Monte Carlo on compact complex manifolds using Bergman kernels

Thibaut Lemoine*[1] and Rémi Bardenet[1]

[1]Université de Lille, CNRS, Centrale Lille, UMR 9189 – CRIStAL, F-59000 Lille, France

Abstract

In this paper, we propose a new randomized method for numerical integration on a compact complex manifold with respect to a continuous volume form. Taking for quadrature nodes a suitable determinantal point process, we build an unbiased Monte Carlo estimator of the integral of any Lipschitz function, and show that the estimator satisfies a central limit theorem, with a faster rate than under independent sampling. In particular, seeing a complex manifold of dimension d as a real manifold of dimension $d_{\mathbb{R}} = 2d$, the mean squared error for N quadrature nodes decays as $N^{-1-2/d_{\mathbb{R}}}$; this is faster than previous DPP-based quadratures and reaches the optimal worst-case rate investigated by Bakhvalov (1965) in Euclidean spaces. The determinantal point process we use is characterized by its kernel, which is the Bergman kernel of a holomorphic Hermitian line bundle, and we strongly build upon the work of Berman that led to the central limit theorem in (Berman, 2018). We provide numerical illustrations for the Riemann sphere.

Keywords— Bergman kernel, complex manifolds, determinantal point processes, Monte Carlo integration

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*Corresponding author: thibaut.lemoine@univ-lille.fr

1 Introduction

1.1 Numerical integration on complex manifolds

Numerical integration, also known as quadrature, is the task of approximating an integral by a weighted sum of evaluations of the integrand. The points at which the integrand is evaluated are called the nodes of the quadrature. One usually distinguishes between Monte Carlo methods (MC; Robert and Casella, 2004), where the nodes are taken to be a random configuration of points, and quasi-Monte Carlo methods (QMC; Dick and Pilichshammer, 2010), which rely on deterministic configurations such as low-discrepancy sequences. Both approaches come with different measures of efficiency: for deterministic configurations, one usually wants to bound the worst-case error over a (large) class of functions. For random configurations, it is typical to consider a single integrand and derive a concentration inequality or a central limit theorem, and characterize their rate of convergence as the number of nodes tends to infinity.

Both MC and QMC methods have their strong points: MC error bounds are usually easier to intepret and estimate, while QMC convergence rates usually decrease faster than

MC for smooth integrands. There is a growing body of literature on methods that try to take the best of both worlds, like randomized QMC (Owen, 1997, 2008) or variants of kernel herding (Chen et al., 2010; Liu and Wang, 2016), to cite only a few influential papers. Similarly, Bardenet and Hardy (2020) and Mazoyer et al. (2020b) proved fast central limit theorems for integration with particular classes of determinantal point processes (DPPs, Macchi, 1972; Soshnikov, 2000). Together with their computational tractability, the tunable negative dependence among the points of a DPP makes this distribution a natural candidate for structured Monte Carlo integration; see also (Gautier et al., 2019b; Belhadji et al., 2019, 2020; Belhadji, 2021). While previous work on Monte Carlo with DPPs has focused on integrals over the Euclidean space, we investigate in this paper the use of a DPP for integrating over a compact complex manifold.

Real and complex manifolds naturally arise in domains as diverse as theoretical physics (Wells, 1979), information geometry (Amari and Nagaoka, 2000; Molitor, 2014), or even computer vision (Turaga and Chellappa, 2008). They often represent a space of states or parameters, and the geometric properties of the manifold reflect the inherent properties of the underlying model. Integrating over manifolds is a key task, for instance, in Bayesian inference when the manifold represents the considered probabilistic models. Numerical integration on manifolds is a relatively recent topic, but the field is growing, see e.g. the QMC methods in (Brandolini et al., 2014; Bittante et al., 2016; De Marchi and Elefante, 2018; Berman, 2024) and the MC methods in (Girolami and Calderhead, 2011; Diaconis et al., 2013; Zappa et al., 2018; Ehler et al., 2019), to cite only a few.

In this article, we follow Bardenet and Hardy (2020) and introduce a well-chosen DPP on a compact complex manifold, namely the DPP with kernel the Bergman kernel of the manifold. This allows us to generalize the orthogonal polynomials-based arguments of Bardenet and Hardy (2020), and actually improve on the rate in their central limit theorem, to now match a lower bound dating back to Bakhvalov (1965). Our paper can also be seen as a Monte Carlo counterpart of (Berman, 2024), where the worst-case error of a DPP over the sphere known as the *spherical ensemble* is investigated, comparing it to the QMC designs studied by Brauchart et al. (2014). Our proofs heavily rely on the seminal papers by Berman (2014, 2018), and a third way to see our paper is as an extension of a central limit theorem in (Berman, 2018) to an unbiased estimator of the integral of interest. We also point to recent work by (Levi, Marzo, and Ortega-Cerdà, 2023), who derive variance estimates for linear statistics under DPPs on the sphere using very different analytic techniques that may generalize to more manifolds.

In the rest of this introduction, we state our main result, give some context about other methods of integration on manifolds, and give the outline of the paper.

1.2 Main result

We now define our setting and state our main result. The setting is fairly technical; the reader not accustomed to the vocabulary of complex manifolds is referred to the more exhaustive development in Section 2.

Let L be a holomorphic line bundle over a compact complex manifold $\mathcal{M}$ of dimension d. If h is an Hermitian metric on L with local weight ϕ, we shall denote respectively by $\langle \cdot, \cdot \rangle_\phi$ and $|\cdot|_\phi$ the associated inner product and norm on each fiber. Given a section

e_U of L that does not vanish on an open subset U (a *trivializing section*), any section $s \in H^0(\mathcal{M}, L^k)$ can be represented on U by a function $f : U \to \mathbb{C}$ as

$$s(x) = f(x)e_U(x), \ \forall x \in U.$$

In particular, the local weight ϕ is characterized by e_U as follows: for any two sections s_1, s_2 respectively represented by f_1, f_2,

$$h_x(s_1(x), s_2(x)) = f_1(x)\overline{f_2}(x)h_x(e_U(x), e_U(x)) = f_1(x)\overline{f_2}(x)\mathrm{e}^{-\phi(x)}.$$

Furthermore, if μ is a finite measure on $\mathcal{M}$, then (ϕ, μ) is called a *weighted measure*. Such a pair induces an inner product on the space $H^0(\mathcal{M}, L)$ of holomorphic sections of L,

$$\langle s^{(1)}, s^{(2)} \rangle_{(\phi,\mu)} = \int_{\mathcal{M}} h_x(s^{(1)}(x), s^{(2)}(x))\mathrm{d}\mu(x). \tag{1}$$

In the present article, we consider a semiclassical setting: we replace L by $L^k = L^{\otimes k}$ and let $k \to \infty$. We endow L^k with the metric h^k with weight $k\phi$, and the corresponding weighted measure is then $(k\phi, \mu)$. The inner product space $(H^0(\mathcal{M}, L^k), \langle \cdot, \cdot \rangle_{(k\phi,\mu)})$ is in fact a finite-dimensional Hilbert space; we denote by N_k its dimension.

There exists a reproducing kernel $B_{(k\phi,\mu)}$ of $H^0(\mathcal{M}, L^k)$ called the *Bergman kernel*, which intuitively is the integral kernel of the projection $L^2(\mathcal{M}, L^k) \to H^0(\mathcal{M}, L^k)$. If $(s_i)_{1 \leqslant i \leqslant N_k}$ is an orthonormal basis of $H^0(\mathcal{M}, L^k)$, $B_{(k\phi,\mu)}$ admits the following decomposition:

$$B_{(k\phi,\mu)}(x, y) = \sum_{i=1}^{N_k} s_i(x) \otimes \overline{s_i}(y), \tag{2}$$

and on the diagonal it even becomes a well-defined function $\mathcal{M} \to \mathbb{R}$,

$$B_{(k\phi,\mu)}(x, x) = \sum_{i=1}^{N_k} h^k_x(s_i(x), s_i(x)), \tag{3}$$

thanks to the isomorphism of fibers $L_x \otimes \overline{L_x} \cong \mathbb{C}$.

Under suitable assumptions on the metric h and the measure μ, the *Bergman measures*

$$\mathrm{d}\mu_{k\phi}(x) = \frac{1}{N_k} B_{(k\phi,\mu)}(x, x)\mathrm{d}\mu(x) \tag{4}$$

converge weakly to an equilibrium measure when k tends to infinity. More precisely, the weighted measure (ϕ, μ) is called *strongly regular*[1] if (i) the weight ϕ is locally $\mathscr{C}^{1,1}$-smooth, *i.e.* it is differentiable and its first partial derivatives $\frac{\partial \phi}{\partial z_i}, \frac{\partial \phi}{\partial \overline{z}_j}$ are locally Lipschitz continuous, and (ii) the measure μ corresponds to the volume form

$$\omega_d = \det(h_0) \left(\frac{i}{2}\right)^d \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}z_d \wedge \mathrm{d}\overline{z}_d,$$

[1] We borrow this terminology from (Berman, 2018), although it does not seem to be standard yet.

with respect to a continuous Hermitian metric h_0 on $\mathcal{M}$ (possibly different from the metric with local weight ϕ). For any $k \geqslant 1$, let $\mu_{k\phi}$ be the measure on $\mathcal{M}$ defined by

$$\mathrm{d}\mu_{k\phi}(x) = \frac{1}{N_k} B_{(k\phi,\mu)}(x,x)\mathrm{d}\mu(x). \tag{5}$$

If the weighted measure is strongly regular, Berman (2009a) proved the weak convergence of measures

$$\mu_{k\phi} \underset{k\to\infty}{\longrightarrow} \mu_{\mathrm{eq}}, \tag{6}$$

where μ_{eq} is the pluripotential equilibrium measure (or *Monge–Ampère measure*)

$$\mu_{\mathrm{eq}} = \frac{1}{\mathrm{vol}(L)} \left(\mathrm{dd}^c \phi_e\right)^d. \tag{7}$$

Here, ϕ_e denotes the upper semicontinuous regularization of the plurisubharmonic enveloppe of ϕ, and $\mathrm{vol}(L)$ denotes the volume of the line bundle, defined by

$$\mathrm{vol}(L) = \limsup_{k\to\infty} \frac{d!N_k}{k^d}, \tag{8}$$

while the operator dd^c in (7) is defined as follows. The Dolbeault operators ∂ and $\overline{\partial}$ on L give rise to the operator $\partial\overline{\partial}$, which maps (p,q)-forms to $(p+1,q+1)$-forms on $\mathcal{M}$. Consider the real operators $\mathrm{d} = \partial + \overline{\partial}$ and $\mathrm{d}^c = \frac{1}{4\mathrm{i}\pi}(\partial - \overline{\partial})$, now we obtain

$$\mathrm{dd}^c = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}.$$

In our setting, $\partial\overline{\partial}\phi_e$ is then a complex $(1,1)$-form, and $\mathrm{dd}^c\phi_e$ a real 2-form, whose dth exterior power leads to a volume form $\omega_\phi = (\mathrm{dd}^c\phi_e)^d$ that can be normalized to produce the equilibrium measure μ_{eq}. Note that in the particular case where $\mathrm{dd}^c\phi > 0$, we have $\phi_e = \phi$. All $\mu_{k\phi}$, as well as μ_{eq}, are absolutely continuous with respect to μ, with respective densities[2]

$$\beta_k(x) = \frac{1}{N_k} B_{(k\phi,\mu)}(x,x) \text{ and } \beta_{\mathrm{eq}}(x) = \frac{\det(\mathrm{dd}^c\phi_e)(x)}{\mathrm{vol}(L)}.$$

There is a subset of $\mathcal{M}$ called the *weak bulk*, or simply *bulk*, such that the weak convergence (6) can be replaced by a pointwise convergence $\beta_k(x) \to \beta_{\mathrm{eq}}(x)$ for all x in the bulk. For almost every point x outside the bulk, $\beta_k(x) \to 0$. In the simpler case $\mathrm{dd}^c\phi > 0$, the weak bulk is the whole manifold. Note that, according to Berman (2018), we always have $\mathrm{dd}^c\phi > 0$ in the weak bulk. The 2-form $\mathrm{dd}^c\phi$ also induces a metric whose associated inner product and norm are denoted by $\langle\cdot,\cdot\rangle_{\mathrm{dd}^c\phi}$ and $|\cdot|_{\mathrm{dd}^c\phi}$.

Remark 1.1. Most of the ideas and results about the convergence of Bergman measures are related to the idea of having (at least locally) a positive curvature form $\mathrm{dd}^c\phi$. It might seem restrictive with respect to the choice of the manifold, but in fact one can endow any compact complex manifold with a positive line bundle: for instance, although the standard torus $\mathbb{C}/\mathbb{Z}^2$ endowed with the metric inherited by $\mathbb{C}$ is flat (in the sense that

[2]See (Berman, 2018), Thm. 3.1 for a discussion on this assertion.

its tangent bundle has zero curvature), one can still consider another line bundle which has positive curvature. It is important to make a distinction between the geometry of the underlying manifold (which is related to the geometry of its (co)tangent bundle, and captured by the Borel measure μ) and the geometry of the line bundle.

Beside this purely deterministic construction, one can consider a family $(X_1, \ldots, X_{N_k})$ of random variables on $\mathcal{M}$, whose joint density with respect to $\mu^{\otimes N_k}$ is proportional to

$$|\det(s_i(x_j))|^2_{k\phi} = \sum_{\sigma,\tau\in\mathfrak{S}_{N_k}} \varepsilon(\sigma)\varepsilon(\tau)\langle s_{\sigma(1)}(x_1), s_{\tau(1)}(x_1)\rangle_{k\phi} \cdots \langle s_{\sigma(N_k)}(x_{N_k}), s_{\tau(N_k)}(x_{N_k})\rangle_{k\phi},$$

where (s_i) is an orthonormal basis of $H^0(\mathcal{M}, L^k)$ with respect to the inner product (1). This density is symmetric with respect to the x_i's and vanishes as soon as there are i, j such that $i \neq j$ and $x_i = x_j$: the family $(X_1, \ldots, X_{N_k})$ therefore defines a random configuration, or a simple point process, and belongs to the subclass of *determinantal point processes* (DPPs); see (Berman, 2018; Charles and Estienne, 2020; Lemoine, 2022) for some generalities and relations with models of noninteracting fermions in quantum mechanics. The Bergman kernel contains most of the required information to study the distribution of the point process, and complex geometers provided many powerful asymptotic results about this kernel. In particular, Berman proved the following central limit theorem.

Theorem 1.1 (Berman, 2018, Theorem 1.5)**.** *Let L be a holomorphic line bundle over a compact complex manifold $\mathcal{M}$. Let (ϕ, μ) be a strongly regular weighted measure and μ_{eq} be the associated Monge–Ampère measure. For any $k \in \mathbb{N}^*$, let $(X_1, \ldots, X_{N_k})$ be a DPP with kernel $B_{(k\phi,\mu)}$. For any Lipschitz continuous $f : \mathcal{M} \to \mathbb{R}$ with compact support included in the weak bulk,*

$$\sqrt{N_k^{1+\frac{1}{d}}}\left(\frac{1}{N_k}\sum_{i=1}^{N_k} f(X_i) - \mathbb{E}\left[\frac{1}{N_k}\sum_{i=1}^{N_k} f(X_i)\right]\right) \xrightarrow[k\to\infty]{(d)} \mathcal{N}(0, \tfrac{1}{2}\|\mathrm{d}f\|^2_{\mathrm{dd}^c\phi}), \tag{9}$$

where $\|\mathrm{d}f\|^2_{\mathrm{dd}^c\phi}$ is the Dirichlet norm

$$\|\mathrm{d}f\|^2_{\mathrm{dd}^c\phi} = \int_{\mathcal{M}} |\nabla f|^2_{\mathrm{dd}^c\phi}(\mathrm{dd}^c\phi)^d. \tag{10}$$

Just like (Bardenet and Hardy, 2020, Theorem 2.1), this theorem is not a result on Monte Carlo integration as it stands, since the linear statistics in the left-hand side has no reason to converge to the target integral $\int f \mathrm{d}\mu$ in any useful sense. Actually,

$$\mathbb{E}\left[\frac{1}{N_k}\sum_i f(X_i)\right] = \frac{1}{N_k}\int_{\mathcal{M}} f(x)B_{(k\phi,\mu)}(x,x)\mathrm{d}\mu(x)$$

depends on k and converges to the integral of f with respect to the equilibrium measure μ_{eq}. The main result of the present paper is a variant of Theorem 1.1 akin to (Bardenet and Hardy, 2020, Theorem 2.2), where we introduce the inverse of a kernel diagonal as a weight in the linear statistic. To state our result, let the *equilibrium weight* be

$$w^{\phi}_{\mathrm{eq}} = \frac{\mathrm{vol}(L)}{d!\det(\mathrm{dd}^c\phi)},$$

which is well-defined in the weak bulk.

Theorem 1.2. *Let L be a holomorphic line bundle over a compact complex manifold $\mathcal{M}$, (ϕ,μ) be a strongly regular weighted measure, and F be a line bundle endowed with a continuous local weight ψ. For any $k\in\mathbb{N}^*$ let $(X_1,\dots,X_{N_k})$ be a DPP with kernel $B_{(k\phi+\psi,\mu)}$. For any Lipschitz continuous $f:\mathcal{M}\to\mathbb{C}$ with compact support included in the weak bulk and such that $\sigma^2_{f,\phi}:=\frac{1}{2}\|\mathrm{d}(w^\phi_{\mathrm{eq}}f)\|^2_{\mathrm{dd^c}\phi}<\infty$,*

$$\sqrt{N_k^{1+\frac{1}{d}}}\left(\sum_{i=1}^{N_k}\frac{f(X_i)}{B_{(k\phi+\psi,\mu)}(X_i,X_i)}-\int_{\mathcal{M}}f(x)\mathrm{d}\mu(x)\right)\xrightarrow[k\to\infty]{(d)}\mathcal{N}\left(0,\sigma^2_{f,\phi}\right). \tag{11}$$

Unlike in Theorem 1.1, the expectation of the estimator in the left-hand side does not depend on k, and is directly the integral of f with respect to the target measure μ. It is also interesting to remark that the assumptions on f slightly differ between the two theorems, because we need to ensure that the asymptotic variance is finite, which was trivially true in Theorem 1.1. The proof of Theorem 1.2 will be given in Section 5. It mostly follows the steps of the proof of Theorem 1.1 by Berman (2018), with different estimates due to the dependence in k of the integrand.

Finally, let us stress the importance of the "universality" of Theorem 1.2: if a DPP with kernel $B_{(k\phi+\psi,\mu)}$ can produce an estimator of $\int_{\mathcal{M}}f(x)\mathrm{d}\mu(x)$, then it can also produce an estimator of $\int_{\mathcal{M}}f(x)\mathrm{e}^{-V(x)}\mathrm{d}\mu(x)$ for many weight functions $V:\mathcal{M}\to\mathbb{R}$, because the asymptotic variance does not depend on ψ.

Corollary 1.3. *Let L be a holomorphic line bundle over a compact complex manifold $\mathcal{M}$, (ϕ,μ) be a strongly regular weighted measure, and $\psi:\mathcal{M}\to\mathbb{R}$ be a continuous local weight. If $(X_1,\dots,X_{N_k})$ is a DPP with kernel $B_{(k\phi,\mu)}$, then it is also a DPP with kernel $B_{(k\phi-\psi,\mathrm{e}^{-\psi}\mu)}$, and for any Lipschitz continuous $f:\mathcal{M}\to\mathbb{C}$ with compact support included in the weak bulk and such that $\|\mathrm{d}(w^\phi_{\mathrm{eq}})f\|^2_{\mathrm{dd^c}\phi}<\infty$,*

$$\sqrt{N_k^{1+\frac{1}{d}}}\left(\sum_{i=1}^{N_k}\frac{f(X_i)}{B_{(k\phi,\mathrm{e}^{-\psi}\mu)}(X_i,X_i)}-\int_{\mathcal{M}}f(x)\mathrm{e}^{-\psi(x)}\mathrm{d}\mu(x)\right)\xrightarrow[k\to\infty]{(d)}\mathcal{N}\left(0,\widehat{\sigma}^2_{f,\phi}\right). \tag{12}$$

In particular, the asymptotic variance remains the same as in Theorem 1.2, even though we use the same DPP but a different target measure. This invariance was expected since a similar property holds for multivariate orthogonal ensembles under the so-called *Nevai condition*; see (Bardenet and Hardy, 2020, Theorem 2.3 and Remark 4). However, it remains suprising if we compare the situation to classical (independent) importance sampling, where sampling the quadrature nodes from a distribution different from the target μ has an impact on the asymptotic variance (Robert and Casella, 2004).

1.3 Comparison with other methods

In a previous paper, the second author and Hardy provided a method for integration over the hypercube $I^d=[-1,1]^d\subset\mathbb{R}^d$ in any dimension for a measure μ. They started by a general CLT analogous to Theorem 1.1 based on a DPP with kernel

$$K_N(x,y)=\sum_{k=0}^{N-1}\phi_k(x)\phi_k(y),$$

where (ϕ_i) is a family of orthonormal functions in $L^2(\mu)$, and they expressed the asymptotic variance in terms of Fourier coefficients. When f is a linear combination of monomials $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ with $\alpha_i \in \{0,1\}$, this asymptotic variance coincides with the Dirichlet norm $\frac{1}{2}\|\mathrm{d}f\|^2_{\omega_{\mathrm{eq}}^{\otimes d}}$, where $\omega_{\mathrm{eq}}^{\otimes d}$ is the equilibrium weight corresponding to $\mathrm{dd}^c\phi$ in our setting. They finally proved the following convergence result.

Theorem 1.4 (Bardenet and Hardy, 2020, Theorem 3)**.** *Let $\mu(\mathrm{d}x) = \omega(x)\mathrm{d}x$ be a positive measure absolutely continuous with respect to the Lebesgue measure, with density $\omega(x) = \prod_{i=1}^d \omega_i(x_i)$, such that $\mathrm{supp}(\mu) \subset I^d$ and ω is $\mathscr{C}^1$ and positive on $(-1,1)^d$. Assume further that for any $\varepsilon > 0$,*

$$\frac{1}{N}\sup_{x\in[-1+\varepsilon,1-\varepsilon]^d} |\nabla K_N(x,x)| < \infty.$$

If $(X_1,\dots,X_N)$ is the multivariate orthonormal ensemble with respect to μ, that is, the DPP with kernel K_N, then for every $f \in \mathscr{C}^1(I^d,\mathbb{R})$ supported in $[-1+\varepsilon, 1-\varepsilon]^d$ for a fixed $\varepsilon > 0$, we have the following central limit theorem:

$$\sqrt{N^{1+\frac{1}{d}}}\left(\sum_{i=1}^N \frac{f(X_i)}{K_N(X_i,X_i)} - \int_{I^d} f(x)\mathrm{d}\mu(x)\right) \xrightarrow[N\to\infty]{law} \mathcal{N}(0,\sigma_f^2),$$

where σ_f^2 is an asymptotic variance that depends on f and ω.

Although Theorem 1.2 seems to simply generalize this result (and that was what we first expected), it happens not to be the case. Indeed, in (Bardenet and Hardy, 2020) the authors considered *real* d-dimensional spaces. if $\mathcal{M}$ is a complex manifold of complex dimension $d_{\mathbb{C}}$, it is in particular a real manifold of dimension $d_{\mathbb{R}} = 2d_{\mathbb{C}}$, therefore the speed of convergence that appears in Theorem 1.2 is

$$\frac{1}{\sqrt{N^{1+\frac{1}{d_{\mathbb{C}}}}}} = \frac{1}{\sqrt{N^{1+\frac{2}{d_{\mathbb{R}}}}}},$$

which is actually better than the speed obtained in (Bardenet and Hardy, 2020). It is in fact the same order as the worst-case mean square error of any randomized integration algorithm on $\mathbb{R}^d$ for $\mathscr{C}^1$ functions, according to a result by Bakhvalov (1965), see also (Novak, 2016, Theorem 3). Note that a key assumption of Theorem 1.4 is the positivity of the density ω on $(-1,1)^d$, which is similar to the property $\mathrm{dd}^c\phi > 0$ satisfied in the weak bulk in Theorem 1.2.

Quasi-Monte Carlo methods on a compact real Riemannian manifold $\mathcal{M}$ of dimension d, endowed with a Riemannian measure vol, have been studied by Brandolini et al. (2014). The authors provide upper bounds of quadrature errors in the following setting: assume that $\mathcal{M} = U_1 \cup \cdots \cup U_N$ is a partition of $\mathcal{M}$ in disjoint subsets. For any $1 \leqslant i \leqslant N$, set $w_i = \mathrm{vol}(U_i)$.

Theorem 1.5 (Brandolini et al., 2014)**.** *For every $d/2 < \alpha < d/2+1$ there exists a constant $c > 0$ and points $z_i \in U_i$, $1 \leqslant i \leqslant N$, such that*

$$\left|\sum_{i=1}^N w_i f(z_i) - \int_{\mathcal{M}} f(x)\mathrm{d}x\right| \leqslant c \max_{1\leqslant i\leqslant N}\{\mathrm{diameter}(U_i)^\alpha\}\|f\|_{W^{\alpha,2}(\mathcal{M})}.$$

They improved this result by controlling the difference between any probability measure on $\mathcal{M}$ and the uniform measure $\mathrm{d}x$, so that their method works for any probability measure, provided that the integrand is regular enough. An important difference between our approach and theirs is the fact that their bound relies on the maximum diameter of the subsets appearing in the partition of $\mathcal{M}$, whereas ours relies on the volume of the line bundle over $\mathcal{M}$.

At the intersection of Monte Carlo methods and QMC guarantees, in the case of the sphere $\mathbb{S}^2$, there is a result due to Berman (2024) that we now explain. Let $X = (x_1, \ldots, x_{N_k})$ be a N_k-point configuration on $\mathbb{S}^2$. The *worst-case error* for the Monte Carlo method with respect to the smoothness parameter $s \in (1, \infty)$ is defined by

$$\mathrm{wce}(X, s) = \sup_{f : \|f\|_{H^s(\mathbb{S}^2)} \leqslant 1} \left| \int_{\mathbb{S}^2} f(x) \mathrm{dvol}_{S^2}(x) - \sum_{i=1}^{N_k} \frac{f(x_i)}{B_k(x_i, x_i)} \right|, \tag{13}$$

where $\|f\|_{H^s(\mathbb{S}^2)}$ is the norm of the Sobolev space $H^s(\mathbb{S}^2)$. Recalling[3] that $B_k(x, x) = k+1$ for all $x \in \mathbb{S}$, we remark that if X is the DPP with kernel the Bergman kernel on $\mathbb{S}^2$ (the so-called *spherical ensemble*) and $\mathrm{d}\hat{\mu}_k$ is its empirical measure, then

$$\sum_{i=1}^{N_k} \frac{f(x_i)}{B_k(x_i, x_i)} = \int_{\mathbb{S}^2} f(x) d\hat{\mu}_k(x).$$

We also notice that $\mathrm{d}\mu_{\mathrm{eq}}^{\phi} = \mathrm{dvol}_{S^2}$, therefore the statements of Theorems 1.1 and 1.2 are equivalent, and in this case Berman already estimated the worst-case error.

Theorem 1.6 (Berman, 2024, Theorem 1.1)**.** *Let $X = (x_1, \ldots, x_N)$ be the spherical ensemble with N particles. For any $s \in (1, 2]$, there exists a constant $C(s)$ such that for any $R \in [\log(N)^{-\frac{1}{2}}, N \log(N)^{-\frac{1}{2}}]$,*

$$\mathbb{P}_N \left(\mathrm{wce}(X, s) \leqslant R^{\frac{s}{2}} \frac{\log(N)^{\frac{s}{4}}}{N^{\frac{s}{2}}} \right) \geqslant 1 - \frac{1}{N^{R^2/C(s)} - C(s)}. \tag{14}$$

Finally, a string of works have investigated Markov chain Monte Carlo (MCMC) on manifolds embedded in Euclidean spaces, for integration with respect to the Hausdorff measure; see e.g. the seminal (Diaconis et al., 2013) and (Zappa et al., 2018), as well as references in the latter paper. Although not explicitly mentioned, with the right assumptions, we expect a central limit theorem to hold for these chains, but with the usual rate of the inverse of the square root of the number of integrand evaluations. Convergence will thus be slower than our DPP-based method, when measured in number of integrand evaluations. Let us also emphasize that our method is a paradigm shift in the sense of requiring only minimal geometric information on the underlying space.

1.4 Outline of the article

The rest of the paper is organized as follows. In Section 2, we introduce the necessary background in complex geometry, culminating with the Bergman kernel. In Section 3, we

[3] Alternately, we prove the claim in Section 6.

introduce the DPPs that we will be using as quadrature nodes, which we call Bergman ensembles. Section 4 contains kernel estimates that are necessary in the proof of Theorem 1.2, to which Section 5 is dedicated. We specify all notions in the case of the Riemann sphere in Section 6, and we give experimental illustrations of Theorem 1.2 in that case, where the reference process can be easily sampled using random matrix models. Finally, we discuss limitations of the method, as well as current and future work, in Section 7.

2 Complex manifolds and Bergman kernels

In this section, we recall enough notions on complex manifolds and Hermitian line bundles to define the Bergman kernel, which is the main geometric object involved in our model. A reader accustomed to the vocabulary of smooth (real) manifolds (Lee, 2013) should quickly see the modifications, essentially adding holomorphicity requirements. Everything in section is standard; we refer to (Huybrechts, 2005) for complex manifolds and line bundles, and (Ma and Marinescu, 2007) for Bergman kernels. In the specific context of determinantal point processes, the reader may also refer to (Lemoine, 2022). We provide a few examples at the end of the section, but the special case of the Riemann sphere will be treated in more detail, and numerically illustrated, in Section 6.

2.1 Basic definitions

We start with the notion of complex manifold.

Definition 2.1. A *complex manifold* of dimension d is a topological space $\mathcal{M}$ endowed with a family $(U_i, \varphi_i)_{i\in I}$ of open subsets $U_i \subset \mathcal{M}$ and homeomorphisms $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^d$ such that, if $U_i \cap U_j \neq \varnothing$,

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is a biholomorphism[4] between open subsets of $\mathbb{C}^d$. The open subsets U_i are called *charts*, and the maps φ_i *local coordinates*.

The fundamental idea of complex manifolds is that the compatibility of charts and coordinates, which is illustrated in Figure 1, enables the use of the usual tools of complex analysis on $\mathbb{C}^d$. For instance, a function $f : \mathcal{M} \to \mathbb{C}$ is holomorphic (resp. $\mathscr{C}^s$) if for any $i \in I$ the function $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{C}$ is holomorphic (resp. $\mathscr{C}^s$).

Definition 2.2. Let $\mathcal{M}$ be a complex manifold. A *holomorphic vector bundle of rank r* over $\mathcal{M}$ is a complex manifold E endowed with a holomorphic surjective map $\pi : E \to \mathcal{M}$ such that, for any $x \in \mathcal{M}$, the *fiber* $E_x = \pi^{-1}(x)$ is an r-dimensional vector space over $\mathbb{C}$. A holomorphic vector bundle of rank r is *locally trivial* if there exist an open covering $(V_j)_{j\in J}$ of $\mathcal{M}$ and biholomorphic maps $\psi_j : \pi^{-1}(V_j) \to V_j \times \mathbb{C}^r$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(V_j) \arrow[rr, "\psi_j"] \arrow[dr, "\pi"'] & & V_j \times \mathbb{C}^r \arrow[dl, "\mathrm{pr}_1"] \\ & V_j & \end{tikzcd}$$

[4]That is, a holomorphic bijection whose inverse is also holomorphic.

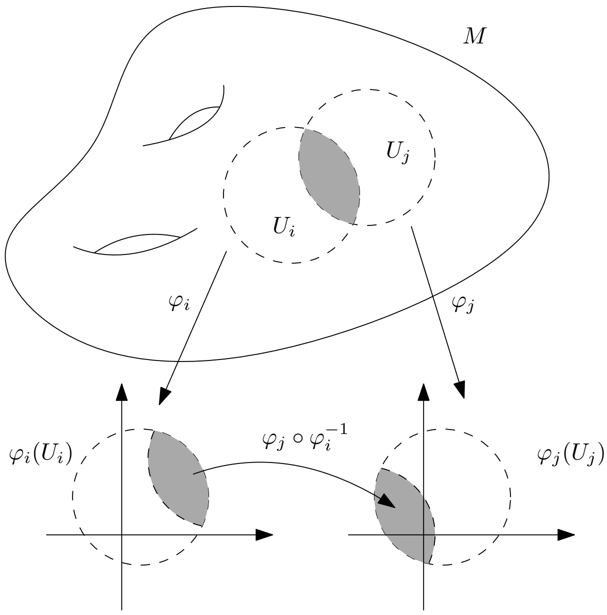


Figure 1: The interaction between two charts and local coordinates.

commutes, and such that the restriction of ψ_j to E_x is a $\mathbb{C}$-linear map for all $x \in \mathcal{M}$. The maps ψ_j are called *trivialization functions*.

Vector bundles are usually denoted as $E \to \mathcal{M}$. A graphical depiction of a vector bundle of rank 1 (a so-called *line* bundle) is given in Figure 2. In words, in a locally trivial vector bundle, on each V_j, π is akin to the canonical projection of $V_j \times \mathbb{C}^r$ onto V_j. Finally, we define *sections*, for which we can talk of holomorphicity.

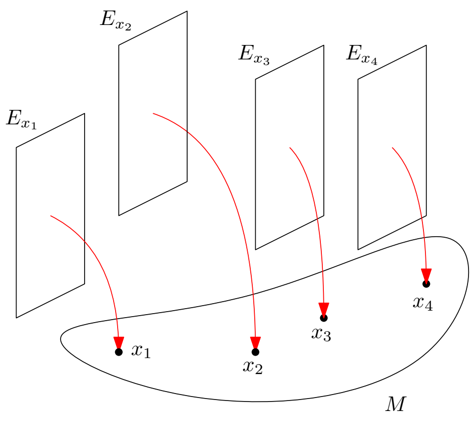


Figure 2: A vector bundle of rank 1 over a complex manifold, made of copies of the complex plane over each point of $\mathcal{M}$.

Definition 2.3. Let E be a holomorphic vector bundle over a complex manifold $\mathcal{M}$. A *local section* of E is a continuous map $s : U \to E$, for $U \subset \mathcal{M}$ open, such that $\pi \circ s = \mathrm{Id}_{\mathcal{M}}$ on U. A local section defined on $\mathcal{M}$ is called a *global section*.

Heuristically, taking a section of the vector bundle is equivalent to taking a continuous family of vectors indexed by an open subset of $\mathcal{M}$, that is, a vector field on $\mathcal{M}$. We denote by $\mathscr{C}^s(\mathcal{M}, E)$ (resp. $H^0(\mathcal{M}, E)$) the space of $\mathscr{C}^s$ (resp. holomorphic) sections of E, for any $0 \leqslant s \leqslant \infty$.

2.2 Operations on vector bundles

As a holomorphic vector bundle E over a complex manifold $\mathcal{M}$ induces a complex vector space on each fiber, one can leverage this linear algebraic structure in two ways: to perform algebraic transformations (direct sum, tensor product, wedge product) or to enrich the line bundle. The idea is that any such operation can be done on a fiber, in a way which is explicit whenever one works in a local trivialization.

Hermitian metrics. If we endow E with a family of Hermitian inner products $h_x : E_x \times E_x \to \mathbb{C}$, the vector bundle is said to be *Hermitian*, and h is called a *Hermitian metric*. The regularity (e.g. continuous, differentiable, smooth) of the metric, by convention, will be the regularity of the map $x \mapsto h_x$. This regularity can be made more explicit by using the notion of local weight: let $U \subset \mathcal{M}$ be an open subset where E can be trivialized. There exists a section $e_U : U \to E$ such that $e_U(x) \neq 0$ for all $x \in U$, called *local frame*, such that for all $s \in H^0(\mathcal{M}, E)$, there exists $f : U \to \mathbb{C}$ that satisfies

$$s(x) = f(x)e_U(x), \ \forall x \in U.$$

In this case, the Hermitian metric h reads

$$h_x(s_1(x), s_2(x)) = f_1(x)\overline{f_2}(x)h_x(e_U(x), e_U(x)) = f_1(x)\overline{f_2}(x)\mathrm{e}^{-\phi(x)},$$

where $\phi : U \to \mathbb{R}$ is defined by $\phi(x) = -\log h_x(e_U(x), e_U(x))$ and is called the *local weight* of h. The regularity of the metric h is then equivalent to the regularity of the weight ϕ as a function on (an open subset of) the complex manifold $\mathcal{M}$. We will often identify the metric h with its local weight for the sake of simplicity.

Pullback bundle. If $E \to \mathcal{M}_2$ is a vector bundle with projection π and $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a holomorphic function, then we define the *pullback bundle* $f^*E \to \mathcal{M}_1$ as

$$f^*E = \{(m, v) \in \mathcal{M} \times E : f(m) = \pi(v)\} \subset \mathcal{M} \times E,$$

with projection $\tilde{\pi}(m, v) = m$.

Dual bundle. If $E \to \mathcal{M}$ is a vector bundle, one can define its *dual bundle* $E^* \to \mathcal{M}$ as the vector bundle whose fibers are the dual fibers of E: $(E^*)_x = (E_x)^*$. If E is endowed with a Hermitian metric h, we shall denote $\overline{E}$ its dual. For any section s_1 of E, there is a unique[5] section $\overline{s_1}$ of $\overline{E}$ such that

$$(\overline{s_1}(x), s_2(x)) = h_x(s_2(x), s_1(x)), \quad \forall s_2 \in H^0(\mathcal{M}, E), \forall x \in \mathcal{M}, \tag{15}$$

where parentheses in the left-hand side denote the duality pairing.

Tensor products of bundles. If $E_1 \to \mathcal{M}$ and $E_2 \to \mathcal{M}$ are two vector bundles over the same manifold, their *tensor product* is the vector bundle $E_1 \otimes E_2 \to \mathcal{M}$ whose fibers are defined by tensor products of the fibers of E_1 and E_2.

[5] This is a well-known consequence of the Riesz representation theorem.

In a similar fashion, if $E_1 \to \mathcal{M}_1$ and $E_2 \to \mathcal{M}_2$ are two vector bundles over two separate manifolds, their *external tensor product* is the vector bundle

$$E_1 \boxtimes E_2 = \mathrm{pr}_1^* E_1 \otimes \mathrm{pr}_2^* E_2 \to \mathcal{M}_1 \times \mathcal{M}_2,$$

where we denoted by $\mathrm{pr}_1 : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_1$ and $\mathrm{pr}_2 : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_2$ the standard projections. It must be distinguished from the previously defined tensor product of two bundles over the same manifold $\mathcal{M}$.

2.3 Differential forms and integration on complex manifolds

Any complex manifold $\mathcal{M}$ of dimension d is also a smooth real manifold of dimension $2d$, *i.e.* it can be locally modelled on $\mathbb{R}^{2d}$ by the natural identification $\mathbb{C} \cong \mathbb{R}^2$. The *tangent bundle* $T\mathcal{M}$ of $\mathcal{M}$ is the smooth vector bundle[6] of rank $2d$ such that for any $x \in \mathcal{M}$, the fibre $T_x\mathcal{M}$ is the tangent space of $\mathcal{M}$ at x. There is a morphism of line bundles $J : T\mathcal{M} \to T\mathcal{M}$ such that $J^2 = -\mathrm{Id}_{T\mathcal{M}}$, defined on each fibre by $J_x : T_x\mathcal{M} \to T_x\mathcal{M}$, which satisfies the Cauchy–Riemann equations on open subsets $U \subset \mathcal{M}$

$$\mathrm{d}f_x(J_x v) = \mathrm{i} \times \mathrm{d}f_x(v), \ \forall v \in T_x\mathcal{M}, \ \forall f \in \mathscr{O}(U), \ \forall x \in U.$$

The cotangent bundle $T^*\mathcal{M} = \mathrm{Hom}_{\mathbb{R}}(T\mathcal{M}, \mathbb{C})$ splits into $\mathrm{Hom}_{\mathbb{C}}(T\mathcal{M}, \mathbb{C}) \oplus \mathrm{Hom}_{\mathbb{C}}(T\mathcal{M}, \overline{\mathbb{C}})$ of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear maps, where $T\mathcal{M}$ is endowed with the complex structure induced by the morphism J. We denote respectively by $T^{*(1,0)}\mathcal{M}$ and $T^{*(0,1)}\mathcal{M}$ the subspaces of this decomposition. If $(z_1, \ldots, z_d)$ is a local holomorphic coordinate system in an open subset $U \subset \mathcal{M}$ (for instance, in the atlas (U_i, φ_i), it means that we note $\varphi_i(x) = (z_1, \ldots, z_d) \in \mathbb{C}^d$ for any $x \in U_i$), and if we set $z_j = x_j + \mathrm{i}y_j$, then (x_j, y_j) is a local smooth coordinate system of $\mathcal{M}$ as a real manifold, and $(\mathrm{d}z_1, \ldots, \mathrm{d}z_d)$ is a local frame of $T^{*(1,0)}\mathcal{M}$, where $\mathrm{d}z_j = \mathrm{d}x_j + \mathrm{i}\mathrm{d}y_j$. Analogously, $(\mathrm{d}\overline{z}_1, \ldots, \mathrm{d}\overline{z}_d)$ is a local frame of $T^{*(0,1)}\mathcal{M}$, where $\mathrm{d}\overline{z}_j = \mathrm{d}x_j - \mathrm{i}\mathrm{d}y_j$.

Definition 2.4. The *bundle of* (p,q)*-forms* on a complex manifold $\mathcal{M}$ is the vector bundle $\Lambda^{p,q}(T^*\mathcal{M}) = \Lambda^p(T^{*(1,0)}\mathcal{M}) \otimes \Lambda^q(T^{*(0,1)}\mathcal{M})$. We denote by $\Omega^{p,q}(\mathcal{M})$ the subspace of smooth (p,q)-forms on $\mathcal{M}$.

Any (p,q)-form ω on $\mathcal{M}$ can be expressed as follows in local coordinates:

$$\omega(x) = \sum_{\substack{1 \leqslant i_1 < \cdots < i_p \leqslant d \\ 1 \leqslant j_1 < \cdots < j_q \leqslant d}} u_{i_1,\ldots,i_p,j_1,\ldots,j_q}(x) \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{j_q}, \tag{16}$$

and in particular any $(0,0)$-form is simply a function $f : \mathcal{M} \to \mathbb{C}$. There are two differential operators of interest, called the *Dolbeault operators*:

$$\partial : \left\{ \begin{array}{ccc} \Omega^{(p,q)}(\mathcal{M}) & \to & \Omega^{(p+1,q)}(\mathcal{M}) \\ f(x)\omega(x) & \mapsto & \sum_i \frac{\partial f(x)}{\partial z_i} \mathrm{d}z_i \wedge \omega(x) \end{array} \right. , \ \overline{\partial} : \left\{ \begin{array}{ccc} \Omega^{(p,q)}(\mathcal{M}) & \to & \Omega^{(p+1,q)}(\mathcal{M}) \\ f(x)\omega(x) & \mapsto & \sum_i \frac{\partial f(x)}{\partial \overline{z}_i} \mathrm{d}\overline{z}_i \wedge \omega(x) \end{array} \right. ,$$

[6] Replace holomorphic by smooth and 1-dimensional complex vector space by $2d$-dimensional real vector space in Definition 2.1.

and they can be turned into operators $\mathrm{d} = \partial + \overline{\partial}$ and $\mathrm{d}^c = \frac{1}{4i\pi}(\partial - \overline{\partial})$, where d coincides with the exterior derivative.

A *volume form* on $\mathcal{M}$ is a volume form in the differential sense, therefore a nonvanishing section of $\Lambda^{2d}(T^*\mathcal{M}) \simeq \Lambda^{d,d}(T^*\mathcal{M})$, and it can be written locally on $U \subset \mathcal{M}$ as

$$\omega(x) = u(x)\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_d \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_d,$$

where $u : \mathcal{M} \to \mathbb{C}$ is a function that does not vanish. The volume form is continuous (resp. smooth, holomorphic) if and only if u is continuous (resp. smooth, holomorphic).

Any volume form on $\mathcal{M}$ can be identified to a Borel measure $\mathrm{d}\mu$ on $\mathcal{M}$ by setting $\int_U \mathrm{d}\mu = \int_U \omega$ for any Borel set $U \subset \mathcal{M}$.

2.4 Line bundles and Bergman kernel

In the sequel, we will consider vector bundles of rank 1, also called *line bundles*. We shall usually denote L such a line bundle, rather than E.

Definition 2.5. Let L be a holomorphic line bundle over $\mathcal{M}$, endowed with an Hermitian metric h. Let μ be a continuous volume form on $\mathcal{M}$ and ϕ be the local weight corresponding to h. The *Bergman kernel* $B_{(\phi,\mu)}$ of L with respect to the weighted measure (ϕ, μ) is the Schwartz kernel (cf. (Le Floch, 2018, Chapter 6) for a detailed introduction of such kernels) of the orthogonal projection

$$P_{(\phi,\mu)} : L^2(\mathcal{M}, L) \longrightarrow H^0(\mathcal{M}, L)$$

with respect to μ.

Namely, it is a section of $L \boxtimes \overline{L} \to \mathcal{M} \times \mathcal{M}$, and it can be written as

$$B_{(\phi,\mu)}(x, y) = \sum_{i=1}^{N} s_i(x) \otimes \overline{s_i(y)}, \ \forall x, y \in \mathcal{M}, \tag{17}$$

where (s_i) is an orthornormal basis of $H^0(\mathcal{M}, L)$ for the inner product $\langle \cdot, \cdot \rangle_{(\phi,\mu)}$. By construction, $B_{(\phi,\mu)}$ is the reproducing kernel of the Hilbert space $(H^0(\mathcal{M}, L), \langle \cdot, \cdot \rangle_{(\phi,\mu)})$, which means that

$$\int_{\mathcal{M}} B_{(\phi,\mu)}(x, y) \cdot s(y)\mathrm{d}\mu(y) = s(x), \ \forall s \in H^0(\mathcal{M}, L), \ \forall x \in \mathcal{M}. \tag{18}$$

The dot in (18) represents the contraction between the Bergman kernel and the section s induced by (15), so that the left-hand side of (18) is the decomposition of s onto the orthogonal basis (s_i); see (Le Floch, 2018, Lemma 6.3.2) for more details.

As we shall see later, the correlation functions of our point processes will be expressed as determinants of the Bergman kernel. Let us stress that such a determinant is not obvious to define: for instance, formally,

$$\begin{vmatrix} B_\phi(x,x) & B_\phi(x,y) \\ B_\phi(y,x) & B_\phi(y,y) \end{vmatrix} = B_\phi(x,x) \otimes B_\phi(y,y) - B_\phi(x,y) \otimes B_\phi(y,x), \ \forall x, y \in \mathcal{M}, \tag{19}$$

but this equality does not really make sense because we are adding elements of two different vector spaces: one is a section of $L_x \otimes \overline{L}_x \otimes L_y \otimes \overline{L}_y$ and the other is a section of $L_x \otimes \overline{L}_y \otimes L_y \otimes \overline{L}_x$. We circumvent this difficulty by using contractions on tensor products of each fiber and its dual, thanks to the following isomorphism of vector spaces, for any finite-dimensional vector space E:

$$\begin{cases} L_x \otimes E \otimes \overline{L_x} & \xrightarrow{\sim} & E \\ u_x \otimes w \otimes \overline{v_x} & \longmapsto & h_x(u_x, v_x) w \end{cases}.$$

In particular, the Bergman kernel on the diagonal can be identified with a function $\mathcal{M} \to \mathbb{C}$ by

$$B_\phi(x,x) = \sum_{i=1}^N h_x(s_i(x), s_i(x)),\ \forall x \in \mathcal{M},$$

and Equation (19) becomes

$$\begin{vmatrix} B_\phi(x,x) & B_\phi(x,y) \\ B_\phi(y,x) & B_\phi(y,y) \end{vmatrix} = B_\phi(x,x)B_\phi(y,y) - B_\phi(x,y)\cdot B_\phi(y,x),\ \forall x,y \in \mathcal{M}.$$

Note the $\cdot$ in the right-hand side, which is a contraction in the sense of (15). It leads to the following definition of the determinant of the Bergman kernel, cf. (Lemoine, 2022).

Definition 2.6. The *determinant* $\det(B_\phi(x_i,x_j))_{1\leqslant i,j\leqslant n}$ is defined by

$$\det(B_\phi(x_i,x_j))_{1\leqslant i,j\leqslant n} = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) \sum_{i_1,\dots,i_n=1}^N \prod_{j=1}^n h_{x_j}(s_{i_j}(x_j), s_{i_{\sigma^{-1}(j)}}(x_j)). \tag{20}$$

We finish this section with a result that will be instrumental later, the so-called *extremal property* of the Bergman kernel:

$$B_\phi(x,x) = \sup\{|s(x)|_\phi^2 :\ s \in H^0(\mathcal{M}, L), \|s\|^2_{(\phi,\mu)} \leqslant 1\}. \tag{21}$$

It is a direct consequence of the reproducing property (18) of the kernel.

2.5 Examples

Let us give a few examples without proofs, which are all standard and can be found for instance in (Le Floch, 2018).

The plane. The complex plane $\mathbb{C}$ is obviously a complex manifold, but it is not compact. However, most of the aforementioned constructions still hold, and are intuitive. We endow the real plane $\mathbb{R}^2$ with the standard symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$, and identify it with $\mathbb{C}$ by setting $z = \frac{x+\mathrm{i}y}{\sqrt{2}}$, so that $\omega = \mathrm{i}\mathrm{d}z \wedge \mathrm{d}\overline{z}$. We can consider the trivial line bundle $L = \mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$, endowed with its standard Hermitian metric

$$h_{x,y}(z,w) = z\overline{w},\ \forall (x,y) \in \mathbb{R}^2,\ \forall z,w \in \mathbb{C}.$$

A global nonvanishing section of L is given by

$$\psi : \begin{cases} \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & e^{-\frac{|z|^2}{2}}. \end{cases}$$

As $\mathbb{C}$ is not compact, the space $H^0(M, L^k)$ is infinite-dimensional, and it is not even a subspace of $L^2(\mathbb{C}, L^k)$. However, if we replace it by

$$\mathcal{H}_k = H^0(\mathbb{C}, L^k) \cap L^2(\mathbb{C}, L^k),$$

we obtain the so-called *Bargmann spaces*, which are Hilbert spaces for all $k \geqslant 1$. They have the following explicit description:

$$\mathcal{H}_k = \{f\psi^k : \ f : \mathbb{C} \to \mathbb{C} \text{ is holomorphic}, \int_{\mathbb{C}} f(z)e^{-k|z|^2} \mathrm{d}z \wedge \mathrm{d}\overline{z} < +\infty\}.$$

This Hilbert spaces is generated by the functions $(z^n\psi^k)_{n\geqslant 0}$, and one can check that they form an orthonormal family of $\mathcal{H}_k$, hence an orthogonal basis. A straightforward computation of their norm yields the orthonormal basis $(s_{k,n})_{n\geqslant 0}$ with

$$s_{k,n}(z) = \sqrt{\frac{k^{n+1}}{2\pi n!}} z^n \psi^k(z), \ \forall z \in \mathbb{C}, \ \forall n \geqslant 0.$$

The Bergman kernel is

$$B_k(z,w) = \sum_{n\geqslant 0} s_{k,n}(z)\overline{s_{k,n}}(w) = \frac{k}{2\pi} \sum_{n\geqslant 0} \frac{1}{n!} (kz\overline{w})^n \psi^k(z)\psi^k(w) = \frac{k}{2\pi} e^{kz\overline{w} - \frac{k}{2}|z|^2 - \frac{k}{2}|w|}.$$

In this context, B_k is rather called the *Christoffel–Darboux kernel*, and the orthonormal family $(s_{k,n})$ is related to the Hermite polynomials. If we fix $k = 1$, the kernel is related to the *infinite Ginibre ensemble* in random matrix theory (Hough et al., 2009).

Projective spaces. For any $d \geqslant 1$, the complex projective space $\mathbb{CP}^d$ is defined as the quotient $\mathbb{C}^{d+1} \setminus \{0\}/\mathbb{C}^*$ for the action

$$\lambda \cdot (z_0, \dots, z_d) = (\lambda z_0, \dots, \lambda z_d), \ \forall \lambda \in \mathbb{C}^*, \ \forall (z_0, \dots, z_d) \in \mathbb{C}^{d+1} \setminus \{0\}.$$

We denote by $[Z_0 : \dots : Z_d]$ the equivalence class of $(Z_0, \dots, Z_d)$ in $\mathbb{CP}^d$, known as the *homogeneous coordinates*. We also denote by π the projection $\mathbb{C}^{d+1} \setminus \{0\} \to \mathbb{CP}^d$ induced by the action of $\mathbb{C}^*$. The structure of complex manifold is for instance given by the atlas $(U_i, \varphi_i)_{0\leqslant i\leqslant d}$, with

$$U_i = \{[Z_0 : \dots : Z_d] \in \mathbb{CP}^d, Z_i \neq 0\},$$

$$\varphi_i : [Z_0 : \dots : Z_d] \mapsto \left(\frac{Z_0}{Z_i}, \dots, \widehat{\frac{Z_i}{Z_i}}, \dots, \frac{Z_d}{Z_i}\right) = (z_1, \dots, z_d) \in \mathbb{C}^d \setminus \{0\},$$

where the hat indicates that the term is omitted.

Consider the set

$$\mathcal{O}(-1) = \{([u], \lambda u), u \in \mathbb{C}^{d+1} \setminus \{0\}, \lambda \in \mathbb{C}\},$$

together with the projection $\pi : \mathcal{O}(1) \to \mathbb{CP}^d$ given by $\pi([u], v) = [u]$. It defines a holomorphic line bundle, called the *tautological line bundle*. The fibers over $\mathbb{CP}^d$ are exactly the lines generated by the homogeneous coordinates. It is endowed with a Hermitian metric induced by the one from $\mathbb{C}^{d+1}$:

$$h_{[u]}(v, w) = \sum_{i=1}^{d+1} v_i \overline{w_i}, \ \forall [u] \in \mathbb{CP}^d, \ \forall v, w \in \mathbb{C}^{d+1}.$$

The complex projective spaces can be somehow related to the unit spheres $\mathbb{S}^{2d+1} \subset \mathbb{R}^{2d+2}$ by the following construction: the action of $\mathbb{C}^*$ on $\mathbb{C}^{d+1} \setminus \{0\} \simeq \mathbb{R}^{2d+2}$ can be decomposed into two successive actions of $\mathbb{R}_+^*$ and $\mathbb{S}^1$, thanks to the polar decomposition of nonzero complex numbers. Let us denote by $\pi_1 : \mathbb{C}^{d+1} \to \mathbb{S}^{2d+1}$ and $\pi_2 : \mathbb{S}^{2d+1} \to \mathbb{S}^{2d+1}/\mathbb{S}^1$ the corresponding projections. The following diagram commutes:

$$\begin{tikzcd} \mathbb{C}^{d+1} \setminus \{0\} \arrow[r, "\pi_1"] \arrow[d, "\pi"'] & \mathbb{S}^{2d+1} \arrow[d, "\pi_2"] \\ \mathbb{CP}^d \arrow[r, dashed, "\simeq"] & \mathbb{S}^{2d+1}/\mathbb{S}^1. \end{tikzcd}$$

The bottom arrow is indeed a bijection. The *Fubini–Study* metric on $\mathbb{CP}^d$ is the Hermitian metric on $\mathbb{CP}^d \simeq \mathbb{S}^{2d+1}/\mathbb{S}^1$ induced by the round metric (which is a Riemannian metric) on $\mathbb{S}^{2d+1}$. In the case $d = 1$, there is an additional relationship $\mathbb{CP}^1 \simeq \mathbb{S}^2$ that we will develop in Section 6.

3 Bergman ensembles

3.1 Definitions

Let $\mathcal{M}$ be a compact complex manifold of dimension d endowed with a Borel probability measure μ, L be a holomorphic line bundle over $\mathcal{M}$ with Hermitian metric h, represented by a local weight function ϕ. Set $N = \dim H^0(\mathcal{M}, L)$, and denote by $\mathcal{P}_\phi$ the probability measure on $\mathcal{M}^N$ defined by

$$\mathrm{d}\mathcal{P}_\phi(x_1, \ldots, x_N) = \frac{1}{Z_N(\phi)} |\det(s_i(x_j))|_\phi^2 \mathrm{d}\mu^{\otimes N}(x_1, \ldots, x_N), \tag{22}$$

where (s_i) is an orthonormal basis of $(H^0(\mathcal{M}, L), \langle \cdot, \cdot \rangle_{(\phi,\mu)})$, and $Z_N(\phi)$ is a normalization constant called *partition function*. Using the generalized Cauchy–Binet identity (Johansson, 2006, Proposition 2.10), the partition function satisfies $Z_N(\phi) = N!$ for any weight ϕ. We will denote by $\mathbb{E}_\phi$ the expectation with respect to $\mathcal{P}_\phi$, i.e., for any bounded measurable $F : \mathcal{M}^N \to \mathbb{C}$,

$$\mathbb{E}_\phi[F(X_1, \ldots, X_N)] = \int_{\mathcal{M}^N} F \mathrm{d}\mathcal{P}_\phi.$$

Recall that a (simple) point process on $\mathcal{M}$ is a random configuration on $\mathcal{M}$, or equivalently the counting measure of this configuration. In particular, given a family $(X_1, \ldots, X_N)$

of almost-surely distinct random variables on $\mathcal{M}$, the random measure $\sum_{i=1}^N \delta_{X_i}$ defines almost-surely a simple point process. The n-point correlation function $\rho_n : \mathcal{M}^n \to \mathbb{R}$ of such point process, when it exists, is characterized by the following property: for any bounded measurable function $f : \mathcal{M}^n \to \mathbb{R}$,

$$\mathbb{E}_\phi \left[\sum_{i_1 \neq \dots \neq i_n} f(X_{i_1}, \dots, X_{i_n}) \right] = \int_{\mathcal{M}^n} f(x_1, \dots, x_n) \rho_n(x_1, \dots, x_n) \mathrm{d}\mu^{\otimes N}(x_1, \dots, x_n).$$

Definition 3.1. The *Bergman ensemble* for the weighted measure (ϕ, μ) is the simple point process $\sum_{i=1}^N \delta_{X_i}$, where $(X_1, \dots, X_N)$ is a family of random variables on $\mathcal{M}$ with distribution $\mathcal{P}_\phi$.

It was proved in (Lemoine, 2022) that such ensemble is a *determinantal point process* with kernel $B_{(\phi,\mu)}$, which means by definition that

$$\rho_n(x_1, \dots, x_N) = \det(B_{(\phi,\mu)}(x_i, x_j)), \ \forall n \geqslant 1, \ \forall x_1, \dots, x_n \in \mathcal{M}, \tag{23}$$

where the determinant of the Bergman kernel is given in Definition 2.6. Although this construction holds for any Hermitian line bundle, we shall focus on the case where L is replaced by $L^k \otimes F$, for large k, with corresponding weight $k\phi + \phi_F$.

3.2 Convergence of the Bergman measures

The first macroscopic estimation of the Bergman ensemble for the weighted measure $(k\phi, \mu)$ for large k is given by the asymptotics of the *Bergman measures* $\mathrm{d}\beta_k(x) = \frac{1}{N_k} B_{(k\phi,\mu)}(x, x) \mathrm{d}\mu(x)$. Indeed, as

$$\mathbb{E}_{k\phi} \left[\frac{1}{N_k} \sum_{i=1}^{N_k} f(X_i) \right] = \int_{\mathcal{M}} f(x) \mathrm{d}\beta_k(x),$$

the weak convergence in expectation of the empirical measures of the DPP is equivalent to the weak convergence of the Bergman measures. In the case of a compact complex manifold $\mathcal{M}$ endowed with a positive Hermitian line bundle L, the Bergman measures converge pointwise (hence weakly) to an equilibrium measure μ_{eq} because of the diagonal expansion of the Bergman kernel. In the more general case studied by Berman (2018) that we consider here, the (weak or pointwise) convergence of Bergman measures is not automatic.

If $\Omega \subset \mathbb{C}^d$ is an open set, a function $f : \Omega \to [-\infty, +\infty]$ is called *plurisubharmonic* (psh) if it is upper semicontinuous, and if for all $z \in \Omega$ and $\xi \in \mathbb{C}^d$ such that $|\xi| < d(z, \Omega^c)$,

$$f(z) \leqslant \frac{1}{2\pi} \int_0^{2\pi} f(z + e^{i\theta}\xi) \mathrm{d}\theta. \tag{24}$$

Given a complex manifold $\mathcal{M}$ of dimension d, a function $f : \mathcal{M} \to [-\infty, +\infty]$ is called plurisubharmonic if for any chart (U, φ), $f \circ \varphi^{-1} : \varphi^{-1}(U) \to [-\infty, +\infty]$ is plurisubharmonic. A function $f \in \mathscr{C}^{1,1}(\mathcal{M})$ is plurisubharmonic if and only if $\partial\overline{\partial} f$ is a nonnegative $(1,1)$-form, namely the coefficients of the decomposition of $\partial\overline{\partial} f$ in (16) are nonnegative for all $x \in \mathcal{M}$. Equivalently, it means that $\mathrm{dd}^c f$ is a nonnegative 2-form. Now, let us go back to the Hermitian line bundle $L \to \mathcal{M}$ endowed with a local weight ϕ. If ϕ is only continuous, one can define its *plurisubharmonic envelope* $\phi_e : \mathcal{M} \to [-\infty, +\infty]$ by

$$\phi_e(x) = \sup\{\psi(x) : \psi \text{ continuous and psh s.t. } \psi \leqslant \phi\}.$$

Definition 3.2. The *weak bulk* is the largest open subset of $\mathcal{M}$ where the following equality holds pointwise:

$$\det(\mathrm{dd}^c\phi_e) = \det(\mathrm{dd}^c\phi).$$

See (Berman, 2018, Theorem 3.1) for a description of the weak bulk and its properties. A noticeable fact is that it is contained in the set $\{x \in \mathcal{M} : \mathrm{dd}^c\phi(x) > 0\}$.

The Bergman measure β_k has density $\frac{1}{N_k} B_{(k\phi,\mu)}$ with respect to $\mathrm{d}\mu$; we will actually control the convergence of the inverse of this density, for a reason that will appear later. Let $w^\phi_{\mathrm{eq}} = \frac{\mathrm{vol}(L)}{d! \det(\mathrm{dd}^c\phi)}$ be the *equilibrium weight.* The following lemma is an analog of a classical result about the Christoffel–Darboux kernel for orthogonal polynomials on the unit circle (Simon, 2011, Theorem 2.15.1) and on the real segment $[-1, 1]$ (Simon, 2011, Theorem 3.11.1).

Lemma 3.1. *Let $\mathcal{M}$ be a compact complex manifold of dimension d endowed with a continuous volume form ω, and μ be the Borel measure on $\mathcal{M}$ corresponding to ω. Let L be a holomorphic line bundle over $\mathcal{M}$ endowed with an Hermitian metric h corresponding to a local weight ϕ such that (ϕ, μ) is strongly regular. For all x in the weak bulk,*

$$\frac{N_k}{B_{(k\phi,\mu)}(x, x)} = w^\phi_{\mathrm{eq}} + O(k^{-1}), \tag{25}$$

where the remainder is uniformly bounded in $x \in \mathcal{M}$.

Proof. The result will follow from two separate estimates that we pull together from the literature: one for $N_k = \dim H^0(\mathcal{M}, L^k)$ and one for $B_{(k\phi+\phi_k,\mu)}$ on the diagonal. Both are quite standard but we expose them for readers who are less familiar with complex geometry.

We define the space of L^k-valued (p, q)-forms as

$$\Omega^{p,q}(L^k) = \mathscr{C}^\infty(\Omega^{p,q}(\mathcal{M}) \otimes L^k).$$

The Dolbeault operator $\overline{\partial}$ extends to an operator $\overline{\partial}_{L^k} : \Omega^{p,q}(L^k) \to \Omega^{p,q+1}(L^k)$, and we define for all $q \geqslant 0$ the vector space

$$H^q(M, L^k) = \ker(\overline{\partial}_{L^k} : \Omega^{0,q}(L^k) \to \Omega^{0,q+1}(L^k))/\overline{\partial}_{L^k}(\Omega^{0,q-1}(L^k).$$

Note that for $q = 0$, it coincides with the space of holomorphic sections $H^0(M, L^k)$ which is at the center of the present paper. The dimensions of these spaces are involved in the definition of the *Euler characteristic* of the line bundle:

$$\chi(\mathcal{M}, L^k) := \sum_{q=0}^{d} (-1)^q \dim H^q(\mathcal{M}, L^k).$$

The Kodaira–Serre vanishing theorem (Ma and Marinescu, 2007, Theorem 1.5.6) states that $\dim H^q(\mathcal{M}, L^k) = 0$ for all $q > 0$ and k large enough, leaving

$$\chi(\mathcal{M}, L^k) = \dim H^0(\mathcal{M}, L^k) = N_k.$$

We can then apply the Hirzebruch–Riemann–Roch formula (see (Demailly, 1985) for instance), and there is a polynomial $P_{d-1} \in \mathbb{Q}[X]$ of degree at most $d-1$ such that

$$\chi(\mathcal{M}, L^k) = \frac{\mathrm{vol}(L)}{d!} k^d + P_{d-1}(k) = N_k. \tag{26}$$

On the diagonal, the Bergman kernel admits the following expansion (Berman, 2009a, Theorem 4.14): for any x in the weak bulk,

$$B_{(k\phi,\mu)}(x, x) = \det(\mathrm{dd}^c\phi) k^d + O(k^{d-1}), \tag{27}$$

where the big O only depends on k and is uniform in x. Equation (25) results trivially from (26) and (27). □

3.3 Laplace transform of linear statistics

DPPs are known to have tractable Laplace transforms of linear statistics, and Berman (2018) has used this Laplace transform to obtain central limit theorems for $\mathrm{d}\mathcal{P}_\phi$. For any nonnegative and measurable $\psi : \mathcal{M} \to \mathbb{R}$, define the log Laplace transform of the linear statistics $\sum_i \psi(X_i)$ as

$$K_\phi^\psi(t) = \log \mathbb{E}_\phi \left[\mathrm{e}^{-t\sum_{i=1}^N \psi(X_i)} \right].$$

The first result used by Berman is the following, relating the derivatives of K_ϕ^ψ and the expectation and variance of the linear statistics.

Proposition 3.2 (Berman, 2018)**.** *The log Laplace transform K_ϕ^ψ is at least twice derivable with respect to t, and satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} K_\phi^\psi(t) = -\mathbb{E}_{\phi+t\psi}\left[\sum_{i=1}^N \psi(X_i)\right] = -\int_{\mathcal{M}} \psi(x) B_{(\phi+t\psi,\mu)}(x, x) \mathrm{d}\mu(x), \tag{28}$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} K_\phi^\psi(t) &= \mathbb{V}\mathrm{ar}_{\phi+t\psi}\left[\sum_{i=1}^N \psi(X_i)\right] \\ &= \frac{1}{2}\int_{\mathcal{M}^2} (\psi(x) - \psi(y))^2 |B_{(\phi+t\psi,\mu)}(x, y)|^2_{\phi+t\psi} \mathrm{d}\mu^{\otimes 2}(x, y).\end{aligned} \tag{29}$$

The second one is a control of the asymptotics of the integral in the expression of the variance in Proposition 3.2, when ϕ is replaced by $k\phi$ and $k \to \infty$.

Theorem 3.3 (Berman, 2018, Theorem 5.8). *Let $\mathcal{M}$ be a compact complex manifold of dimension d endowed with a Borel measure μ associated with a continuous volume form, L be a big line bundle over $\mathcal{M}$ endowed with a $\mathscr{C}^{1,1}$ metric ϕ, F be a line bundle endowed with a continuous metric with weight ϕ_F, and $B_{k\phi+\phi_F}$ be the Bergman kernel of $H^0(\mathcal{M}, L^k \otimes F)$. If f is a Lipschitz function with compact support included in the bulk, then*

$$\lim_{k\to\infty} \frac{1}{2} \iint_{\mathcal{M}^2} k^{1-d} |B_{k\phi+\phi_F}(x,y)|^2 (f(x)-f(y))^2 \mathrm{d}\mu^{\otimes 2}(x,y) = \|\mathrm{d}f\|^2_{\mathrm{dd}^c\phi}. \tag{30}$$

Let $f : \mathcal{M} \to \mathbb{C}$ be a Lipschitz continuous function with compact support included in the weak bulk, and set

$$f_k : x \mapsto \frac{N_k}{B_{(k\phi,\mu)}(x,x)} f(x).$$

Our proof of Theorem 1.2 will follow the steps of Theorem 1.1 by replacing f with f_k. In particular, we will prove a generalization of Theorem 3.3 taking into account the dependence on k in both f and ϕ_F. See Theorem 4.7 for the precise statement. As we will see, almost all arguments from Berman (2018) will remain unchanged.

4 Bergman kernel estimates

Before we dive into the proofs of our main Theorem, let us state a few technical estimates of the Bergman kernel. They generalize slightly some results by Berman (2018), yet their proofs are almost identical. They are based on *local* properties of the kernel, meaning that they are proved by using adequate sets of coordinates.

Definition 4.1. Local coordinates $z_1, \dots, z_d$ are called *normal* if the $(1,1)$-form ω that induces the continuous volume form ω^d associated with the measure μ has the following expression when $|z| \to 0$:

$$\omega(z) = \frac{i}{2} \sum_{i,j=1}^{d} h_{ij}^{(0)}(z) \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j, \quad h_{ij}^{(0)}(z) = \delta_{ij} + O(|z|^2). \tag{31}$$

A local trivialization e_U with weight $\phi = -\partial\bar{\partial} \log |e_U|_\phi$ is *normal* if the weight ϕ has the following expression when $|z| \to 0$:

$$\phi(z) = \sum_{j=1}^{d} \lambda_j |z_j|^2 + O(|z|^3). \tag{32}$$

See Griffiths and Harris (1994) or Berman (2009b) for an explanation of why such coordinates exist. In this setting, we have an explicit expression of the curvature $\mathrm{dd}^c\phi$ in the centered coordinate:

$$\mathrm{dd}^c\phi(0) = \frac{\mathrm{i}}{2\pi} \sum_{j=1}^{d} \lambda_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j,$$

so that

$$\omega_\phi = (\mathrm{dd}^c\phi)^d = \frac{\det \lambda}{\pi^d} \omega^d,$$

where we denote by $\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$ the diagonal matrix of eigenvalues of the curvature of ϕ. Given a family (ϕ_k) of local weights on F that converges uniformly to a weight ϕ_F, we will study the convergence of the Bergman kernel $B_{(k\phi+\phi_k,\mu)}$ at two different scales: near the diagonal, and far from the diagonal. All results are actually variants of results by Berman (2018), which were stated for $B_{(k\phi+\phi_F,\mu)}$, and their proofs are extremely similar, but we shall recall them for the sake of completeness. They will rely on (31), as well on the the following estimate.

Lemma 4.1. *Let $\mathcal{M}$ be a compact complex manifold of dimension d endowed with a continuous volume form ω^d, and μ be the Borel measure on $\mathcal{M}$ corresponding to ω^d. Let L be a holomorphic line bundle over $\mathcal{M}$ endowed with an Hermitian metric h corresponding to a local weight ϕ such that (ϕ, μ) is strongly regular. For all x in the weak bulk, in a normal trivialization and a normal coordinate system z centered at x,*

$$k\phi(z) + \phi_k(z) = k\left(\sum_i \lambda_i |z_i|^2 + \underset{|z|\to 0}{O}(|z|^3)\right). \tag{33}$$

Proof. One can choose the trivialization such that $\phi_F(0) = 0$, and in this case the uniform convergence of (ϕ_k) and the continuity of ϕ_F yield

$$\phi_k(z) = \phi_F(z) + o_{k\to\infty}(1) = o_{|z|\to 0}(|z|) + o_{k\to\infty}(1),$$

so that $k\phi(z) + \phi_k(z)$ and $k\phi(z)$ have the same asymptotic expression for $|z| \to 0$ and $k \to \infty$ (which actually does not depend on ϕ_F). $\square$

In particular, we have the following rescaled uniform convergence for $|z| \leqslant R$, with $R > 0$ fixed:

$$k\phi\left(\frac{z}{\sqrt{k}}\right) + \phi_k\left(\frac{z}{\sqrt{k}}\right) - \sum_i \lambda_i |z_i|^2 = \frac{1}{\sqrt{k}} O(|z|^3). \tag{34}$$

4.1 Scaling limit near the diagonal

Theorem 4.2. *Let L be a big line bundle with a weight ϕ which is locally $\mathscr{C}^{1,1}$, and F be another line bundle endowed with a continuous local weight ϕ_F. Let $(\phi_k)_k$ be a sequence of continuous local weights on F that converges uniformly to ϕ_F. Assume that μ is a continuous volume form on $\mathcal{M}$. Let x be a fixed point in the weak bulk and take normal local coordinates z centered at x and a normal trivialization of $L \otimes F$. Then*

$$k^{-d} B_{(k\phi+\phi_k,\mu)}\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right) \underset{k\to\infty}{\longrightarrow} \frac{\det \lambda}{\pi^d} \mathrm{e}^{\langle \lambda z, w\rangle} \tag{35}$$

in the $\mathscr{C}^\infty$-topology on compact subsets of $\mathbb{C}^d \times \mathbb{C}^d$.

In order to prove this Theorem, we will need two inequalities that generalize well-known estimates in the case of $B_{(k\phi+\phi_F,\mu)}$.

Lemma 4.3 (Holomorphic Morse inequality)**.** *Let $x \in \mathcal{M}$ be a point such that the second order derivatives of ϕ exist, and let z be a normal local coordinate system centered at x. Then*

$$\limsup_k k^{-d} B_{(k\phi+\phi_k,\mu)}\left(\frac{z}{\sqrt{k}}, \frac{z}{\sqrt{k}}\right) \leqslant \frac{\det(\lambda)}{\pi^d}. \tag{36}$$

Proof. Let $\varepsilon > 0$ be fixed. Using the local normal coordinate system z and a normal trivialization e_U in a local neighborhood of x, given an orthonormal basis (s_i) of $H^0(\mathcal{M}, L^k \otimes F)$, there are holomorphic functions (f_i) such that for all i

$$\left| s_i\left(\frac{z}{\sqrt{k}}\right)\right|^2_{k\phi+\phi_k} = \left| f_i\left(\frac{z}{\sqrt{k}}\right)\right|^2 e^{-k\phi\left(\frac{z}{\sqrt{k}}\right)-\phi_k\left(\frac{z}{\sqrt{k}}\right)},$$

therefore

$$B_{(k\phi+\phi_k,\mu)}\left(\frac{z}{\sqrt{k}}, \frac{z}{\sqrt{k}}\right) = \sum_i \left| f_i\left(\frac{z}{\sqrt{k}}\right)\right|^2 e^{-k\phi\left(\frac{z}{\sqrt{k}}\right)-\phi_k\left(\frac{z}{\sqrt{k}}\right)}.$$

Analogously, we have

$$B_{(k\phi+\phi_k,\mu)}(0,0) = \sum_i |f_i(0)|^2 e^{-k\phi(0)-\phi_k(0)} = \sum_i |f_i(0)|^2.$$

For all $1 \leqslant i \leqslant N_k$ there are integers n_i such that for all $k \geqslant n_i$,

$$\left| f_i\left(\frac{z_i}{\sqrt{k}}\right)\right|^2 \leqslant |f_i(0)|^2(1+\varepsilon).$$

For k large enough, we also have

$$\left| k\phi\left(\frac{z}{\sqrt{k}}\right) + \phi_k\left(\frac{z}{\sqrt{k}}\right)\right| \leqslant \varepsilon,$$

therefore, for k large enough,

$$B_{(k\phi+\phi_k,\mu)}\left(\frac{z}{\sqrt{k}}, \frac{z}{\sqrt{k}}\right) \leqslant \sum_i |f_i(0)|^2(1+\varepsilon)e^{\varepsilon} = B_{(k\phi+\phi_k,\mu)}(0,0)(1+\varepsilon)e^{\varepsilon}.$$

Using the extremal property of the Bergman kernel, for any fixed $R > 0$,

$$\begin{aligned} k^{-d} B_{(k\phi+\phi_k,\mu)}(0,0) &= \sup_{s \in H^0(\mathcal{M}, L^k \otimes F)} \frac{|s(0)|^2_{k\phi+\phi_k}}{k^d \int_{\mathcal{M}} |s(x)|^2_{k\phi+\phi_k} \mathrm{d}\mu(x)} \\ &\leqslant \sup_{s \in H^0(\mathcal{M}, L^k \otimes F)} \frac{|s(0)|^2_{k\phi+\phi_k}}{k^d \int_{|z| \leqslant \frac{R}{\sqrt{k}}} |s(z)|^2_{k\phi+\phi_k} \mathrm{d}\mu(z)}. \end{aligned}$$

We can replace z by $\frac{z}{\sqrt{k}}$ in the integral, giving

$$k^d \int_{|z| \leqslant \frac{R}{\sqrt{k}}} |s(z)|^2_{k\phi+\phi_k} \mathrm{d}\mu(z) = \int_{|z| \leqslant R} \left| f\left(\frac{z}{\sqrt{k}}\right)\right| e^{-k\phi\left(\frac{z}{\sqrt{k}}\right)-\phi_k\left(\frac{z}{\sqrt{k}}\right)} \mathrm{d}\mu(z),$$

where f is a holomorphic function such that locally $s(z) = f(z)e_U(z)$. From (34) we deduce that

$$\sup_{|z|\leqslant R}\left|k\phi\left(\frac{z}{\sqrt{k}}\right)+\phi_k\left(\frac{z}{\sqrt{k}}\right)-\sum_i \lambda_i|z_i|^2\right| \underset{k\to\infty}{\longrightarrow} 0,$$

and by continuity of f we get

$$\lim_{k\to\infty}\int_{|z|\leqslant R}\left|f\left(\frac{z}{\sqrt{k}}\right)\right| e^{-k\phi\left(\frac{z}{\sqrt{k}}\right)-\phi_k\left(\frac{z}{\sqrt{k}}\right)}\mathrm{d}\mu(z) = \int_{|z|\leqslant R} e^{-\sum_i \lambda_i|z_i|^2}\prod_i \frac{\mathrm{i}}{2}\mathrm{d}z_i\wedge \mathrm{d}\overline{z}_i.$$

So far, we have obtained that, for $\varepsilon > 0$ fixed, for k large enough,

$$k^{-d}B_{(k\phi+\phi_k,\mu)}\left(\frac{z}{\sqrt{k}},\frac{z}{\sqrt{k}}\right) \leqslant \sup_{s\in H^0(\mathcal{M},L^k\otimes F)} \frac{|s(0)|^2_{k\phi+\phi_k}(1+\varepsilon)e^{\varepsilon}}{k^d\int_{|z|\leqslant \frac{R}{\sqrt{k}}}|s(z)|^2_{k\phi+\phi_k}\mathrm{d}\mu(z)}.$$

Taking the limsup over k and then letting $R\to\infty$ yields

$$\limsup_k k^d B_{(k\phi+\phi_k,\mu)}\left(\frac{z}{\sqrt{k}},\frac{z}{\sqrt{k}}\right) \leqslant \frac{(1+\varepsilon)e^{\varepsilon}}{\int_{\mathbb{C}^d} e^{-\sum_i\lambda_i|z_i|^2}\prod_i\frac{\mathrm{i}}{2}\mathrm{d}z_i\wedge\mathrm{d}\overline{z}_i} = \frac{\det(\lambda)}{\pi^d}(1+\varepsilon)e^{\varepsilon}.$$

As it holds for any $\varepsilon > 0$, letting $\varepsilon\to 0$ yields the lemma. □

Lemma 4.4. *Let $x \in \mathcal{M}$ be a point in the weak bulk, and z be a normal local coordinate system centered at x. Then*

$$\liminf_k \frac{1}{k^d}\left|B_{(k\phi+\phi_k,\mu)}\left(\frac{z}{\sqrt{k}},\frac{z}{\sqrt{k}}\right)\right|_{k\phi+\phi_k} \geqslant \frac{\det(\lambda)}{\pi^d}. \tag{37}$$

Proof. **Step 1: construction of a smooth extremal section.** Let $x \in \mathcal{M}$ be in the weak bulk. We will prove that there exists a smooth section σ_k of $L^k\otimes F$ such that for z_0 fixed and for (ϕ_k) a sequence of weights on F such that $\phi_k(0) = 0$ for all k and $\phi_k \to \phi_F$ uniformly,

$$(i)\ \lim_{k\to\infty}\frac{|\sigma_k|^2_{k\phi}\left(\frac{z_0}{\sqrt{k}}\right)}{k^d\|\sigma_k\|^2_{k\phi+\phi_k}} = \det_{\omega}(\mathrm{dd}^c\phi)(x), \quad (ii)\ \|\overline{\partial}\sigma_k\|^2_{k\phi+\phi_k} \leqslant Ce^{-k/C}. \tag{38}$$

In order to do so, we take a smooth function χ which is constant and equal to 1 on $\{z : |z| \leqslant \delta/2\}$ and supported in $\{z : |z| \leqslant \delta\}$ for a given δ that we will fix later. Let $\sqrt{\lambda} = \mathrm{diag}(\sqrt{\lambda_1},\dots,\sqrt{\lambda_d})$ be the squared root of the matrix λ, and set for any z such that $|z|\leqslant\lambda$

$$\sigma_k(z) = \chi(z)e^{-\frac{k}{2}\left(\left|\sqrt{\lambda}(z-\frac{z_0}{\sqrt{k}})\right|^2-|\sqrt{\lambda}z|^2\right)}e_U(z) = \chi(z)e^{k\left(\langle\sqrt{\lambda}z,\sqrt{\lambda}\frac{z_0}{\sqrt{k}}\rangle-\frac{1}{2k}|\sqrt{\lambda}z_0|^2\right)}e_U(z),$$

where e_U is a local frame such that $|e_U(z)|^2_{k\phi} = e^{-k\phi(z)}$. We extend σ_k to the value 0 for $|z| > \delta$, which indeed defines a smooth section. We claim that σ_k satisfies (38). Indeed,

$$|\sigma_k|^2_{k\phi}\left(\frac{z_0}{\sqrt{k}}\right) = \chi\left(\frac{z_0}{\sqrt{k}}\right)^2 e^{|\sqrt{\lambda}z_0|^2}e^{-k\phi\left(\frac{z_0}{\sqrt{k}}\right)},$$

and for k large enough we have $|z_0/\sqrt{k}| \leqslant \delta/2$, so that (32) yields

$$\lim_{k\to\infty} |\sigma_k|^2_{k\phi}\left(\frac{z_0}{\sqrt{k}}\right) = 1.$$

Likewise, let us compute the denominator of (i) in (38). Using the fact that the support of σ_k is included in $\{|z| \leqslant \delta\}$,

$$\begin{aligned}\|\sigma_k\|^2_{(k\phi+\phi_k,\mu)} &= \int_{\mathcal{M}} |\sigma_k|^2_{k\phi+\phi_k}(x)\omega_d(x) \\ &= \int_{|z|\leqslant\delta} \chi(z)^2 e^{k\left(\left|\sqrt{\lambda}(z-\frac{z_0}{\sqrt{k}})\right|^2-|\sqrt{\lambda}z|^2\right)} e^{-k\phi(z)-\phi_k(z)} \det h^{(0)}(z) \bigwedge_i \frac{\mathrm{i}}{2}\mathrm{d}z_i\wedge\mathrm{d}\overline{z}_i.\end{aligned}$$

We split the domain of integration into two regions:

$$A_k = \{|z| \leqslant R/\sqrt{k}\}, \quad B_k = \{R/\sqrt{k} \leqslant |z| \leqslant \delta\},$$

and we perform a change of variables $\zeta = \sqrt{k}z$. First,

$$\|\sigma_k\|^2_{(k\phi+\phi_k,\mu),A_k} = k^{-d}\int_{|\zeta|\leqslant R} \chi\left(\frac{\zeta}{\sqrt{k}}\right)^2 e^{|\sqrt{\lambda}(\zeta-z_0)|^2} e^{k|\sqrt{\lambda}\zeta|^2-k\phi\left(\frac{\zeta}{\sqrt{k}}\right)-\phi_k\left(\frac{\zeta}{\sqrt{k}}\right)} \det h^{(0)}\left(\frac{\zeta}{\sqrt{k}}\right)\bigwedge_i \frac{\mathrm{i}}{2}\mathrm{d}\zeta_i\wedge\mathrm{d}\overline{\zeta}_i$$

and a dominated convergence combined with (34) yields

$$\lim_{k\to\infty} k^d\|\sigma_k\|^2_{(k\phi+\phi_k,\mu),A_k} = \int_{|\zeta|\leqslant R} e^{|\sqrt{\lambda}(\zeta-z_0)|^2}\left(\frac{\mathrm{i}}{2}\right)^d \mathrm{d}\zeta_1\wedge\mathrm{d}\overline{\zeta}_1\wedge\cdots\wedge\mathrm{d}\zeta_d\wedge\mathrm{d}\overline{\zeta}_d.$$

The RHS converges then to $(2\pi)^d(\det\lambda)^{-1}$ as $R\to\infty$. Using similar arguments we see that $k^d\|\sigma_k\|^2_{(k\phi+\phi_k,\mu),B_k}$ converges, as $k\to\infty$, to the tail of a multidimensional Gaussian integral, and letting $R\to\infty$ make it finally vanish. We have proved (38) (i).

Now, let us prove the point (ii). We have for any z

$$\overline{\partial}\sigma_k(z) = \left(\overline{\partial}\chi(z)\right) e^{-\frac{k}{2}\left(\left|\sqrt{\lambda}(z-\frac{z_0}{\sqrt{k}})\right|^2-|\sqrt{\lambda}z|^2\right)} e_U(z)$$

because the exponential part is holomorphic. In partiular, it means that $\overline{\partial}\sigma_k(z) = 0$ for all $|z| \leqslant \delta/2$, and we get

$$\begin{aligned}\|\overline{\partial}\sigma_k\|^2_{(k\phi+\phi_k,\mu)} &= \int_{\delta/2\leqslant|z|\leqslant\delta} |\overline{\partial}\sigma_k|^2_{k\phi+\phi_k}(z)\det h^{(0)}(z)\bigwedge_i\frac{\mathrm{i}}{2}\mathrm{d}z_i\wedge\mathrm{d}\overline{z}_i \\ &= \int_{\delta/2\leqslant|z|\leqslant\delta}\left(\overline{\partial}\chi(z)\right)^2 e^{-k|\sqrt{\lambda}(z-\frac{z_0}{k})|^2+k|\sqrt{\lambda}z|^2-k\phi(z)-\phi_k(z)}\det h^{(0)}(z)\bigwedge_i\frac{\mathrm{i}}{2}\mathrm{d}z_i\wedge\mathrm{d}\overline{z}_i.\end{aligned}$$

By (33) we know that there exists a constant $C_1 > 0$ such that

$$\left|k|\sqrt{\lambda}z|^2 - k\phi(z) - \phi_k(z)\right| \leqslant C_1 k|z|^3,$$

so that if we take δ small enough we get, for $|z| \leqslant \delta$,

$$\Big|k|\sqrt{\lambda}z|^2 - k\phi(z) - \phi_k(z)\Big| \leqslant \frac{k}{4}\inf_i \lambda_i |z|^2 \leqslant \frac{k}{4}|\sqrt{\lambda}z|^2.$$

For k large enough, we also have

$$|\sqrt{\lambda}(z - \frac{z_0}{k})|^2 \geqslant \frac{1}{4}|\sqrt{\lambda}z|^2.$$

Let us combine these estimates and set $C_2 = \sup_{|z|\leqslant\delta}\left(\overline{\partial}\chi(z)\det h^{(0)}(z)\right)^2$, yielding

$$\|\overline{\partial}\sigma_k\|^2_{(k\phi+\phi_k,\mu)} \leqslant C_2 \int_{\delta/2\leqslant|z|\leqslant\delta} e^{-\frac{k}{2}|\sqrt{\lambda}z|^2} \bigwedge_i \frac{\mathrm{i}}{2}\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i.$$

Equation (38) (ii) follows.

Step 2: perturbation to a holomorphic extremal section. We equip $L^k \otimes F$ with a strictly positively curved modification ψ_k of the weight $k\phi + \phi_k$ (see Lemma 2.5 in (Berman, 2009a)). Let $g_k = \overline{\partial}\sigma_k$. According to the Hörmander–Kodaira estimate (see Theorem 4.1 in (Berman, 2018)), there exists for all k a smooth section u_k with values in $L \otimes K_{\mathcal{M}}$, where $K_{\mathcal{M}}$ is the canonical bundle of $\mathcal{M}$, such that

$$\overline{\partial}u_k = g_k, \quad \|u_k\|_{(\psi_k,\mu)} \leqslant C\|g_k\|_{(\psi_k,\mu)}.$$

We conclude by setting $\alpha_k = \sigma_k - u_k$, which is indeed holomorphic, and satisfies (38). □

Proof of Theorem 4.2. It follows more or less directly from Lemmas 4.3 and 4.4. The proof of (Berman, 2018, Theorem 1.1) can also be adapted verbatim using our estimates. □

4.2 Off-diagonal decay

The following result is adapted from (Berman, 2018, Theorem 5.7), and has more or less the same proof.

Theorem 4.5. *Let L be a big line bundle with a $\mathscr{C}^{1,1}_{loc}$ weight ϕ and F be another line bundle endowed with a continuous local weight ϕ_F. Assume that μ is the Borel measure associated to a continuous volume form on $\mathcal{M}$, and that $(\phi_k)_k$ is a sequence of weights on F that converge uniformly to ϕ_F. Let E be a compact subset of the interior of the bulk. There is a constant C such that for any k, any $x \in E$ and $y \in \mathcal{M}$,*

$$k^{-2d}|B_{k\phi+\phi_k}(x,y)|^2 \leqslant C\mathrm{e}^{-\frac{\sqrt{k}d(x,y)}{C}}, \tag{39}$$

where $d(x,y)$ is the distance with respect to a smooth metric on $\mathcal{M}$.

4.3 Convergence to the equilibrium weight

A sequence (f_k) of functions on $\mathcal{M}$ with values in $\mathbb{R}$ is said to *converge uniformly* to $f : \mathcal{M} \to \mathbb{R}$ *with speed* u_k, where (u_k) is an increasing sequence of positive real numbers with limit ∞, if there exists $C > 0$ such that

$$\sup_{x\in\mathcal{M}} |f_k(x) - f(x)| \leqslant \frac{C}{u_k}. \tag{40}$$

If (v_k) is another sequence of positive real numbers with limit ∞ such that $u_k > v_k$ for all k, then we also say that f_k converges uniformly to f *faster than* v_k. Note that she speed of convergence is not unique: if (u_k) and (v_k) are two sequences of positive numbers such that $u_k \leqslant v_k$ and if f_k converge to f uniformly with speed u_k, then *a fortiori* it also converges with speed v_k. The following proposition is a trivial consequence of Lemma 3.1.

Proposition 4.6. *Let $\mathcal{M}$ be a compact complex manifold of dimension d endowed with a Borel measure μ associated with a continuous volume form, L be a line bundle over $\mathcal{M}$ endowed with a $\mathscr{C}^{1,1}$ weight ϕ such that (ϕ, μ) is strongly regular, F be a line bundle endowed with a continuous weight ϕ_k, and $B_{k\phi+\phi_k}$ be the Bergman kernel of $H^0(\mathcal{M}, L^k \otimes F)$. Assume that the sequence (ϕ_k) converges uniformly to a continuous weight ϕ_F. Let $f : \mathcal{M} \to \mathbb{C}$ be a Lipschitz function with compact support included in the weak bulk. If we set $f_k : x \mapsto \frac{N_k}{B_{(k\phi,\mu)}(x,x)} f(x)$ for any k, then all f_k are Lipschitz and they converge uniformly to $f_{\mathrm{eq}} = w^{\phi}_{\mathrm{eq}} f$ with speed $\frac{1}{k}$.*

We are now able to state and prove a variant of Theorem 3.3.

Theorem 4.7. *Let $\mathcal{M}$ be a compact complex manifold of dimension d endowed with a Borel measure μ associated with a continuous volume form, L be a line bundle over $\mathcal{M}$ endowed with a $\mathscr{C}^{1,1}$ weight ϕ such that (ϕ, μ) is strongly regular, F be a line bundle endowed with a continuous weight ϕ_k, and $B_{k\phi+\phi_k}$ be the Bergman kernel of $H^0(\mathcal{M}, L^k \otimes F)$. Let (f_k) be a sequence of Lipschitz functions with compact support included in the weak bulk. Assume that:*

(i) the sequence (ϕ_k) converges uniformly to a continuous weight ϕ_F;

(ii) the sequence (f_k) converges uniformly to a Lipschitz continuous function f faster than $\frac{1}{\sqrt{k}}$.

Then,

$$\lim_{k\to\infty} \frac{1}{2} \iint_{\mathcal{M}^2} k^{1-d} |B_{k\phi+\phi_k}(x, y)|^2 (f_k(x) - f_k(y))^2 \mathrm{d}\mu^{\otimes 2}(x, y) = \|\mathrm{d}f\|^2_{\mathrm{dd}^c\phi}. \tag{41}$$

Proof. We follow closely the proof of Theorem 5.8 in (Berman, 2018), because most of the arguments still apply. Let d be the distance on $\mathcal{M}$ induced by any continous metric. For fixed $k \geqslant 1$ and $R > 0$, we split the integral in (41) into three parts $A_{k,R}, B_{k,R}, C_{k,R}$, corresponding to integrating respectively over $d(x, y) \geqslant 1$, $k^{-\frac{1}{2}}R \leqslant d(x, y) \leqslant 1$ and $0 \leqslant d(x, y) \leqslant k^{-\frac{1}{2}}R$. The idea is to let $k \to \infty$ then $R \to \infty$. The first two contributions vanish in the large k limit thanks to the off-diagonal decay of the Bergman kernel (Theorem 4.5); we will then focus on the third one.

The key point is to prove that for x in the bulk and $z \in \mathbb{C}^d$ a normal coordinate, as well as normal trivializations of L and F, all centered at x,

$$\sup_{|z|<R} \left| \sqrt{k} \left(f_k \left(\frac{z}{\sqrt{k}} \right) - f_k(0) \right) - \nabla f(0) \cdot z \right| \longrightarrow 0. \tag{42}$$

We have, for any $|z| < R$,

$$\begin{aligned}\sqrt{k} \left(f_k \left(\frac{z}{\sqrt{k}} \right) - f_k(0) \right) - \nabla f(0) \cdot z =& \sqrt{k} \left(f_k \left(\frac{z}{\sqrt{k}} \right) - f \left(\frac{z}{\sqrt{k}} \right) \right) - \sqrt{k}(f_k(0) - f_k(0)) \\ &+ \sqrt{k} \left(f \left(\frac{z}{\sqrt{k}} \right) - f(0) \right) - \nabla f(0) \cdot z\end{aligned}$$

First, as f is differentiable at 0, we have

$$\left| \sqrt{k} \left(f \left(\frac{z}{\sqrt{k}} \right) - f(0) \right) - \nabla f(0) \cdot z \right| = |z| \varepsilon \left(\frac{1}{\sqrt{k}} \right) \underset{k\to\infty}{\longrightarrow} 0.$$

Then, using the uniform convergence faster than $\frac{1}{\sqrt{k}}$, we get that for all $\varepsilon > 0$, there exists k_0 such that for all $k \geqslant k_0$, for all $z \in \mathbb{C}^d$,

$$\sqrt{k} |f_k(z) - f(z)| \leqslant \varepsilon.$$

It is in particular true if $z = 0$, and if one replaces z by $\frac{z}{\sqrt{k}}$ because of the uniformity. We conclude by the triangle inequality that (42) is satisfied. Once we get this estimate, the rest of the proof is identical to the proof of Theorem 3.3. $\square$

5 Proof of the main results

We are now in position to prove the main results of this paper.

Proof of Theorem 1.2. We want to prove that if $(X_1, \ldots, X_{N_k})$ is distributed according $\mathcal{P}_{k\phi+\psi}$,

$$\sqrt{N_k^{1+\frac{1}{d}}} \left(\sum_i \frac{1}{N_k} f_k(X_i) - \int_{\mathcal{M}} f(x) \mathrm{d}\mu(x) \right) \Rightarrow \mathcal{N}(0, \sigma^2).$$

It is equivalent to consider the convergence of the rescaled random variables

$$\Xi_k = N_k^{\frac{1-d}{2d}} \left(\sum_i f_k(X_i) - N_k \int f(x) \mathrm{d}\mu(x) \right). \tag{43}$$

We set $u_k : x \mapsto N_k^{\frac{1-d}{2d}} \left[f_k(x) - \int f(y) \mathrm{d}\mu(y) \right]$, so that $\Xi_k = \sum_i u_k(X_i)$ is a linear statistic of the point process.

We set, for any $t \in \mathbb{C}$,

$$\begin{aligned} F_k(t) &= -\log \mathbb{E}_{k\phi+\psi+tu_{k,\phi,d}}\left[\mathrm{e}^{-t\sum_i u_{k,\phi,d}(X_i)}\right] \\ &= -\log \mathbb{E}_{k\phi+\psi+tu_{k,\phi,d}}\left[e^{tN_k^{\frac{1-d}{2d}}\sum_i f_k(X_i)}\right] + N_k^{\frac{1-d}{2d}} N_k \int_{\mathcal{M}} f(x)\mathrm{d}\mu(x). \end{aligned}$$

It clearly defines a holomorphic function on $\mathbb{C}$, and it is uniformly bounded on any compact subset of $\mathbb{C}$. Our goal is to demonstrate that $F_k(i\xi)$ converges to the Fourier transform of the right Gaussian, for all $\xi \in \mathbb{R}$; the proof will be decomposed into three steps.

Step 1: convergence on $\mathbb{R}$. Let $t \in \mathbb{R}$. According to Proposition 3.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F_k(t) = \mathbb{E}_{k\phi+\psi+tu_k}[\Xi_k] = N_k^{\frac{1-d}{2d}}\mathbb{E}_{k\phi+\psi+tu_k}\left[\sum_i f_k(X_i)\right] - N_k^{\frac{1-d}{2d}} N_k \int_{\mathcal{M}} f(x)\mathrm{d}\mu(x),$$

and in particular it vanishes at $t = 0$ because

$$\mathbb{E}_{k\phi+\psi}\left[\sum_i f_k(X_i)\right] = N_k \mathbb{E}_{k\phi}\left[\sum_i \frac{f(X_i)}{B_{(k\phi,\mu)}(X_i, X_i)}\right] = N_k \int_{\mathcal{M}} f(x)\mathrm{d}\mu(x).$$

We also have

$$\begin{aligned} \frac{\mathrm{d}^2 F_k(t)}{\mathrm{d}t^2} &= \mathbb{V}\mathrm{ar}_{k\phi+\psi+tu_k}[\Xi_k] \\ &= N_k^{\frac{1-d}{d}}\frac{1}{2}\iint_{\mathcal{M}^2}(f_k(x) - f_k(y))^2 |B_{(k\phi+\psi+tu_k,\mu)}(x,y)|^2_{k\phi+\psi+tu_k}\mathrm{d}\mu^{\otimes 2}(x,y) \end{aligned}$$

On the one hand, we know by Proposition 4.6 that the sequence (f_k) converges uniformly to f^{ϕ}_{eq}, which is Lipschitz continuous. On the other hand, we also have the uniform convergence of (u_k). If $d = 1$, then $N_k^{\frac{1-d}{d}} = 1$ and the uniform limit is f^{ϕ}_{eq}, otherwise the uniform limit is 0. In any case, we can apply Theorem 4.7 and get

$$\lim_{k\to\infty}\frac{\mathrm{d}^2 F_k(t)}{\mathrm{d}t^2} = \|\mathrm{d}f^{\phi}_{\mathrm{eq}}\|^2_{\mathrm{dd}^c\phi}, \ \forall t \geqslant 0.$$

Now, we can rewrite F_k as

$$F_k(t) = F_k(0) + \int_0^t \frac{\mathrm{d}F_k(s)}{\mathrm{d}t}\mathrm{d}s = \int_0^t\left(\frac{\mathrm{d}F_k(0)}{\mathrm{d}t} + \int_0^s \frac{\mathrm{d}^2F_k(u)}{\mathrm{d}t^2}\mathrm{d}u\right)\mathrm{d}s.$$

Since $F_k(0) = \frac{\mathrm{d}F_k(0)}{\mathrm{d}t} = 0$, this simplifies to

$$F_k(t) = \int_0^t\int_0^s \frac{\mathrm{d}^2F_k(u)}{\mathrm{d}t^2}\mathrm{d}u\mathrm{d}s, \tag{44}$$

and dominated convergence (induced by the uniform convergence and integration on compact sets) yields

$$\lim_{k\to\infty} F_k(t) = -\int_0^t\int_0^s \|\mathrm{d}f^{\phi}_{\mathrm{eq}}\|^2\mathrm{d}u\mathrm{d}s = -\frac{t^2}{2}\|\mathrm{d}f_{\mathrm{eq}}\|^2.$$

Step 2: analytic continuation. As stated earlier in the proof, we know that (F_k) is locally uniformly bounded, so that Montel's theorem states that the family $(F_k)_{k\geqslant 0}$ is normal. Thus, from any subsequence of the family (F_k) we can extract a subsubsequence that converges uniformly to some holomorphic function F_∞ on any compact of $\mathbb{C}$; it is in particular the case for the sequence (F_k). At that point, F_∞ might depend on the subsequence; however we know that all these limits coincide on $\mathbb{R}$, according to Step 1. From the analytic extension Theorem, we obtain that these limits also coincide on $\mathbb{C}$, hence we have the uniform convergence $F_k \to F_\infty$ on all compacts subsets of $\mathbb{C}$, where $F_\infty(t) = -\frac{t^2}{2}\|\mathrm{d}f_{\mathrm{eq}}\|^2$.

Step 3: restriction to $i\mathbb{R}$. If we restrict the previous convergence to the imaginary line $i\mathbb{R}$, we obtain

$$\lim_{k\to\infty} \mathbb{E}_{k\phi+\psi+tu_k}\left[\mathrm{e}^{i\xi\sum_i u_k(X_i)}\right] = \exp\left(-\frac{\xi^2}{2}\|\mathrm{d}f_{\mathrm{eq}}\|^2\right),$$

and this holds for all compact subsets of $i\mathbb{R}$. We recognize, in the right-hand side, the characteristic function of a Gaussian distribution, and the convergence (43) follows. □

Proof of Corollary 1.3. Let $(X_1,\dots,X_N)$ be a DPP with kernel $B_{(k\phi+\psi,\mu)}$. Its distribution is given by

$$\frac{1}{N_k!}|\det(s_i(x_j))|^2_{k\phi+\psi}\mathrm{d}\mu^{\otimes N_k}(x_1,\dots,x_{N_k}) = \frac{1}{N_k!}|\det(s_i(x_j))|^2_{k\phi}e^{-2\sum_i\psi(x_i)}\mathrm{d}\mu^{\otimes N_k}(x_1,\dots,x_{N_k}).$$

Here, (s_i) is an orthonormal basis of $H^0(M,L^k)$ for the inner product $\langle\cdot,\cdot\rangle_{(k\phi+\psi,\mu)}$, which is equal to the inner product $\langle\cdot,\cdot\rangle_{(k\phi,e^{-2\psi}\mu)}$: for any sections s_1, s_2,

$$\begin{aligned}\langle s_1,s_2\rangle_{(k\phi+\psi,\mu)} &= \int_{\mathcal{M}}\langle s_1(x),s_2(x)\rangle_{k\phi+\psi}\mathrm{d}\mu(x)\\ &= \int_{\mathcal{M}}\langle s_1(x),s_2(x)\rangle_{k\phi}e^{-2\psi(x)}\mathrm{d}\mu(x)\\ &= \langle s_1,s_2\rangle_{(k\phi,e^{-2\psi}\mu)}.\end{aligned}$$

Let $V:\mathcal{M}\to\mathbb{R}$ be a fixed continuous local weight. The result follows then from a direct application of Theorem 1.2 for the weighted measure $(k\phi,e^{-2\psi}\mu)$ where $\psi:\mathcal{M}\to\mathbb{R}$ is defined by $\psi(x)=\frac{1}{2}V(x)$. □

6 Application to the Riemann sphere

We shall illustrate our result to the simplest possible model, where computations can be made explicit and simulations are affordable.

6.1 Complex structure and Bergman kernel

Consider the unit sphere $\mathbb{S}^2\subset\mathbb{R}^3$; we will simultaneously see it as a submanifold of $\mathbb{R}^3$ and as a complex manifold of dimension 1. As a submanifold, it is defined by $\mathbb{S}^2 = F^{-1}(0)$, where $F : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$. It can be endowed with the atlas (U_0, U_1) such that U_0 is the sphere without the North pole $(0, 0, 1)$ and U_1 the sphere without the South pole $(0, 0, -1)$. The corresponding charts are given by stereographic projections:

$$\varphi_0 : \left\{ \begin{array}{ccc} S^2 \setminus \{(0,0,1)\} & \longrightarrow & \mathbb{C} \\ (x,y,z) & \longmapsto & \frac{x+iy}{1-z} \end{array} \right. , \ \varphi_1 : \left\{ \begin{array}{ccc} S^2 \setminus \{(0,0,-1)\} & \longrightarrow & \mathbb{C} \\ (x,y,z) & \longmapsto & \frac{x+iy}{1+z} \end{array} \right. .$$

We denote by ζ the local complex coordinate given by those charts. Note that φ_0 (resp. φ_1) is centered in the South pole (resp. the North pole). We will usually stick to U_0 but everything is similar in U_1. If we take $Z \in \varphi_0(U_0 \cap U_1)$, then

$$\varphi_1 \circ \varphi_0(\zeta) = \frac{\zeta}{|\zeta|^2},$$

which is holomorphic on $\mathbb{C}^* = \varphi_0(U_0 \cap U_1)$, and it is an involution, hence a biholomorphism. It follows that $\mathbb{S}^2$ is a complex manifold of dimension 1, *i.e.* a Riemann surface. Note that, as $F^{-1}(0)$, it is a closed subset of $\mathbb{R}^3$, and it is obviously bounded, therefore it is compact. We endow this manifold with the following volume form in the local chart U_0:

$$\omega(\zeta) = \frac{\mathrm{i}\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}}{2\pi(1 + |\zeta|^2)^2}. \tag{45}$$

A quick computation shows that

$$\int_{\mathbb{C}} \omega = 1, \tag{46}$$

therefore it corresponds to a probability measure that we will denote by $\mathrm{dvol}_{\mathbb{S}^2}$. We will denote by $\mathrm{d}m(\zeta) = \frac{\mathrm{i}}{2}\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}$ the Lebesgue measure on $\mathbb{C}$, so that

$$\mathrm{dvol}_{\mathbb{S}^2}(\zeta) = \frac{\mathrm{d}m(\zeta)}{\pi(1 + |\zeta|^2)^2}.$$

Let us consider the line bundle[7] L defined as follows: for any point $P = (x, y, z) \in \mathbb{S}^2$, the fiber L_P is the line in $\mathbb{R}^3$ generated by P. We have the open covering (U_0, U_1) of S^2, and the associated trivialization functions are

$$\psi_0 : (P, \lambda P) \in \pi^{-1}(U_0) \mapsto (P, \lambda)$$

and

$$\psi_1 : (P, \lambda P) \in \pi^{-1}(U_1) \mapsto (P, \lambda).$$

The transition function γ_{10} is then the identity. Let us restrict again to U_0: a section of L_{U_0} is a function $f : U_0 \to \mathbb{C}$ that maps P to $\lambda = f(P)$, that is, a choice of a coordinate λ in the complex line $L_P = \mathbb{C}P$. The restriction to U_0 of a holomorphic section of L is then a holomorphic function $f : U_0 \to \mathbb{C}$, that we can identify to a holomorphic function $f_0 : \mathbb{C} \to \mathbb{C}$ *via* the stereographic projection $f_0 = f \circ \varphi_0^{-1}$. Let us endow L^k with the metric h^k given in the local coordinate ζ on U_0 by

$$h^k(s_1(\zeta), s_2(\zeta)) = f_1(\zeta)\overline{f_2}(\zeta)\mathrm{e}^{-k\phi(\zeta)},$$

[7] It is actually equivalent to the tautological line bundle of the complex projective plane $\mathbb{CP}^1$.

with a weight $\phi(\zeta) = \log(1+|\zeta|^2)$ that has positive curvature:

$$\mathrm{dd}^c\phi(\zeta) = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\phi(\zeta) = \omega(\zeta).$$

The inner product $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{(k\phi,\mathrm{dvol}_{\mathbb{S}^2})}$ is then given by

$$\langle s_1, s_2\rangle = \int_{\mathbb{C}} f_1(\zeta)\overline{f_2}(\zeta)\frac{\mathrm{d}m(\zeta)}{\pi(1+|\zeta|^2)^{2+k}}. \tag{47}$$

An orthonormal basis of $H^0(\mathbb{S}^2, L^k), \langle\cdot,\cdot\rangle_{(k\phi,\mathrm{dvol}_{\mathbb{S}^2})}$ is given by $(s_\ell)_{0\leqslant\ell\leqslant k}$, where the sections s_ℓ are the spherical harmonics, given in complex stereographic coordinates in U_0 by

$$s_\ell(\zeta) = \sqrt{k+1}\sqrt{\binom{k}{\ell}}\zeta^\ell,\ \forall 0\leqslant\ell\leqslant k.$$

In particular, we find that $H^0(\mathbb{S}^2, L^k)$ has dimension $N_k = k+1$. The Bergman kernel can be written in local coordinates

$$B_{(k\phi,\mathrm{dvol}_{\mathbb{S}^2})}(\zeta,\xi) = (k+1)\mathrm{e}^{-\frac{k}{2}(\phi(\zeta)+\phi(\xi))}\sum_{\ell=0}^{k}\binom{k}{\ell}(\zeta\overline{\xi})^\ell = (k+1)\frac{(1+\zeta\overline{\xi})^k}{(1+|\zeta|^2)^{\frac{k}{2}}(1+|\xi|^2)^{\frac{k}{2}}}.$$

In particular, the Bergman kernel is constant on the diagonal:

$$B_{(k\phi,\mathrm{dvol}_{\mathbb{S}^2})}(\zeta,\zeta) = k+1,\ \forall\zeta\in\mathbb{C}.$$

6.2 The spherical ensemble

Now we can describe the DPP whose kernel is B_k: it is the point process associated to the $(k+1)$-tuple of random variables on $\mathbb{S}^2$ whose joint distribution is given, in the stereographic coordinate ζ on U_0, by

$$\frac{1}{(k+1)!}|\det(s_{i-1}(\zeta_j))_{1\leqslant i,j\leqslant k+1}|^2\prod_{i=1}^{k+1}\frac{\mathrm{d}m(\zeta_i)}{(1+|\zeta_i|^2)^{k+2}}.$$

Krishnapur (2006) proved that the *spherical ensemble*, defined as the distribution of the eigenvalues of AB^{-1} where A and B are independent standard Gaussian complex matrices, has joint distribution

$$\frac{1}{Z_N}\prod_{i<j}|\zeta_i-\zeta_j|^2\prod_{i=1}^{N}\frac{dm(\zeta_i)}{(1+|\zeta_i|^2)^{N+1}}.$$

We see that in the case of $N = N_k = k+1$, we obtain the DPP on the sphere associated to the normalized volume form $\mathrm{dvol}_{\mathbb{S}^2}$ and the metric h with Kähler potential $\phi(\zeta) = \log(1+|\zeta|^2)$, which has a curvature everywhere positive (in particular, it means that the bulk is the whole sphere). If we denote by $(X_1,\ldots,X_{k+1})$ the spherical ensemble, Theorem 1.2 states that for any Lipschitz function $f:\mathbb{S}^2\to\mathbb{R}$, the integral

$$\int_{\mathbb{S}^2} f(x)\mathrm{dvol}_{\mathbb{S}^2}(x) = \int_{\mathbb{C}} f\circ\varphi_0^{-1}(\zeta)\frac{\mathrm{d}m(\zeta)}{\pi(1+|\zeta|^2)^2}$$

can be approximated by

$$\sum_{i=1}^{k+1} \frac{f \circ \varphi_0^{-1}(X_i)}{B_k(X_i, X_i)}.$$

6.3 Numerical experiments

We shall now proceed to a comparison of our Monte Carlo method, which reduces to using the empirical measure of the spherical ensemble as a Monte Carlo estimator, with a few other estimators. We consider a standard Monte Carlo estimator with a i.i.d. uniform sample, a Monte Carlo estimator based on a DPP in $[-1,1]^2$ mapped to the sphere, and a randomized Quasi Monte Carlo estimator. The latter two are now introduced in more detail, before showing the experimental results.

6.3.1 Legendre DPP

The *Jacobi measure* of parameters $\alpha_1, \beta_1, \ldots, \alpha_d, \beta_d > -1$ is the measure on $(-1,1)^d$ given by

$$\mathrm{d}\mu_{\alpha,\beta}(x_1, \ldots, x_d) = \prod_{j=1}^{d} (1 - x_j)^{\alpha_j} (1 + x_j)^{\beta_j} \mathrm{d}x_j.$$

The corresponding orthonormal polynomials are the so-called multivariate *Jacobi polynomials*; see e.g. (Dunkl and Xu, 2014). It has been shown by Bardenet and Hardy (2020) that, using a suitable ordering (p_k) of these multivariate Jacobi polynomials, the projection determinantal point process with kernel $\sum_{k=1}^{N} p_k(x)p_k(y)$ satisfies the assumptions of Theorem 1.4. An interesting fact is that the integration on $\mathbb{S}^2$ with respect to the uniform measure boils down to an integration on $(-1,1)^2$ with respect to the uniform measure, which is actually the Jacobi measure of parameters $(0,0)$, as explained in the following proposition.

Proposition 6.1. *For any $f : \mathbb{S}^2 \to \mathbb{R}$ measurable and bounded,*

$$\int_{\mathbb{S}^2} f(x,y,z)\mathrm{dvol}_{\mathbb{S}^2}(x,y,z) = \frac{1}{4} \int_{(-1,1)^2} f \circ \Phi(x,y)\mathrm{d}x\mathrm{d}y, \tag{48}$$

where $\Phi : [-1,1]^2 \to \mathbb{S}^2$ is the function defined by

$$\Phi(x,y) = (\sqrt{1-x^2}\cos(\pi(y+1)), \sqrt{1-x^2}\sin(\pi(y+1)), x).$$

Proof. Let $\Phi_1 : [0, 2\pi] \times [0, \pi] \to \mathbb{S}^2$ be the change of variable from spherical to Cartesian coordinates in the sphere, namely

$$\Phi_1(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

It is a diffeomorphism from $(0, 2\pi) \times (0, \pi)$ onto its image, whose complement is negligible in $\mathbb{S}^2$ with respect to $\mathrm{dvol}_{\mathbb{S}^2}$. Moreover,

$$\Phi_1^* \mathrm{dvol}_{\mathbb{S}^2}(\theta, \phi) = \frac{1}{4\pi} \sin\phi \mathrm{d}\theta \mathrm{d}\phi.$$

Hence, for any bounded measurable function $f : \mathbb{S}^2 \to \mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{S}^2} f(x,y,z)\mathrm{dvol}_{\mathbb{S}^2}(x,y,z) &= \int_{\Phi_1((0,2\pi)\times(0,\pi))} f(x,y,z)\mathrm{dvol}_{\mathbb{S}^2}(x,y,z) \\ &= \int_{(0,2\pi)\times(0,\pi)} f\circ\Phi_1(\theta,\phi)\sin\phi\frac{\mathrm{d}\theta\mathrm{d}\phi}{4\pi}.\end{aligned}$$

We also introduce the diffeomorphism

$$\Phi_2 : \begin{cases} (0,2\pi)\times(0,\pi) & \to & (-1,1)^2 \\ (\theta,\phi) & \mapsto & (\cos\phi, \frac{\theta}{\pi}-1). \end{cases}$$

The result follows from the fact that $\Phi = \Phi_1 \circ \Phi_2^{-1}$. □

We conclude with a remark that in our case, where $d = 2$ and $\alpha = \beta = 0$, the multivariate Jacobi polynomials specialize to the Legendre polynomials. To sample the corresponding DPP, we use the classical algorithm by Hough et al. (2006), in the specific implementation of the Python library DPPy (Gautier et al., 2019a) for multivariate Jacobi ensembles.

6.3.2 Randomized spiral points

Following Rakhmanov et al. (1994) or Brauchart et al. (2014), given a fixed parameter $C > 0$ and a fixed sample size N, the *generalized spiral points* are the points of the sphere with spherical coordinates $(\theta_i, \phi_i)_{1\leqslant i\leqslant N}$ defined by an iterative procedure: for any $1 \leqslant i \leqslant N$, set $z_i = 1 - \frac{2i-1}{N}$ and

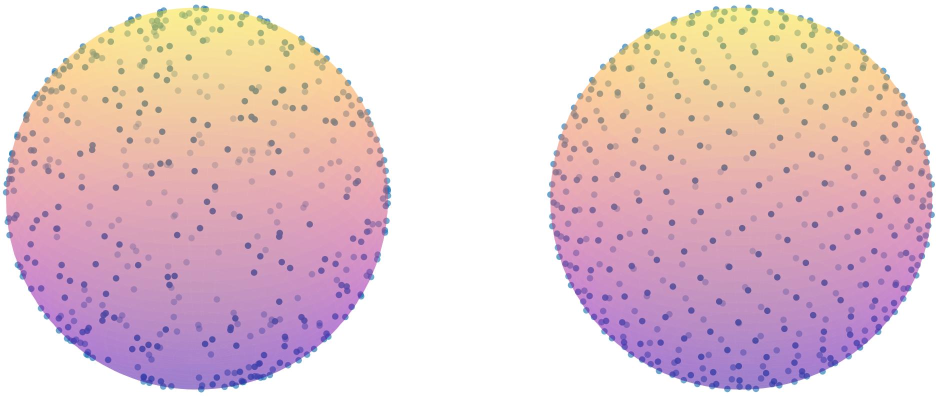
$$\theta_i = \arccos z_i, \;\; \phi_i = C\sqrt{N}\theta_i.$$

It provides a deterministic low-discrepancy family $\{(x_i, y_i, z_i), 1 \leqslant i \leqslant N\}$ of points of $\mathbb{S}^2$, which can be randomized through a random (uniform) rotation $R \in \mathrm{SO}(3)$. Although the QMC method using spiral points was studied in the aforementioned papers, we are unaware of any theoretical estimation of the variance of the corresponding randomized QMC. Yet we expect it to be competitive in our low-dimensional setting.

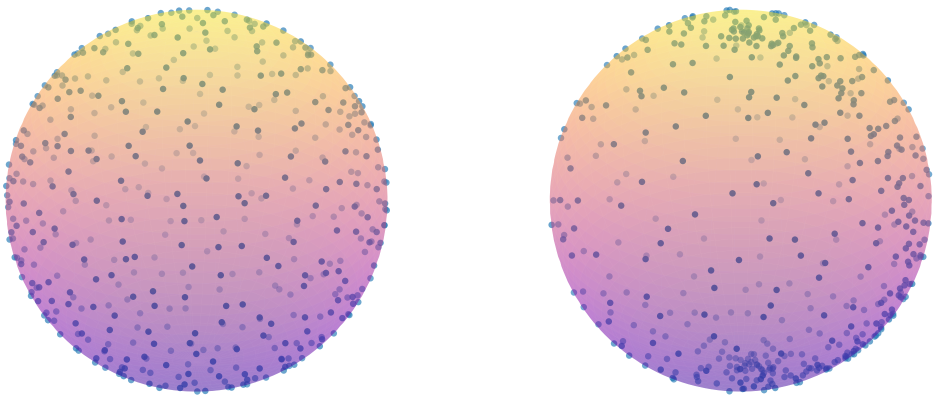
6.3.3 Results

In Figure 3, we display samples of all the models we consider. Note that all spiral points of the sample are randomized through the same rotation, which makes them look like usual spiral points. In the case of a Jacobi ensemble, we take a Jacobi ensemble of parameters $(0,0)$ (or equivalently, a Legendre ensemble) on $[-1,1]^2$ mapped onto the sphere through the diffeomorphism Φ introduced in Proposition 6.1. It corresponds to the method of Bardenet and Hardy (2020). As expected, we see that a cluster appears in the image of the boundary of the square $[-1,1]^2$.

Figure 4 displays the logarithm of each sample variance as a function of $\log N$, across 200 independent repetitions for each model and each N. Keeping in mind that the variance should be proportional to N^α for various values of α depending on the model, all plots

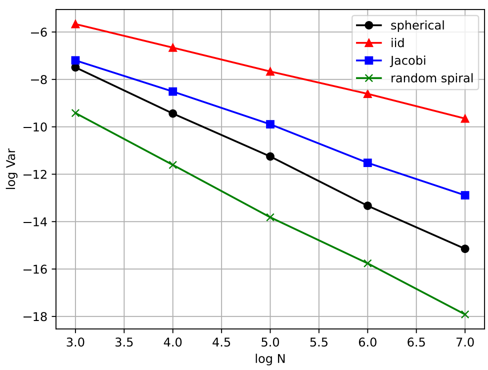


(a) i.i.d. uniform (b) randomized spiral

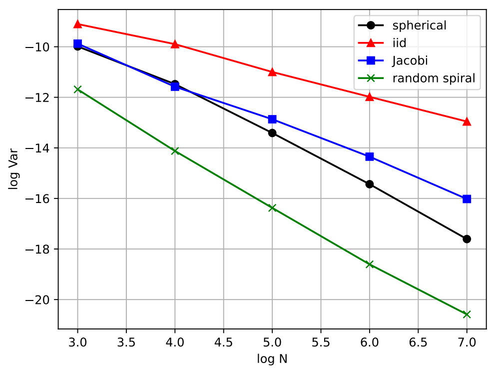


(c) spherical (or Bergman) ensemble (d) Jacobi ensemble

Figure 3: Independent samples of size $N = 500$ from four distributions on the sphere.



(a) f_1



(b) f_2

Figure 4: log variance of four integral estimators wrt. the number N of quadrature nodes.

are supposed to be linear. We consider two functions $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ that we restrict to

$\mathbb{S}^2$, and we choose them with support in

$$\mathbb{S}^2_+ = \{(x, y, z) \in \mathbb{S}^2 : z \geqslant 0\}$$

in order to avoid numerical errors that could happen in stereographic coordinates for points which are close to the South pole (corresponding to the point at infinity). We take

$$f_1(x, y, z) = z^2 \mathbf{1}_{z\geqslant 0}, \quad f_2(x, y, z) = |x|^{3/2} yz \mathbf{1}_{z\geqslant 0}.$$

Both functions are $\mathscr{C}^1$ on $\mathbb{S}^2_+$, but f_1 is actually smooth. Besides, f_2 is supported in the image of the open square $(-1, 1)^2$ and satisfies therefore the assumptions of Bardenet and Hardy (2020), whereas f_1 is nonzero on the image of the boundary of the square. Both functions also naturally satisfy the assumptions of the CLT for the i.i.d. Monte Carlo method. In both cases, the variances of the estimators have the same rankings, and the slopes are quite close to their theoretical values. It is interesting to see that for f_2, on low values of N, the estimator for the Jacobi ensemble has a slightly lower variance than the spherical ensemble, although it decays more slowly when N grows. It does not happen for f_1, which is not surprising because it is an edge case for the method by Bardenet and Hardy (2020). We also remark that the randomized spiral points seem to provide an overall better performing estimator than all other methods, although there is no theoretical result to support that, with a similar slope to the spherical ensemble.

7 Conclusion and perspectives

Building on Berman's seminal work that led to the central limit theorem in (Berman, 2018), we showed that Bergman ensembles can lead to fast Monte Carlo integration on compact complex manifolds, just like multivariate orthogonal polynomial ensembles (OPEs) yield fast quadrature on compacts of the Euclidean space (Bardenet and Hardy, 2020). The take-home message is that, like OPEs, Bergman ensembles come up with a *fast* central limit theorem for Monte Carlo integration that is also *universal*, in the sense that the asymptotic variance of the central limit theorem is invariant to a suitable change of the reference measure and the kernel. Unlike OPEs, however, the dimension in the rate of convergence is now the *complex* dimension of the manifold, which has the important consequence that Bergman ensembles outperform multivariate OPEs for integration in Euclidean spaces of even real dimension. Actually, the error rate for Bergman ensembles matches the optimal worst-case rate by Bakhvalov (1965) for functions of class $\mathscr{C}^1$.

We are currently working on further experiments, to include for instance competitive approaches such as the Dirichlet DPP of Mazoyer et al. (2020a). We also strive to include a reasonable implementation of a Bergman ensemble on the sphere that is *not* the spherical ensemble, by considering a non-uniform reference measure. We expect the performance to be similar to the uniform case.

Future work includes the following tasks, roughly ranked by increasing difficulty. We shall first investigate the influence of the smoothness of the integrand on the error decay, in line with (Belhadji et al., 2019, 2020). Then, if we want to make DPPs on arbitrary complex manifolds practical, we need to circumvent the fact that the Bergman kernel is usually only available in the form of asymptotic estimates. We should thus investigate the statistical effect of working with an approximate DPP built using these estimates. Alternately, it might be possible to use importance sampling with a proposal DPP for which we know how to numerically evaluate the Bergman kernel, for instance, using the Kodaira embedding of a Kähler manifold in a complex projective space of higher dimension, where the Bergman kernel is explicit. Finally, while compact complex manifolds already include important practical settings, such as Bayesian quantum tomography, extending the results to more general manifolds, e.g. with a boundary, would further broaden the applicability of DPP-based Monte Carlo integration.

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