Cyclic Quadrilaterals

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Conjecture 9.4. Let A, B, C and D be four points. The quadrilateral ABDC is cyclic if and only if angle ABC is congruent to ADC.

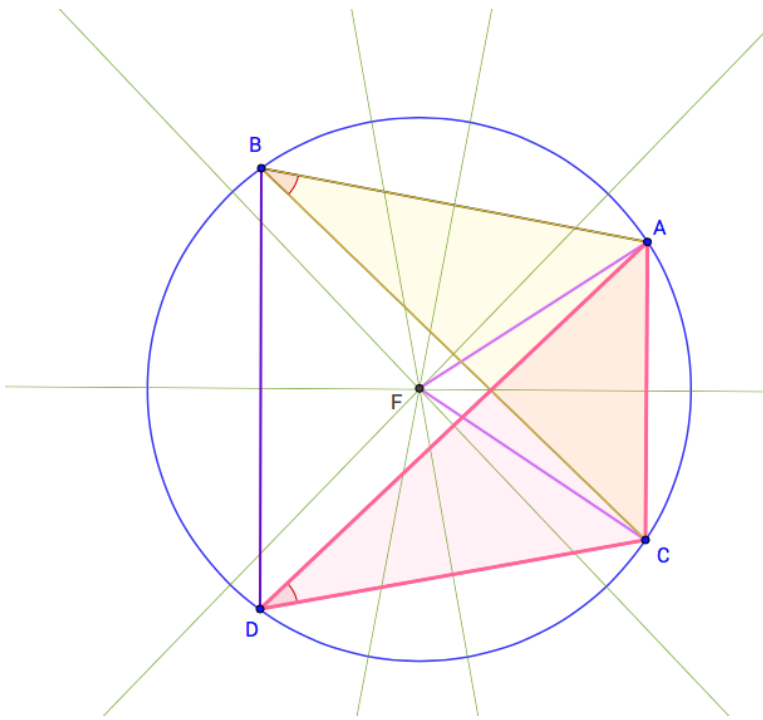

Figure 1: A cyclic quadrilateral

Proof. Part 1: If angle ABC is congruent to angle ADC, then ABDC is cyclic.

Quadrilateral ABCD contains the triangles ABC and ADC. Both of these triangles have a circumcenter, F and F'. For all four points to lie on one circle, the two circumcenters (and therefore the two circumcircles) must be the same.

By Euclid III.20, angle AFC is twice ABC, and angle AF'C is twice ADC. Since angle ADC Is congruent to angle ABC, AFC is congruent to AF'C. Since AF and CF are radii of a circle around F, they are congruent, making AFC an isosceles triangle. Similarly, AF' and

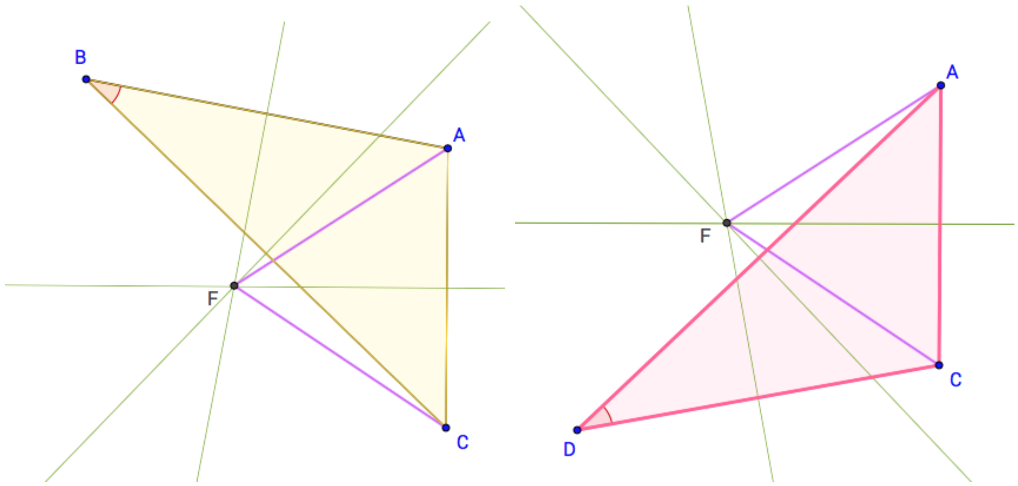

Figure 2: Triangle ABC and triangle ADC

CF' are radii of the circle around F', making AF'C an isosceles triangle.
In the triangle AFC, let's call the point where the altitude meets AC point D. Since FD is the altitude, it meets AC at a right angle, bisects angle AFC, and bisects AC.
In the triangle AF'C, let's call the point where the altitude meets AC point E. Since F'E is the altitude, it bisects the angle AF'C, and meets AC at a right angle. Since AF'C is congruent to AFC, this new angle AF'E is congruent to angle AFD. Since both points E and D are midpoints of AC, AC is congruent to AD. Using Angle-angle-side with the angles AF'E and AFD, angles AEF' and ADF, and sides AE and AD, triangles AF'E and AFD are congruent. Since these are halves of the isosceles triangles AFC and AF'C respectively, AFC and AF'C are congruent. Thus, F is the same point as F', and the triangles have the same circumcircle. Therefore, the quadrilateral ABDC is cyclic.

Part 2: If ABDC is cyclic, then angle ABC is congruent to angle ADC.

By Euclid III.21, two angles on a circle from the same segment are congruent. In this case the shared segment is AC, which is a chord of the circle The vertices of angles ABC and ADC both lie on the circle. Therefore, angle ABC is congruent to angle ADC. □

Refereed by Harmony Van Nevele.