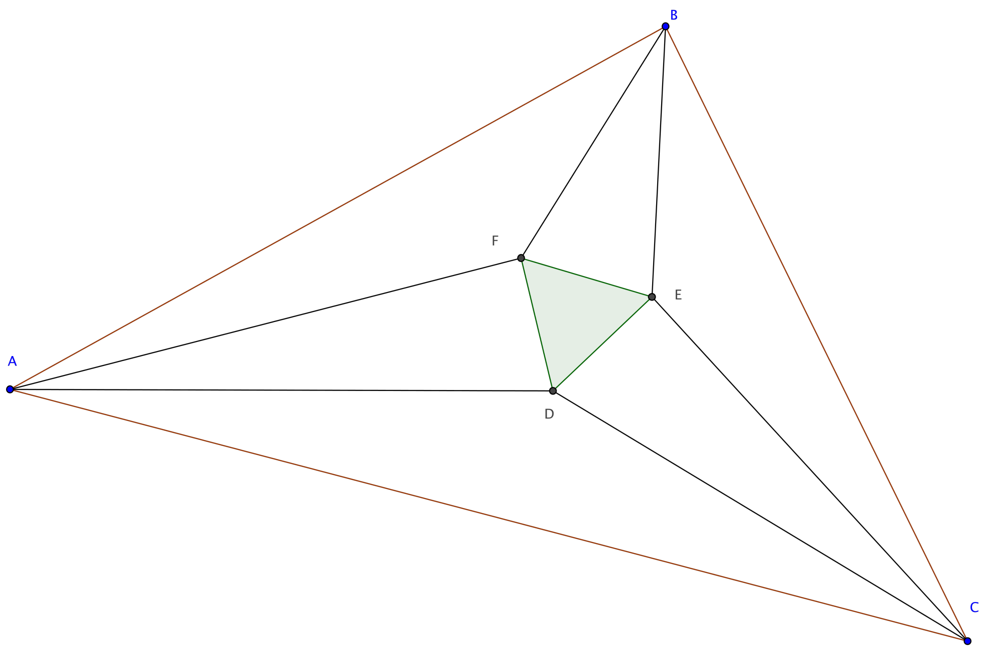
Transactions

in

Euclidean Geometry



Issue # 8

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Regularity of a Pentagon? A Counterexample

Kevin Conger

December 6, 2016

Communicated by Ms. Mitchell.

As Conjecture 6.4 was being proved, a counterexample crossed my mind. The following is the statement of Conjecture 6.4 followed by a construction and a proof of a counterexample, making Conjecture 6.4 false.

Conjecture 6.4. Let ABCDE be an equilateral pentagon. If angle A is congruent to angle B, then ABCDE is regular.

`To construct a counterexample for Conjecture 6.4:`

1. With a straightedge, create any segment EC (Euclid Postulate 1).

2. With a compass, create a circle with center E and radius EC.

3. With a compass, create a circle with center C and radius CE. Let one of the two intersection points of circle E and circle C (Circle-Circle Intersection Property) be point D.

4–5. Construct line segments CD and DE (Euclid Postulate 1).

6. Extend EC to be a line (Euclid Postulate 2), and let the intersection of line EC with circle E be point X.

7–9. Construct the perpendicular bisector of XC and name it l. Let the intersection of l and circle E, on the side of line EC opposite of point D, be point A.

10. With a compass, create a circle with center A and radius AE. Circle A and circle C will intersect at two points (Circle-Circle Intersection Property), one of which will be point E. Let the other intersection be point B.

11–13. Construct line segments CB, BA, and AE.

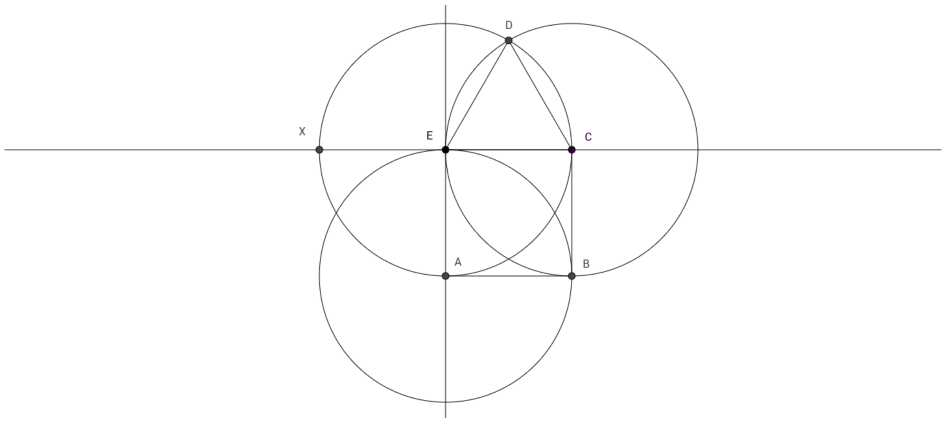

Figure 1: The construction of pentagon ABCDE.

Proof. Since circle E and circle C were constructed to have the same radius, circle E and circle C are congruent. Since circle A and circle E share radius AE, then circle A is congruent to circle E. By Euclid's Common Notion 1, circles A, C, and E are congruent. Since segments AE, ED, DC, CB, and BA are radii of congruent circles, then segments AE, ED, DC, CB, and BA are congruent segments. Thus the pentagon ABCDE is an equilateral pentagon.

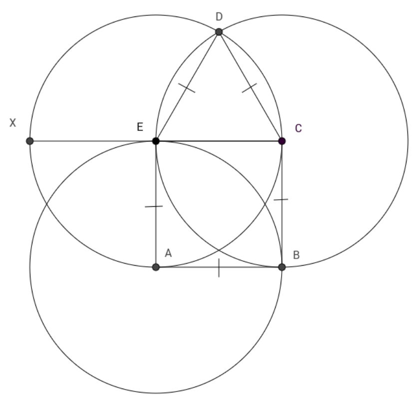

Figure 2: ABCDE is an equilateral pentagon.

By construction, EC is the radius of circle E and is extended to a line which intersects circle E at a point we named X, so EX is the diameter of circle E. Since XC is the diameter of circle E and since E is the center of circle E, then E is the midpoint of XC. Since line l is the perpendicular bisector of XC, then l goes through E. Since A is on l and l is perpendicular to EC, then angle AEC is a right angle. Since CB, BA, and EA are congruent and since EC is congruent to EA, by definition of radii of a circle, then CB, BA, EA, and EC are congruent by Euclid's Common Notion 1. Then quadrilateral ABCE is a rhombus. By Ms. Mitchell's Theorem 1.1, the opposite angles of a rhombus are congruent. Since angle AEC is a right angle, then angle ABC is a right angle. By Ms. Cohen's Theorem 1.6, rhombus ABCE is a parallelogram. Since rhombus ABCE is a parallelogram, then EC is parallel to AB. Since EA is a line which cuts two parallel lines, namely EC and AB, and since AEC is a right angle, then EAB is a right angle by Euclid I.29. Since angle ABC and angle EAB are right angles, then angles ABC and EAB are congruent. Then pentagon ABCDE is an equilateral pentagon with angle A and angle B congruent, as is the assumption of Conjecture 6.4.

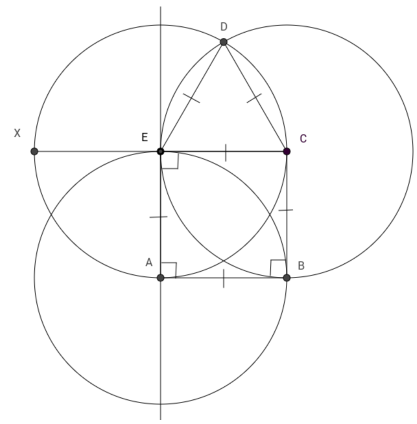

Figure 3: In pentagon ABCDE, angles A and B are right angles and are congruent.

Since angle AEC is a part of the whole angle AED and since angle AEC is a right angle, then AED is greater than a right angle by Euclid's Common Notion 5. Since angle AED is greater than a right angle and angle EAB is a right angle, then angle AED is not congruent to angle EAB. Then pentagon ABCDE is not a regular pentagon, which contradicts the conclusion of Conjecture 6.4.

□

The Three Perpendicular Bisectors of a Triangle are Concurrent

Abigail Goedken

December 6, 2016

Communicated by Ms. Bavido.

Theorem 8.4. The three perpendicular bisectors of a triangle are concurrent.

Proof. To prove this conjecture we will be using three different cases. Ahrens-Goedken theorem 8.3 will be used to state that any two of the perpendicular bisectors of a triangle meet at a point X. We will show that the third perpendicular bisector also goes through the point X.

Case 1: The perpendicular bisectors cross inside of the triangle.

Let ABC be a triangle. Let lines h, l, and m be the perpendicular bisectors of sides AB, BC, and CA respectively. By Ahrens-Goedken theorem 8.3 let X be the point where perpendicular bisectors h and l meet. We will show that perpendicular bisector m also crosses through point X.

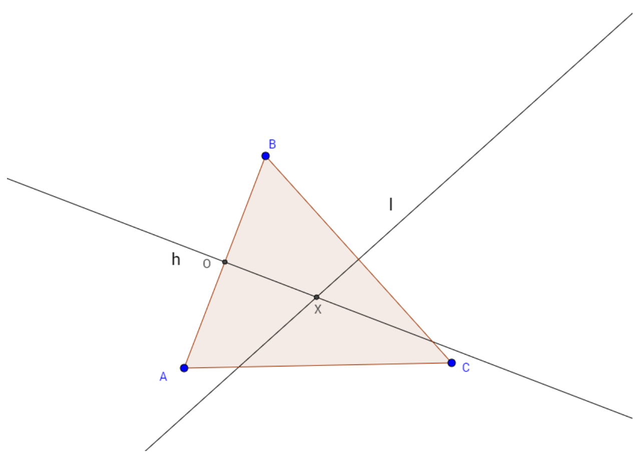

Figure 1: Case 1

By Euclid postulate one the lines from X to A and from X to B can be drawn to create line segments XA and XB. Since line h is the perpendicular bisector of AB there is a point O on line h that lies on segment AB such that AO is congruent to BO by the definition of a bisector. Since line h is the perpendicular bisector of segment AB then angle AOX and angle BOX are right angles. Since segment AO is congruent to segment BO, angles AOX and BOX are congruent, and segment OX is congruent to itself, triangles AOX and BOX are congruent to each other. Since triangle AOX and BOX are congruent then line segments AX and BX are congruent.

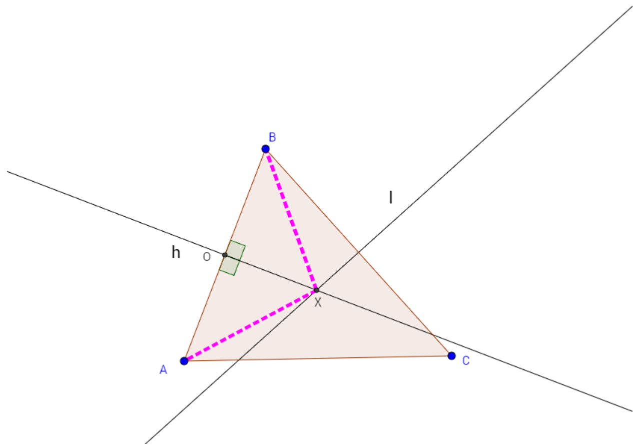

Figure 2: Segment AX is congruent to segment BX

Similarly we can create the line segment from X to C by Euclid postulate 1. Since line l is the perpendicular bisector of BC there is a point P on line l that lies on segment BC such that BP is congruent to CP by the definition of a bisector. Since line l is the perpendicular bisector of segment BC then angle BPX and angle CPX are right angles. Since segment BP is congruent to segment CP, angles BPX and CPX are congruent, and segment PX is congruent to itself, triangles BPX and CPX are congruent to each other. Since triangle BPX and CPX are congruent then line segments BX and CX are congruent. Since AX is congruent to BX and CX is congruent to BX then AX and CX are congruent to each other by Euclid's common notion 1.

The triangle AXC is an isosceles triangle since segments AX and CX are congruent to each other. Therefore, angles XCA and XAC are congruent to each other by Euclid I:5. Line m is the perpendicular bisector of segment CA. Since line m is the perpendicular bisector of segment CA, then there is a point Q on segment CA such that Q lies on line m. So, AQ is congruent to CQ by the definition of perpendicular bisector. Since segment AQ is congruent to segment CQ, angle XAQ is congruent to angle XCQ, and segments AX and CX are congruent, then triangle XAQ is congruent to triangle XCQ by Euclid I:4.

Since triangle XAQ is congruent to triangle XCQ, then angle XQA is congruent to angle XQC. Since angle XQA and XQC fall on line CA and the angles are congruent to each other, they both must be right angles by Euclid I:13. Since point Q lies on line m which is the perpendicular bisector of segment CA, and angles CQX and AQX are both right angles, segment QX lies on the perpendicular bisector m. Therefore, line m also crosses through point X. The perpendicular bisectors h,l and m are concurrent at point X.

Case 2: The perpendicular bisectors cross outside of the triangle.

Let ABC be a triangle. Let lines h, l, and m be the perpendicular bisectors of sides AB, BC, and CA respectively. By Ahrens-Goedken theorem 8.3 let X be the point where

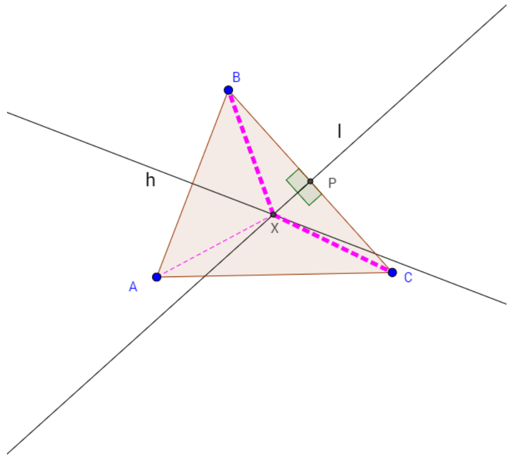

Figure 3: CX is congruent to BX

perpendicular bisectors h and l meet. We will show that perpendicular bisector m also crosses through point X.

By Euclid postulate one the lines from X to A and from X to B can be drawn to create line segments XA and XB. Since line h is the perpendicular bisector of AB there is a point O on line h that lies on segment AB such that AO is congruent to BO by the definition of a bisector. Since line h is the perpendicular bisector of segment AB then angle AOX and angle BOX are right angles. Since segment AO is congruent to segment BO, angles AOX and BOX are congruent, and segment OX is congruent to itself, triangles AOX and BOX are congruent to each other. Since triangle AOX and BOX are congruent then line segments AX and BX are congruent.

Similarly we can create the line segment from X to C by Euclid postulate 1. Since line l is the perpendicular bisector of BC there is a point P on line l that lies on segment BC such that BP is congruent to CP by the definition of a bisector. Since line l is the perpendicular bisector of segment BC then angle BPX and angle CPX are right angles. Since segment BP is congruent to segment CP, angles BPX and CPX are congruent, and segment PX is congruent to itself, triangles BPX and CPX are congruent to each other. Since triangle BPX and CPX are congruent then line segments BX and CX are congruent. Since AX is congruent to BX and CX is congruent to BX then AX and CX are congruent to each other by Euclid's common notion 1.

The triangle AXC is an isosceles triangle since segments AX and CX are congruent to each other. Therefore, angles XCA and XAC are congruent to each other by Euclid I:5. Line m is the perpendicular bisector of segment CA. Since line m is the perpendicular bisector of segment CA, then there is a point Q on segment CA such that Q lies on line m. So, AQ is congruent to CQ by the definition of perpendicular bisector. Since segment AQ is congruent to segment CQ, angle XAQ is congruent to angle XCQ, and segments AX and CX are congruent, then triangle XAQ is congruent to triangle XCQ by Euclid I:4.

Since triangle XAQ is congruent to triangle XCQ, then angle XQA is congruent to angle

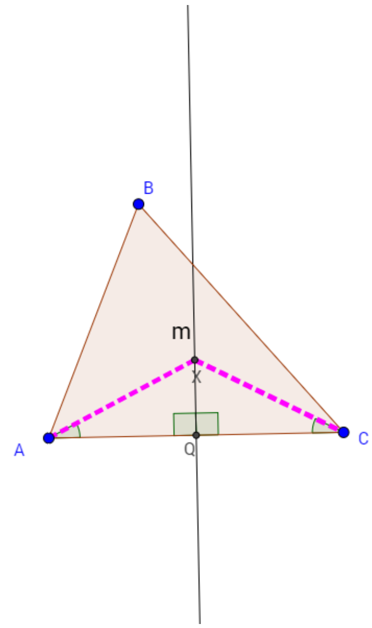

Figure 4: Line m crosses through point X

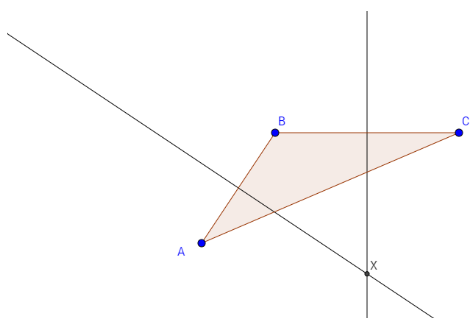

Figure 5: Case 2

XQC. Since angle XQA and XQC fall on line CA and the angles are congruent to each other, they both must be right angles by Euclid I:13. Since point Q lies on line m which is the perpendicular bisector of segment CA, and angles CQX and AQX are both right angles, segment QX lies on the perpendicular bisector m. Therefore, line m also crosses through point X. The perpendicular bisectors h,l and m are concurrent at point X.

Case 3: Triangle DEF is a right triangle.

Let triangle DEF be a right triangle. Let points A, B, and C be the midpoints of sides, DE, EF, and FD respectively. Using midline theorem 3.6 we know that midline AC is parallel to segment EF. Since AC is parallel to EF then angle DAC is congruent to angle AEF by Euclid I:29 so angle DAC is a right angle.

Similarly we know that midline BC is parallel to segment ED. Therefore angle FBC is a right angle by Euclid I:29. Since AC meets side DE at a right angle and point A is the midpoint of DE, then AC is the perpendicular bisector of side DE. Similarly, since BC meets side EF at a right angle and point B is the midpoint of EF then BC is the perpendicular bisector of side EF. Since point C is the same as point C, then the perpendicular bisectors AC

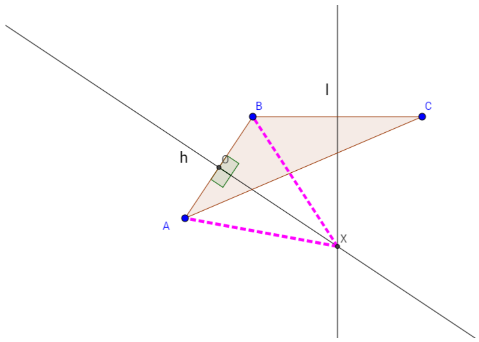

Figure 6: Segment AX is congruent to segment BX

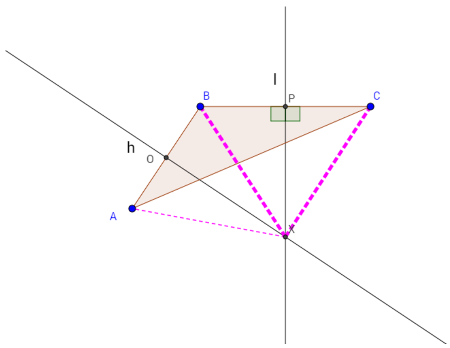

Figure 7: CX is congruent to BX

and BC meet at point C. Since point C is the midpoint of side FD, the perpendicular bisector of FD goes through point C. Therefore all three perpendicular bisectors are concurrent at point C.

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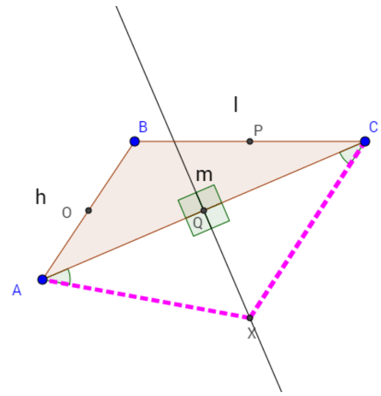

Figure 8: Line m crosses through point X

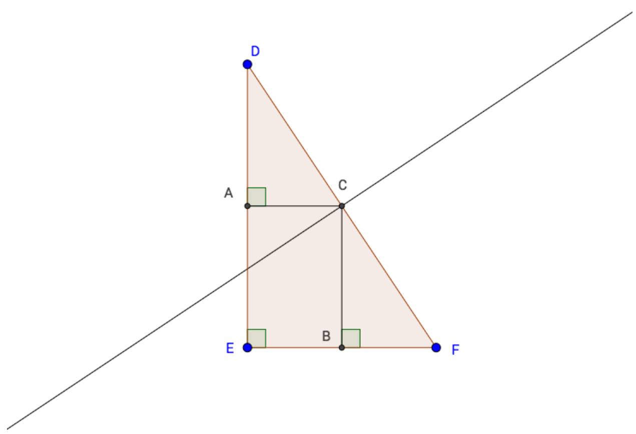

Figure 9: Case 3

Common Chords between two Equal Circles

Duece K Phaly

December 6, 2016

Communicated by Ms. Cohen.

Conjecture AG. In two equal circles one with center A and the other with center W, if a chord BC in circle A is congruent to a chord XY in circle W, chord BC and XY will be the same distance from the center.

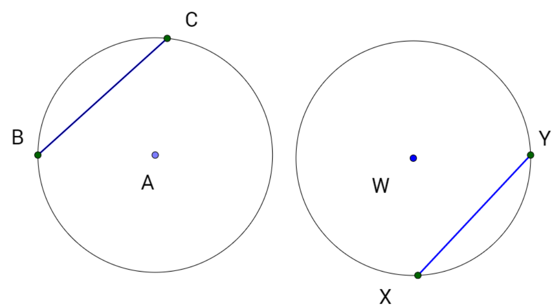

Figure 1: The figure above is a visual representation of Conjecture AG.

Proof. By Euclid's Postulate 1 in book 1, create the radius AB and WX in the respective circles. A perpendicular is drawn from chord BC to point A. Let this intersection point be called point D. Similarly with Chord XY and point W. Let this intersection point be called point Z.

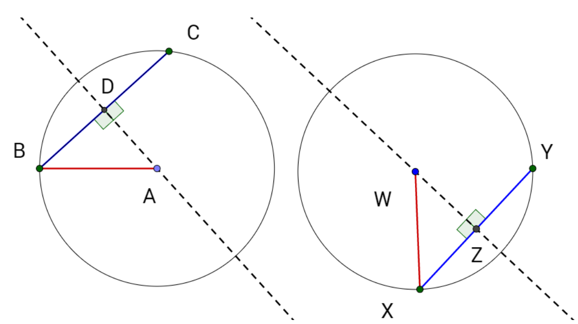

Figure 2: Circles A and W with the radii and perpendiculars drawn

By Euclid's Proposition [33.III], since we have straight lines, perpendiculars AD and WZ, that cut a chord of the circle at right angles, the straight lines (perpendiculars) cut the chords in half respectively. Thus, we have CD congruent to DB and XZ congruent to ZY respectively. We also know that CD, DB, XZ, ZY are all congruent because the original chords BC and XY are congruent. Thus, when dividing equal segments in half, the half will also be equal.

Triangles ABD and WXZ are congruent by the Hypotenuse-Leg Theorem. Having segments BD and AB congruent to XZ and WX respectively allows us to use this theorem.Thus, segment AD is congruent to segment WZ. By definition four in Euclid's book III, since the perpendiculars drawn from the center of each circle are congruent, the lines are equally far from the center; which is exactly what was trying to be proven. □

Theorem AG. If there are two equal circles and they contain a chord(s) which are congruent then they will be equally far from the center of the circle(s).

Bisecting an Angle

Staci Schmeling

November 11, 2016

This report is going to show how to construct an angle bisector in four steps. Following the construction steps, we will prove that our construction works.

Theorem 11.1. Given an angle, construct the angle bisector within 4 steps.

Construction Steps:

Using the given angle ABD.

1. Set the compass to the distance of AB. Place the compass at B and draw a circle with B being the center. The circle should have two intersections, one at A and a new point D.
2. With the same setting on the compass, make a circle with A as the center.
3. Still same setting on your compass, make a circle with D as the center.
4. Connect the intersection of the circles from steps 2 and 3, with the vertex B.

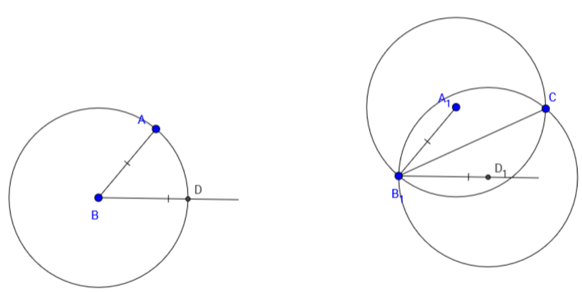

Figure 1: Figure on the left shows that BA is congruent to BD. Figure on the right shows the angle ABD being bisected by BC.

Proof. By using Postulate 1 we can draw line segments AC and CD. We know that line segments BA is congruent to BD because they are the radii of the same circle. We also know that line segments BA is congruent to AC because they are radii of the same circle. By using common notion 1, things which are equal to the same thing are also equal to one another, we know that line segments BA, AC, CD, and DB are all congruent to each other.

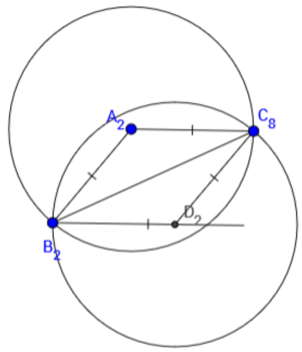

Figure 2: Proving congruent triangles and congruent angles

Since all sides are congruent, we know that we have a rhombus. By Miss Mitchell's Theorem 1.1, We know that angles BAC is congruent to BDC. Then by using Euclid I.4, we know that triangle BAC is congruent to triangle BDC. Since triangle BAC is congruent to triangle BDC, then we know that angle ABC is congruent to angle DBC. Thus we know that BC bisected the given angle using the construction steps above.

□

Constructing a Perpendicular from a Point

Tessa Cohen

December 6, 2016

Communicated by Ms. Schultz.

This paper will outline a 4-step construction for a perpendicular to a given line, l, from a given point, A, which does not lie on l. Following the construction is a proof of why the construction works.

Construction . We are given a line, l, and a point, A, not lying on l.

1. Draw a circle with center A which will intersect l at two points, these new points being called B and C.

2. Draw a circle with center B that goes through point A.

3. Draw a circle with center C that goes through point A.

4. Draw a segment between the two points of intersection of circle B and circle C.

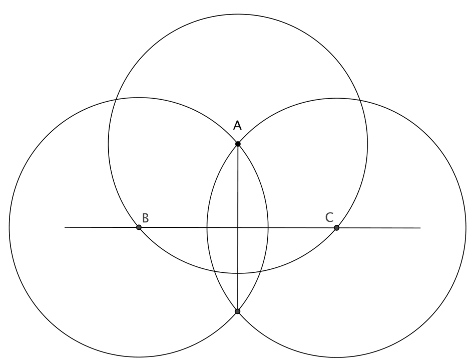

Figure 1: The 4 steps of the construction

Proof. By the Circle-Circle Intersection Theorem, we know there will be two intersection points between circle B and circle C. One of these points is point A, because circle B and C were both drawn through point A. We will let the other intersection point be called D. By Euclid Postulate 1, we will construct segments BA, AC, CD and DB. Since points B and C both lie on circle A, we know by Euclid's definition of a circle that segments BA and AC must be congruent. Since segment BA is also a radius of circle B and segment AC is a radius of circle C, and they are congruent, we know that all of the radii of circle B and circle C are congruent. Since point D lies on the intersection of circle B and circle C, we know that segments BD and CD are radii of circles B and C, respectively. Since we know that the radii of circle B and circle C are all congruent, and segments BA, AC, CD and DB are all radii of either circle B or circle C, we know that segments BA, AC, CD and DB are all congruent. Thus, by the definition of a rhombus, ACDB is a rhombus.

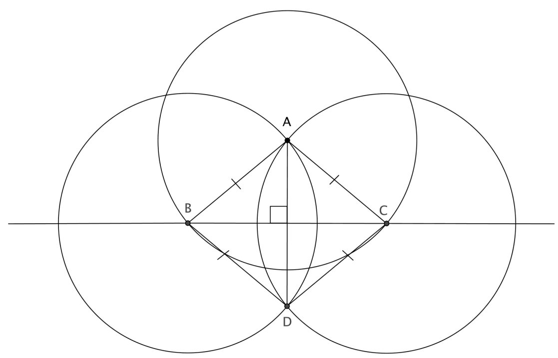

Figure 2: ACDB is a rhombus

By Miss Bavido's Theorem 1.7, we know that the diagonals of a rhombus cross at right angles. Therefore, segment AD is perpendicular to segment BC, and B and C both lie on l, so AD is perpendicular to line l.

$\square$

A Perpendicular Line

Staci Schmeling

December 6, 2016

Communicated by Mr. Merck.

This report is going to show how to construct a line perpendicular to a another line but going through a point not lying on that line in three steps. After the construction steps, we will prove that the two lines are actually perpendicular to one another.

Challenge 11.3 Given a line l and a point A not lying on l, construct and prove a line perpendicular to l through point A.

Construction Steps: Start with the given line l and a point not lying on l. Let this point be point A.

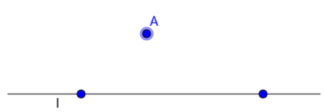

Figure 1: Line l and point A

1. Create two points on the given line l, B and C. Set the compass to the distance of BA. Draw a circle with center at B and radius BA.

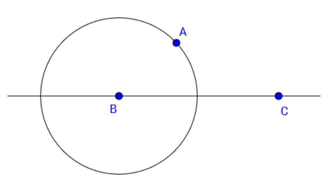

Figure 2: Points B and C, circle with center B and radius BA

2. Set the compass to the distance of CA. Draw a circle with the radius CA centered C.

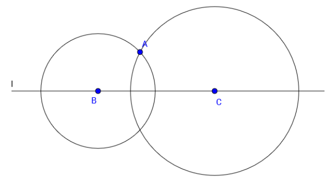

Figure 3: Circle with center C and radius of CA

3. By Euclid's Postulate 1, draw a line from point A to point D, thus making line segment AD. AD is then perpendicular to line l.

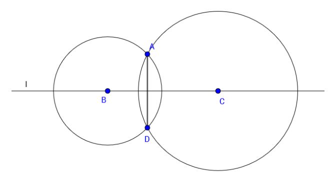

Figure 4: Point D and the perpendicular line through A and line l

Proof. Using the constructions from the steps above, we can use Euclid's Postulate 1 to connect a point to another point. Then we will have the new line segments of BA, AC, BD, and DC. Since line segments BA and BD are radii of circle with center B they are congruent. Also, since line segments AC and DC are radii of circle with center C, we know that they are congruent.

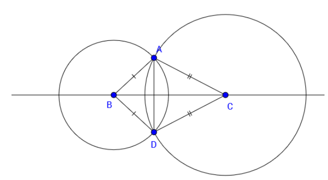

Figure 5: BA is congruent to BD; CA is congruent to CD

Since we have two pairs of adjacent and congruent sides, we have a kite. Using Euclid's Postulate 1 we can connect the diagonals, AD and BC, of our kite. By Miss Worsfold's Theorem 2.5, we know that the diagonals of a kite meet at a right angle. Hence, the construction works, proving that line segment AD is perpendicular to line l.

□

Copying an Angle

Rebecca Shere and Staci Schmeling

December 6, 2016

Communicated by Ms. Bavido.

This report is going to show how to copy a given angle to a given ray. After the construction steps, we will prove that the copied angle is actually congruent to the given angle.

Challenge 11.5 Given an angle at a point A and given a ray emanating from a point B, construct an angle at B congruent to the angle at A having the given ray as a side.

Construction Steps:

Start with the given angle at point A and a given ray emanating from a point B.

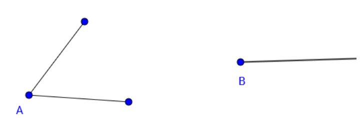

Figure 1: Given angle A and ray with point B

1. Make points C and D on the given angle. Using a compass, set the compass at the distance AC. Set the compass at point B and draw a circle with center B.

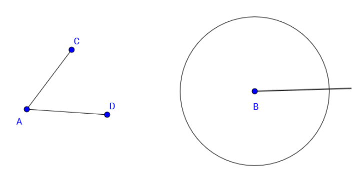

Figure 2: Circle with center B and distance of AC

2. Next we want to create a point on the ray that is congruent to line segment AD. By setting our compass to the distance AD, and drawing a circle with center at B we will have the intersection which is point F on the ray given. Making AD congruent to BF.

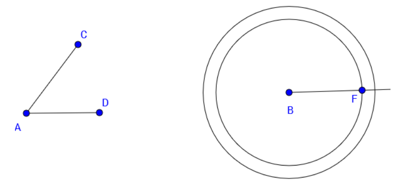

Figure 3: New point F using the circle with center B and distance AD

3. Set your compass to the distance CD. Draw a circle with center F with the distance of CD.

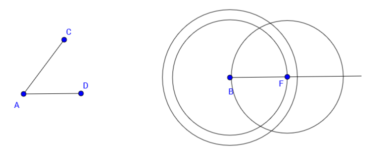

Figure 4: Circle with center F and distance of CD

4. The intersection of circle centered at B with radius congruent to AC and circle F with radius congruent to CD, will be the new point G. Connect point B and point G to create the copied angle GBF from the given angle CAD.

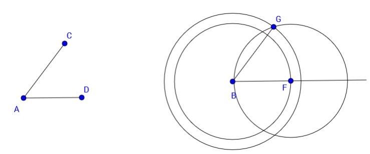

Figure 5: New point G and connection of BG

Proof. Using the construction that we just created, we want to connect points C and D, and G and F by using Euclid's postulate 1. So now we have two triangles. Since we copied the distance from each line segment in triangle ACD, to create triangle BGF we know that the corresponding angles on the triangles are congruent. Thus we know that line segments AC is congruent to BG, AD is congruent to BF, and CD is congruent to GF.

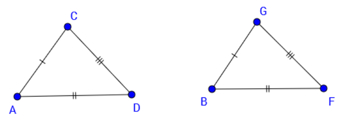

Figure 6: All sides are congruent respectively between triangle ACD and triangle BGF

Since each side of the triangle's, ACD and BGF, are congruent to the corresponding sides on the other one, we can use Euclid's I.8 and say that the two triangles are congruent to each other. Since the triangles are congruent, we know that the angles in the triangles are congruent to each angle in the other triangle. Thus showing that angle CAD is congruent to angle GBF.

□

Construction of a line parallel to line L and through point A

Kandy Schwan

December 6, 2016

Communicated by Ms. King.

I am going to show the construction of a parallel line to L through point A when given a line L and a point A not on that line. I will do this in 4 steps, and then show the proof.

First I will state the construction of the parallel line to L in 4 steps.

Construction: We are given a line L and a point A. We need to find a line parallel to L that is through point A. The first step is to construct with a compass a circle with center A and label the point of intersection of circle A and line L point B. We will call this circle c. Then, the second step is to fix the compass with the radius of AB and construct another circle with center B. We will call this circle d. Then, we label the point of intersection of line L and circle with center B point C. The third step is using our same setting on the fixable compass to create a circle with center C with the same radius as the circles with center A and center B. We will call this circle e. Then, we label the intersection of circle with center C and circle with center A point D. The fourth and final step is to use the straight edge to draw a line through AD, which is our parallel line to line L.

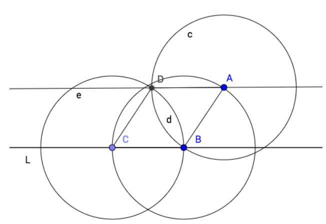

Figure 1: Line L is parallel to line AD.

Theorem 11.6. Given a line L and a point A not lying on L, construct a line parallel to L which passes through A. In my construction, Line AD is parallel to line L.

Proof. Since line segments AD, AB, BC, and CD are all radii of congruent circles c, d, and e, they are all congruent to each other. By the definition of a rhombus, the quadrilateral ABCD is a rhombus. By a theorem 1.6, the opposite sides of a rhombus are parallel. Since we have constructed a line AD that is parallel to line L, we constructed parallel lines.

□

Alternate angle copy proof for the construction of parallel lines

Juliana Herran

December 6, 2016

Theorem 11.6. Given a line l and a point A not lying on l, construct a line parallel to l which passes through A.

Proof. Let a line l and a point A not lying on l exist, we will show a construction of a line parallel to l passing through A and provide a proof for it.

First, create a transversal through line l and point A. Let the point of intersection of the transversal with l be point B. Then, draw the arc from a circle centered at A and repeat the process keeping the same radius of the drawn circle but now center the new circle at B. Now set the compass equal to the distance between C and D as shown in the figure below. Set the compass at E and draw the arc of a circle that intersects with the first drawn circle centered at A. Draw a line through A and F. The constructed line AF is parallel to line l.

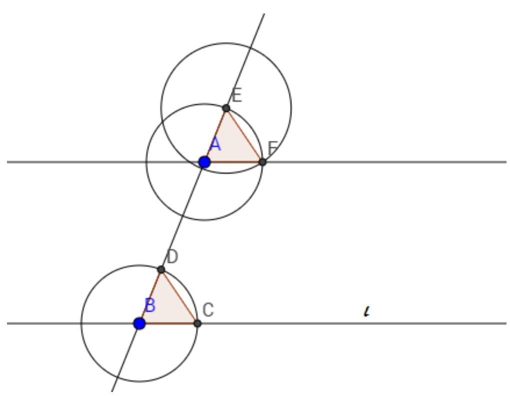

Figure 1: Proposed construction

Since the circle centered at A is of equal radius as the circle centered at B by construction, line segment AF is congruent to BC which are also congruent to line segments BD and AE as the length of these radii is equal. Similarly, line segment DC is congruent to EF by construction as EF is a radius of the circle centered at E. Then, triangles BDC and AEF have three congruent sides and by Euclid 1.8, the contained angles in these triangles are also congruent to one another. Furthermore, by Euclid 1.28, since angle EAF is congruent to angle DBC, line l is parallel to the constructed line through A and F. $\square$

Constructing a Tangent

Heather Bavido

December 6, 2016

Communicated by Ms. Maus.

Challenge 11.8. Given a circle with center O, and given a point A outside the circle, construct a line through A which is tangent to the circle.

Construction 11.8. There are six steps in constructing a tangent line to circle O from point A.
Step 1: With a straight edge, draw a line segment from point O to point A

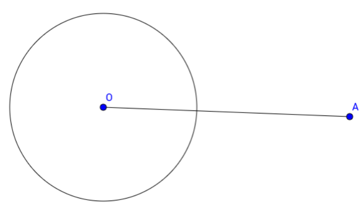

Figure 1: Step 1: Drawing line segment OA

Step 2-4: is finding the midpoint of the line segment OA

With a compass, draw a circle with center O and radius OA.

Next, draw a circle with center A with radius AO. These circles create two points of intersection C1 and C2

With a straight edge, draw a line segment from C1 to C2. This creates an intersection point with OA. I will label this point M, for the midpoint.

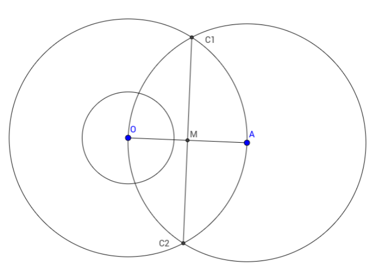

Figure 2: Steps 2-4: Finding the midpoint of line OA

Step 5: With a compass, create circle centered at M with radius MO. This creates a point of intersection between circle M and the original circle O. I will label this point T.

Step 6: With a straight edge, draw a line from point A to point T. This line is tangent to circle O and goes through point A.

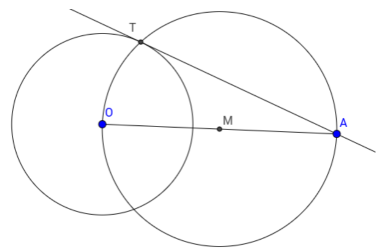

Figure 3: Step 5-6: Drawing circle MO and tangent line AT

Proof. Circle M with radius MO has a diameter of OA. From Thales' Theorem, if two segments are drawn from each of the diameter end points to a point on the circle, it creates a right angle. Therefore, since OA is the diameter of circle M and AT and OT meet at point T which lies on circle M, AT and OT will create a right angle OTA.

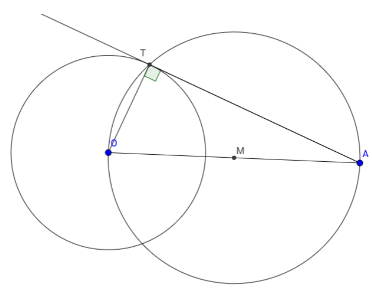

Figure 4: Line segments AT and OT meet at a right angle

By III.18, since the line OT comes from the center of circle O and meets the line AT at point T at a right angle, then line AT is tangent to the circle O.

□