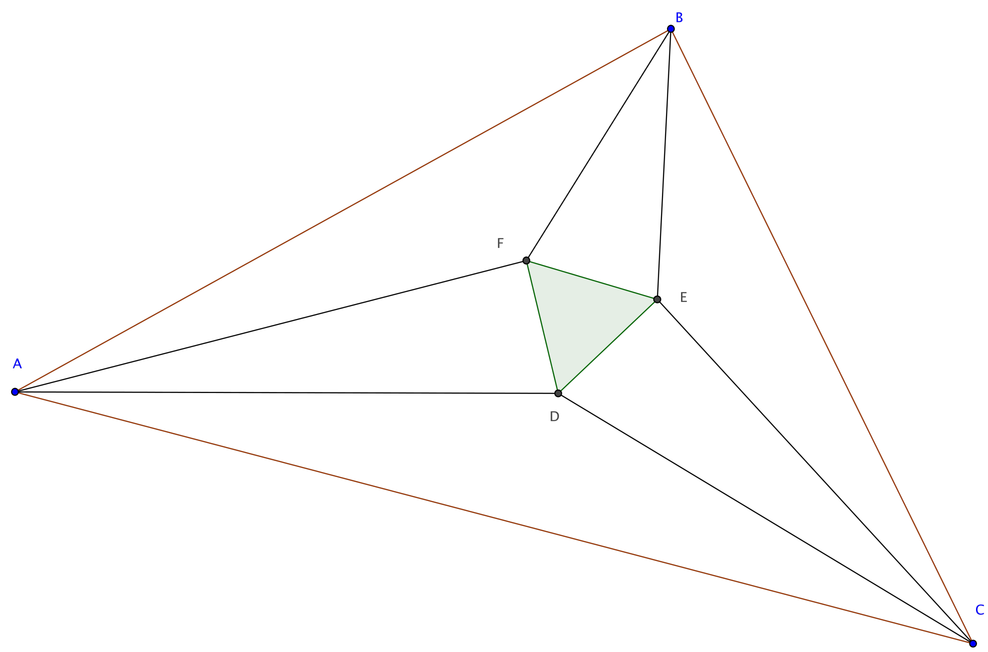
Transactions
in
Euclidean Geometry



Volume 2018F

Issue # 5

Table of Contents

It Has to Be a Square

Lexis Wiegmann

September 10, 2018

Communicated by: The Editor.

Conjecture 1.1 b. Let ABCD be a rhombus. Then, angle BAC is congruent to angle BDC.

This statement is false for when ABCD is a general rhombus.

Lemma . If ABCD is a square, then angle BAC is congruent to angle BDC.

Proof. Let ABCD be a square. The definition of a square is that all four sides are congruent and all four angles are right angles. Line AC and line BD bisect each other from Ms. Falck's Theorem of bisectors. If all of the angles are congruent, right angles, then each of the bisected angles will be congruent as well. For example, angle BAC is congruent to angle ABD. Since all of the bisected angles are congruent, then angle BAC is congruent to angle BDC. □

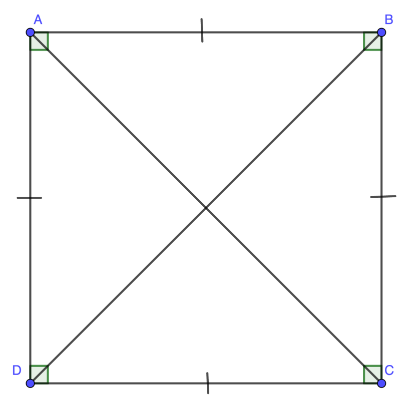


Figure 1: Square ABCD has four right angles and four congruent sides.

Conjecture 1.1 b is true only when it is a square.

Theorem . Let ABCD be a rhombus. If angle BAC is congruent to angle BDC, then ABCD is a square.

Proof. Let ABCD be a rhombus. We know that line segment AC and line segment BD are bisectors from Ms. Falck's Theorem of bisectors. From the definition of bisectors, we know that the angles on each side of the bisector are congruent. For example, angle DAC is congruent with angle BAC. This goes for all of the other angles as well.

From conjecture 1.6, we know that a rhombus is a parallelogram. By Euclid I.29, the opposite interior angle BAC is congruent to angle DCA. Angle D and angle C will add up to two right angles because of Euclid I.29 as well. For a general rhombus, we know angle D and angle C are not right angles, so they can not be congruent.

Since angle D and angle C are not congruent, the angles that are bisected can't be congruent either. For example, angle BDC is not congruent to ACD. Because of Euclid I.29, we know that angle BAC and angle DCA are congruent. If BDC in not congruent to ACD, then angle BAC can not be congruent to BDC. If they aren't congruent, rhombus ABCD can not be a square. Therefore, if it is not a square, angle BAC and angle BDC are not congruent. □

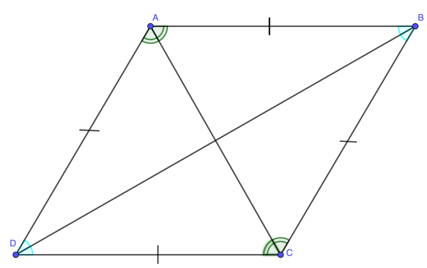


Figure 2: Rhombus ABCD has angle B and angle D congruent, as well as angle A and angle C congruent.

By the way, this still leaves the question, is it possible to construct a rhombus which is not a square, and fails to have the condition of this theorem?

The Diagonals of a Rhombus

Lauren Falck

September 28, 2018

Communicated by: The Editor.

Theorem B. Let ABCD be a rhombus. Assume that the diagonals AC and BD meet at point E. Then diagonals BD and AC bisect each other.

Proof. By definition of a rhombus, lines AB, BC, CD, AD are congruent to each other. By postulate number 1, points A and C can be connected, and points B and D can be connected.

By Euclid I.8, since side AD, BC, DC, and AB are congruent and triangle ADC and triangle ABC share the diagonal AC, then angle ADC is congruent to ABC. Similarly, since side AD, BC, DC, and AB are congruent and triangle BAD and triangle BCD share the diagonal BD, then angle BAD is congruent to BCD.

The diagonals DB and AC cross at point E. Since angle BAD is congruent to angle BCD and angle ABD is congruent to angle ADC, angle EAD is congruent to angle to angle ECB and angle EBC is congruent to angle EDA. By Euclid I.26, since angle EAD is congruent to ECB and side AD is congruent to BC and angle EDA is congruent to angle EBC, triangle AED is congruent to triangle BEC. Since triangle AED is congruent to triangle BEC, line AE is congruent to line EC and line DE is congruent to EB. Thus, point E is the mid point of both of the diagonals BD and AC. Therefore, the diagonals BD and AC bisect each other.

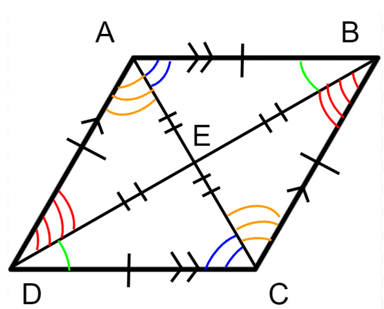


Figure 1: This is a picture of Rhombus ABCD.

□

Congruent Angles in a Rhombus

Jason Stine

December 4, 2018

Communicated By Ms.Miller

Theorem 1.3. Let ABCD be a rhombus. Then angle ABC is congruent to angle ADC.

Proof. Let ABCD be a rhombus. Due to the definition of a rhombus, we know sides AB, BC, CD, and DA are congruent. Let AC and BD be diagonals, such that they intersect at point X. Because of Falck's Theorem B, we know that AC and BD bisect each other. Since BD bisects AC at point X, then AX is congruent to XC. Since AC bisects BD at point X, then BX is congruent to XD. By Euclid Proposition I.8 we know that the four triangles, AXB, BXC, CXD, and DXA are congruent because they have three congruent corresponding sides. Since the four triangles are congruent, they also have congruent corresponding angles. Thus angle ABX is congruent to XDC. Similarly, angles XBC and ADX are congruent. We know that angle ABC is congruent to the sum of angles ABX and XBC. We also know that angle ADC is congruent to the sum of angles ADX and XDC. Recall that angles ADX and XBC are congruent and angles ABX and XDC are congruent. Since angles ABC and ADC are equal to the sum of congruent angles, the two angles ABC and ADC are congruent.

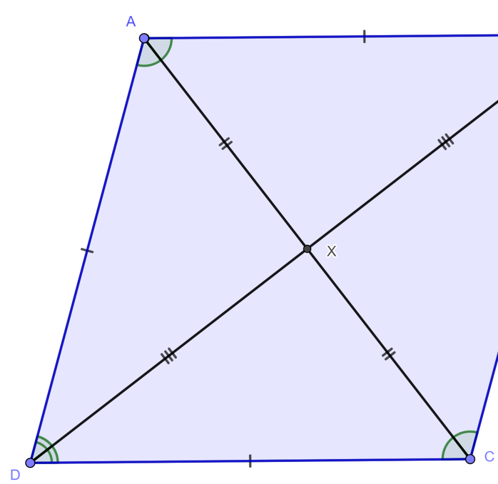


Figure 1: Rhombus ABCD

□

Varignon's Theorem

Payson VandeLune

December 4, 2018

Communicated by: Jaclyn Miller.

Theorem 3.6 (Miller). Let ABC be a triangle, D the midpoint of AB and E the midpoint of AC. Then the line through E and D, called a midline, is parallel to the line through B and C.

Theorem 3.7. Let ABCD be a quadrilateral. The midpoints of the four sides are the vertices of a parallelogram.

Proof. Let ABCD be a quadrilateral with W, X, Y, and Z the midpoints of sides AB, BC, CD, and DA, respectively. We will show that WXYZ is a parallelogram. Draw diagonal BD by Euclid Postulate I.1. Notice BD forms two triangles within ABCD, namely, triangle ABD and triangle CBD. Since Z is the midpoint of segment DA and W the midpoint of segment AB, segment ZW is the midline of triangle ABD. Therefore, by Theorem 3.6, midline ZW is parallel to side BD of triangle ABD. Similarly by Thereom 3.6, notice midline XY of triangle CBD is parallel to side BD. Since segments ZW and XY are mutually parallel to a common line, they are also parallel to one another.
Now draw diagonal AC of ABCD by Euclid Postulate I.1. Notice AC also forms two triangles within ABCD, namely, triangle ABC and triangle ADC. By an argument similar to the one stated previously, we can conclude that midlines WX and YZ are mutually parallel to diagonal AC, and thus, parallel to one another.
Since ZW is parallel to XY, and WX is parallel to YZ, WXYZ is a parallelogram. □

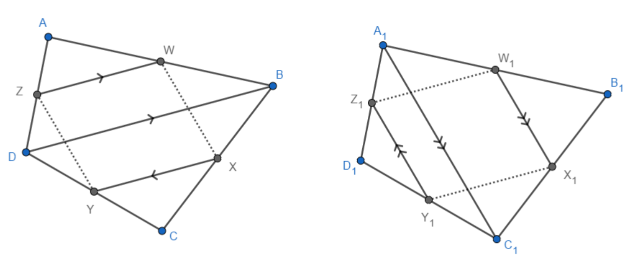


Figure 1: On the left is ABCD with diagonal BD, and midlines ZW and XY emphasized. On the right is ABCD with diagonal AC, and midlines WX and YZ emphasized.

Not Always Regular

Alexa DeVore and Lexis Wiegmann

November 16, 2018

Communicated by: Mr. VandeLune.

This conjecture is proved false by a counterexample.

Conjecture 6.3. Let ABCD be a rhombus. If angle A is congruent to angle C, then ABCD is regular.

Proof. Let line segment AC be the radius of a circle with A being the center. Let AC be the radius of another circle with point C being the center. There are two points where the first circle meets the second circle, make those points B and D. By Euclid Postulate I.1, draw lines CB, AB, AD, and CD. Since these segments are all the radii of congruent circles, we knew they themselves must be congruent. Because they are all congruent we know that this figure is a rhombus. From this construction of a rhombus, we know that angles A and C are twice as big as angles B and D, based off Euclid Proposition I.1. Euclid Proposition I.1 gives a construction of an equilateral triangle. Since our construction uses these steps to create two equilateral triangles, we know that angles A and C are composed of two of these triangle angles . Because of this reason, all of the angles of the rhombus are not congruent. Because the angles aren't congruent, it can't be equiangular. If this figure isn't equiangular, than rhombus ABCD is not regular. □

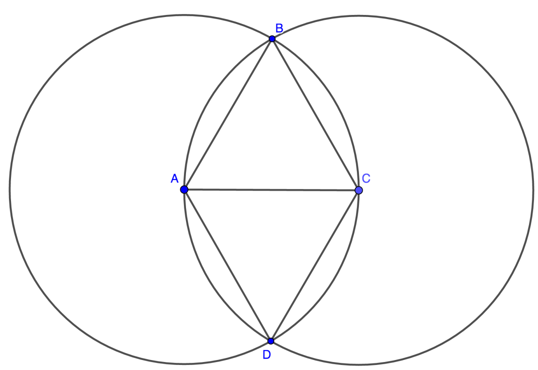


Figure 1: This is a picture of the rhombus.

Inconstructible Triangle

Mr. Stine

November 19, 2018

Communicated by: Ms. Falck

Theorem 7.1. It is possible to construct three line segments which are not congruent to the sides of any triangle.

Proof. Given the circle centered at B with radius BC and diameter AC, the segments AB, BC, and AC are not congruent to the sides of any triangle. Notice that segments AB and BC taken together are equal to, but not greater than, segment AC. By Euclid 1.22, in order for three segments to construct a triangle, it is necessary that any two segments taken together be greater than the remaining side. Since we don't have two segments taken together that are greater than the third, a triangle cannot be constructed with sides congruent to segments AB, BC, and AC.

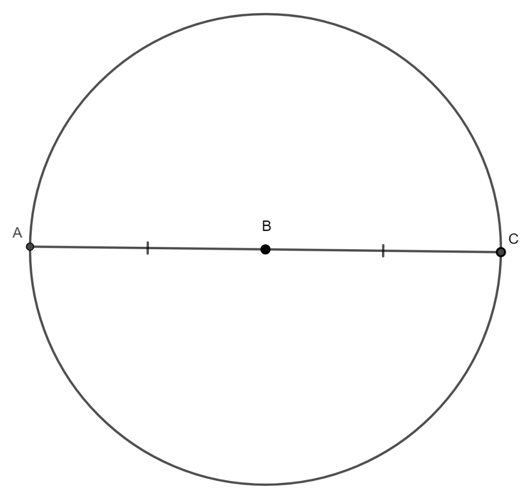


Figure 1: Circle B with radius BC and diameter AC.

□

Cyclic Quadrilaterals

Jaclyn Miller

November 26, 2018

Communicated by: Mr. Warner.

Theorem 9.4a. Let ABCD be a cyclic quadrilateral. Then, angle DAC is congruent to angle DBC.

Proof. Let ABCD be a cyclic quadrilateral. Angle DAC stands on chord CD. Angle DBC also stands on chord CD. Thus, by Euclid Proposition III.21, angle DBC is congruent to angle DAC.

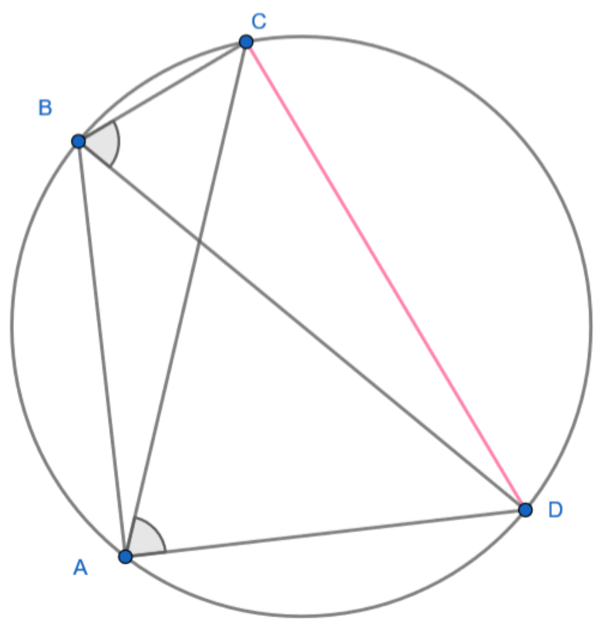


Figure 1: Quadrilateral ABCD.

□

Same Angle

Lexis Wiegmann

November 27, 2018

Communicated by: Ms. Falck.

Question 11.5. Given an angle at a point A and given a ray emanating from a point B, construct an angle at B congruent to the angle at A having the given ray as the side. (Par 4)

Proof. Let there be an angle at point A and a point B with a ray emanating from point B. Pick point C on one of the lines emanating from angle A.

1. By Euclid Postulate I.3, create a circle centered at A with radius AC. Call the intersection point D.
2. Make the same circle with radius AC at point B. Where that circle intersects the ray at B, call that point E.
3. By Euclid I.3, make a circle centered at E with radius CD. By Circle Circle Intersection Theorem, we get two points, make one of them F.
4. By Euclid Postulate I.1 draw line BF.

By construction, we know line segments AC and BE are congruent and line segments DC and FE are congruent as well. Since line segments AD and BF are radii of the same circle (by construction) they are congruent to each other. By Euclid Proposition I.8, triangles ACD and BEF are congruent. Since they are congruent, we know that the corresponding angles are congruent as well. Therefore, angle CAD and angle EBF are congruent. □

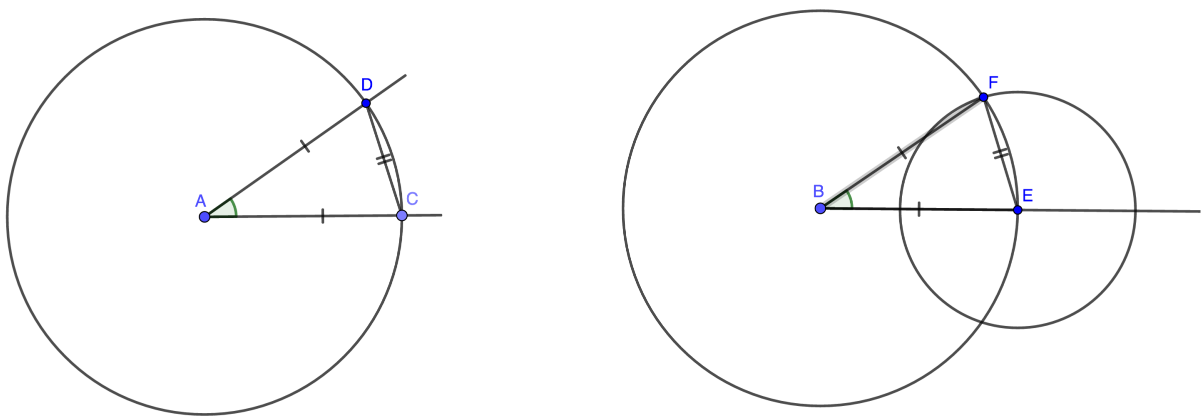


Figure 1: Angle CAD is congruent to angle EBF.

These Specific Triangles Have Equal Content

Mr. Stine

December 4, 2018

Communicated by:Ms. Miller

Theorem 13.2. Let ABC and DEF be triangles. Let X be the midpoint of DE, Y the midpoint of BC. If AB is congruent to DX, EF is congruent to BY, and angle ABC is congruent to angle DEF, then ABC and DEF have equal content.

Proof. Let ABC and DEF be triangles such that DE is twice the length of AB, BC is twice the length of EF, and angle ABC is congruent to angle DEF. On triangle DEF, let X be the midpoint of DE. Let segment EF be extended to point C such that point F is the midpoint of EC. By Euclid Postulate 1 create segments XF and XC. Triangle XEC is congruent to triangle ABC by Euclid Proposition 1.4 in which the two triangles have two congruent sides and the contained angle congruent. By EC_1 we know that congruent triangles XEC and ABC have equal content. By Ms. Miller's Midline Theorem, segment XF is parallel to segment DC. By Euclid Proposition 1.37 triangles DXF and CXF have equal content because they have the same base and lie within the same parallels. Notice that triangle XEF has equal content to itself. Notice that triangle XEC is made of the sum of triangles XEF and XFC. Similarly triangle DEF is made of the sum of triangles XEF and DXF. Since triangles XEC and DEF are both equal to the sum of equal contents, where XEF is common, XEC and DEF have equal content. By Euclid Common Notion 1, triangles DEF and ABC have equal content.

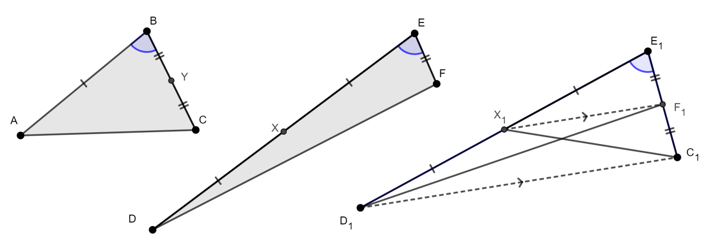


Figure 1: Triangles ABC, DEF, and DEF with triangle XEC falling on it.

□

Hypotenuse Leg Congruence

Alexa DeVore

December 2, 2018

Communicated by: Mr. Stine.

Conjecture AA. Prove the hypotenuse leg theorem using properties of circles.

Proof. Let AC be the diameter of a circle. Place point B on the circumference of the circle and connect to points A and C by Euclid Postulate 1 to create triangle ABC. From Theorem 7.4, we know that triangle ABC must have a right angle at B. Now construct a circle with radius AB, centered at C by Euclid Postulate 3. Let where this circle centered at C intersects our original circle, be named point D. Now connect point D to C and A respectively by Euclid Postulate 1, and create another right triangle named CDA.

Since angles BCA, and DAC subtend congruent arcs on the same circle, these angles must be congruent from Euclid Proposition 3.21. Similarly, since both angles BCD and BAD subtend segment BD, these must also be congruent from Euclid Proposition 3.21. Therefore, angle BAC and DCA must be congruent since they are composed of congruent angles. Therefore by the ASA property from Euclid Propostion 1.4, triangle ABC must be congruent to triangle CDA and the hypotenuse leg theorem holds.

□

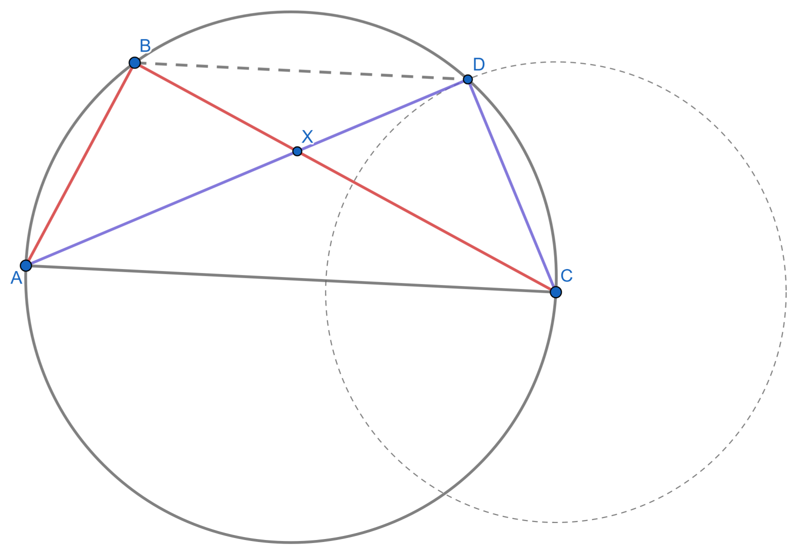


Figure 1: This is a picture of the construction.

Proving Hypotenuse Leg using Area

Mr. Stine

December 4, 2018

Communicated by: Ms. Wiegmann

Theorem 13.4. Let ABC and DEF be triangles. If side AC is congruent to DF, side BC congruent to EF, angle BAC and EDF be right, and triangles ABC and DEF have equal content, then ABC and DEF are congruent.

Proof. Let ABC and DEF be triangles such that angle BAC and angle EDF are right angles, side BC and EF congruent, AC and DF congruent, and ABC and DEF have equal content. Since triangles ABC and DEF have equal content and congruent bases, by Euclid Proposition 1.40 these triangles are within the same parallels. This means that line BE is parallel to line ACDF. By Euclid Postulate 1 let line segment BE and AD be created. Notice that quadrilateral ABED is a parallelogram because BE and AD are parallel and AB and ED are parallel. We know that AB and ED are parallel because of Euclid Proposition 1.28 where angles BAC and EDF are both right angles and all right angles are congruent. By Euclid Proposition 1.34, parallelograms have opposite sides congruent, meaning side BA and ED are congruent. Since triangles ABC and DEF have three corresponding congruent sides, by Euclid Proposition 1.8, triangles ABC and DEF are congruent.

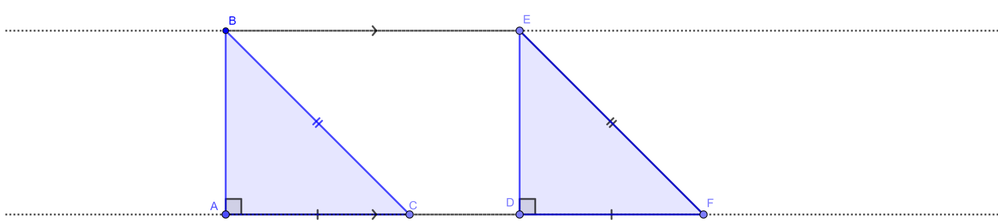


Figure 1: Triangles ABC and DEF

□

The Parallelogram Law

Jaclyn Miller

December 3, 2018

Communicated by: Ms. Falck.

Theorem 13.6. Let ABCD be a parallelogram. Then the squares on the diagonals taken together have equal content with the squares on the four sides taken together.

Proof. Let ABCD be a parallelogram. Then, by Euclid Proposition I.34, opposite sides AB and CD are congruent and parallel, and opposite sides AD and BC are congruent and parallel. Then, since AB and CD are congruent, the squares on AB and CD have equal content. Similarly, since AD and BC are congruent, the squares on AD and BC have equal content. Angles ADC and ABC are congruent, and angles BCD and BAD are congruent, by Euclid Proposition I.34.

Case One: Without loss of generality, suppose that angles ABC and ADC are obtuse, and angles BCD and BAD are acute.

By Euclid Postulate I.1, extend AD to a line. By Euclid Proposition I.11, draw a perpendicular line from point C to line AD. Call the point where the perpendicular and AD intersect point E. Similarly, by Euclid Postulate I.1, extend AB into a line. By Euclid Proposition I.11, draw a perpendicular line from point C to line AB. Call the point where the perpendicular and AB intersect point F.

Since angles ABC and ADC are obtuse, and AC subtends both angles, the square on AC has equal content with the square on CD, the square on AD, and twice the rectangle on (AD, DE) taken together by Euclid Proposition II.12. Similarly, the square on AC also has equal content with the square on AB, the square on BC, and twice the rectangle on (AB, BF) taken together by Euclid Proposition II.12. When combined, twice the square on AC has equal content with the square on AB, the square on BC, the square on CD, the square on AD, twice the rectangle on (AB, BF), and twice the rectangle on (AD, DE).

By Euclid Proposition I.11, draw a perpendicular line from point D to BC. Call the point where the perpendicular intersects BC point H. Similarly, draw a perpendicular line from point B to AD. Call the point where the perpendicular intersects AD point G.

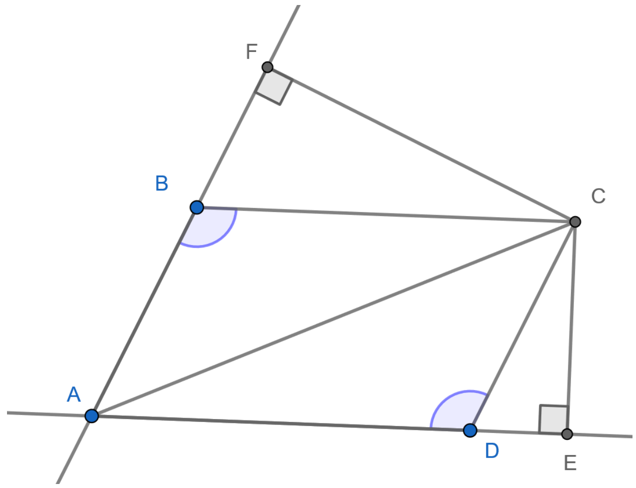


Figure 1: Parallelogram ABCD: Examining Obtuse Angles.

Since angles BCD and BAD are acute and BD subtends both angles, the square on BD has equal content with the square on AB taken together with the square on AD, but without twice the rectangle on (DA, AG) by Euclid Proposition II.13. Similarly, the square on BD has equal content with the square on BC taken together with the square on CD, but without twice the rectangle on (BC, CH). When combined, twice the square on BD has equal content with the square on AB, the square on AD, the square on BC, and the square on CD taken together, but without twice the rectangle on (DA, AG) and twice the rectangle on (BC, CH).

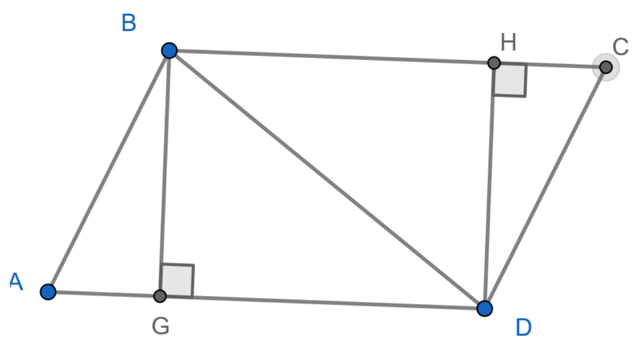


Figure 2: Parallelogram ABCD: Examining Acute Angles.

Therefore, twice the square on BD taken together with twice the square on AC has equal content with twice the square on AB, twice the square on AD, twice the square on BC, twice the square on CD, twice the rectangle on (AB, BF), twice the rectangle on (AD, DE) taken together; but without twice the rectangle on (DA, AG) and twice the rectangle on (BE, CH) taken together. Then, factoring out a two, the square on AC taken together with the square on BD has equal content with the square on AB, the square BC, the square CD, the square on AD, the rectangle (AB, BF), and the rectangle (AD, DE) taken together; but without the rectangle on (DA, AG) and the rectangle on (BC, CH).

Recall that the square on AC has equal content with the square on CD, the square on AD, and twice the rectangle on (AD, DE) taken together; and the square on AC also has equal content with the square on AB, the square on BC, and twice the rectangle on (AB, BF) taken together. Then the square on CD, the square on AD, and twice the rectangle on (AD, DE) taken together has equal content with the square on AB, the square on BC, and twice the rectangle on (AB, BF) taken together. Also, recall that the square on AB has equal content with the square on CD; and the square on BC has equal content with the square on AD. Thus, twice the rectangle (AB, BF) has equal content with twice the rectangle on (AD, DE). Hence, after factoring out a two, the rectangle (AB, BF) has equal content with the rectangle on (AD, DE).

Similarly, recall that the square on BD has equal content with the square on AB taken together with the square on AD, but without twice the rectangle on (DA, AG); and the square on BD has equal content with the square on BC taken together with the square on CD, but without twice the rectangle on (BC, CH). Then by a similar argument as stated previously, the rectangle on (DA, AG) has equal content with the rectangle on (BC, CH).

Now, focus on triangles DCH and CDE. DC is a shared side, and angles DHC and DEC are congruent because they are both right angles. Recall that ABCD is a parallelogram, so BC is parallel to line AD. Then DC is a transversal cutting two parallel lines, thus alternate interior angles DCH and CDE are congruent by Euclid Proposition I.29. Then, because angles DCH and CDE are congruent, angles DHC and DEC are congruent, and side DC is shared, triangles DCH and CDE are congruent by Euclid Proposition I.26. Corresponding parts of congruent triangles are congruent, so segment CH is congruent to segment DE.

Recall that since ABCD is a parallelogram, segment BC is congruent to segment AD. Since segments DE and CH are congruent, and segments BC and AD are congruent, the rectangle on (AD, DE) has equal content with the rectangle on (BC, CH). But the rectangle on (AD, DE) already has equal content with rectangle (AB, BF) and the rectangle on (BC, CH) already has equal content with the rectangle on (DA, AG). Thus, all four rectangles have equal content.

Recall that the square on AC taken together with the square on BD has equal content with the square on AB, the square BC, the square CD, the square on AD, the rectangle (AB, BF), and the rectangle (AD, DE) taken together; but without the rectangle on (DA, AG) and the rectangle on (BC, CH). All four rectangles have equal content; thus, the the square on AC taken together with the square on BD has equal content with the square on AB, the square BC, the square CD, and the square on AD.

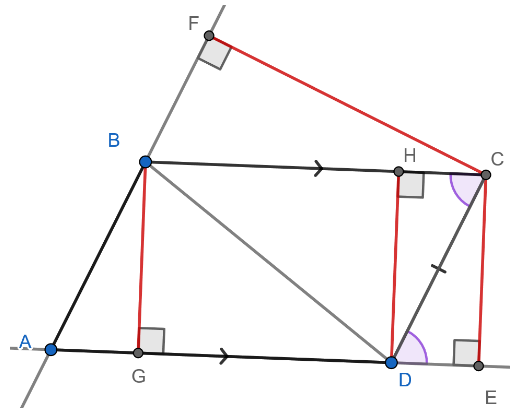


Figure 3: Parallelogram ABCD.

Case Two: Suppose ABCD is a rectangle.

By Euclid Proposition I.47, the square on AC has equal content with the square on CD and the square on AD taken together. The square on AC also has equal content with the square on AB and the square on BC by Euclid Proposition I.47. Thus, twice the square on AC has equal content with the square on BC, the square on AB, the square on CD, and the square on AD taken together.

Similarly, the square on BD has equal content with the square on CD and the square on BC taken together, by Euclid Proposition I.47. The square on BD also has equal content with the square on AB and the square on AD taken together, by Euclid Proposition I.47. Thus, twice the square on BD has equal content with the square on BC, the square on AB, the square on CD, and the square on AD taken together.

Then twice the square on BD and twice the square on AC taken together has equal content with twice the square on AB, twice the square on BC, twice the square on CD, and twice the square on AD taken together. After factoring out a two, the square on AC and the square on BD taken together have equal content with the square on BC, the square on AB, the square on CD, and the square on AD taken together.

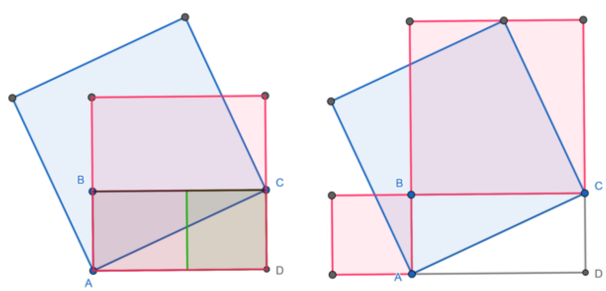


Figure 4: Parallelogram ABCD.

□