DEF be two right triangles, with the angles at A and D right angles. Suppose that BC is congruent to EF and AB is congruent to DE. Then the triangles are congruent.

Proof. By using Euclid 1.2, we can construct a straight line congruent to segment AC from point D on triangle DEF. Let this new segment be DG. Then we know that segment DG is congruent to segment AC. By Postulate 1, we will create the segment EG.

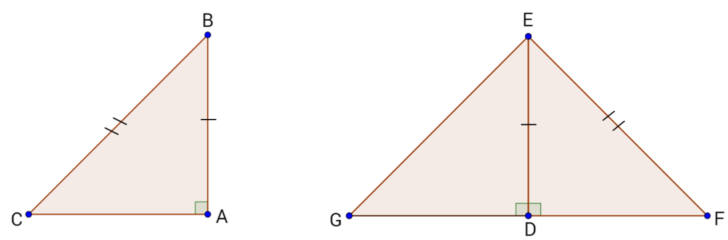


Figure 1: Triangles ABC, DEG, and DEF

By Euclid 1.13, the angles EDF and EDG must add up to two right angles and since we already know that EDF is a right angle, EDG must be a right angle as well. By Postulate 4, right angles BAC and EDG are congruent to one another. Since we are also given that segments AB and DE are congruent, we now know that triangle DEG is congruent to triangle ABC by Euclid 1.4. Since these two triangles are congruent to one another, we know that segments EG and BC are congruent to one another, so segments EG and EF are congruent to one another as well by Common Notion 1. Now we have angles EFD and EGD congruent to one another by Euclid 1.5. By Euclid 1.26, we can now conclude that triangles DEF and DEG are congruent to one another. Then by Common Notion 1, triangles ABC and DEF are congruent to one another. $\square$

Congruency Within Triangles

Erica Schultz

December 9, 2016

Theorem 8.1. Let ABC be a triangle, with rays r and s the angle bisectors at A and B, respectively. Suppose that r and s meet at the point I which lies inside the triangle. Draw lines l and m through I that are perpendicular to AC and BC respectively. If l meets AC at point X and m meets BC at Y, then triangle IXC is congruent to triangle IYC.

Proof. By Postulate 1, draw segment IC. By Postulate 2, we can extend segment AB and use Euclid 1.12 to draw a perpendicular line from point I to segment AB. Let Z be the point at which the perpendicular segment meets AB. Then we know that angles BZI and AZI are right angles. It is also known that angles IYB and IXA are right angles because segments IX and IY are perpendicular to segments AC and BC, respectively. Considering triangles BIY and BIZ, we know that angles IYB and IZB are congruent to one another by Postulate 4, angles YBI and ZBI are congruent to one another because angle B is bisected by the ray BI, and segment BI is congruent to itself. Then triangles BIY and BIZ are congruent to one another by Euclid 1.26. Similarly for triangles AIX and AIZ, we know that angles IXA and IZA are congruent to one another by Postulate 4, angles ZAI and XAI are congruent to one another because angle A is bisected by the ray AI, and segment AI is congruent to itself. Then triangles AIX and AIZ are congruent to one another by Euclid 1.26. Figure 1 can help clarify the triangles within ABC that we have mentioned.

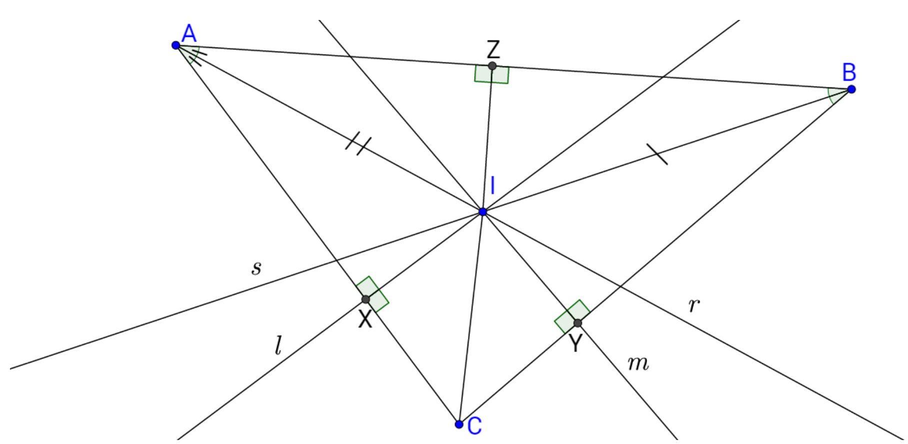


Figure 1: Triangle ABC, with inner triangles shown

Given that triangles BIY and BIZ are congruent to one another, we know that the segments IY and IZ are congruent to one another. Given that triangles AIX and AIZ

are congruent to one another, we know that the segments IX and IZ are congruent to one another. By Common Notion 1, segments IX and IY are congruent to one another because they are both congruent to segment IZ. Considering triangles IXC and IYC, we now know that sides IX and IY are congruent to one another. We also know that angles CXI and CYI are congruent right angles by Postulate 4, given that segments IX and IY are perpendicular to segments AC and BC, respectively, and that segment CI is congruent to itself. Then triangles IXC and IYC are congruent to one another by the Hypotenuse Leg Theorem.

□

The Multiplicative Identity of Segment Classes

Samantha Ancona

December 10, 2016

Assume that the definition for the multiplication of segment classes is true (Conjecture B.1). Then the following theorem is also true.

Theorem B.2. For any segment class a, it is true that a $\times$ 1 = a.

Proof. Construct right triangle ABC such that AB is an element of the unit class 1, BC is an element of the class of a, and B is a right angle. In addition, construct right triangle DEF such that angle D is congruent to angle A, angle E is congruent to angle B, and DE is an element of the segment class b. Then by the definition B.1, EF is an element of the segment class a $\times$ b. Let b = 1, then by Euclid I.26 (ASA), triangles ABC and DEF are congruent. Thus by substitution, a $\times$ 1 = a.

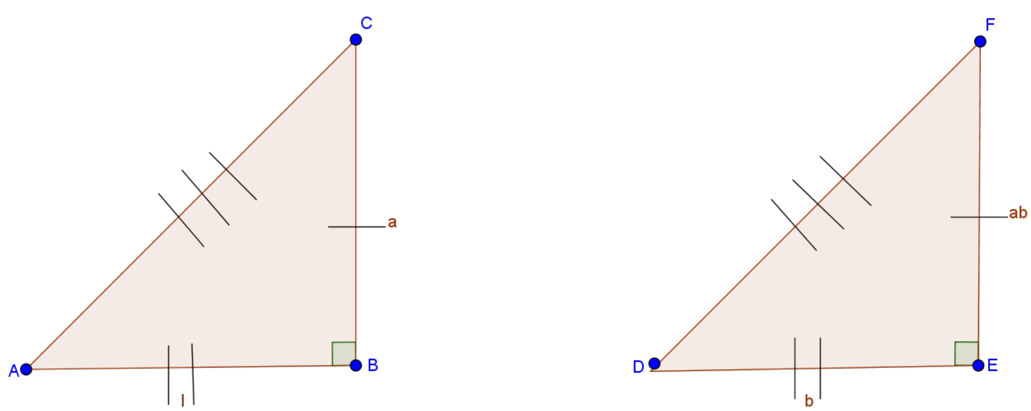


Figure 1: Triangles ABC and DEF have congruent side lengths.

□

Method for Splitting Simple, Convex Pentagons Into Three Non-Overlapping Triangles

Maria Ahrens

December 10, 2016

Communicated by Ms. Bavido.

Recall that Mr. Phaly had made a conjecture T for a method to split a Pentagon ABCDE into three, non-overlapping triangles. This paper will use a two-part argument to describe a specific method that will allow for such a result when the pentagon is simple and convex. The first part of the proof will show that the created diagonals do not consist of any of the original five sides of the pentagon. Then it will be proven why the diagonals will never cross to validate the theorem.

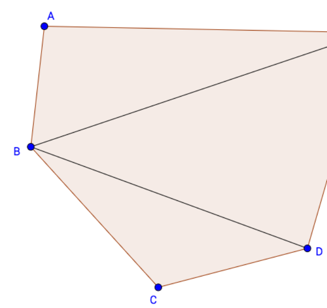


Figure 1: Pentagon ABCDE containing triangles ABD, DBE, and CBD.

Theorem Ahrens' T. Given a simple, convex pentagon ABCDE, three non-overlapping triangles can be formed by drawing diagonals from a distinct vertex to every non-consecutive vertex from that distinct chosen point.

Proof. Let ABCDE be a simple, convex pentagon.

By Miss Shere's definition of a simple, convex polygon, ABCDE has five distinct vertices with two sides extending from each vertex. It will be shown that by drawing diagonals using this method will not result in a segment that is one of the five original sides of ABCDE. Line segments can be drawn to each non-consecutive vertex from vertex B by Euclid Postulate 1, which creates BD and BE. To show that BD and BE are not an original sides of ABCDE, we will compare the original sides that extend from the endpoints comprising each of the two created line segments, BD and BE. If it is the case that none of the original sides matches with either of the new line segments, then we have verified that BD and BE are diagonals.

If it is the case that BD or BE matches one of the original sides from particular endpoints, then the segment with the match will not be considered a diagonal. For the segment BD, we will focus on the sides extending from vertices B and D. Since neither two sides from the vertex B, BA and BC, nor either of the sides extended from vertex D, DC and DE, include the segment BD, BD is a diagonal. Similarly, we know that the diagonal BE is not a side of the pentagon since the two sides extended from the vertex E are ED and EA and is not either sides BA or BC extended from Vertex B. Thus BD and BE are diagonals.

The second part to this argument is to show that diagonals BD and BE will not cross within the polygon. Since Pentagon ABCDE is convex, diagonals can only cross within the figure by Ms. Worsfold's definition of a convex polygon, so it is not possible in this case for diagonals to cross outside the figure. Points D and E are two separate and consecutive endpoints of the segment DE of a simple, convex pentagon ABCDE. Thus, BD and BE will be two segments that meet at one side, segment DE. Each diagonal would be drawn to one of the two consecutive vertices of the side DE, forming a triangle BDE, and never crossing inside the pentagon ABCDE.

As a result of BD and BE being diagonals within the pentagon and never crossing within or outside such a figure, the Ahrens' T Theorem is a valid method for forming three, non-overlapping triangles. □

Finding Centers of Circles

Erica Schultz

December 9, 2016

This paper will give five steps to find the center of a circle using a compass and straightedge, given the circumference of a circle. Following the construction is a proof that will show why this construction works.

Construction

1. Choose any points A and B on the given circumference. With a compass, construct a circle with radius AB centered at point A.

2. With a compass, construct a circle with radius AB centered at point B. One intersection point of the circle with center B and the given circumference is point A, label the other intersection point as C.

3. With a compass, construct a circle with radius BC centered at point C.

4. Let the intersection points of circles with centers A and B be called D and E. With a straightedge, create a line that goes through points D and E.

5. Let the intersection points of circles with centers B and C be called F and G. With a straightedge, create a line that goes through points F and G. The intersection point of the two lines created will be the center of the circle with given circumference.

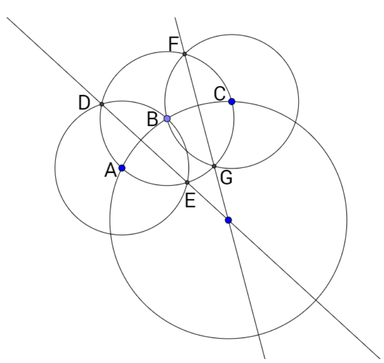


Figure 1: Constructions to find the center of a circle, given the circumference

Proof. Let the intersection of lines DE and FG be point O. By Postulate 1, we want to create segments AB and BC. We can see that these are chords of the circle O. By Postulate 1, we also want to create segments AD, AE, DB, EB, BF, BG, FC, and GC.

Segments AB, AD, and AE are congruent to one another because they are all radii of circle A. Segments AB, DB and EB are congruent to one another because they are all radii of circle B. By Common Notion 1, segments AD, AE, DB, and EB are congruent to one another. Then by definition of a rhombus, $ADBE$ is a rhombus. By Theorem 1.7, the diagonals of the rhombus $ADBE$, namely AB and DE, meet at right angles. Let this intersection point of the diagonals be called H. Consider the two right triangles DHA and DHB. We know that segments DA and DB are congruent to one another, segment DH is congruent to itself, and angles DHA and DHB are congruent right angles. Then by the Hypotenuse Leg Theorem, triangles DHA and DHB are congruent to one another. Since line DE cuts segment AB at right angles, and into congruent segments AH and HB, then line DE is the perpendicular bisector of segment AB.

A similar argument can be made to prove that line FG is the perpendicular bisector of the segment BC.

By the Porism in Euclid 3.1, since AB is a chord of the circle we are given, then the center of our given circle must lie on the line DE. Similarly by the Porism in Euclid 3.1, since BC is a chord of the circle we are given, then the center of our given circle must lie on the line FG. Thus, the center of the circle given must be point O, the intersection of lines DE and FG.

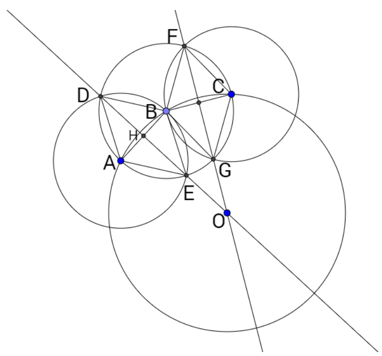


Figure 2: Construction proof visual

□

Constructing a Cord in a Given Circle with Given Length

Taryn VanRyswyk and Heather Bavido

December 10, 2016

Communiated by Ms. Goedken.

Challenge 12.5. Given a segment AB, a circle with center O and a point P inside the circle, construct a line through P on which the circle cuts off a segment congruent to AB. When exactly is this construction possible?

Construction 12.5. Let circle O with point P inside of it and segment length AB be given.
Step 1: Pick a point C on circle O.
Step 2: Draw a circle centered at C with radius AB. This creates two points of intersection. Label one of the points D.
Step 3: Draw line segment CD. This will create a cord of circle O that is congruent to the given segment AB.

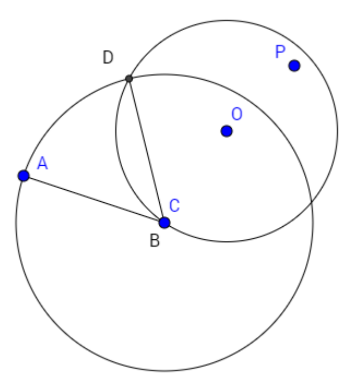


Figure 1: Step 1-3: Finding a cord congruent to AB

Step 4: Draw a circle with center O and radius OP. This circle will intersect the cord CD. Label one of these intersection points E.

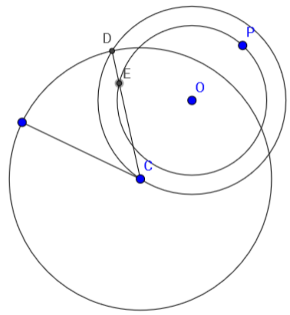


Figure 2: Step 4: Finding point E on the cord

Step 5: Draw a circle with center P with radius EC. This will intersect the original circle centered at O at 2 points. Label one of them point Q.
Step 6: Draw a ray from point Q through point P. This will intersect the given circle O. Label this point R.
Note: Cord RQ is congruent to line segment AB and goes through the given point P.

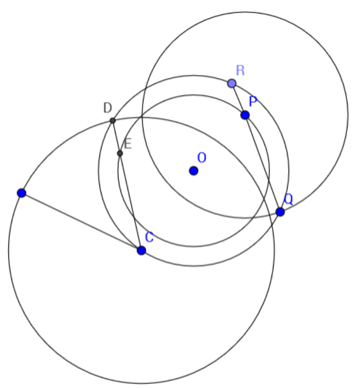


Figure 3: Steps 5-6: Finding the cord RQ which is congruent to line segment AB.

When is this construction possible?
The previous construction is only possible when the following two situations do not occur:

1) In Step 2, if circle C with radius AB does not create a point of intersection on circle O, then line segment AB is too large for the construction to work.

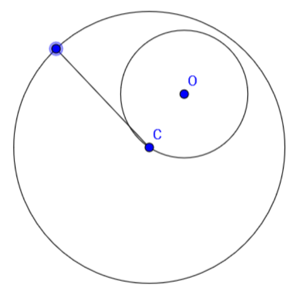


Figure 4: An example where Step 2 does not create a point of intersection on circle O

2) In Step 4, if circle O with radius OP does not intersect the cord CD, then line segment AB is too short for the construction to work.

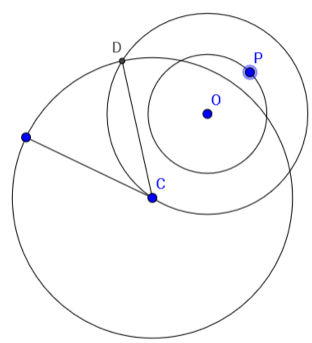


Figure 5: An example where Step 4 does not create a point of intersection on cord CD

Proof. Congruencies:
1) Line segment OP and OE are congruent because they are radii of circle O with radius OP.
2) Line segments OC, OQ, OR, and OD are congruent because they are radii of the given circle O.
3) Line segment PQ is congruent to line segment EC because point Q is created from circle P with radius EC.

Due to the congruencies obtained from 1, 2, 3, by Euclid proposition 1.8, triangle EOC is congruent to triangle POQ. Since triangle EOC and triangle POQ are congruent, the angles of the triangles are congruent. Particularly, angle ECO and PQO are congruent.

Because of congruency 3, triangles ROQ and DOC are isosceles triangles. By Euclid 1.5, the base angles of triangle ROQ are congruent to each other, and the base angles of DOC are congruent to each other. Since angle ECO and angle PQO are congruent to each other, this makes the base angles of triangle DOC and ROQ congruent to each other.

Since the base angles of triangles DOC and ROQ are congruent, and line segment DO is congruent to line segment OR, by Euclid 1.26, triangle DOC is congruent to triangle ROQ. Since the triangles are congruent, then the corresponding sides are congruent.

Thus, cord CD is congruent to cord QR. Since cord CD is congruent to the given line segment AB because it was created from circle C with radius AB, and CD is congruent to cord QR, by Common Notion 1, cord QR is congruent to the given line segment AB.

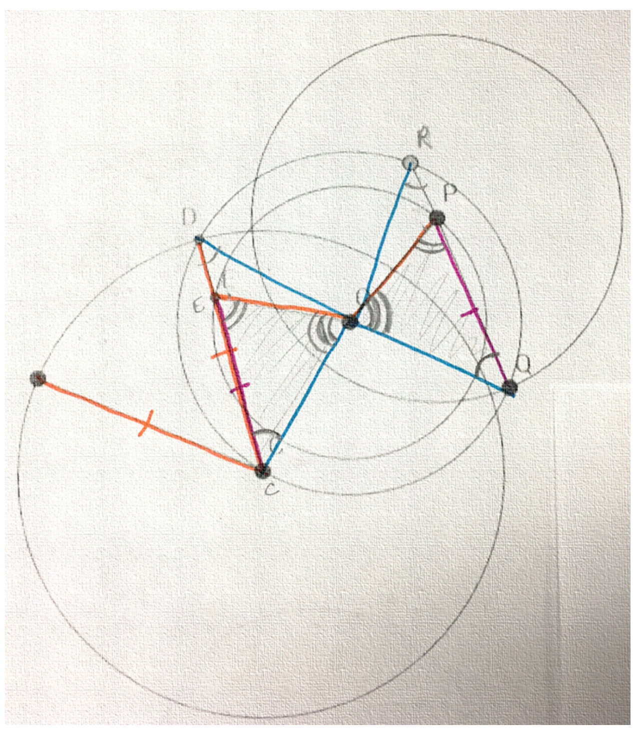


Figure 6: Construction showing congruencies

□

Triangles of Equal Content

Tessa Cohen

December 10, 2016

Communicated by Ms. Ancona.

Note: For this particular argument, it will be necessary for triangles ABC and DEF to share one congruent angle.

Theorem 13.2. Let ABC and DEF be triangles. Let X be the midpoint of DE, Y the midpoint of BC. If AB is congruent to DX and EF is congruent to BY, then ABC and DEF have equal content.

Proof. Let BC be a segment with midpoint Y. Let DB be a segment off of point B with midpoint X. By *Euclid Postulate 1*, create segments XC and DY. We have triangles DBY and XBC. For purposes of this proof, We will consider point X to also be point A, point B to also be point E and point Y to also be point F. Therefore we have triangle ABC and triangle DEF with angle ABC and angle DEF in common.

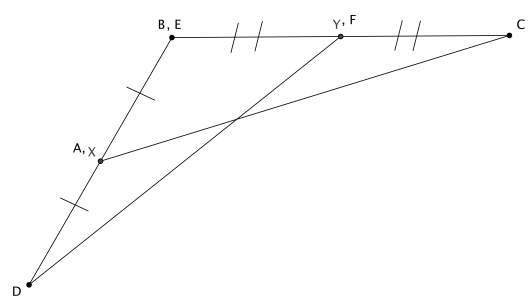


Figure 1: Triangles ABC and DEF. X is the midpoint of DE, also the vertex A. Y is the midpoint of BC, also the vertex F.

By *Euclid Proposition I.31*, construct a line, l, off of point C that is parallel to segment DB. Similarly, construct a line, r, off of point D that is parallel to segment BC. By *Cohen's Lemma*, we know that since DB is parallel to line l and line r cuts segment DB then line r also cut line l. We will call point of intersection between the lines l and r, point G. We now have quadrilateral BCGD with pairs of parallel sides DB, CG and BC, DG. Thus, BCGD is a parallelogram. By *Euclid Postulate 1*, create segment DC.

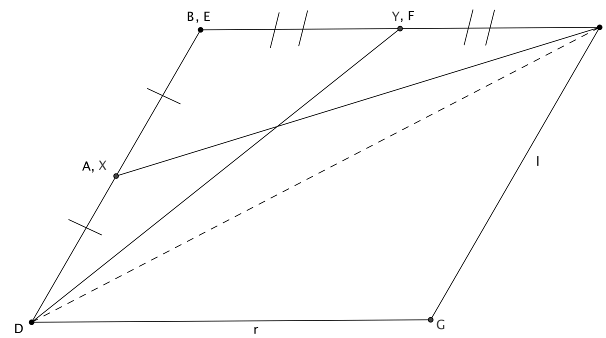


Figure 2: Parallelogram BCGD.

By Euclid Proposition I.34, since BCGD is a parallelogram, diameter DC will bisect the area. Thus, triangle DBC and triangle DGC will have equal content. Since X is the midpoint of segment DB, segment DX will be congruent to segment AB. If we consider triangles ABC and DXC, we see that they share a congruent base, DX and AB, and a point C on the same parallel, CG. Therefore, by Euclid Proposition I.38, triangle ABC is of equal content to triangle DXC. Since triangle ABC and triangle DXC make up triangle DBC, and are of equal content, triangle ABC will be half of the content of triangle DBC. Triangle DBC is also half of the content of parallelogram BCGD, therefore, triangle ABC will be one quarter of the content of BCGD. Similarly, since Y is the midpoint of segment BC, segment EF will be congruent to segment YC. Considering triangles EFD and FCD, we see they have congruent bases, EF and YC, and share point D on the same parallel, DG. Therefore, by Euclid Proposition I.38, triangles EFD and DYC will be of equal content. Since triangles EFD and DYC make up triangle DCB, we know that triangle DEF will be half the content of triangle DBC. Since triangle DBC is half the content of parallelogram BCGD, triangle DEF will be one quarter of the content of BCGD. Triangle ABC and triangle DEF were both found to be one quarter of the content of BCGD therefore they are of equal content. □

The Parallelogram Law

Tessa Cohen

December 10, 2016

Communicated by Ms. Ahrens.

Theorem 13.6. Let ABCD be a parallelogram. Then the squares on the diagonals taken together have equal content with the squares on the four sides taken together.

Proof. There will be two different cases to this argument. The first case is only applicable to rectangles, whereas the second case is applicable to any other parallelogram.

Case 1. Given a rectangle ABCD, we want to prove that the squares on AC and BD taken together are of equal content to the squares on AB, BC, CD and DA taken together. Draw diagonals AC and DB by Euclid Postulate 1.

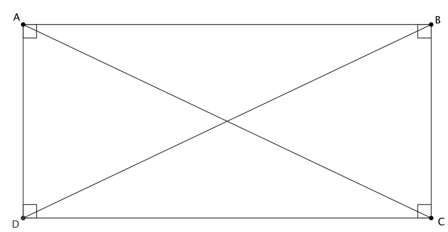


Figure 1: Rectangle ABCD

Euclid Proposition I.47 says that in right-angled triangles the square on the side subtending the right angle will be of equal content to the squares on the sides containing the right angle. Since ABCD is a rectangle, all four angles will be right by the definition of a rectangle. Therefore, we have right triangles ABC and DBC. Thus, by Euclid Proposition I.47, the square on segment AC subtending the right angle ABC will be of equal content to the square on segment AB and the square on segment BC taken together. Similarly, the square on DB will be of equal content to the square on segment BC and the square on segment DC taken together. When the square on AC and the square on DB are taken together they are of equal content to the square on AB, the square on DC, and twice the square on BC taken together. In a rectangle, there are two pairs of congruent sides by definition. In this case, AB and DC make up one pair with AD and BC being the other pair. Since segment AD is congruent to segment BC, the square on BC will be of equal content to the square on

AD. Therefore, we can say that the square on AC and the square on DB taken together is of equal content to the squares on AB, BC, CD, and DA taken together.

Case 2. Given any parallelogram ABCD that is not a rectangle, we want to prove that the squares on AC and BD taken together are of equal content to the squares on AB, BC, CD and DA taken together. Draw diagonals AC and BD by Euclid Postulate 1. Now, by Euclid Postulate 2, extend segment DC into a ray in the direction from D to C. By Euclid I.12, drop perpendiculars from point A to point F on ray DC and from point B to point E on ray DC.

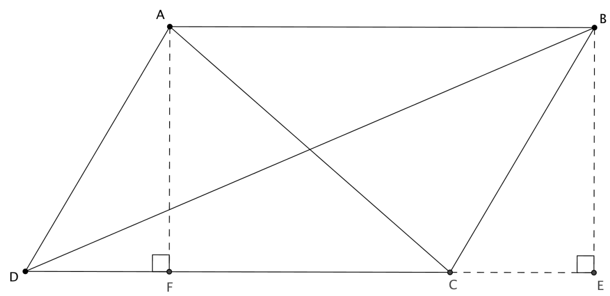


Figure 2: Parallelogram ABCD with diagonals AC and DB drawn and perpendiculars dropped from points A and B.

Since segment DA is parallel to segment BC, by definition of a parallelogram, we know that angles BCD and ADC will add up to two rights by Euclid Proposition I.29. Since ABCD is not a rectangle, neither angle BCD or angle ADC will be right. Therefore, one of the angles will be obtuse and the other will be acute. We will consider triangle BCD with the obtuse angle BCD. By Euclid Proposition II.12, the square on BD will be of equal content to the square on BC, the square on CD and twice the rectangle of DC, CE taken together. Next we will consider triangle ACD with acute angle ADC. By Euclid Proposition II.13, the square on segment AC and twice the rectangle DC, DF taken together will be of equal content to the square on AD and the square on DC taken together. We want to see the content of the squares on AC and BD taken together, so we will add the contents of the square on BD and the contents of the square on AC and twice the rectangle DC, CE taken together. Then we have the square on BD, the square on AC, and twice the rectangle DC, DF taken together will be of equal content to twice the square on DC, the square on BC, the square on AD and twice the rectangle DC, CE taken together.

When perpendiculars were dropped from points A and B, a quadrilateral ABEF was created. Since segments BE and AF are perpendicular to segment DE, angle AFC and angle BEC will be right. Then, by Euclid I.28, segment AF will be parallel to segment BE. Segment AB is also parallel to segment FE, since FE is on ray DC of the original parallelogram. With two pairs of parallel sides, AB, FE and AF, BE, ABFE is a parallelogram, by definition. Then, by Euclid I.34, segment AF will be congruent to segment BE. Now we will consider triangle AFD and triangle BEC. We have congruent sides BE, AF. Since ABCD is a parallelogram, segment AD will be congruent to segment BC, by Euclid I.34. Since

AFD and BEC are right triangles, segment AF is congruent to segment BE, and they have congruent hypotenuses, by the Hypotenuse-Leg Theorem, triangle AFD will be congruent to triangle BEC. Therefore, segment DF will be congruent to segment CE. Since segment DF is congruent to segment CE, rectangle DC, DF will be congruent to rectangle DC, CE.

We also know that since ABCD is a parallelogram, by Euclid Proposition I.34, segment AB is congruent to segment DC. Therefore, the square on AB is of equal content to the square on DC.

Then we have the square on AC, the square on BD, and twice the rectangle DC, CF taken together will be of equal content to the square on BC, the square on AD, the square on AB, the square on CD and twice the rectangle DC, CF. By EC4, when subtracting figures of equal content from figures with equal content, the resulting figures will be of equal content. When we subtract twice the rectangle DC, CF from each of the quantities, we will have the square on AC and the square on BD taken together to be of equal content to the square on BC, the square on AD, the square on AB, and the square on CD taken together. Therefore, in any parallelogram, the squares on the diagonals taken together will be of equal content to the squares on the four sides taken together.

□