DPPs on complex manifolds: construction and applications to numerical integration

Thibaut Lemoine

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Abstract

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***Keywords*—** —

Contents

1 Crash course on complex geometry

This section is here to redefine most objects in detail, and can be skipped at first. It is mostly taken from [Lem22].

1.1 Complex manifolds and Hermitian line bundles

A *complex manifold* is a topological space M endowed with a family $(U_i, \phi_i)_{i\in I}$ of open subsets $U_i \subset M$ and homeomorphisms $\phi_i : U_i \to \mathbb{C}^d$ such that, if $U_i \cap U_j \neq \varnothing$, then

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is a biholomorphism between subsets of $\mathbb{C}^d$. Similarly, one defines a d-dimensional smooth real manifold by replacing $\mathbb{C}^d$ by $\mathbb{R}^d$ and biholomorphism by smooth diffeomorphism. In particular, if M is a d-dimensional complex manifold, then it is also a $2d$-dimensional real manifold using the bijection $\mathbb{C} \simeq \mathbb{R}^2$. A *holomorphic line bundle* over a compact complex manifold M is a complex manifold L endowed with a holomorphic map $\pi : L \to M$ such that any fiber $L_x = \pi^{-1}(\{x\})$ is a one-dimensional complex vector space for $x \in M$, with the following extra condition: there exists an open covering $(U_i)_{i\in I}$ of M and biholomorphic maps $\psi_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{C}$ commuting with the projection π and so that the induced map $L_x \cong \mathbb{C}$ is $\mathbb{C}$-linear.

If $L \to M$ is a holomorphic line bundle with the associated open covering (U_i, ψ_i), then the functions ψ_i are called *trivialization functions* and the $\mathbb{C}$-linear functions

$$\gamma_{ji}(x) : z \mapsto (\psi_j \circ \psi_i^{-1})(x, z)$$

are called the *transition functions* for any i, j and any $x \in U_i \cap U_j$. A *holomorphic section* of $L \to M$ is a holomorphic function $s : M \to L$ such that $\pi \circ s = id_M$. The space of holomorphic sections of L is denoted by $H^0(M, L)$ and is a finite-dimensional complex vector space. A section $s \in H^0(M, L)$ is entirely characterized by local holomorphic functions $f_i : U_i \to \mathbb{C}$ that satisfy the compatibility conditions

$$f_j(x) = \gamma_{ji}(x) f_i(x), \ \forall x \in U_i \cap U_j. \tag{1}$$

For instance, for any $i \in I$ set $f_i = \mathrm{pr}_2 \circ \psi_i \circ s|_{U_i}$, where $\mathrm{pr}_2 : U_i \times \mathbb{C} \to \mathbb{C}$ is the projection on the second coordinate: these functions satisfy (1). By construction, we have then

$$\psi_i \circ s(x) = (x, f_i(x)), \ \forall x \in U_i. \tag{2}$$

Another way of dealing with sections is to use *local frames*: if $e^{(i)}$ is a nonvanishing section of L over U_i, then there are holomorphic functions $f_i : U_i \to \mathbb{C}$ such that

$$s(x) = f_i(x)e^{(i)}(x), \ \forall x \in U_i.$$

In this case, the compatibility condition (1) implies

$$f_j(x)e^{(j)}(x) = \gamma_{ji}(x)f_i(x)e^{(j)}(x) = f_i(x)e^{(i)}(x), \ \forall x \in U_i \cap U_j,$$

so that $f_i, f_j, e^{(i)}$, and $e^{(j)}$ are related through γ_{ji}. If two line bundles $\pi_1 : L_1 \to M$ and $\pi_2 : L_2 \to M$ are isomorphic, then so are $H^0(M, L_1)$ and $H^0(M, L_2)$: assuming that for any $x \in M$ there is an isomorphism $\Phi_x : (L_1)_x \simeq (L_2)_x$, then we have the isomorphism $\Phi : H^0(M, L^1) \to H^0(M, L^2)$ defined by

$$\Phi \circ s(x) = \Phi_x(s(x)), \ \forall s \in H^0(M, L^1), \ \forall x \in M.$$

If L_1 and L_2 are two holomorphic line bundles over M, we have then the canonical isomorphism

$$H^0(M, L_1 \otimes L_2) \cong H^0(M, L_2 \otimes L_1). \tag{3}$$

It also holds for the external tensor product: if $L_1, \dots, L_n$ are line bundles over M and $\mathrm{pr}_i : M^n \to M$ is the projection onto the i-th coordinate, then the *external tensor bundle* $L_1 \boxtimes \cdots \boxtimes L_n$ is the line bundle over M^n defined by

$$(\mathrm{pr}_1^* L_1) \otimes \cdots \otimes (\mathrm{pr}_n^* L_n).$$

We have the canonical isomorphism (of finite-dimensional Hilbert spaces)

$$H^0(M^n, L_1 \boxtimes \cdots \boxtimes L_n) \cong H^0(M, L_1) \otimes \cdots \otimes H^0(M, L_n), \tag{4}$$

and the canonical isomorphism (3) generalizes, for any $\sigma \in \mathfrak{S}_n$, into

$$H^0(M^n, L_1 \boxtimes \cdots \boxtimes L_n) \cong H^0(M^n, L_{\sigma(1)} \boxtimes \cdots \boxtimes L_{\sigma(n)}). \tag{5}$$

1.2 Kähler potentials

Let M be a closed compact complex manifold of dimension d and $L \to M$ be a holomorphic line bundle. A standard procedure of geometric quantization is to consider the space $H^0(M, L^k)$ of holomorphic sections of L^k as the *space of quantum states*. In the context of integer quantum Hall effect (IQHE), it corresponds to particles in the lowest Landau level (LLL) under a uniform magnetic field of strength k [Kle16].

A *Hermitian structure* on L is the data of a Hermitian inner product h_x on each fiber L_x, for $x \in M$. The inner product h is called a *Hermitian metric*. If h is a smooth Hermitian metric on L and μ is a finite measure on M, the space $H^0(M, L^k)$ can be equipped with the following Hermitian inner product:

$$\langle s_1, s_2 \rangle_{L^2(\mu), h^k} = \int_M h_x^k(s_1(x), s_2(x)) d\mu(x). \tag{6}$$

Let us explain how this expression translates in local coordinates. If (U_i) is an open covering on M and $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ are associated local trivialization functions, a section $s \in H^0(M, L)$ can be locally identified to holomorphic functions $f_i : U_i \to \mathbb{C}$ according to (2). The Hermitian metric h then reads on U_i

$$h_x(s^{(1)}(x), s^{(2)}(x)) = \langle \phi_i \circ s^{(1)}(x), \phi_i \circ s^{(2)}(x)\rangle_i = f_i^{(1)}(x)\overline{f_i^{(2)}}(x)e^{-\phi_i(x)}, \tag{7}$$

where $\phi_i : U_i \to \mathbb{R}$ is the *local weight* of the inner product $\langle \cdot, \cdot \rangle_i$ on $\mathbb{C}$ corresponding to h on U_i. Using a local frame, we can also write

$$h_x(s^{(1)}(x), s^{(2)}(x)) = f_i^{(1)}(x)\overline{f_i^{(2)}}(x)\|e^{(i)}(x)\|^2_{h^k},$$

and one recovers (7) by setting $\phi_i(x) = -2\log\|e^{(i)}(x)\|_{h^k}$.

Analogously, a section $f \in H^0(M, L^k)$ can be described by local functions $f_i : U_i \to \mathbb{C}$: if $e^{(i)}$ is a local frame of L over U_i, then $(e^{(i)})^{\otimes k}$ is a local frame of L^k and a section $s \in H^0(M, L^k)$ satisfies

$$s(x) = f_i(x)(e^{(i)})^{\otimes k}(x), \ \forall x \in U_i.$$

The metric h^k on L^k induced by h corresponds then to the local weight

$$-2\log\|(e^{(i)}(x))^{\otimes k}\|^2_{h^k} = -2\log\|e^{(i)}(x)\|^{2k}_h = k\phi_i(x).$$

Proposition 1.1. *For any i, j and any $x \in U_i \cap U_j$,*

$$\phi_i(x) = \phi_j(x) - 2\log|\gamma_{ji}(x)|. \tag{8}$$

An immediate consequence is that for any $s^{(1)}, s^{(2)} \in H^0(M, L^k)$, the inner product $h^k(s^{(1)}, s^{(2)})$ defines a global function on M, which justifies the definition of the inner product (6). In particular, the associated norm is also globally defined: for any $x \in U_i \cap U_j$,

$$\|s\|^2_{h^k}(x) = |f_i(x)|^2 e^{-k\phi_i(x)} = |f_j(x)|^2 e^{-k\phi_j(x)}.$$

The *curvature form* Θ_h of the metric h is defined locally by $\Theta_h = -\partial\overline{\partial}\phi$ for a local weight ϕ, and one can turn it into a real $(1,1)$-form $\omega = \frac{i}{2}\Theta_h$. We said that ω is a *Kähler form* if it is closed, *i.e.* $d\omega = 0$, where $d = \partial + \overline{\partial}$, and ϕ is called a *Kähler potential* for ω. Let us insist that the Kähler potential is only locally defined most of the time; a necessary and sufficient condition for the Kähler potential to be global is given by the global $\partial\overline{\partial}$-lemma [Huy05, Cor. 3.2.10].

If we denote by N_k the dimension of $H^0(M, L^k)$, then the *volume* of L is defined by

$$\mathrm{vol}(L) = \limsup_{k\to\infty} \frac{d!}{k^d} N_k, \tag{9}$$

and L is *big* if its volume is positive. It follows from the asymptotic Riemann–Roch theorem [MM07, Thm 1.7.1] and the Kodaira–Serre vanishing theorem [MM07, Thm 1.5.6] that for k large enough,

$$N_k = \frac{k^d}{d!}\int_M \frac{1}{\pi^d}\omega^d + o(k^d), \tag{10}$$

hence the volume of L satisfies

$$\mathrm{vol}(L) = \frac{1}{\pi^d}\int_M \omega^d. \tag{11}$$

In particular, we see that the total volume of M with respect to dv_M is $\frac{\pi^d \mathrm{vol}(L)}{d!}$.

1.3 Tensor product and duality

Let M be a complex manifold of dimension d. If $L \to M$ is a holomorphic line bundle defined by the transition functions γ_{ji} endowed with a Hermitian metric h, then its dual bundle $\overline{L} \to M$ is defined by $\overline{L}_x = (L_x)^*$ and by the transition functions γ_{ji}^{-1}. From the Riesz representation theorem we know that for any $\overline{v} \in \overline{L}_x$ there exists a unique $v = \varphi_{\overline{v}} \in L_x$ such that for all $w \in L_x$

$$(\overline{v}, w) = h_x(w, v).$$

For any finite-dimensional[1] Hilbert space E, there is a canonical isomorphism of Hilbert spaces $L_x \otimes E \otimes \overline{L_x} \cong E$, given by the contraction

$$u_x \otimes e \otimes \overline{v_x} \mapsto h_x(u_x, v_x)e.$$

By induction, for any $\sigma \in \mathfrak{S}_n$ we obtain the canonical isomorphism

$$L_{x_1} \otimes \overline{L}_{x_{\sigma(1)}} \otimes \cdots \otimes L_{x_n} \otimes \overline{L}_{x_{\sigma(n)}} \cong \mathbb{C}. \tag{12}$$

In particular, any tensor product of the form

$$s^{(1)}(x_1) \otimes \overline{s^{(1)}}(x_{\sigma(1)}) \otimes \cdots \otimes s^{(n)}(x_n) \otimes \overline{s^{(n)}}(x_{\sigma(n)})$$

can be canonically identified with the product of functions

$$h_{x_1}(s^{(1)}(x_1), s^{(\sigma^{-1}(1))}(x_1)) \cdots h_{x_n}(s^{(n)}(x_n), s^{(\sigma^{-1}(n))}(x_n)).$$

Moreover, if e_U is a local nonvanishing section of L on U and ϕ is a local weight of h on U, then for any $x, y \in U$, $e_U(x) \otimes \overline{e_U}(y)$ can be identified with $e^{-\frac{k}{2}(\phi(x)+\phi(y))}$ by contraction: we have the canonical[2] isomorphisms $L_x \otimes \overline{L_y} \cong \mathrm{Hom}(L_x, L_y) \cong \mathbb{C}$ and

$$\|e_U(x) \otimes \overline{e_U}(y)\|_{h^k}^2 = e^{-k(\phi(x)+\phi(y))}.$$

1.4 The Bergman kernel and its determinant

Let M be a complex manifold of dimension d and (L, h) be a Hermitian holomorphic line bundle over M such that $\omega = -\frac{i}{2}\partial\overline{\partial}\log h$ is a Kähler form. Consider for any integer k the space $\mathscr{C}^\infty(M, L^k)$ of smooth sections of L^k: it can be endowed with the inner product (6) introduced in Section 1.2, and its completion for such inner product is $L^2(M, L^k)$. Let us start by recalling the Schwartz kernel theorem.

Theorem 1.2 ([MM07], Theorem B.2.7). *Let E, F be two vector bundles on M and $A : \mathscr{C}_0^\infty(M, E) \to \mathscr{D}'(M, F)$ be a linear continuous operator. There exists a unique distribution $K \in \mathscr{D}'(M^2, F \boxtimes E^*)$, called the* Schwartz kernel distribution*, such that*

$$(Au, v) = (K, v \otimes u), \ \forall u \in \mathscr{C}_0^\infty(M, E^*), \ \forall v \in \mathscr{C}_0^\infty(M, F). \tag{13}$$

[1] We assume finite dimensions so that the algebraic tensor product of Hilbert spaces is automatically a Hilbert space.

[2] The second isomorphism is canonical because the bases of L_x and L_y are given in the same trivialization.

Moreover, for any volume form $d\mu$, the Schwartz kernel of A is represented by a smooth kernel $K \in \mathscr{C}^\infty(M^2, F \boxtimes E^*)$, called the Schwartz kernel of A *with respect to $d\mu$, such that*

$$(Au)(x) = \int_M K(x,y)u(y)d\mu(y), \ \forall u \in \mathscr{C}_0^\infty(M,E). \tag{14}$$

Finally, A can be extended as a linear continuous operator $A : \mathscr{D}_0'(M,E) \to \mathscr{C}^\infty(M,F)$ by setting

$$(Au)(x) = (u(\cdot), K(x,\cdot)), \ \forall x \in M, \ \forall u \in \mathscr{D}_0'(M,E). \tag{15}$$

Definition 1.1. The *Bergman projection* is the orthogonal projection

$$P_k : L^2(M, L^k) \longrightarrow H^0(M, L^k), \tag{16}$$

and the Schwartz kernel B_k of P_k with respect to $d\mu$ is called the *Bergman kernel.*

Note that the kernel B_k is in particular the *reproducing kernel* of the Hilbert space $H^0(M, L^k)$ endowed with the inner product (6), as a consequence of (14). It is the geometric version of the Christoffel–Darboux kernel introduced in Section **??**. Although it does not appear in the notation B_k, the Bergman kernel does depend on the choice of the inner product, thus on the geometric structure induced by h.

Proposition 1.3. *Let (s_ℓ) be an orthonormal basis of $H^0(M, L^k)$. The Bergman kernel can be written*

$$B_k(x,y) = \sum_{\ell=1}^{N_k} s_\ell(x) \otimes \overline{s_\ell}(y) \in L_x^k \otimes \overline{L_y^k}. \tag{17}$$

In particular, on the diagonal, it defines a global function

$$B_k(x,x) = \sum_{\ell=1}^{N_k} \|s_\ell(x)\|_{h^k}^2. \tag{18}$$

The reproducing property of the Bergman kernel has a few direct consequences. For instance,

$$\int_M B_k(x,x)dv_M(x) = \sum_\ell \|s_\ell\|_{h^k}^2 = N_k, \tag{19}$$

and for any $x, z \in M$

$$\int_M B_k(x,y)B_k(y,z)dv_M(y) = B_k(x,z). \tag{20}$$

Lemma 1.4 ([Lem22])**.** *Let B_k be the Bergman kernel of $H^0(M, L^k)$. For any $n < N_k$, and any $(x_1, \dots, x_n) \in M^n$,*

$$\int_M \det_{n+1}(B_k(x_i,x_j))dv_M(x_{n+1}) = (N_k - n)\det_n(B_k(x_i,x_j)). \tag{21}$$

Theorem 1.5 ([BSZ00], Theorem 3.1)**.** *Let (L,h) be a positive Hermitian line bundle on a d-dimensional compact complex manifold M with Kähler form $\omega = \frac{i}{2}\Theta_h$. Let $z_0 \in M$ and choose normal local coordinates in a neighborhood of z_0. Then the Bergman kernel B_k of $H^0(M, L^k)$ satisfies, as $k \to \infty$:*

$$k^{-d}B_k(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}) = B_\infty(u,v) + O(k^{-\frac{1}{2}}). \tag{22}$$

2 DPPs on complex manifolds

2.1 Construction

Consider N_k free fermions on a complex manifold M of dimension d with a positive Hermitian line bundle (L,h), and set $\omega = -\frac{i}{2}\partial\overline{\partial}\log h$. If (s_ℓ) is an orthonormal basis of $H^0(M,L^k)$, the N_k-particle wavefunction is given by their Slater determinant

$$\Psi(x_1,\dots,x_N) = \frac{1}{\sqrt{N_k!}}\det(s_\ell(x_m))_{1\leqslant \ell,m\leqslant N_k}.$$

To be precise, this determinant is in fact a holomorphic section of the external tensor power bundle $(L^k)^{\boxtimes N_k}\to M^{N_k}$, and $H^0(M^{N_k},(L^k)^{\boxtimes N_k})$ is endowed with the inner product

$$\langle s_1^{(1)}\otimes\cdots s_{N_k}^{(1)}, s_1^{(2)}\otimes\cdots s_{N_k}^{(2)}\rangle_{h^k}(x_1,\dots,x_{N_k}) = \prod_{i=1}^{N_k}(h^k)_{x_i}(s_i^{(1)}(x_i),s_i^{(2)}(x_i)), \tag{23}$$

that induces a L^2 inner product $\langle\cdot,\cdot\rangle_{L^2(dv_M^{\otimes N_k}),h^k}$ with respect to $\mu^{\otimes N_k}$ and h^k. The squared norm of Ψ defines a probability density function on M^{N_k} with respect to $dv_M^{\otimes N_k}$:

$$p(x_1,\dots,x_{N_k}) = \|\Psi(x_1,\dots,x_N)\|_{h^k}^2. \tag{24}$$

Theorem 2.1 ([Lem22]). *Let $(X_1,\dots,X_{N_k})$ be a family of random variables with joint density (24) with respect to $dv_M^{\otimes N_k}$. It is a determinantal process on (M,dv_M) with kernel B_k.*

2.2 Asymptotics

We want to discuss several convergence results.

Theorem 2.2 ([Lem22]). *Let M be a compact complex manifold of dimension d, and (L,h) be a Hermitian holomorphic line bundle over M such that $\omega = \frac{i}{2}\Theta_h$ is a Kähler form that induces a measure dv_M on M. Set $N_k = \dim H^0(M,L^k)$ and consider an orthonormal basis $(S_1,\dots,S_{N_k})$ of $H^0(M,L^k)$ for the L^2 inner product*

$$\langle s_1,s_2\rangle_{L^2(dv_M),h^k} = \int_M h_x^k s_1(x),s_2(x))dv_M(x).$$

Let $(X_1,\dots,X_{N_k})$ be a random family on M^d with joint density

$$\frac{1}{N_k!}\|\det(S_i(z_j))\|_{h^k}^2 dv_M^{\otimes N_k}(z_1,\dots,z_{N_k}).$$

Then, the following assertions hold:

(i) $(X_1,\dots,X_{N_k})$ is a determinantal point process with kernel B_k, the Bergman kernel associated to $H^0(M,L^k)$.

(ii) As $k \to \infty$, its correlation functions admit the following scaling limit on normal coordinates:

$$\frac{1}{k^{nd}}\rho_n(\frac{u_1}{\sqrt{k}}, \dots, \frac{u_n}{\sqrt{k}}) = \det(B_\infty(u_i, u_j)) + O(\frac{1}{\sqrt{k}}),\ \forall u, v \in \mathbb{C}^d,$$

where B_∞ is the kernel defined locally by

$$B_\infty(u, v) = \frac{1}{(\pi)^d} e^{u\cdot\bar{v} - \frac{1}{2}|u|^2 - \frac{1}{2}|v|^2},\ \forall u, v \in \mathbb{C}^d.$$

Corollary 2.3 ([Lem22]). *Let $(X_1, \dots, X_{N_k})$ be as in Theorem 2.2, and μ be the probability measure $(\int_M dv_M)^{-1} dv_M$. As $k \to \infty$, the empirical measures $\hat{\mu}_k = \frac{1}{N_k}\sum_{i=1}^{N_k} \delta_{X_i}$ converge in probability to μ, with respect to the weak topology.*

Below are convergence results obtained by Berman under weaker assumptions: we do not assume the manifold to be Kähler anymore, and we rather put (loose) assumptions on the metrics.

Theorem 2.4 (Universality, [Ber18]). *Let L be a holomorphic line bundle over a compact complex manifold M of dimension d, h be a Hermitian metric on L with associated weight ϕ and a finite measure μ on M. Assume that ϕ is $\mathscr{C}^{1,1}_{\text{loc}}$ and that the volume form $\frac{1}{d!}\omega^d$ is continuous. Let x be a fixed point in the weak bulk and take normal local coordinates z centered at x and a normal trivialization of L. Then,*

$$\lim_{k\to\infty} k^{-d} B_k(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}) = \frac{\det \lambda}{\pi^d} e^{\langle \lambda z, w\rangle} \tag{25}$$

in the $\mathscr{C}^\infty$-topology on compact subsets of $\mathbb{C}^d_z \times \mathbb{C}^d_w$. In particular, the 2-point correlation function has the following scaling asymptotics

$$\lim_{k\to\infty} -k^{-2d}\rho_2(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}) = \left(\frac{\det \lambda}{\pi^d}\right)^2 e^{-\lambda_i |z_i - w_i|^2} \tag{26}$$

uniformly on compacts of $\mathbb{C}^d_z \times \mathbb{C}^d_w$.

A measure on M is said to satisfy the *Bernstein–Markov property* with respect to the weighted set (K, ω) if for any $\varepsilon > 0$, there exists C_ε such that

$$\sup_{x\in K} \|s(x)\|^2 \leqslant C_\varepsilon e^{k\varepsilon} \int_M \|s_k\|^2 d\nu,\ \forall s \in H^0(M, L^k). \tag{27}$$

The norm $\|s\|^2$ is taken with respect to the Hermitian metric h on L whose curvature is ω, cf. [Ber14, § 3.6].

Theorem 2.5 (Large deviation principle, [Ber14]). *Let (ν, ω) be a weighted measure and assume that ν has the Bernstein–Markov property and that its support K is non-pluripolar. Then the distributions of the normalized empirical measures of the DPP satisfy a LDP with good rate functional $H = H_{(K,\omega)}$ and speed Vk^{d+1}, where V is the volume of L. The rate functional is defined by*

$$H(\mu) = E_\omega(\mu) - C,$$

where E_ω is the pluricomplex energy and C the pluricomplex capacity. On the space $\mathcal{P}(K)$, the rate functional is minimized (and vanishes) on the pluripotential equilibrium measure μ_{eq}.

If $f \in \mathscr{C}^1$, then its *Dirichlet norm* $\|\nabla f\|$ is defined by

$$\|\nabla f\|^2 = \int_M h_x^k(\nabla f(x), \nabla f(x)) dv_M(x). \tag{28}$$

Theorem 2.6 (Central limit theorem, [Ber18])**.** *Let L be a holomorphic line bundle over a compact complex manifold M of dimension d, h be a Hermitian metric on L with associated weight ϕ and a finite measure μ on M. Assume that ϕ is $\mathscr{C}^{1,1}_{\mathrm{loc}}$ and that the volume form $\frac{1}{d!}\omega^d$ is continuous. Denote by S the support of the equilibrium measure of (M, ϕ). For any Lipschitz function f on M supported in a compact subset of the bulk, then*

$$\lim_{k\to\infty} \mathbb{E}[e^{-tk^{-(d-1)/2}(\sum_i f(X_i) - \mathbb{E}[\sum_i f(X_i)])}] = e^{\frac{t^2}{8\pi}\|\nabla f\|^2} \tag{29}$$

in the $\mathscr{C}^\infty$-topology when t is restricted to a compact subset of $\mathbb{C}$. In particular, the variance of $\sum_i f(x_i)$ has the following asymptotics:

$$\mathrm{Var}(\sum_i f(X_i)) = \frac{k^{d-1}}{4\pi}(\|\nabla f\|^2) + o(k^{d-1}),$$

and we have the following convergence:

$$\sqrt{N_k^{1+\frac{1}{d}}}\left(\frac{1}{N_k}\sum_i f(X_i) - \mathbb{E}\left[\frac{1}{N_k}\sum_i f(X_i)\right]\right) \xrightarrow[k\to\infty]{(d)} \mathcal{N}(0, \frac{1}{4\pi}\|\nabla f\|^2).$$

Note that in Theorems 2.5 and 2.6, the convergence is of order $Vk^{d+1} \sim N_k^{1+\frac{1}{d}}$. Let us compare this to the convergences of i.i.d. uniform random variables on M.

Theorem 2.7. *Precise assumptions?Let $(X_1, \ldots, X_{N_k})$ be a N_k-tuple of i.i.d. random variables on M of law $\mu = \frac{d!}{\pi^d \mathrm{vol}(L)} dv_M$ (i.e. the uniform probability measure on M). The sequence of empirical measures $\widehat{\mu}_k$ satisfies a LDP with rate functional [...] and speed N_k.*

Before we prove this theorem, let us recall the abstract Gärtner–Ellis theorem for Gibbs measures as in [REF].

Theorem 2.8. *Let $(H^{(N)})$ be a sequence of Hamiltonians*[3] *on M^N and (β_N) a sequence of positive numbers such that $\beta_N \to \beta \in (0, \infty]$. Assume that, for any $u \in \mathscr{C}^0(M)$, as $N \to \infty$,*

$$-\frac{1}{N\beta_N}\log\int_{M^N} e^{-\beta_N(H^{(N)}(x_1,\ldots,x_N)+\sum_i u(x_i))} dv_M^{\otimes N}(x_1, \ldots, x_N) \longrightarrow F_\beta(u),$$

where F_β is a Gateaux-differentiable function on $\mathscr{C}^0(M)$. Then the measures Γ_N satisfy a LDP with speed $\beta_N N$ and good rate functional $f_\beta^*(\mu)$, where $f_\beta(u) = -F_\beta(-u)$.

Proof of Theorem 2.7. We apply the Gärtner–Ellis theorem to the zero Hamiltonians

$$H^{(N)}(x_1, \ldots, x_N) = 0.$$

[3] Here, a Hamiltonian on M^N is simply a symmetric lower semi-continuous function on M^N.

If we take $\beta_N = 1$ for all N, we see that

$$-\frac{1}{N_k}\log\int_M e^{-\sum_i u(x_i)} dv_M^{\otimes N_k}(x_1,\dots,x_{N_k}) = -\log\int_M e^{-u(x)} dv_M(x),$$

which defines a functional $F(u)$ that does not depend on k, so that we automatically obtain the convergence. Now it remains to prove that F is Gateaux-differentiable. Let $u, v \in \mathscr{C}^0(M)$ and $t > 0$. We first remark that the functional $\tilde{F} : u \mapsto \int_M e^{-u(x)} dv_M(x)$ is Gateaux-differentiable, provided that

$$\begin{aligned}\frac{\tilde{F}(u+tv) - \tilde{F}(u)}{t} &= \frac{1}{t}\int_M e^{-u(x)}\left(e^{-tv(x)} - 1\right) dv_M(x)\\ &= \int_M e^{-u(x)} \frac{e^{-tv(x)} - 1}{t} dv_M(x)\end{aligned}$$

and using dominated convergence to intervert the limit and the integral. Then, as $\tilde{F}$ takes values in $(0, +\infty)$ and the function $-\log$ is derivable on this interval, we get by composition that F is Gateaux-differentiable on $\mathscr{C}^0(M)$. □

We can probably? extend this result to the case of Bernstein–Markov measures.

Theorem 2.9. *Let (ν, ω) be a weighted measure and assume that ν has the Bernstein–Markov property and its support K is non-pluripolar. If $(X_1, \dots, X_{N_k})$ is a N_k-tuple of i.i.d. random variables on M of law ν, then their empirical measure $\widehat{\mu}_k$ satisfy a LDP with rate functional [...] and speed N_k.*

Proof. Use the version of Gärtner–Ellis theorem given in appendix. □

As a consequence, the approximation of integrals using DPPs is much better on low dimensions, but the difference progressively vanishes in high dimensions.

3 A basic example: the sphere

3.1 Manifold structure

Consider the unit sphere $S^2 \subset \mathbb{R}^3$; we will simultaneously see it as a submanifold of $\mathbb{R}^3$ and as a (real) differentiable manifold of dimension 2. As a submanifold, it is defined by $S^2 = F^{-1}(0)$, where $F : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$. It can be endowed with the atlas (U_0, U_1) such that U_0 is the sphere without the North pole $(0, 0, 1)$ and U_1 the sphere without the South pole $(0, 0, -1)$. The corresponding charts are given by stereographic projections:

$$\varphi_0 : \left\{\begin{array}{ccc} S^2 \setminus \{(0,0,1)\} & \longrightarrow & \mathbb{R}^2 \\ (x,y,z) & \longmapsto & (\frac{x}{1-z}, \frac{y}{1-z}) \end{array}\right., \quad \varphi_1 : \left\{\begin{array}{ccc} S^2 \setminus \{(0,0,-1)\} & \longrightarrow & \mathbb{R}^2 \\ (x,y,z) & \longmapsto & (\frac{x}{1+z}, \frac{y}{1+z}) \end{array}\right..$$

We denote by X, Y the local real coordinates given by those charts, or equivalently by $Z = X + iY$ the local complex coordinate. We will usually stick to U_0 but everything is similar in U_1. The inverse stereographic projections are

$$\varphi_0^{-1}(X, Y) = \left(\frac{2X}{1 + X^2 + Y^2}, \frac{2Y}{1 + X^2 + Y^2}, \frac{-1 + X^2 + Y^2}{1 + X^2 + Y^2}\right),$$

and

$$\varphi_1^{-1}(X,Y) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{1-X^2-Y^2}{1+X^2+Y^2}\right).$$

Let us make a few remarks on these coordinate systems: as $\varphi_0(0,0,-1) = \varphi_1(0,0,1) = (0,0)$ we see that φ_0 (resp. φ_1) is centered in the South pole (resp. the North pole). If we take $(X,Y) \in \varphi_0(U_0 \cap U_1)$, then

$$\varphi_1 \circ \varphi_0^{-1}(X,Y) = \left(\frac{X}{X^2+Y^2}, \frac{Y}{X^2+Y^2}\right),$$

and it translates in terms of $Z = X + iY$ as

$$\varphi_1 \circ \varphi_0(Z) = \frac{Z}{|Z|^2},$$

which is holomorphic on $\mathbb{C}^* = \varphi_0(U_0 \cap U_1)$, and it is an involution, hence a biholomorphism. It follows that S^2 is a complex manifold of dimension 1, *i.e.* a Riemann surface. Note that, as $F^{-1}(0)$, it is a closed subset of $\mathbb{R}^3$, and it is obviously bounded, therefore it is compact.

For any $P = (x_0, y_0, z_0) \in S^2$, the (real) tangent space $T_P S^2$ is given by $\ker(d_P F)$. It is equal to

$$T_P S^2 = \{(x,y,z) \in \mathbb{R}^3 : x_0 x + y_0 y + z_0 z = 0\} = (\mathbb{R}P)^\perp \simeq \mathbb{R}^2 \simeq \mathbb{C}.$$

We will denote by u_P the complex coordinate of a tangent vector with respect to some basis.

3.2 Kähler structure

The SO(3)-invariant 2-form of $\mathbb{R}^3$ are given by

$$\omega_\lambda = \lambda x dy \wedge dz + \lambda y dz \wedge dx + \lambda z dx \wedge dy,$$

and we will denote by $\omega = \omega_{-1}$ the one that we take as a reference. Using the stereographic projection φ_0, the restriction of ω to U_0 can be pulled back to the following form on $\mathbb{R}^2 \cong \mathbb{C}$:

$$\varphi_0^* \omega = \frac{4 dX \wedge dY}{(1+X^2+Y^2)^2} = \frac{2i dZ \wedge d\overline{Z}}{(1+|Z|^2)^2}.$$

It corresponds to the Hermitian metric

$$g_Z(u,v) = \frac{u\overline{v}}{(1+|Z|^2)^2} = u\overline{v} e^{-\phi(Z)},$$

with $\phi(Z) = 4\log(1+|Z|^2)$. Indeed, its curvature is

$$\begin{aligned}\Theta_g = -\partial\overline{\partial}\phi(Z) &= -4\partial \frac{Z d\overline{Z}}{1+|Z|^2} \\ &= -4\frac{(1+|Z|^2) - |Z|^2}{(1+|Z|^2)^2} dZ \wedge d\overline{Z},\end{aligned}$$

and we obtain $\varphi_0^*\omega = \frac{i}{2}\Theta_g$. Here, one can check that $d\omega = 0$ (where $d = \partial + \overline{\partial}$) so that (S^2, ω) is a Kähler manifold, and ϕ is then a Kähler potential.

It is not hard to see that the total volume is

$$\int_{\mathbb{C}} \frac{2idZ \wedge d\overline{Z}}{(1+|Z|^2)^2} = 4\pi,$$

so that we can take as normalized volume the form

$$\frac{idZ \wedge d\overline{Z}}{2\pi(1+|Z|^2)^2}.$$

Recall that the standard Euclidean volume form on $\mathbb{C}$ is given by $\frac{i}{2}dZ \wedge d\overline{Z}$, and the corresponding measure is the Lebesgue measure on $\mathbb{C}$ that we will denote by $dm(Z)$. The probability measure on $\mathbb{C}$ associated with $\varphi_0^*\omega$ is therefore

$$dv_{S^2}(Z) = \frac{dm(Z)}{\pi(1+|Z|^2)^2}.$$

3.3 Bergman kernel

Let us consider the line bundle L defined as follows: for any point $P = (x, y, z) \in S^2$, the fiber L_P is the line in $\mathbb{R}^3$ generated by P. Note: it is the equivalent of the tautological bundle on the complex projective plane $\mathbb{CP}^1$. We have the open covering (U_0, U_1) of S^2, and the associated trivialization functions are

$$\psi_0 : (P, \lambda P) \in \pi^{-1}(U_0) \mapsto (P, \lambda)$$

and

$$\psi_1 : (P, \lambda P) \in \pi^{-1}(U_1) \mapsto (P, \lambda).$$

The transition function γ_{10} is then the identity and the line bundle is trivial: $L \cong S^2 \times \mathbb{C}$. A section of L is then a function $f : S^2 \to \mathbb{C}$ that maps P to $\lambda = f(P)$, that is, a choice of a coordinate λ in the complex line $L_P = \mathbb{C}P$. A holomorphic section of L is then a holomorphic function $f : S^2 \to \mathbb{C}$. Note that one can also define the global frame $e_0 : S^2 \to \mathbb{C}$ such that $e_0(P) = 1$ for all $P \in S^2$.

Let us endow L^k with the metric h^k given in the local coordinate Z on U_0 by

$$h^k(s_1(Z), s_2(Z)) = f_1(Z)\overline{f_2}(Z)e^{-k\phi(Z)},$$

where $\phi(Z) = 4\log(1 + |Z|^2)$ is the Kähler potential of the round metric. The inner product $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{h^k, L^2(dv_{S^2}}$ is then given by

$$\begin{aligned}
\langle s_1, s_2\rangle &= \int_{S^2} h^k(s_1(P), s_2(P))dv_{S^2}(P) \\
&= \int_{S^2\setminus\{(0,0,1)\}} h^k(s_1(P), s_2(P))dv_{S^2}(P) \\
&= \int_{\mathbb{C}} h^k(s_1(Z), s_2(Z))\frac{dm(Z)}{\pi(1+|Z|^2)^2} \\
&= \int_{\mathbb{C}} f_1(Z)\overline{f_2}(Z)\frac{dm(Z)}{\pi(1+|Z|^2)^{2+k}}.
\end{aligned}$$

Proposition 3.1. *An orthonormal basis of* $H^0(S^2, L^k), \langle\cdot,\cdot\rangle_{h^k, L^2(dv_{S^2})}$ *is given by* $(s_\ell)_{0\leqslant \ell\leqslant k}$*, where the sections* s_ℓ *are given in complex stereographic coordinates in* U_0 *by*

$$s_\ell(Z) = \sqrt{k+1}\sqrt{\binom{k}{\ell}} Z^\ell, \ \forall 0 \leqslant \ell \leqslant k.$$

Proof. Let $0 \leqslant \ell, m \leqslant k$. Using polar coordinates, the inner product satisfies

$$\begin{aligned}
\langle s_\ell, s_m\rangle =&(k+1)\sqrt{\binom{k}{\ell}\binom{k}{m}} \int_{\mathbb{C}} \frac{z^\ell \overline{z}^m dm(z)}{\pi(1+|z|^2)^{2+k}} \\
=&(k+1)\sqrt{\binom{k}{\ell}\binom{k}{m}} \int_{\mathbb{C}\setminus\mathbb{R}_+} \frac{z^\ell \overline{z}^m dm(z)}{\pi(1+|z|^2)^{2+k}} \\
=&(k+1)\sqrt{\binom{k}{\ell}\binom{k}{m}} \int_0^{2\pi} \frac{e^{i\theta(\ell-m)}d\theta}{2\pi} \int_0^\infty \frac{2r^{\ell+m+1}dr}{(1+r^2)^{2+k}} \\
=&(k+1)\sqrt{\binom{k}{\ell}\binom{k}{m}} \delta_{\ell m} \int_0^\infty \frac{2r^{2\ell+1}dr}{(1+r^2)^{2+k}}.
\end{aligned}$$

The last integral can be computed explicitly using integrations by parts:

$$\int_0^\infty \frac{2r^{2\ell+1}dr}{(1+r^2)^{2+k}} = \frac{\ell}{k+1}\int_0^\infty \frac{2r^{2\ell-3}dr}{(1+r^2)^{k+1}} = \cdots = \frac{\ell!(k+1-\ell)!}{(k+1)!}\int_0^\infty \frac{2rdr}{(1+r^2)^{k-\ell+2}} = \frac{1}{(k+1)\binom{k}{\ell}},$$

which concludes the proof. □

Note that the sections s_ℓ actually correspond to the spherical harmonics. As a consequence of the previous proposition, the Bergman kernel can be written

$$B_k(Z_1, Z_2) = (k+1)e^{-\frac{k}{2}(\phi(Z_1)+\phi(Z_2))}\sum_{\ell=0}^k \binom{k}{\ell}(Z_1\overline{Z_2})^\ell,$$

and we have from the binomial formula:

$$B_k(Z_1, Z_2) = (k+1)\frac{(1+Z_1\overline{Z}_2)^k}{(1+|Z_1|^2)^{\frac{k}{2}}(1+|Z_2|^2)^{\frac{k}{2}}}.$$

We have therefore the following scaling limit:

$$\frac{1}{k+1}B_k(\frac{Z_1}{\sqrt{k}}, \frac{Z_2}{\sqrt{k}}) = \frac{(1+\frac{Z_1\overline{Z}_2}{k})^k}{(1+\frac{|Z_1|^2}{k})^{\frac{k}{2}}(1+\frac{|Z_2|^2}{k})^{\frac{k}{2}}} \longrightarrow e^{Z_1\overline{Z}_2 - \frac{1}{2}|Z_1|^2-\frac{1}{2}|Z_2|^2},$$

which corresponds to the kernel of infinite Ginibre ensemble as expected.

3.4 The corresponding DPP

In this section, we describe the DPP whose kernel is B_k: it is the point process associated to the $(k+1)$-tuple of random variables on S^2 whose joint distribution is given (in the sterographic coordinate Z on U_0) by:

$$\frac{1}{(k+1)!}|\det(Z_i^{j-1})_{1\leqslant i,j\leqslant k+1}|^2\prod_{i=1}^{k+1}\frac{dm(Z_i)}{(1+|Z_i|^2)^{k+2}}.$$

The *spherical ensemble*, defined as the distribution of the eigenvalues of $A^{-1}B$ where A and B are independent standard Gaussian complex matrices, is given by

$$\frac{1}{Z_N}\prod_{i<j}|Z_i-Z_j|^2\prod_{i=1}N\frac{dm(Z_i)}{(1+|Z_i|^2)^{N+1}}.$$

We see that in the case of $N=N_k=k+1$, we obtain in fact the DPP on the sphere associated to the normalized volume form dv_{S^2} and the metric h with Kähler potential $\phi(Z)=4\log(1+|Z|^2)$.

4 A more advanced example: the projective space

4.1 Definition

Let us consider the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}\setminus\{0\}$ given by

$$(\lambda, z=(z_0,\dots,z_n))\mapsto \lambda\cdot z=(\lambda z_0,\dots,\lambda z_n).$$

Definition 4.1. The n-dimensional *complex projective space* is the quotient

$$\mathbb{CP}^n=(\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*.$$

Its elements are denoted by $[z]$, or in homogeneous coordinates $[Z_0:\dots:Z_n]$ (so that the vector $(Z_0,\dots,Z_n)\in\mathbb{C}^n$ belongs to the equivalence class).

The action of $\mathbb{C}$ on $\mathbb{C}^{n+1}\setminus\{0\}$ is proper and free, so that:

- The quotient $\mathbb{CP}^n$ is Hausdorff;
- The projection
$$\pi:\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{CP}^n$$
is a G-principal bundle, *i.e.* for all $[z]\in\mathbb{CP}^n$, there is a neighborhood $U\subset\mathbb{CP}^n$ of $[z]$ and a G-equivariant homeomorphism
$$\Phi:\pi^{-1}(U)\xrightarrow{\simeq}U\times\mathbb{C}^*,$$
which means that for any $\lambda\in\mathbb{C}^*$, $[z]\in U$, we have $\Phi(\lambda\cdot z)=\lambda\cdot\Phi(z)$.

- The orbit space $\mathbb{CP}^n$ has a unique structure of complex manifold such that π has local holomorphic sections.

One can make the manifold structure explicit using the atlas $(V_i, \phi_i)_{0\leqslant i\leqslant n}$ defined by:

$$V_i = \{[Z_0 : \cdots : Z_n] \in \mathbb{CP}^n : Z_i \neq 0\}, 0 \leqslant i \leqslant n,$$

$$\phi_i : [Z_0 : \cdots : Z_n] \in V_i \mapsto \left(\frac{Z_0}{Z_i}, \ldots, \frac{\widehat{Z_i}}{Z_i}, \ldots, \frac{Z_n}{Z_i}\right) = (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

In the definition of ϕ_i, the hat indicates that the element is omitted. By convention, to avoid mixing local coordinates and homogeneous coordinates, we will write coordinates $(z^1, \ldots, z^n)$ in lower case and the homogeneous coordinates $[Z_0 : \cdots : Z_n]$ in upper case[4].

Proposition 4.1. *For any $i < j$, the transition function $\phi_j \circ \phi_i^{-1}$ is given by*

$$\phi_j \circ \phi_i^{-1} : (z^1, \ldots, z^n) \mapsto \left(\frac{z^1}{z^{j+1}}, \ldots, \frac{z^j}{z^{j+1}}, \frac{z^{j+2}}{z^{j+1}}, \ldots, \frac{z^i}{z^{j+1}}, \frac{1}{z^{j+1}}, \frac{z^{i+1}}{z^{j+1}}, \ldots, \frac{z^n}{z^{j+1}}\right).$$

Note that $\mathbb{CP}^n$ is the compactified of $\mathbb{C}^n$ obtained by added a hyperplane H at infinity. In coordinates, we have the inclusion

$$\begin{cases} \mathbb{C}^n & \to & \mathbb{CP}^n \\ (z_1, \ldots, z_n) & \mapsto & [1 : z_1 : \cdots : z_n] \end{cases}$$

and the hyperplane H has equation $Z_0 = 0$. We have the identification $H \cong \mathbb{CP}^{n-1}$ by considering H as the set of possible directions to infinity in $\mathbb{C}^n$.

Proposition 4.2. *For any $n \in \mathbb{N}^*$ we denote by $G_{\mathbb{C}}(1, n+1)$ the vector space of complex lines in $\mathbb{C}^{n+1}$, and S^n the n-dimensional sphere in $\mathbb{R}^n + 1$. We have canonical bijections*

$$G_{\mathbb{C}}(1, n+1) \cong \mathbb{CP}^n \cong S^{2n+1}/S^1,$$

where the quotient of S^{2n+1} by S^1 is given by the action induced by the one of S^1 on $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$: $e^{i\theta} \cdot z = (e^{i\theta} z_0, \ldots, e^{i\theta} z_n)$.

4.2 Tautological bundle and homogeneous polynomials

The set $\mathscr{O}(-1)$ is defined by

$$\mathscr{O}(-1) = \coprod_{[z]\in\mathbb{CP}^n} D_z = \{([z], v), [z] \in \mathbb{CP}^n, v \in D_z\}.$$

It has a natural projection on $\mathbb{CP}^n$, given by $\Phi : ([z], v) \mapsto [z]$.

[4] That is infortunate because it was the opposite in the sphere...

Proposition 4.3. *The projection $\Phi : \mathscr{O}(-1) \to \mathbb{CP}^n$ defines a structure of line bundle on $\mathbb{CP}^n$. The vector bundle structure of Φ is given in the atlas (V_i, ϕ_i) by the trivialization functions*

$$\psi_i : \left\{ \begin{array}{ccc} \Phi^{-1}(V_i) & \to & V_i \times \mathbb{C} \\ ([z], \lambda(\frac{Z_0}{Z_i}, \dots, \frac{Z_n}{Z_i})) & \mapsto & ([z], \lambda) \end{array} \right. ,$$

and the changes of trivializations are the following cocycles:

$$\gamma_{ji}([z]) = \frac{Z_j}{Z_i}.$$

Definition 4.2. The vector bundle $\mathscr{O}(-1) \to \mathbb{CP}^n$ is called the *tautological bundle* on $\mathbb{CP}^n$. Its dual bundle, whose fibers are the dual lines of the ones of $\mathscr{O}(-1)$ and the cocycles are given by

$$\gamma^*_{ji}([z]) = \frac{Z_i}{Z_j} = \gamma^{-1}_{ji}([z]),$$

is called *hyperplane bundle* of $\mathbb{CP}^n$, and denoted by $\mathscr{O}(1)$.

We denote by $\mathscr{O}(k)$ the k-th tensor power of the hyperplane bundle. It is the line bundle on $\mathbb{CP}^n$ whose cocycles are

$$\gamma_{ji}([z]) = \left(\frac{Z_i}{Z_j}\right)^k.$$

Proposition 4.4. *There is a canonical bijection between the space of holomorphic sectioins of $\mathscr{O}(k)$ and the set of homogeneous polynomials of degree k on $\mathbb{C}^{n+1} \setminus \{0\}$.*

4.3 Fubini–Study metric

Let us endow $\mathbb{CP}^n$ with a Hermitian metric.

Proposition 4.5. *The standard Hermitian metric on $\mathbb{C}^{n+1}$ is not invariant by the diagonal action of $\mathbb{C}^*$ therefore the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ is not an isometry. However, the diagonal action of S^1 on $\mathbb{C}^{n+1}$ is an isometry.*

Definition 4.3. The *Fubini–Study metric* on $\mathbb{CP}^n$ is the metric induced on the quotient S^{2n+1}/S^1, where S^{2n+1} is endowed with the round metric..

Proposition 4.6. *In an affine chart (V_i, ϕ_i), if we set*

$$\phi([Z_0 : \dots : Z_n]) = (z^1, \dots, z^n),$$

the Fubini–Study metric reads $ds^2 = g_{i\bar{j}} dz^i d\bar{z}^j$, where g is the Hermitian matrix defined by

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = \frac{(1+|z|^2)\delta_{i\bar{j}} - \bar{z}^i z^j}{(1+|z|^2)^2}. \tag{30}$$

Assume that $[Z_0 : \dots : Z_n] \in V_0$, by construction $[Z_0 : \dots : Z_n] = [1 : z^1 : \dots : z^n]$, and the Fubini–Study metric reads

$$ds^2 = \frac{|Z|^2|dZ|^2 - (\overline{Z} \cdot dZ)(Z \cdot d\overline{Z}}{|Z|^4}.$$

4.4 Kähler structure

The aim of this section is to get more familiar with Kähler geometry. It is important to understand the case of $\mathbb{CP}^n$ for two reasons:

- it is the simplest example of n-dimensional compact Kähler manifold,
- thanks to the celebrated Kodaira embedding theorem, any compact Kähler manifold endowed with a positive Hermitian line bundle can be embedded in $\mathbb{CP}^N$ for a given N, so that the embedding transforms the line bundle in $\mathscr{O}(1)$.

Proposition 4.7. *The hermitian form associated to the Fubini–Study on $\mathbb{CP}^n$ is given by*

$$\omega([Z]) = \frac{i}{2}\partial\overline{\partial}\log(|Z|^2). \tag{31}$$

It is closed, therefore $\mathbb{CP}^n$ is a Kähler manifold.

Note that the function

$$\rho : \begin{cases} \mathbb{C}^{n+1}\setminus\{0\} & \to & \mathbb{R} \\ (Z_0,\dots,Z_n] & \mapsto & \log(|Z_0|^2+\cdots+|Z_n|^2) \end{cases}$$

is not constant on the equivalence classes of $\mathbb{C}^{n+1}\setminus\{0\}$, therefore does not define a function on $\mathbb{CP}^n$. Indeed, let $(Z_0,\dots,Z_n)\in\mathbb{C}^{n+1}$ be a nonzero vector, then for instance $(\lambda Z_0,\dots,\lambda Z_n)$ is in the same class for any $\lambda\in\mathbb{C}^*$ but

$$\rho((\lambda Z_0,\dots,\lambda Z_n)) = \log|\lambda|^2 + \rho(Z_0,\dots,Z_n).$$

However, when the operator $\partial\overline{\partial}$ is applied, the term $\log|\lambda|^2$ vanishes, and therefore ω is well-defined.

Definition 4.4. The function $\rho : \mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{R}$ defined by

$$\rho(Z_0,\dots,Z_n) = \log(|Z_0|^2+\cdots+|Z_n|^2)$$

is called *Kähler potential*.

As stated above, the Kähler potential does not define a global function on $\mathbb{CP}^n$. Fortunately, it unambiguously defines a function in the local charts (V_i,ϕ_i): let $0\leqslant i\leqslant n$, for all $(z^1,\dots,z^n)\in\phi_i(V_i)$ we set

$$K(z^1,\dots,z^n) = \rho(\phi_i^{-1}(z^1,\dots,z^n)) = \log(1+|z^1|^2+\cdots+|z^n|^2),$$

and this expression does not depend on i. By abuse of notation, we will also denote by K the Kähler potential on $\mathbb{CP}^n$.

4.5 Bergman kernel

The sections

$$S_P = \sqrt{\frac{(k+n)!}{p_0! \cdots p_n!}} z^P e_L,$$

for any $P = (p_0, \dots, p_n)$ such that $|P| = \sum_i p_i = k$, form an orthonormal basis of $H^0(\mathbb{CP}^n, \mathscr{O}(k))$. The Bergman kernel is then given by

$$B_k(x,y) = \sum_{|P|=k} \frac{(k+n)!}{P!} x^P \overline{y}^P e_L \otimes \overline{e_L}.$$

Using the multinomial formula, it appears that

$$B_k(x,y) = \frac{k!}{(k+n)!} (x_0\overline{y}_0 + \cdots + x_n\overline{y}_n)^k e_L \otimes \overline{e_L}.$$

If we rescale the x_i and y_i by $\frac{1}{\sqrt{k}}$, we obtain

$$B_k(\frac{x}{\sqrt{k}}, \frac{y}{\sqrt{k}}) = \frac{k!}{(k+n)!k^k} (x_0\overline{y}_0 + \cdots + x_n\overline{y}_n)^k e_L \otimes \overline{e_L}.$$

Then, by Stirling's formula,

$$k! \sim \sqrt{2\pi k} \left(\frac{k}{e}\right)^k,$$

$$B_k(\frac{x}{\sqrt{k}}, \frac{y}{\sqrt{k}}) \sim \frac{\sqrt{2\pi k}}{(k+n)!e^k} (x_0\overline{y}_0 + \cdots + x_n\overline{y}_n)^k$$

Work in progress

5 Applications

We list here a few potential applications of this theory, and compare it to other approaches.

5.1 Riemannian Hamiltonian Monte Carlo

Find some refs.

5.2 Quasi Monte Carlo on compact Riemann surfaces

See Berman's paper and Masatake's paper.

5.3 Quadrature on the sphere

See HealPix and related schemes.

Spoiler: it seems that our model generalizes theoretically the algorithm by McEwen and Wiaux, provided that the orthonormal basis of $H^0(M, L^k)$ is given. However, what might be interesting would be to use the approximation of the Bergman kernel (from the asymptotic expansion) rather than its decomposition on the orthonormal basis. The idea is that such basis is, in general, very hard to find (there is no equivalent of the Rodrigues' formula for orthogonal polynomials)

The standard procedure for sampling on the sphere uses spherical harmonic transforms. The first approach using a fast Legendre transform is due to Orszag and was based on a Wentzel–Kramers–Brillouin (WKB) approximation. Later, other approaches were done using the fast multipole method (FMM) by Alpert and Rokhin, as well as Suda and Takani. Other sampling algorithms with WKB frequency estimates were developed by Mohlenkamp. All these approaches are approximates and do not give exact sampling theorems on the sphere.

Other approaches, such that HEALPix and IGLOC, are based on the separation of variables, and use approximate quadrature rules.

5.4 Quadrature on complex manifolds

In [BH20], an estimator of $\int f(x)d\mu(x)$ is given by

$$\sum_{i=1}^{N} \frac{f(x_i)}{K_N(x_i, x_i)},$$

where K_N is the Christoffel–Darboux kernel for the measure μ. They prove that this estimator is unbiased:

$$\mathbb{E}\left[\sum_{i=1}^{N} \frac{f(x_i)}{K_N(x_i, x_i)}\right] = \int f(x)d\mu(x).$$

There is even a CLT: under some assumptions,

$$\sqrt{N^{1+1/d}}\left(\sum_{i=1}^{N} \frac{f(x_i)}{K_N(x_i, x_i)} - \int f(x)d\mu(x)\right) \xrightarrow[N\to\infty]{(d)} \mathcal{N}(0, \Omega_f^2).$$

We may try to reproduce this result on complex manifolds. Indeed, we have as for classical DPPs

$$\mathbb{E}[\sum_i \varphi(x_i)] = \int_M \varphi(x)B_k(x,x)dv_M(x),$$

and if we take $\varphi(x) = \frac{f(x)}{B_k(x,x)}$ (can we check that $B_k(x,x) \neq 0$?), it follows that

$$\mathbb{E}\left[\sum_i \frac{f(x_i)}{B_k(x_i, x_i)}\right] = \int_M f(x)d\mu(x).$$

The estimator remains unbiased, but we need to control its variance.

Proposition 5.1.

$$\mathrm{Var}(\sum_i f(x_i)) = \frac{1}{2}\iint_{M^2}(f(x)-f(y))^2|B_k(x,y)|^2 d\mu^{\otimes 2}(x,y). \tag{32}$$

Proof. The proof is actually the same as it is for standard orthonormal polynomials.

$$\begin{aligned}\iint_{M^2}(f(x)-f(y))^2 B_k(x,y)B_k(y,x)d\mu^{\otimes 2}(x,y) =& 2\iint_{M^2} f(x)^2 B_k(x,y)B_k(y,x)d\mu^{\otimes 2}(x,y)\\ &-2\iint_{M^2} f(x)f(y)B_k(x,y)B_k(y,x)d\mu^{\otimes 2}(x,y),\end{aligned}$$

and if we use (20) we obtain

$$\iint_{M^2} f(x)^2 B_k(x,y)B_k(y,x)d\mu^{\otimes 2}(x,y) = \int_M f(x)^2 B_k(x,x)d\mu(x) = \mathbb{E}[\sum_i f(x_i)^2].$$

Furthermore, we have

$$\begin{aligned}\mathrm{Var}(\sum_i f(x_i)) =& \mathbb{E}[\sum_{i,j} f(x_i)f(x_j)] - \mathbb{E}[\sum_i f(x_i)]^2\\ =& \mathbb{E}[\sum_i f(x_i)^2] + \mathbb{E}[\sum_{i\neq j} f(x_i)f(x_j)] - \mathbb{E}[\sum_i f(x_i)]^2.\end{aligned}$$

Recall that

$$\begin{aligned}\mathbb{E}[\sum_{i\neq j} f(x_i)f(x_j)] =& \iint_{M^2} f(x)f(y)B_k(x,x)B_k(y,y)d\mu^{\otimes 2}(x,y)\\ &- \iint_{M^2} f(x)f(y)B_k(x,y)B_k(y,x)d\mu^{\otimes 2}(x,y)\\ =& \mathbb{E}[\sum_i f(x_i)]^2 - \iint_{M^2} f(x)f(y)B_k(x,y)B_k(y,x)d\mu^{\otimes 2}(x,y),\end{aligned}$$

so that

$$\mathrm{Var}(\sum_i f(x_i)) = \frac{1}{2}\iint_{M^2}(f(x)-f(y))^2 B_k(x,y)B_k(y,x)d\mu^{\otimes 2}(x,y)$$

as expected. □

We may apply Theorem 2.6 to the function $F : x \mapsto \frac{N_k f(x)}{B_k(x,x)}$. **If we can prove that it is Lipschitz with compact support included in the bulk,** then we would have

$$\sqrt{N_k^{1+\frac{1}{d}}}\left(\frac{1}{N_k}\sum_i F(X_i) - \mathbb{E}\left[\frac{1}{N_k}\sum_i F(X_i)\right]\right) \xrightarrow[k\to\infty]{(d)} \mathcal{N}(0, \frac{1}{4\pi}\|\nabla F\|^2).$$

However,

$$\mathbb{E}\left[\frac{1}{N_k}\sum_i F(X_i)\right] = \int F(x)\frac{B_k(x,x)}{N_k}\omega_d = \int f(x)\omega_d,$$

and

$$\frac{1}{N_k}\sum_i F(X_i) = \sum_i \frac{f(X_i)}{B_k(X_i,X_i)},$$

so that

$$\sqrt{N_k^{1+\frac{1}{d}}}\left(\sum_i \frac{f(X_i)}{B_k(X_i,X_i)} - \int f(x)\omega_d\right) \xrightarrow[k\to\infty]{(d)} \mathcal{N}(0, \frac{1}{4\pi}\|\nabla F\|^2).$$

A Large deviation principles

A.1 Definitions and general results

- LDP
- Legendre–Fenchel transform

A.2 LDP for empirical measures

Let $\mathcal{E}$ be a locally convex Hausdorff topological vector space, and μ a Borel probability measure on $\mathcal{E}$. Consider $(X_1,\dots,X_n)$ a family of i.i.d. random variables on $\mathcal{E}$ with law μ. The *logarithmic moment-generating function*[5] associated with μ is the function

$$\Lambda_\mu : \lambda \in \mathcal{E}^* \mapsto \log \mathbb{E}[e^{\langle\lambda, X_1\rangle}] = \log \int_{\mathcal{E}} e^{\langle\lambda,x\rangle} d\mu(x).$$

In what follows, we consider $\mathcal{E} = \mathcal{M}_1(K)$, where K is a compact subset of a Polish space. It is in fact Hausdorff and locally convex; let us endow it with the distribution of a random measure. More precisely, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $\mu : \omega \to \mu_\omega$ be a $\mathcal{E}$-valued random variable. Its distribution is given by

$$\mathbb{E}[\mu(f)] = \int_\Omega\int_K f(x)d\mu_\omega(x)d\mathbb{P}(\omega).$$

Given a probability measure $d\mathcal{P}$ on K^N, the empirical measure $\widehat{\mu}_N$ of a N-tuple $(X_1,\dots,X_N)$ of random variables with values in K and joint distribution $d\mathcal{P}$ defines a probability measure on $\mathcal{E}$. Let us explicit the corresponding probability measure on $\mathcal{E}$: it is the measure $dm(\nu) = \mathbb{E}[\delta_{\widehat{\mu}_N}(\nu)]$, in the sense that for any measurable function $F : \mathcal{E} \to \mathbb{R}$,

$$\int_{\mathcal{E}} F(\nu)dm(\nu) = \mathbb{E}[F(\widehat{\mu}_N)].$$

For instance, for $f \in \mathscr{C}^0(K)$, set $F_f(\nu) = \langle f,\nu\rangle = \int_K f(x)\nu(x)$ the integral of f with respect to ν. We have in this case

$$\mathbb{E}[\widehat{\mu}_N(f)] = \int_\Omega\int f(x)d\widehat{\mu}_{N,\omega}(x)d\mathbb{P}(\omega) = \int_\Omega \frac{1}{N}\sum_{i=1}^N f(X_i(\omega))d\mathbb{P}(\omega),$$

[5] Also known as the cumulant-generating function.

and it follows that

$$\mathbb{E}[\widehat{\mu}_N(f)] = \int_{K^N} \frac{1}{N} \sum_{i=1}^{N} f(x_i) d\mathcal{P}(x_1, \dots, x_N).$$

In particular, if $d\mathcal{P} = \nu^{\otimes N}$ for a probability measure ν on K, we get

$$\mathbb{E}[\widehat{\mu}_N(f)] = \int_K f(x) d\nu.$$

Let us describe the logarithmic moment-generating function in this setting. It is a function $\Lambda : \mathscr{C}^0(K) \to \mathbb{R}$ given by

$$\Lambda_{\widehat{\mu}_N}(u) = \log \int_{\mathcal{E}} e^{\int_K u(x) d\widehat{\mu}_N(x)} d\mu = \log \mathbb{E}[e^{\frac{1}{N}\sum_i u(X_i)}] = N \log \mathbb{E}[e^{\frac{1}{N} u(X_1)}].$$

Theorem A.1 (Abstract Gärtner–Ellis theorem). *Let (μ_ε) be an exponentially tight family of Borel probability measures on a locally convex Hausdorff topological vector space $\mathcal{E}$. Suppose $\Lambda(\cdot) = \lim_{\varepsilon \to 0} \varepsilon \Lambda_{\mu_\varepsilon}(\cdot/\varepsilon)$ is finite and Gateaux differentiable. Then (μ_ε) satisfies a LDP in $\mathcal{E}$ with speed $\frac{1}{\varepsilon}$ and with good rate function Λ^*.*

Let us set $\varepsilon = \frac{1}{N}$. If $d\mathcal{P} = \nu^{\otimes N}$, we see that

$$\frac{1}{N} \Lambda_{\widehat{\mu}_N}(Nu) = \log \mathbb{E}[e^{u(X_1)}] = \log \int_K e^{u(x)} d\nu(x).$$

It indeed defines a finite function, and one can show that it is Gateaux-differentiable. Furthermore, the exponential tightness is implied by the compactness of K.

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