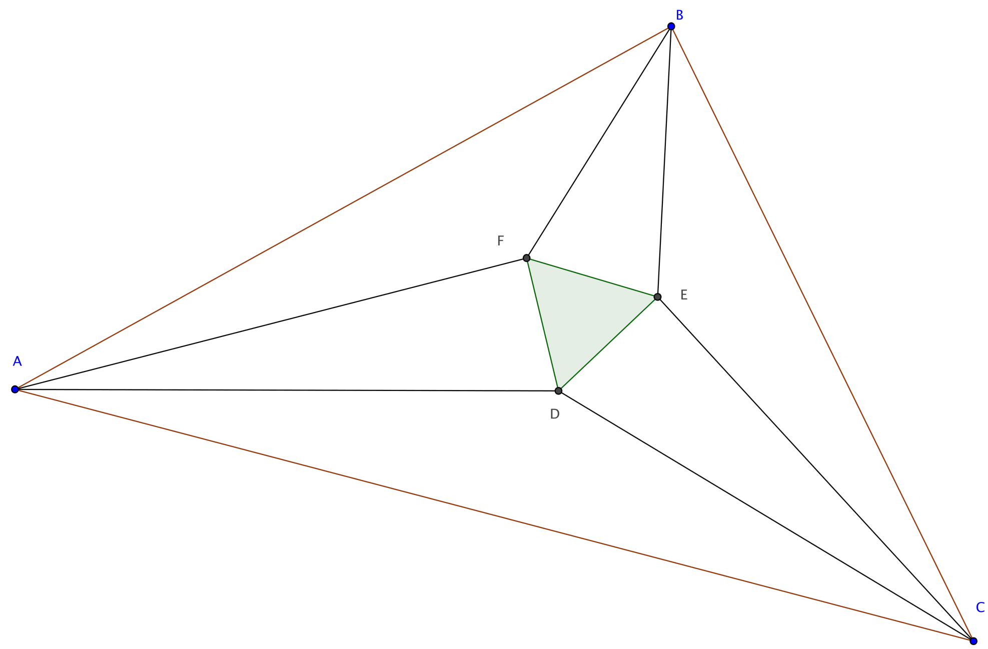
Transactions

in

Euclidean Geometry



Issue # 7

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Transitive Property of Segments

Heather Bavido

December 5, 2016

Communicated by Ms. Cohen.

Conjecture A1. If AB, CD, and EF are line segments such that AB is less than CD and CD is less than EF, then AB is less than EF.

Proof. By the definition of less than for segments, since line segment AB is less than line segment CD, there exists a point M1 on line CD where line segment CM1 is congruent to line segment AB. Thus, CD is the sum of CM1 and M1D. Likewise, there exists a point M2 on line segment EF where EM2 is congruent to CD.
Since CD is the sum of line segments CM1 and M1D, and EM2 is congruent to line segment CD, then EM2 can be written as the sum of line segments CM1 and M1D. Therefore, there exists a point M3 on EM2, which is located at the same point as M1. Since point M3 lies on line segment EF, then line segment EM3 plus M3M2 is congruent to line segment CD. Further, line segment EM3 is congruent to line segment AB. Thus, AB is less than EF. □

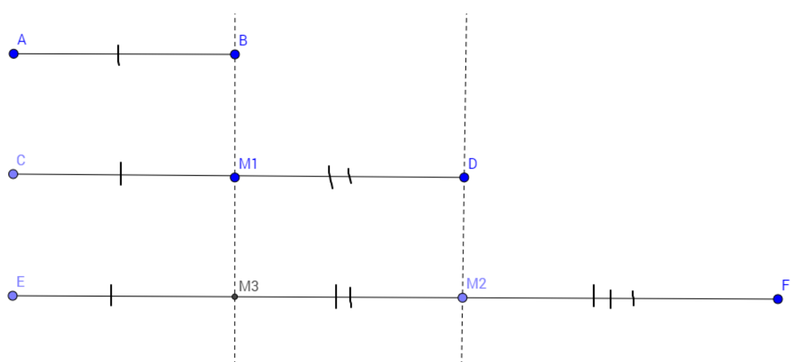

Figure 1: Demonstrating how M points are found

Segment Addition

Erica Schultz

December 5, 2016

Communicated by Ms. Cohen.

Theorem A.2. If AB is a segment, then $AB + AB$ may not be equal to $BA + BA$ but these two new segments are congruent.

Proof. Let AB be a segment. We want to construct the segments $AB + AB$ and $BA + BA$, and show that these two new segments are congruent. We will first look at the construction of $AB + AB$. By the definition given to us on segment addition, we will extend the segment AB to a ray from A through B. We will now choose a point E on the ray so that B lies between A and E, and BE is congruent to AB. We have a new segment, AE. Now we will look at the construction of $BA + BA$. By the definition given to us on segment addition, we will extend the segment BA to a ray from B through A. We will now choose a point F on the ray so that A lies between B and F, and AF is congruent to BA. We now have a new segment, BF. In Figure 1, we can see that $AB + AB$ and $BA + BA$ have given us two new segments, AE and BF, respectively. These segments are not equal considering that AE is not the exact same segment as BF. We may start with the same segment AB, but end with two unequal segments because the way we add the segments give us different end points. However, because AB is congruent to BA, then $AB + AB$ will create a segment AE that is of congruent length of the segment BF created by $BA + BA$.

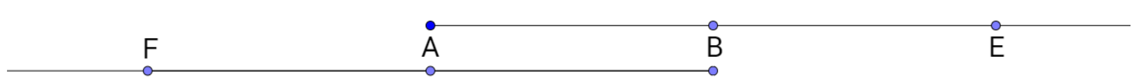

Figure 1: Visual for segment addition $AB + AB$ and $BA + BA$

□

Comparing Two Segment Classes

Heather Bavido

December 5, 2016

Communicated by Ms. Maus.

Conjecture A3. Let a and b be segment classes. Then exactly one of the following holds:

- $a \cap b = \emptyset$ or
- a = b

Proof. Case 1: Let a be the segment class of AB and b be the segment class of CD where AB is not congruent to CD.

$$a = \{AB_1, AB_2, AB_3, AB_4, ...\} b = \{CD_1, CD_2, CD_3, CD_4, ...\}$$

Figure 1: Line segment AB is not congruent to line segment CD

Since the segment class of AB contains all of the segments that are congruent to AB, and segment AB and segment CD are not congruent, then the segments that are congruent to segment CD are not elements of the segment class a.

Similarly, since the segment class of CD contains all of the segments that are congruent to CD, and segment AB and segment CD are not congruent, then the segments that are congruent to segment AB are not elements of the segment class b.

Since the segments that are congruent to segment CD are not in the segment class a, and the segments that are congruent to segment AB are not in the segment class b, then the segment classes a and b have no common elements between them. Thus, $a \cap b = \emptyset$

Case 2: Let a be the segment class of AB and b be the segment class of CD where AB is congruent to CD.

$$a = \{AB_1, AB_2, AB_3, AB_4, ...\}b = \{CD_1, CD_2, CD_3, CD_4, ...\}$$

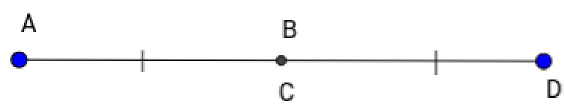

Figure 2: Line segment AB is congruent to line segment CD

Since the segment class of AB contains all of the segments that are congruent to AB, and segment AB and segment CD are congruent, then the segments that are congruent to segment CD are elements of the segment class a.

Similarly, since the segment class of CD contains all of the segments that are congruent to CD, and segment AB and segment CD are congruent, then the segments that are congruent to segment AB are elements of the segment class b.

Since the segments that are congruent to segment CD are in the segment class a, and the segments that are congruent to segment AB are in the segment class b, the segment classes a and b share all of their elements. Thus, a = b $\square$

Commutative Addition: Segment Classes

Heather Bavido

December 5, 2016

Communicated by Ms. Worsfold.

Problem A.5. Let a and b be segment classes. Then a + b = b + a.

Proof. Let line segment AB be an element of a. Let line segment CD be an element of b.

First I will manipulate the left hand side of the equation. The addition of segment classes a + b is equal to the addition of the line segments of lengths AB + CD. The summation of line segments AB + CD make up the line segment AD. Since both AB and CD are less than AD, there exists a point (B=C) on AD where line segment AB is congruent to AB and line segment CD is congruent to line segment CD.

Figure 1: Line segment AD = AB + CD

Next I will manipulate the right hand side of the equations. The addition of segment classes b + a is equal to the addition of the line segments of lengths CD + AB. The summation of line segments CD + AB make up the line segment CB. Since both CD and AB are less than CB, there exists a point (A=D) on CB where line segment CD is congruent to CD and line segment AB is congruent to line segment AB.

Figure 2: Line segment CB = CD + AB

Lastly, I will show that the prior manipulations are congruent to the other. Since line segment AD is the summation of AB and CD, and the line segment CB is the summation of CD and AB, AD is congruent to BC. Thus a + b = b + a.

□

Associative Property of Addition for Segment Classes

Rebecca Shere

December 5, 2016

Communicated by Ms. Bavido.

Problem A.6 Let a,b,c be segment classes. Then (a+b) + c = a + (b+c)

Proof. Let a be the segment class of all line segments congruent to AB, b be the segment class of all line segments congruent to CD, and c be the segment class of all line segments congruent to EF. We will show the sum of (a+b)+c is equal to the sum of a+(b+c) by doing the addition of both sides proving the sums are congruent and thus in the same segment class.

First, we will preform the addition of (a+b)+c. We take a line segment AB from segment class a, and draw a ray from A through B. Then, we pick line segment CD from b, and make a new point G on ray AB, where BG is congruent to CD. Next, we pick line segment EF from class c and place point H on ray AB such that line segment GH is congruent to line segment EF. We have now created a new line segment AH that is the sum of (a+b)+c.

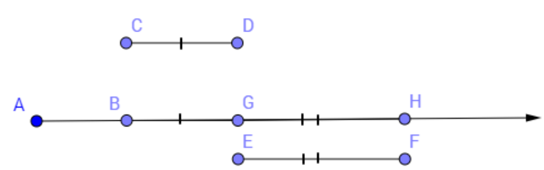

Figure 1: Shows the construction of (a+b)+c

Next, we will preform the addition of a+(b+c). We begin with line segment CD from the segment class b. Then we will draw a ray from C through D and make a point J; such that, DJ is congruent to EF. Then, we take line segment AB from segment class a, and make a ray from A through B. We place a point K on ray AB where line segment BK is congruent to CJ(the sum of b+c). Now, we have created the line segment AK as the sum of a+(b+c).

Now, line segment AH is made up of line segments that are congruent to AB,CD,and EF. Line segment AK is made up of line segments that are congruent to AB,CD, and EF.Since the parts parts are all congruent, the wholes will be congruent, by Euclid's Common Notion 2. Therefore, line segment AH and AK are congruent and will belong to the same segment class. We shall call this segment class d. Thus when the segment addition is preformed, on both sides, we end up with d=d.

□

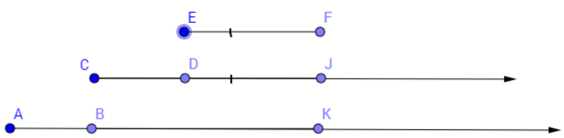

Figure 2: Shows the construction of a+(b+c)

Segment Class Properties

Kandy Schwan

December 5, 2016

Communicated by Ms. King.

Theorem :. Let a and b be segment classes. Then one and only one of the following holds: (i) a = b; (ii) There is a class c such that a + c = b; (iii) There is a class c such that a = b + c.

Proof. First, I will address property (i). This says that segment classes a and b are congruent. If segment class a consists of all of the line segments congruent to AB and segment class b consists of all of the line segments congruent to CD, then segment class a is congruent to segment class b and all of the segments congruent to AB are congruent to all of the line segments congruent to CD.

The next property (ii) says that there is a class c such that $a + c = b$. Let a new segment class a consist of all of the line segments congruent to AB, such that line segment A1G1 belongs to segment class a. Let a new segment class c consist of all of the line segments congruent to EF, where line segment G1F1 belongs to a segment class c. Then, let a new segment class b consist of all of the line segments congruent to CD such that $a + c = b$, where line segment A1F1 belongs to segment class b. Therefore, line segment A1G1 taken together with line segment G1F1 is congruent to line segment A1F1. Thus, segment classes a and c are less than segment class b.

The last property (iii) says that there is a new segment class c such that $b + c = a$. Let a new segment class b consist of all of the line segments congruent to CD, such that line segment AG belongs to a. Let a new segment class c consist of all of the line segments congruent to EF, where line segment GB belongs to segment class c. Then, let a new segment class a consist of all of the line segments congruent to AB such that $a + c = b$, where line segment AB belongs to segment class b. Therefore, line segment AG taken together with line segment GB is congruent to line segment AB. Thus, segment classes b and c are less than segment class a.

Therefore, proving segment classes a and b are congruent in case (i), segment class a is less than segment class b in case (ii), and segment class b is less than segment class a in case (iii), only one of the properties can hold at a time for segment classes a, b, and c. □

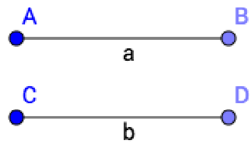

Figure 1: Property (i).

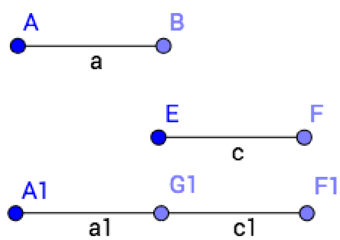

Figure 2: Property (ii).

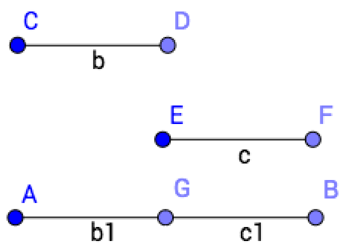

Figure 3: Property (iii).

"Less than" with Segment Classes

Mackenzie Mitchell

December 5, 2016

Communicated by Ms. Bavido.

Theorem A.8. The notion of "less than" extends unambiguously to segment classes.

Proof. In a previous definition, segment AB is less than segment CD when there exists a point M lying between C and D such that AB is congruent to CM. We want to show this definition extends to segment classes.

Let a and b be segment classes such that segment class a contains segments that are congruent to AB and segment class b contains segments that are congruent to CD. Without loss of generality, let AB be less than CD by the previous definition. With a holding all segments congruent to AB and b holding all segments congruent to CD, any segment in class a will always be less than any segment in class b. This is due to the specification of AB being less than CD.

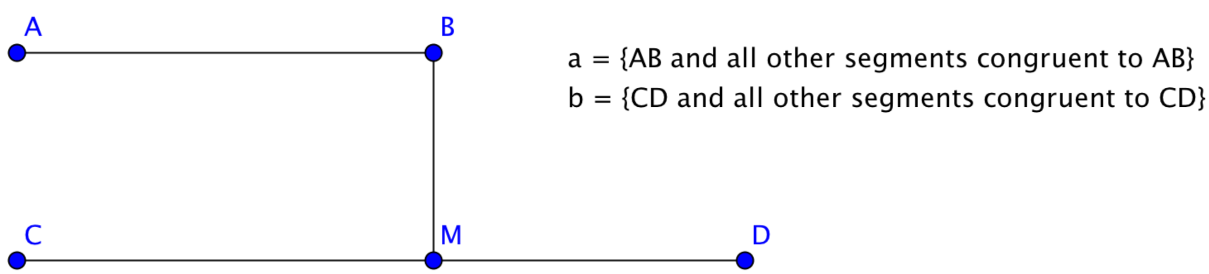

Figure 1: Original definition incorporated into segment classes

Since segment class a will always be less than segment class b, we can extend this definition of "less than" to segment classes.

□

Adding a Constant Segment

Mackenzie Mitchell

December 5, 2016

Communicated by Ms. Bavido.

Theorem A.9. Let a,b,c be segment classes such that a is less than b. Then a+c is less than b+c.

Proof. Let a,b,and c be segment classes such that a contains all segments congruent to AB, b contains all segments congruent to CD, and c contains all segments congruent to EF. Also, let AB be less than CD. Thus all segments in segment class a are less than all segments in segment class b by Theorem A.8. We want to show that when we add c to both a and b, a+c will be less than b+c.

Referring to figure 1, we have segment AB and CD which belongs to segment classes a and b respectively. Since AB is less than CD, we can find a point M on CD in which CM is congruent to AB, by the definition. Using the definition of "sum" with segments, we can add segment EF to AB with EF belonging to segment class c. When adding EF to AB, this creates line segment AX, where BX is congruent to EF. In figure 1, this segment is labeled i. We can also add EF to CM and locate a point M_2 where line segment CM_2 is congruent to AX. Now we add EF to CD to create CY by the definition, where DY is congruent to EF. In figure 1, this segment is labeled j.

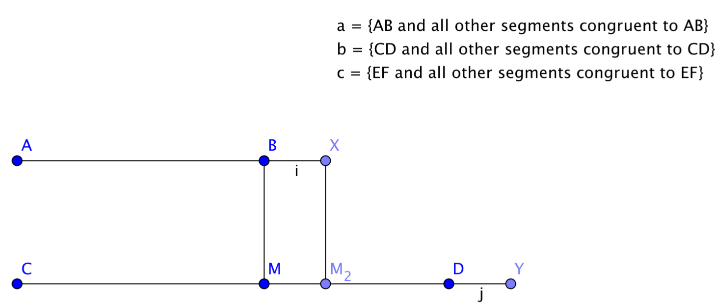

Figure 1: Adding c to both a and b with corresponding congruent points M and M_2

Since AX is congruent to CM_2 and M_2 is in between CY by Ms. Bavido's definition, AX is less than CY. This occurs because when adding a constant length to two segments in which one segment is less than the other, the original lesser segment will continue to be less than the other. This is shown by the point M_2 by the definition of "less than" with segments and segment classes. So when adding class c to class a and b, with a being less than b, a+c will be less than b+c.

□

Angle of Intersecting Chords and Center Angle Formula

Duece K Phaly

December 5, 2016

Communicated by Ms. Goedken.

Conjecture 10.1 Let Γ be a circle with center O. Let X be a point in the interior of the circle, and suppose that two line l and m intersect at X so that l meets Γ at points A and A' and m meets Γ at B and B'. Then 2$\langle$AXB is congruent to $\langle$AOB and $\langle$A'OB' taken together.

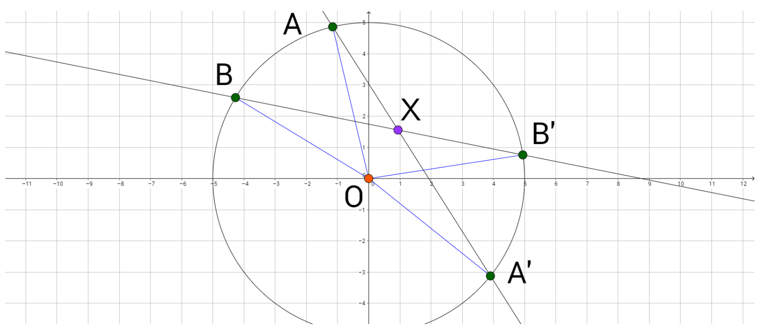

Figure 1: The set up for challenge 10.1.

Proof. We want to show 2$\langle$AXB = $\langle$AOB + $\langle$A'OB'. By Postulate 1 we will create line segments AB, A'B'. Proposition I.15 states that the vertical angles BXA and A'XB are congruent. We will refer to both of these angles as $\langle x$. By Proposition III.21 in Euclid's Elements, since $\langle$ABB' and $\langle$AA'B' lie on the same segment of circle O (ABA'B') they are congruent. We will refer to both those angles as $\langle b$. Similarly, $\langle$ BAA' and $\langle$BB'A are congruent. These angles will both be referred to as $\langle a$

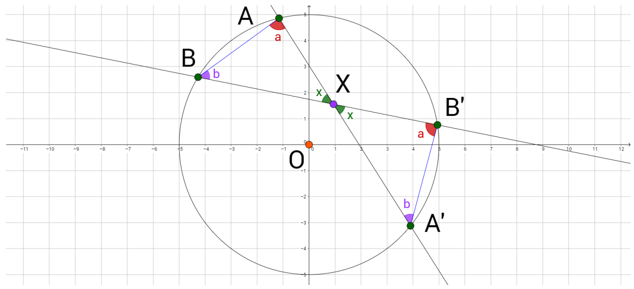

Figure 2: This image shows the angles that are proved to be congruent.Radii OB,OA,OB’,OA’ are removed for simplicity of the figure.

Since $\langle$BOA’ and $\langle a$ share the same circumference base (BA) and $\langle$BOA’ is the angle at the center and $\langle a$ is on the circumference; $\langle$BOA’ $= 2\langle a$. This is by Euclid Proposition III.20. Similarly, $\langle$AOB’ $= 2\langle b$.

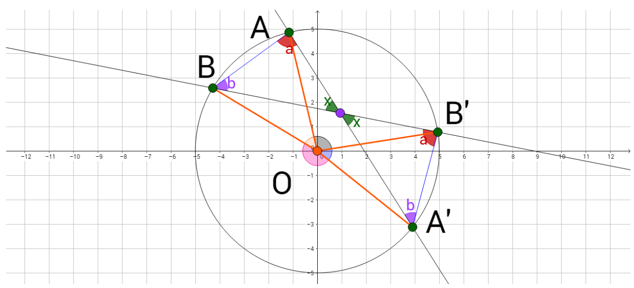

Figure 3: This image shows the angles created at the center for reference to the angles involved in the above and below paragraphs

By VanRyswyk Conjecture AE, we know that sum of the four angles ($\langle$AOB’,$\langle$A’OB’,$\langle$BOA’, $\langle$AOB) formed by the four rays (OA,OB’,OA’,OB) is equal to four right angles. Thus, $\langle$AOB’+ $\langle$A’OB’ + $\langle$BOA’ + $\langle$AOB = 4 Right Angles. By Euclid Proposition I.31 the sum of the angles in triangles AXB is equal to two right angles. Similarly for triangle A’XB’. Thus, in triangle AXB, the angles a,b,x add up to equal two right angles. Since, we proved that the angles of both triangles are congruent respectively, combining the angle equations gives us $2\langle a + 2\langle b + 2\langle$AXB = Four Right Angles.

Since, there are two equations that both equal four right angles we can set them equal to one another. Obtaining, $\langle$ AOB’+ $\langle$A’OB’ + $\langle$BOA’ + $\langle$AOB $= 2\langle a + 2\langle b + 2\langle$AXB. We know that $2\langle a = \langle$BOA’ and $2\langle b = \langle$AOB’ so by substitution our new equation is $2\langle b + \langle$A’OB’ $+ 2\langle a + \langle$AOB $= 2\langle a + 2\langle b + 2\langle$AXB. Simplifying the equation, we have $\langle$A’OB’ + $\langle$AOB $= 2\langle$AXB. Which is what we were trying to show. □

Theorem 10.1. Given a circle Γ centered at point O and a point x (x lies inside the circle). If two intersecting chords l and m which intersect each other at x and intersect the circle at points A,A’ and B,B’ respectively, then the we can say that $\langle$A’OB’ + $\langle$AOB $= 2\langle$AXB.

Relationship Between Angles and Chords

Danielle Maus

December 5, 2016

Communicated by Ms. Van Ryswyk.

Theorem 10.3. If two chords of a circle subtend different acute angles at points of a circle, then the smaller angle belongs to the shorter chord.

Proof. Let O be the center of the circle with radius AO. Let angle DEC and angle AFB be two acute angles in our circle O subtending chords CD and AB respectively. Without loss of generality let angle AFB be less than angle DEC. I want to show that the chord AB is shorter than the chord CD.

By Euclid's Postulate 1 create line segments AO BO, CO, and DO. Because these four line segments connect the center O to a point on the circle, they are all radii of our circle. Thus, line segments AO, BO, CO, and DO are all congruent to one another. By Euclid's III.20 which states, "In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base." Thus, angle AOB is congruent to two of angle AFB because they share the same base circumference (arc AB). Similarly, angle COD is congruent to two of angle DEC because they share the same base circumference (arc CD).

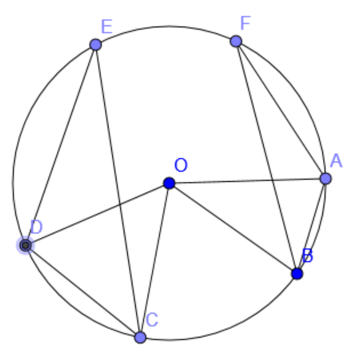

Figure 1: Circle O with Acute Angles AFB and DEC

Since angle AFB is less than angle DEC, then two of angle AFB is less than two of angle DEC. Since above we proved that two of angle AFB is congruent to angle AOB, and two of angle DEC is congruent to angle COD. By substitution, angle AOB is less than angle COD.

Looking at triangles AOB and COD, by Euclid's I.24, "If two triangles have the two sides equal to two sides respectively, but have the one angle contained by the equal straight lines greater than the other, they will also have the base greater than the base." Thus, since line segments AO, BO, CO, and DO are all congruent to one another and angle AOB is less than angle COD, chord AB will be less than chord CD.

Since acute angle AFB is less than acute angle DEC, chord AB is shorter than chord CD. Thus, smaller angles subtend shorter chords.

□