Does Volatility Harvesting Really Work?

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Abstract

A large number of academic papers claim that there is a consistent advantage to rebalancing an investment portfolio relative to a simple Buy&Hold strategy, while other academic papers claim the opposite. The papers on both sides are typically based on unclear mathematical arguments with only few real-world examples. This paper does not try to give a mathematical explanation but is entirely empirical. We use daily stock-returns for more than 900 U.S. stocks between the years 2007 and 2021, where we generate thousands of random portfolios of different sizes and start/end-dates, to compare the Rebalanced and Buy&Hold strategies on different performance metrics. Although the Rebalanced portfolios may seem to perform better than the Buy&Hold portfolios in terms of e.g. the arithmetic mean daily return, it actually depends on which period we consider. We also make preliminary experiments with intraday data, where it does appear that Rebalancing is consistently better than Buy&Hold, although these data-sets are so small that we cannot conclude this advantage holds in general.

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1. Introduction

A large number of academic papers claim that it is possible to earn excess returns from merely rebalancing a portfolio of volatile assets at regular intervals; an idea known as “Volatility Harvesting”. This would be an *extremely* attractive proposition in today’s markets, because U.S. stocks now have near-zero trading fees and commissions, so it would be possible to buy e.g. the same stocks as the S&P 500 index and rebalance the portfolio daily or even multiple times per day, in order to earn excess returns from the rebalancing alone. The rebalancing could easily be automated in a stock-broker’s computer system to make it available for all investors.

The claim originated in a paper from 1992 by David Booth and Eugene Fama. Booth later went on to become a very wealthy investor and Fama went on to receive the Nobel prize in finance in the year 2013. So the original authors of this idea are highly reputable. Many papers have made similar claims in the following decades, see for example Willenbrock 2011, and Bouchey et al. 2012 and 2015.

A common problem in these papers is that their mathematical arguments and proofs are really more like assertions and conjecture. The papers typically consider geometric means for the asset returns and use so-called Taylor expansions and log-transforms to arrive at simpler mathematical expressions. But it is unclear how their assumptions and mathematical transformations limit the validity of their claims to certain scenarios such as mean-reverting or trending asset-prices.

Some papers are critical of the above claims and argue that it is exactly the non-linear transformations that cause problems and confusion, and that there is no general advantage to rebalancing a portfolio, except in certain scenarios. See for example Chambers and Zdanowicz 2014, and Cuthbertson et al. 2016. Unfortunately these papers also use unconvincing mathematical arguments, where the exact assumptions, implications and limitations are often unclear.

The papers cited above only have very limited experiments on real-world data, but a few other papers are more focused on empirical data. Jaconetti et al. 2010 study rebalancing intervals for U.S. stocks and bonds, while Meyer-Bullerdiek 2017 studies a mere 15 German stocks. Unfortunately these experiments are rather limited and unconvincing as well, which is the reason for writing this paper.

In this paper we will abstain completely from making any attempts at formal mathematical analysis of this problem. We will instead use real-world stock-data and generate thousands of random portfolios, and then study various performance metrics. Using daily stock-returns for over 900 U.S. stocks between the years 2007 and 2021, we find that there does not appear to be a consistent advantage of rebalancing the portfolio as claimed in much of the literature, with a few counter-intuitive exceptions: It seems that rebalancing under-performs when stock-markets are crashing, but substantially out-performs when the stock-markets are recovering. Furthermore, using intraday stock-returns for 1 and 5-minute intervals, it seems that rebalancing at such short intervals often performs better than Buy&Hold, although the data-set is so small that we cannot conclude this advantage holds in general.

2. The Problem

We first define the problem mathematically, so we know exactly what we need to calculate and compare between the Buy&Hold and Rebalanced portfolios. First let $Asset\ Price_{i,t}$ denote the price of asset i at time-step t.[1] This could be the price of any asset such as a bond, a currency pair, etc., but in this paper we will only consider American stocks that we can easily obtain data for. Then let $Asset\ Return_{i,t}$ denote the asset's return from the previous time-step t-1 until the current time-step t. We do not subtract 1 from the return, because we would then need to add it back in many of the other formulas. So a return of 1.05 means a gain of 5% and a return of 0.9 means a loss of 10% relative to the asset's price in the previous time-step. The formula for the asset's return is:

$$Asset\ Return_{i,t} = \frac{Asset\ Price_{i,t}}{Asset\ Price_{i,t-1}} \tag{1}$$

The compounded or cumulative return between time-steps t=1 and T is denoted $Asset\ Cum\ Return_{i,T}$ and is defined as the product of the returns for all intermediate time-steps, which can be reduced to the end-price $Asset\ Price_{i,T}$ divided by the start-price $Asset\ Price_{i,0}$. It can also be thought of as the value of the asset after T time-steps, normalized to be independent of the start-price:

$$Asset\ Cum\ Return_{i,T} = \prod_{t=1}^{T} Asset\ Return_{i,t} = \frac{Asset\ Price_{i,T}}{Asset\ Price_{i,0}} \tag{2}$$

2.1 Buy & Hold Portfolio

The cumulative return of the Buy&Hold portfolio after T time-steps, is simply the average of the cumulative returns for the individual assets after T time-steps, without any rebalancing in the intermediate time-steps:

$$BH\ Cum\ Return_T = \frac{1}{N}\sum_{i=1}^{N} Asset\ Cum\ Return_{i,T} = \frac{1}{N}\sum_{i=1}^{N}\prod_{t=1}^{T} Asset\ Return_{i,t} \tag{3}$$

The single-period return on the Buy&Hold portfolio between the previous time-step T-1 and the current time-step T is defined similarly to the return for individual assets. The formula cannot really be simplified, so in the computer code we first need to calculate the time-series for $BH\ Cum\ Return_T$ and then calculate the time-series for single-period returns $BH\ Return_T$ using this formula:

$$BH\ Return_T = \frac{BH\ Cum\ Return_T}{BH\ Cum\ Return_{T-1}} \tag{4}$$

1 In the computer code described in Section 12 we have actually reversed the indices i and t because it is more practical to have the asset-numbers in the columns and the time-steps in the rows like this: $Asset\ Price_{t,i}$ and $Asset\ Return_{t,i}$.

2.2 Rebalanced Portfolio

Now consider the Rebalanced portfolio, where we rebalance the portfolio to have equal weights for all the assets at each time-step, so after *T* time-steps the cumulative return on the Rebalanced portfolio is:

$$Rebal\,Cum\,Return_T = \prod_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} Asset\,Return_{i,t} \tag{5}$$

Compare this formula to Eq. (3) for the Buy&Hold portfolio. The difference is the order of summation and multiplication. For the Rebalanced portfolio we average the asset returns at each time-step, corresponding to having all assets with equal weights, and then we take the product of all these averaged returns to obtain the portfolio's cumulative return. For the Buy&Hold portfolio in Eq. (3) we let the individual assets run their course and then average their final values at the end. In this paper we want to investigate whether one of these approaches is consistently better than the other.

The return on the Rebalanced portfolio between the previous time-step *T*-1 and the current time-step *T* is defined similarly to the return on the Buy&Hold portfolio in Eq. (4). But for the Rebalanced portfolio we can simplify the formula so the portfolio's return is just the average of the returns on the individual assets for that time-step *T*. However, in the computer code we will calculate it similarly to the Buy&Hold portfolio, so we first calculate the time-series *Rebal Cum Return*$_T$ and from this we calculate the time-series *Rebal Return*$_T$. But the simpler formula is derived as follows:

$$Rebal\,Return_T = \frac{Rebal\,Cum\,Return_T}{Rebal\,Cum\,Return_{T-1}} = \frac{\prod_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} Asset\,Return_{i,t}}{\prod_{t=1}^{T-1} \frac{1}{N} \sum_{i=1}^{N} Asset\,Return_{i,t}} = \frac{1}{N} \sum_{i=1}^{N} Asset\,Return_{i,T} \tag{6}$$

2.3 Performance Statistics

One way of comparing the Rebalanced and Buy&Hold portfolios is to consider their end-values after T time-steps, so we would compare *Rebal Cum Return*$_T$ to *B&H Cum Return*$_T$ to see which is higher. But we can also compare various performance statistics for the Rebalanced and Buy&Hold portfolios.

One of the main disagreements in the research literature, is whether we should compare arithmetic or geometric averages for the portfolio returns. In this paper we will compare both. We use the Buy&Hold portfolio as an example in these formulas, but the formulas are similar for the Rebalanced portfolio. Let us first define the arithmetic mean for the portfolio returns:

$$E_A[BH\,Return_T] = \frac{1}{T} \sum_{t=1}^{T} BH\,Return_t \tag{7}$$

The geometric mean is defined as follows, which can be reduced to depend only on the portfolio's start and end-values, as follows:

$$E_G[BH\ Return_T] = \left(\prod_{t=1}^{T} BH\ Return_t \right)^{1/T} = \left(\prod_{t=1}^{T} \frac{BH\ Cum\ Return_t}{BH\ Cum\ Return_{t-1}} \right)^{1/T} = \left(\frac{BH\ Cum\ Return_T}{BH\ Cum\ Return_0} \right)^{1/T} \tag{8}$$

The standard deviation is defined from the portfolio's single-period returns and their arithmetic mean:

$$Std[BH\ Return_T] = \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} \left(BH\ Return_t - E_A[BH\ Return_T] \right)^2} \tag{9}$$

The so-called Sharpe ratio is defined here as the ratio between the arithmetic mean of the portfolio's single-period returns and their standard deviation. The idea is essentially to have a measure of the portfolio's average return relative to its volatility, so a higher Sharpe ratio is supposedly better because it indicates a higher average portfolio return with relatively lower volatility. The Sharpe ratio is sometimes defined slightly differently in the literature, and it can also take into account a risk-free return such as a government bond, or the return on another benchmark such as a stock-index. The definition we use here is in line with Eqs. (3)-(6) from Sharpe's 1994 paper, which suggest using single-period returns in the calculation of the mean and standard deviation, while other people apparently use the annualized return divided by the standard deviation of short-term returns. It does not matter so much which definition we use here, because we will only be using the Sharpe ratio for comparison between the Buy&Hold and Rebalanced portfolios, to see if one type of portfolio is consistently better than the other. We also need to subtract 1 from the mean, because we have defined a return of e.g. 1.05 to be a 5% gain but we need that to be a value of 0.05 instead. So this is the formula we use for the Sharpe ratio:

$$Sharpe\ Ratio[BH\ Return_T] = \frac{E_A[BH\ Return_T] - 1}{Std[BH\ Return_T]} \tag{10}$$

The maximum drawdown is more intuitive to understand than the Sharpe ratio, as it tells us the biggest loss a portfolio has suffered from its historic peak to its subsequent lowest value. Although the mathematical definition is a bit complicated, it is actually quite easy to make an efficient implementation using vectorized computer code, see Section 12:

$$Max\ Drawdown[BH\ Cum\ Return_T] = \min_{\tau=1\ldots T} \left(\frac{BH\ Cum\ Return_\tau}{\max\limits_{t=1\ldots\tau} BH\ Cum\ Return_t} \right) - 1 \tag{11}$$

Conversely we define the maximum pull-up as the biggest gain from the previous lowest value. In this paper, we will calculate it for a window of 1 year instead of using the entire past history:

$$Max\ Pullup[BH\ Cum\ Return_T] = \max_{\tau=1\ldots T} \left(\frac{BH\ Cum\ Return_\tau}{\min\limits_{t=\tau-1\,Year\ldots\tau} BH\ Cum\ Return_t} \right) - 1 \tag{12}$$

3. Simple Examples

Before we study the real-world data, let us first consider a few simple examples with only two assets, that show some of the scenarios where either the Rebalanced or Buy&Hold portfolios perform best.

In Figure 1 the returns on Asset A switch between 1.05 and its reciprocal 1/1.05 (which is roughly equal to 0.95238) so their product is exactly 1.0. The returns on Asset B are always equal to 0.9 thus giving a loss of 10% in each time-step. Figure 1 shows that the Buy&Hold portfolio performs slightly better than the Rebalanced portfolio. This is because Asset B loses 10% in each time-step, so when the portfolio is rebalanced back to equal weights between Asset A and B in each time-step, we keep taking more money from Asset A to invest in Asset B which just keeps declining.

In Figure 2 the returns on Asset A also switch between 1.05 and its reciprocal 1/1.05 so their product is exactly 1.0. But now the returns on Asset B are always 1.1 thus gaining 10% in each time-step. Once again, Figure 2 shows that the Buy&Hold portfolio performs slightly better than the Rebalanced portfolio. This is because Asset B gains 10% in each time-step, but instead of keeping the money invested in Asset B, we take some of those money and move them into Asset A so the two assets have equal weights again. But in the next time-step Asset B gains 10% again, so it would have been better to keep the money invested in Asset B instead of doing the rebalancing.

Figure 1 and Figure 2 are two simple examples of what happens when we are rebalancing a portfolio where some of the asset-prices are trending. Whether the trend is up or down, the rebalancing of the portfolio back to equal weights causes the portfolio to under-perform relative to a Buy&Hold portfolio. In case an asset is declining, the rebalancing keeps adding money to the investment in the declining asset thus causing us to lose even more money than without the rebalancing. In the opposite case where an asset is gaining, the rebalancing takes money from the best investment and moves it into the other assets that perform worse, thus causing us to under-perform relative to a Buy&Hold strategy.

The point here is not that you should try and find investments that are currently trending and then try and profit from following the trend – that is extremely hard to do! We are merely demonstrating what happens to a Rebalanced versus a Buy&Hold portfolio during different market conditions.

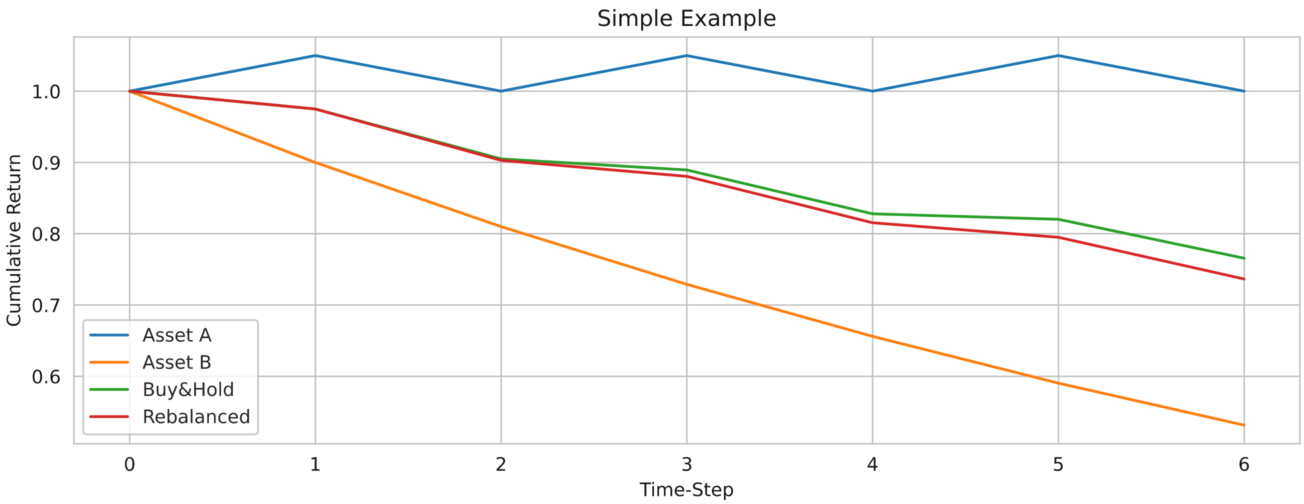

Figure 1: Simple example of cumulative returns. The returns on Asset A switch between 1.05 and 1/1.05 so their geometric mean is 1.0 and the returns on Asset B are always 0.9.

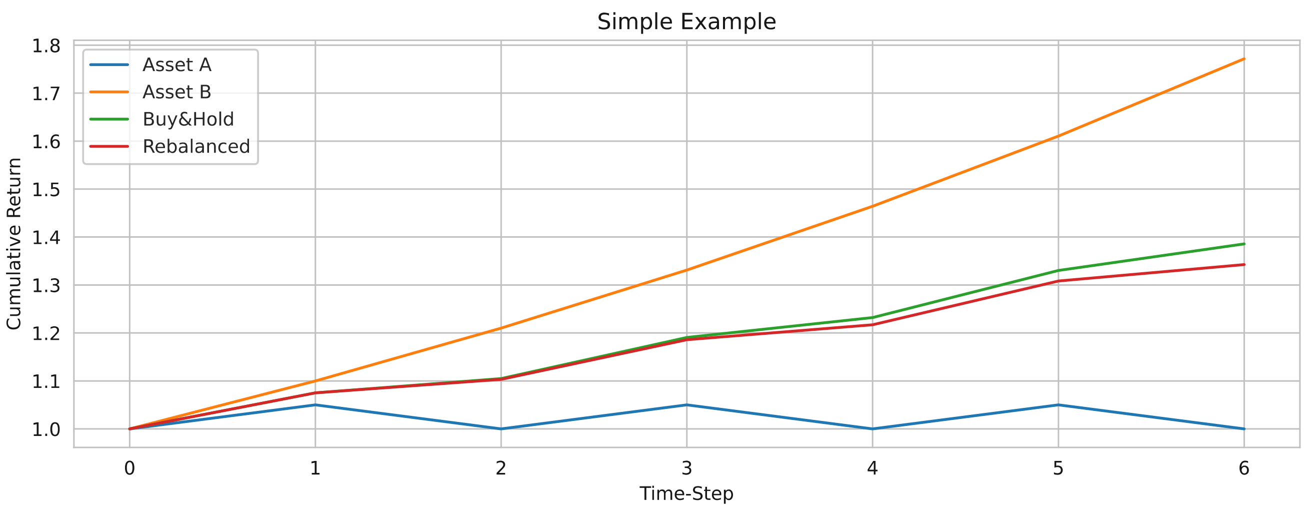

Figure 2: Simple example of cumulative returns. The returns on Asset A switch between 1.05 and 1/1.05 so their geometric mean is 1.0 and the returns on Asset B are always 1.1.

In Figure 3 the returns on both Asset A and B switch between 1.05 and the reciprocal 1/1.05 so the product for each asset's returns is 1.0 over time. But the returns are shifted in time so when Asset A goes up then Asset B goes down, and vice versa. Figure 3 shows that the Rebalanced portfolio now performs better than the Buy&Hold portfolio. This is because the two assets are continually reverting back to their former prices, so when Asset A has gone up and Asset B has gone down, we sell some of Asset A and invest more in Asset B, and then in the next time-step we profit a little more from having invested more in Asset B which goes up next.

Figure 4 is similar but the returns on Asset A and B now switch between 1.05 and 0.95 so their arithmetic mean is 1.0 but their product or geometric mean is actually only 0.9975. Once again the returns are shifted in time so their correlation is -1, that is, when Asset A goes up then Asset B goes down, and vice versa. Figure 4 shows that the Rebalanced portfolio performs better than the Buy&Hold portfolio, for the same reason as in Figure 3.

Also note that in Figure 3 the Buy&Hold portfolio only moves between the values 1.0 and roughly 1.001 which is the arithmetic mean of 1.05 and 1/1.05, while the Rebalanced portfolio has a positive gain in each step because its return in every time-step is also the arithmetic mean of 1.05 and 1/1.05. So the Buy&Hold portfolio stays roughly around the value 1.0 while the Rebalanced portfolio keeps gaining in each time-step.

Compare this to Figure 4 where the Rebalanced portfolio always has the value 1.0 because in each time-step the return on the Rebalanced portfolio is the arithmetic mean of the two asset returns which is $0.5 \cdot (1.05 + 0.95) = 1.0$. But the Buy&Hold portfolio keeps declining because the asset returns switch between 1.05 and 0.95 in each time-step, so the compounded return of two consecutive time-steps is $1.05 \cdot 0.95 = 0.9975$ which results in a small decline for each asset in every two time-steps.

From these simple examples it might seem that the Buy&Hold portfolio performs best for trending asset-prices, and the Rebalanced portfolio performs best for reverting asset-prices. However, real-world data is much more complicated, because the asset-prices trend and revert in very complicated and unpredictable patterns, and the correlation between asset-prices also changes over time. So we need to make proper tests on real-world data to see if either the Buy&Hold or Rebalanced portfolio consistently performs better than the other.

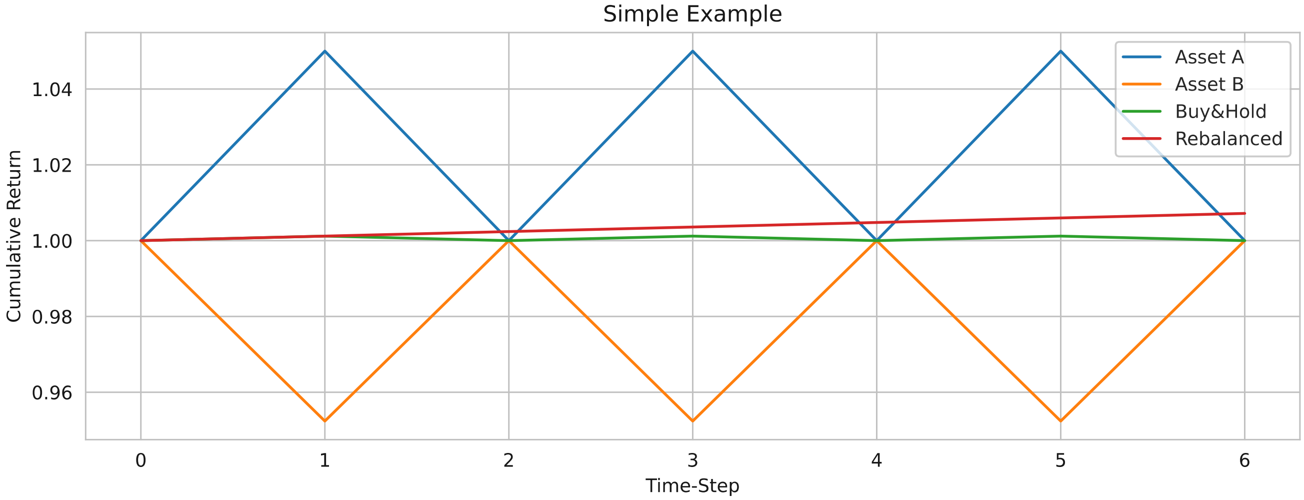

Figure 3: Simple example of cumulative returns. The returns on both Asset A and B switch between 1.05 and 1/1.05 so their geometric mean is 1.0, and their correlation is -1 so they move in opposite direction.

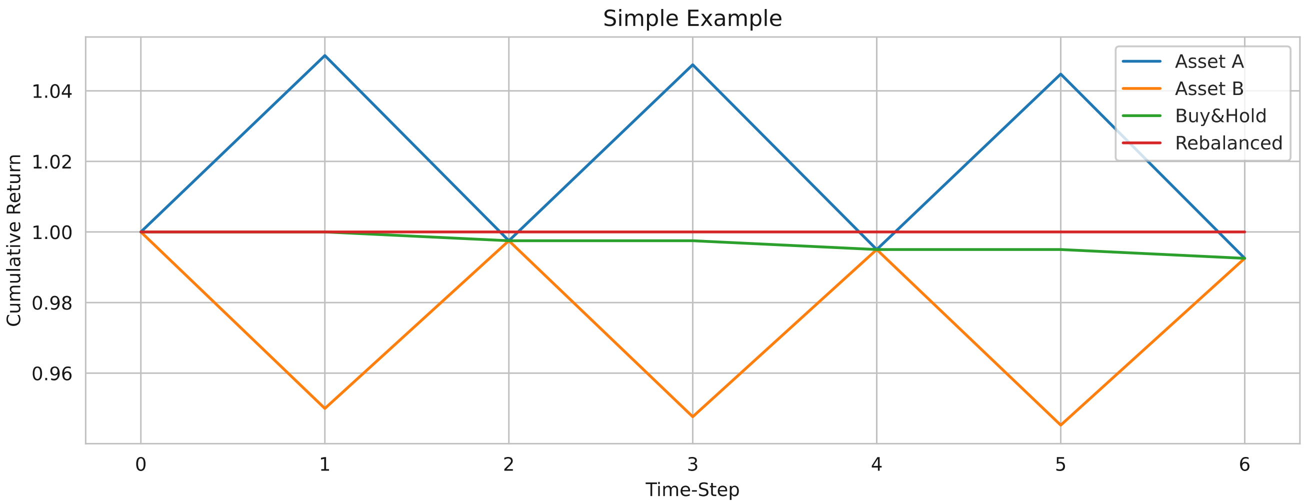

Figure 4: Simple example of cumulative returns. The returns on both Asset A and B switch between 1.05 and 0.95 so their arithmetic mean is 1.0, and their correlation is -1 so they move in opposite direction.

4. Daily Returns

Let us now use real-world data for stocks in USA. We are using a data-set that contains daily stock-data from the year 2007 until early 2021. Figure 5 shows the number of stocks available in this data-set for each day. In the year 2007 there was around 820 stocks available and this number gradually increases until the year 2010 where there was around 920 stocks available until the year 2021.

These stocks come from a larger data-set with around 2600 stocks, where we have excluded nearly 1700 stocks that were either too small or illiquid, had daily returns above 100%, or had too little price-data available. Section 12.5 has a full list of all the stock-tickers in this data-set.

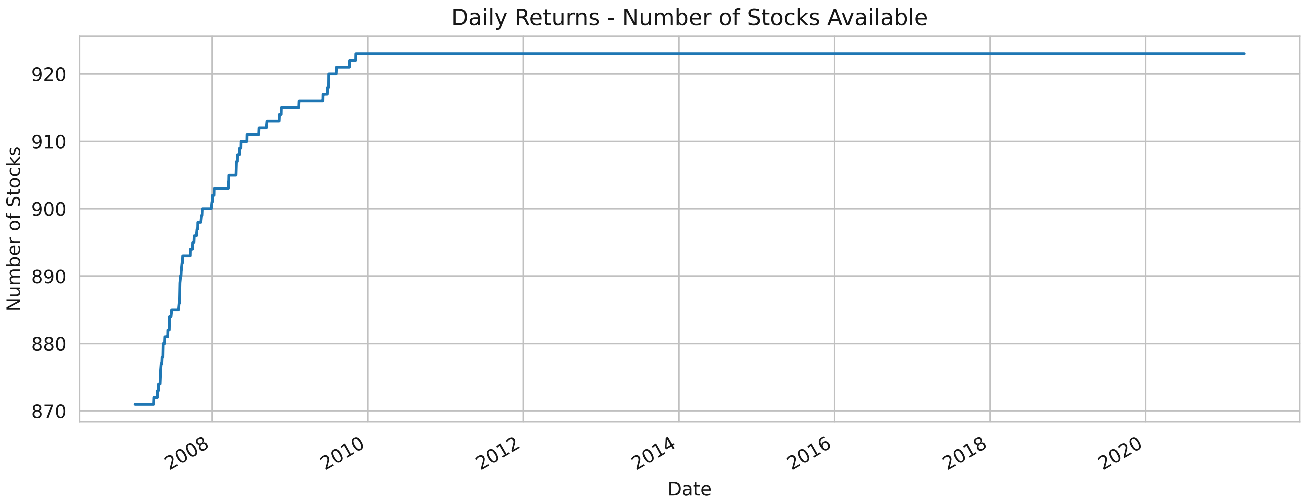

Figure 5: Number of U.S. stocks in the data-set with daily stock-returns. From the year 2007 there were around 870 stocks and this gradually increased until the year 2010 where there were around 920 stocks until the year 2021.

Figure 6 shows the cumulative returns for all the stocks in this data-set. These are basically just the Total Returns of the individual stocks that have been adjusted for both stock-splits and dividends, so the dividends are assumed to have been reinvested immediately without having to pay any taxes. They have also been normalized to all begin at 1.0 so they can easily be compared with each other.

The y-axis in Figure 6 is logarithmic, because many of the stock-prices have had nearly exponential growth in this 14-year period. Note that only a few of the blue lines approach zero, so those stocks have become nearly worthless. One problem with this data-set is that it may not contain enough stocks for companies that have gone bankrupt, so our studies may have so-called “survivorship bias”.

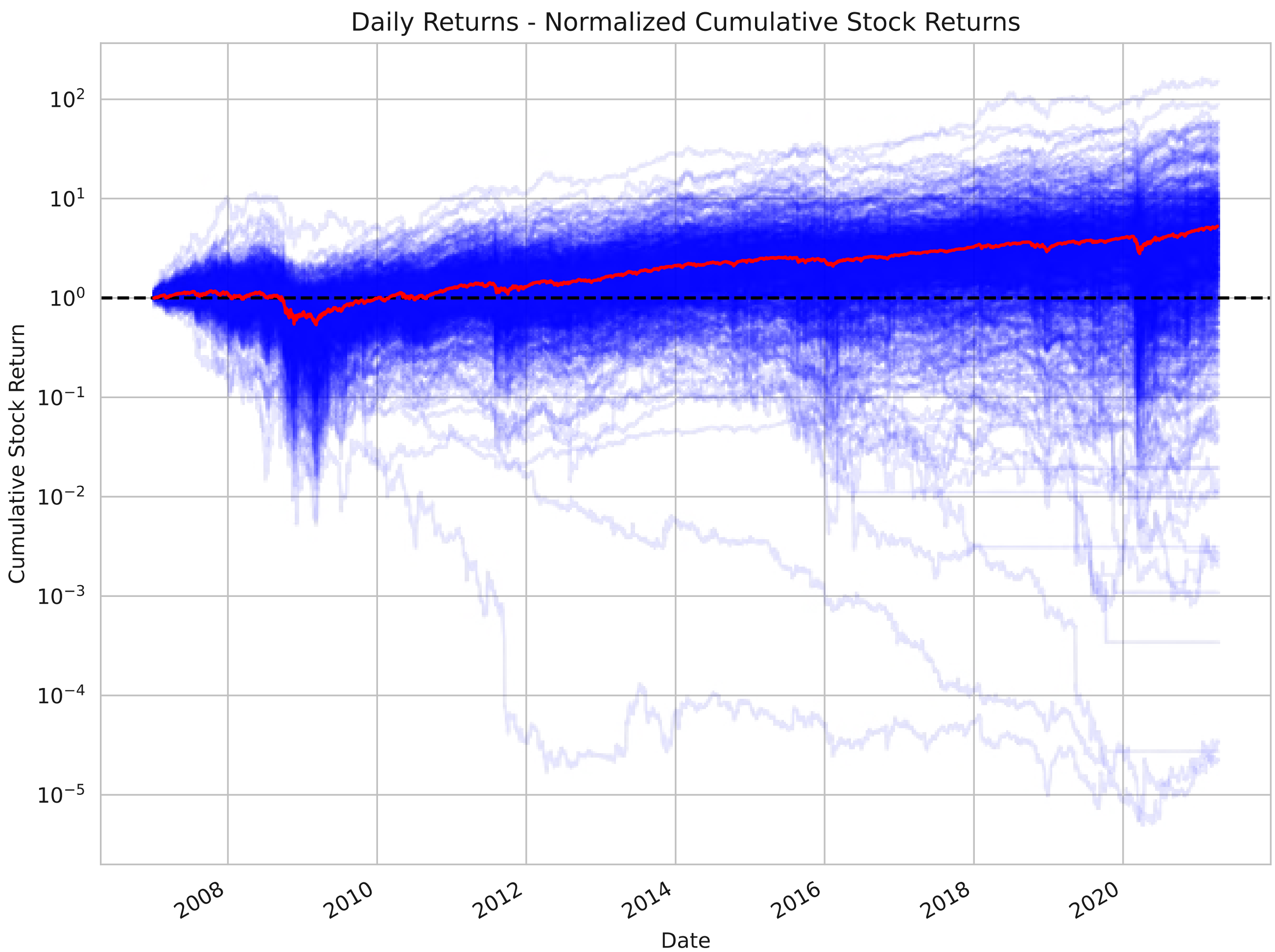

Figure 6: Cumulative returns for all the U.S. stocks in the data-set with daily returns. These are basically just the Total Returns for all the individual stocks plotted as blue lines, which have been normalized to all start at the value 1 so they can easily be compared. The red line is the arithmetic mean for each date. The y-axis is logarithmic.

4.1 Daily Returns – Random Portfolios

When comparing Rebalanced and Buy&Hold portfolios, we ought to compare all possible combinations of stocks and their start/end-dates, because these choices may have an impact on which kind of portfolio performs best. But there is a gigantic number of possible combinations so we will instead do random sampling of the stocks and their start/end-dates.

The top-plot in Figure 7 shows the ratio between the Rebalanced and Buy&Hold portfolios. There are 1000 different portfolios, where each portfolio consists of 5 random stocks and random start/end-dates. The bottom plot in Figure 7 shows a histogram with the distribution of the investment duration for these random portfolios, where it can be seen that there are many more portfolios with short duration.

Because this data-set has daily data-points and there are roughly 252 trading-days per year on average, a portfolio having e.g. 2500 investment periods means that its duration is roughly 10 years. The distribution of investment durations is a result of the sampling function, which starts by uniformly sampling the start-date and then uniformly sampling the end-date from the remaining possible dates.

Figure 8 has plots similar to Figure 7 with the only difference being that the random portfolios now contain 50 stocks each. And Figure 9 has similar plots for random portfolios with 100 stocks each.

Because these plots all show the ratios between the Rebalanced and Buy&Hold portfolios, a ratio of 1.0 would mean that the two portfolios perform equally well. A ratio above 1.0 would mean that the Rebalanced portfolio was best, and a ratio below 1.0 would mean that the Buy&Hold portfolio was best. As can be seen from the plots in Figure 7, Figure 8 and Figure 9 there is no consistent advantage to either the Rebalanced or Buy&Hold portfolios, as their ratios are both above and below 1.0. Although for certain periods such as the end of 2015 and early 2020 where most stocks performed poorly, it seems that the Rebalanced portfolios performed worse than the Buy&Hold portfolios.

There is another important thing we can see from these plots: When the portfolios contain more stocks, the difference between the Rebalanced and Buy&Hold portfolios becomes smaller, as shown by their ratio getting closer to 1.0 for the larger portfolios with 50 and 100 stocks, compared to the portfolios with only 5 stocks. This is presumably because portfolios with a small number of randomly selected stocks may sometimes be particularly suited to either the Rebalanced or Buy&Hold strategies, but when there is a larger number of stocks in the portfolio, it is less likely that many of the stocks will benefit from either type of strategy, so we get a more moderate performance for the larger portfolios.

All of the portfolios we will test in this paper use equal weights for the stocks in the portfolio, except for Figure 10 which is similar to Figure 9 in that it shows 1000 random portfolios with 100 stocks each, but Figure 9 used equal stock-weights and Figure 10 uses random stock-weights that are different for each random portfolio. But the result appears to be the same for equal-weighted and random stock-weights, namely that there does not appear to be a consistent advantage to either the Rebalanced or Buy&Hold portfolios.

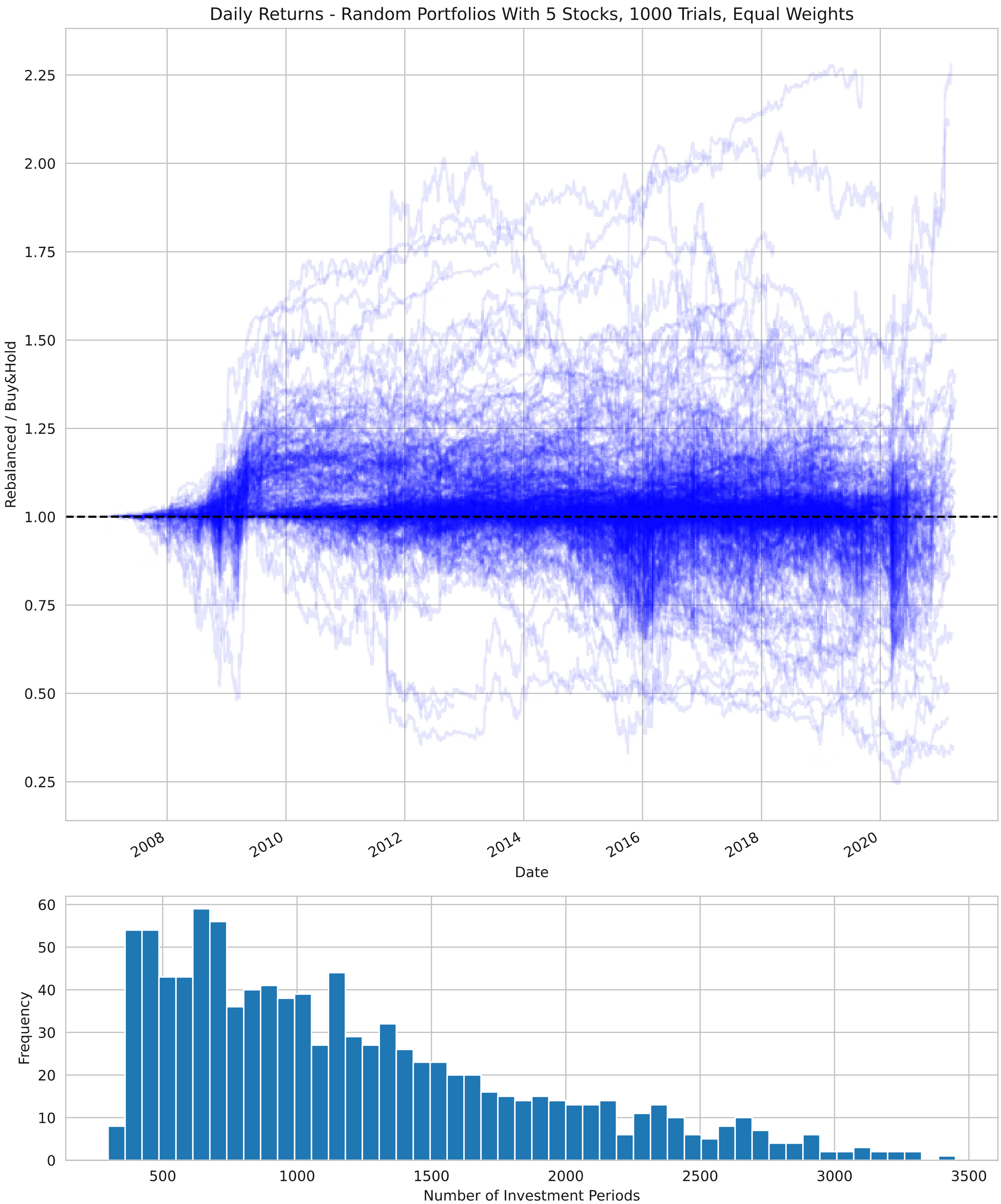

*Figure 7: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using **daily returns**. There are **1000 portfolios each with 5 random stocks and random start/end-dates**. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

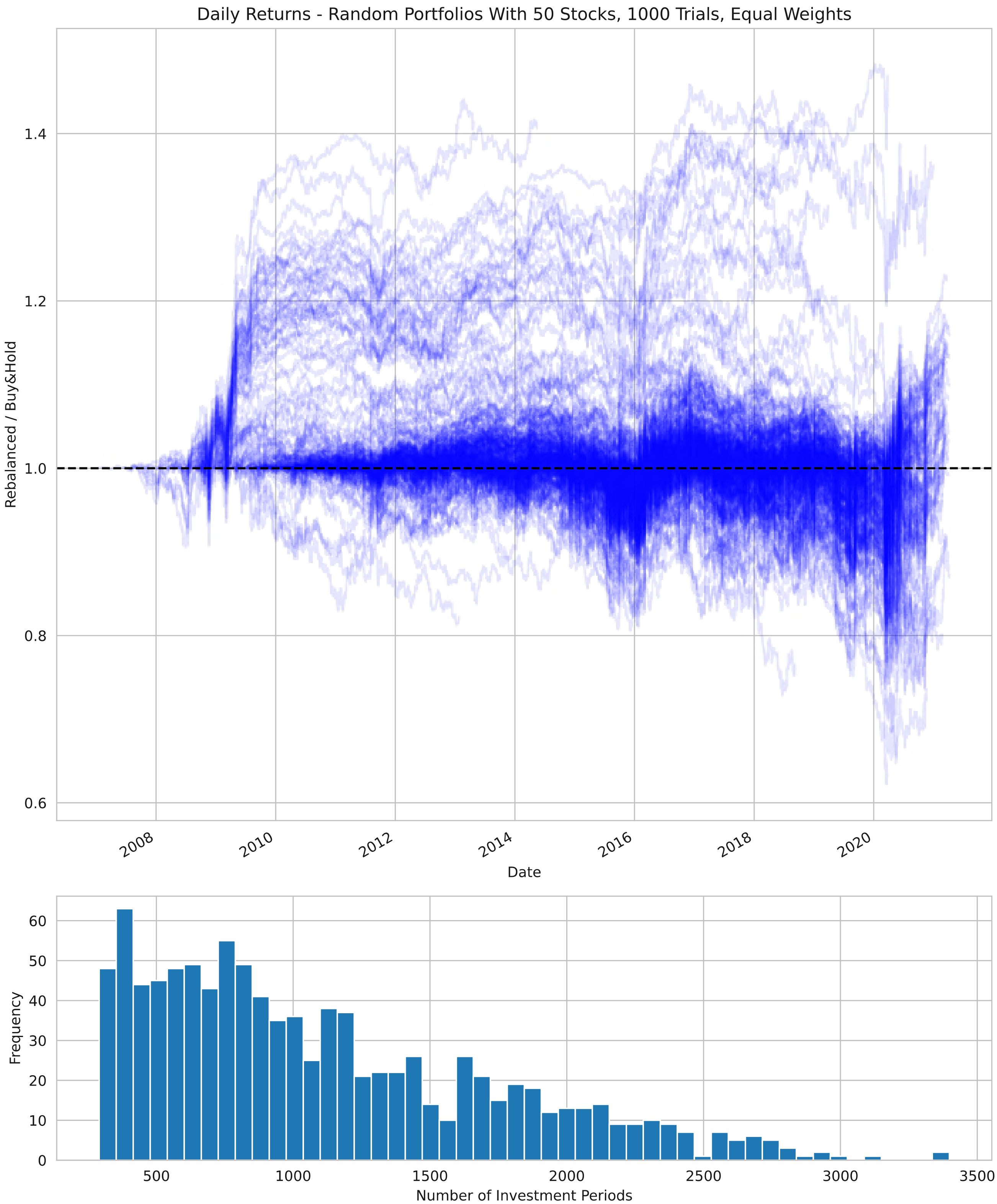

*Figure 8: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using **daily returns**. There are **1000 portfolios each with 50 random stocks and random start/end-dates**. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

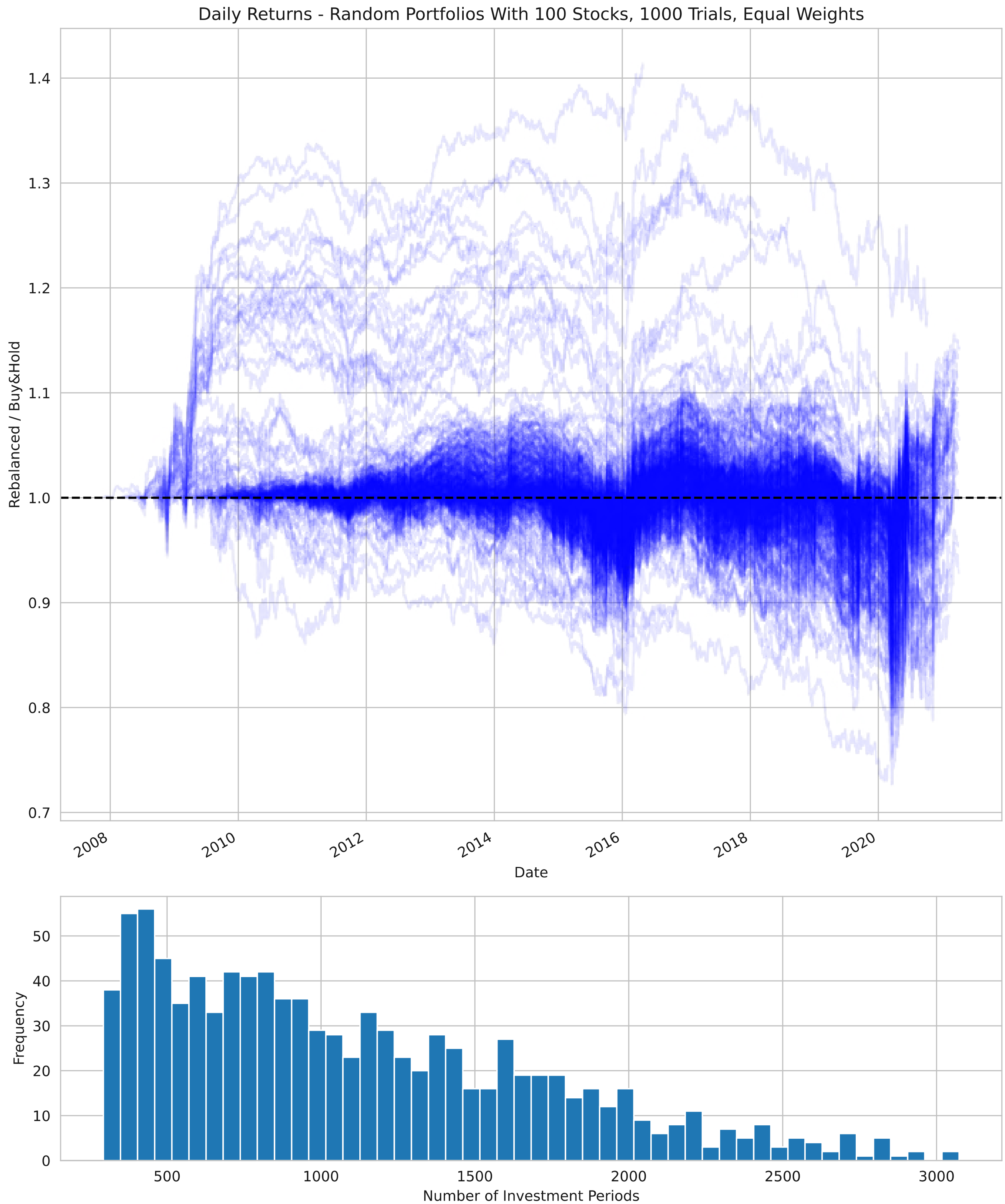

*Figure 9: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using **daily returns**. There are **1000 portfolios each with 100 random stocks and random start/end-dates**. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

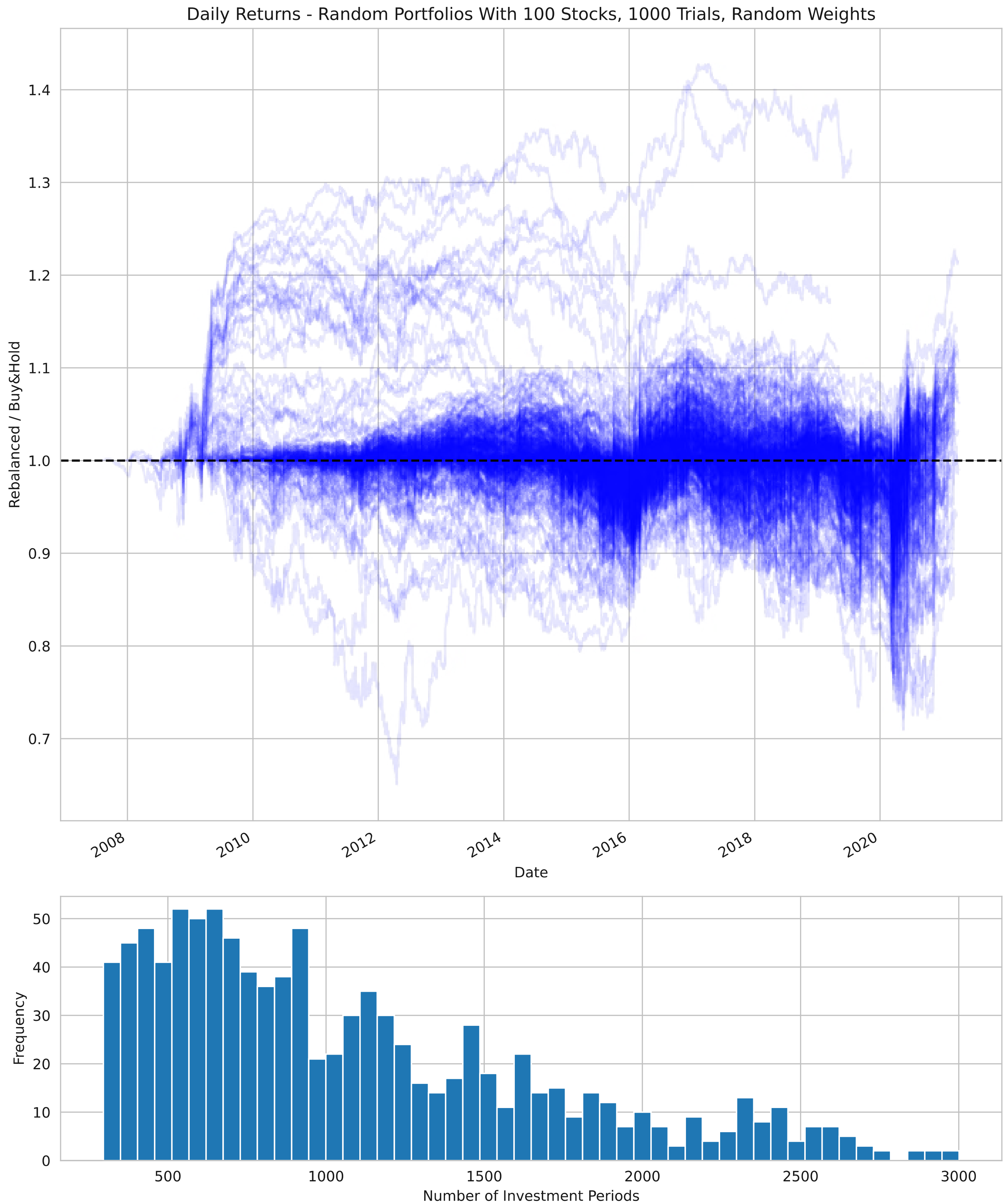

*Figure 10: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios. There are **1000 portfolios each with 100 random stocks and random start/end-dates**. The portfolios in this plot also use **random stock-weights** whereas the portfolios in Figure 9 use equal stock-weights.*

4.2 Daily Returns – Comparing Statistics

In the previous section we generated a large number of random portfolios and plotted the ratio between the Rebalanced and Buy&Hold portfolios. We then looked at the plots and were unable to see any consistent difference between the two types of portfolio strategies – except perhaps for a few particular periods such as the "Corona-Virus Panic" in early 2020, where most stocks crashed simultaneously and the portfolios that were rebalanced daily performed significantly worse than the Buy&Hold portfolios, which would seem to be consistent with the simple trending examples in Section 3.

In this section we will study the performance statistics defined in Section 2.3 and whether there is a significant difference between the Rebalanced and Buy&Hold portfolios regarding those statistics. All these plots are gathered towards the end of this section for easy comparison to each other.

Figure 11 shows the comparison of the arithmetic means for the Rebalanced and Buy&Hold portfolios. The plot has several components and may be a bit confusing, so let us explain it in some detail. The top plot is a so-called box-plot which is used to summarize distributions of numbers by showing their quartiles and outliers. In this case the box-plot is summarizing the distributions of arithmetic means for the Rebalanced and Buy&Hold portfolios. For example, the two left-most boxes show the distributions of arithmetic mean daily returns for 1000 random portfolios that contain 2 random stocks each, where the blue box shows it for the Rebalanced portfolios, and the orange box shows it for the Buy&Hold portfolios. So for each of these 1000 random portfolios, we calculate the arithmetic mean for the daily returns of both the Rebalanced and Buy&Hold strategies. This gives us 1000 data-points for each strategy. The box-plot then shows the distributions of these data-points. This is confusing because e.g. the centre-line in the box-plot then shows the *median of the arithmetic mean* of the daily returns for each strategy. You may have to read this explanation multiple times and think deeply about it, before it is clear what the box-plots show.

The white number written across the two boxes is the difference between the *means of the arithmetic mean* daily returns for the Rebalanced and Buy&Hold portfolios, or to be even more exact: The white number is the difference between the *arithmetic means of the arithmetic means* of the daily returns of the two portfolio strategies. This is probably *very* confusing if you are not a statistician! But the white number essentially shows whether the Rebalanced or Buy&Hold strategy was better on average. If the white number is positive then the Rebalanced portfolios had higher daily return on average, and vice versa, if the white number is negative then the Buy&Hold strategy had higher daily return on average.

Also note that the centre-line in the box-plot shows the *median* of the distributions, but the white number printed above the centre-line is actually for the difference in the *means* of the distributions. The reason for using the *means* in the white numbers instead of the *medians*, is that the p-values shown in the bottom plot in Figure 11 are calculated for the means and not the medians. The p-values are calculated using a so-called paired t-test between the Rebalanced and Buy&Hold portfolios. A p-value close to zero (e.g. below 0.01 or 1.0e-02) indicates that the difference in means between the

Rebalanced and Buy&Hold portfolios is statistically significant and probably not just a coincidence of random sampling.

In the bottom-plot of Figure 11 we see that the p-values are close to zero for portfolio sizes between 2 and 60 stocks, but for portfolios with 80 stocks and above, the p-value is so high that the very small differences shown as the white numbers in the top-plot are probably just a coincidence of random sampling. In other words, the p-values show us that for portfolios up to 60 stocks, there is on average a small advantage of the Rebalanced portfolio over the Buy&Hold portfolio. The white numbers show us that the average difference in daily returns between these two strategies is between 5.8e-06 and 1.8e-05 depending on portfolio size, which corresponds to an annualized excess return between roughly $250 \cdot 5.8\text{e-}06 \simeq 0.00145 = 0.145\%$ and $250 \cdot 1.8\text{e-}05 \simeq 0.0045 = 0.45\%$ when assuming there are 250 trading days in a year. This is a very small gain of not even 0.5% per year from daily rebalancing of the portfolio, and it also does not take trading costs and taxes into account, so it seems unlikely that there will be a net-positive effect of daily rebalancing the portfolio, at least when considering the arithmetic mean returns as the performance metric.

Now consider Figure 12 which is the same type of plot as Figure 11 except that it only considers the daily stock-returns between the years 2010 and 2021, while Figure 11 used the full data-period between 2007 and 2021. Most of the p-values in the bottom-plot of Figure 12 are now very high because the differences in arithemetic mean daily returns between the Rebalanced and Buy&Hold portfolios are so small that they are no longer statistically significant.

Figure 13 shows the same type of plot but for the years between 2007 and 2010. Now the p-values are very close to zero so the differences between the Rebalanced and Buy&Hold portfolios are statistically significant. For portfolios with 2 to 40 stocks the difference in arithmetic mean daily return is between 1.1e-04 and 1.7e-04, which corresponds to an average annualized excess return of the Rebalanced portfolios between 2.75% and 4.25%. As we will see in some of the other experiments below, it seems that the Rebalanced portfolios tend to perform significantly better than Buy&Hold portfolios when the stock-market is recovering from a crash, which might explain why the Rebalanced portfolios were so much better in this period between 2007 and 2010, which was the period of the big “Financial Crisis”.

So it very much depends on the period in time, whether the Rebalanced or Buy&Hold portfolios tend to perform better in terms of their arithmetic mean daily returns. Between 2007 and 2010 the Rebalanced portfolios were significantly better than Buy&Hold portfolios, but in the years between 2010 and 2021 there was no statistically significant difference between the two portfolio strategies.

The next performance metric we consider is the geometric mean of the daily returns which is compared for the two portfolio strategies in Figure 14. Let us first look at the p-values in the bottom-plot. For portfolios with 2, 40, 60, and 80 stocks the p-values are so high that the *means of the geometric means* are probably identical. For most of the other portfolio sizes the p-values are so close to the threshold of statistical significance (e.g. around 0.01 or 1.0e-02) that it is hard to say for certain whether the two portfolio strategies have a different performance on average, when using the geometric means of daily returns as the performance metric. Also note that some of the differences printed as white text are negative and some are positive, compared to the white texts in Figure 11 which were all positive. But even if the average differences between the two strategies were statistically significant, the differences are so small that there is probably no net-positive effect to the Rebalanced strategy when taking trading costs, taxes and the hassle of rebalancing into account. For example, for portfolios with 10 stocks the average difference between Rebalanced and Buy&Hold portfolios is 8.9e-06 which gives an annualized excess return for the Rebalanced portfolio over the Buy&Hold portfolio of only about $(1+8.9\text{e-}06)^{250}-1 \simeq 0.00223 = 0.223\%$ when assuming there is 250 trading days per year.[2] So when considering the geometric mean as the performance metric, there is little or no advantage to rebalancing the portfolio every day.

Figure 15 shows the comparisons of the next performance metric, which is the standard deviation of the daily returns for the two portfolio strategies. First look at the p-values in the bottom-plot, which are all very close to zero, so there is a statistically significant difference between the Rebalanced and Buy&Hold portfolios in regard to the standard deviations of their daily returns. Looking at the box-plot above, we see that the blue boxes are all higher than the orange boxes, and the white numbers are all positive, which means that the Rebalanced portfolios tend to have higher standard deviations for their daily returns than the Buy&Hold portfolios. This is interesting because while the literature cited in the introduction of this paper disagree strongly on whether there is an advantage to rebalancing in regards to the mean daily returns, the papers generally seem to agree that rebalancing lowers the volatility of the portfolio. But these empirical experiments show the opposite, namely that the Rebalanced portfolios are more volatile than the Buy&Hold portfolios, that is, the standard deviation of the portfolio's daily returns is generally higher for the Rebalanced portfolios than for the Buy&Hold portfolios. However, although the difference is statistically significant, it is actually very small and typically around 3e-04 for the larger portfolios and 5.9e-04 for portfolios with only 2 stocks.

The next performance metric is the Sharpe ratio which is compared in Figure 16. The p-values in the bottom-plot are very high for portfolios with 20 and 40 stocks so they can be ignored as statistically insignificant. For portfolios with 5 and 60 stocks the p-values are close to the threshold of statistical significance. For the other portfolio sizes the p-values are very close to zero and the differences between the Rebalanced and Buy&Hold portfolios are therefore statistically significant. The white text in the top-plot is sometimes positive and sometimes negative, so depending on the portfolio size there is sometimes an advantage for the Rebalanced portfolio and sometimes for the Buy&Hold portfolio. It may look like that the Sharpe ratio is higher for Rebalanced portfolios when the portfolio size is small, and vice versa, the Sharpe ratio is higher for Buy&Hold portfolios when the portfolio size is large. For portfolios with 20-60 stocks there is probably no difference in the Sharpe ratio. But even for the

2 Note that this is calculated differently from the examples in the previous paragraph, because we are using geometric means here but arithmetic means in the previous example. But for return-differences so close to zero, the two formulas give nearly identical results anyway.

portfolio sizes where the results are statistically significant, the difference is so small that it seems completely irrelevant in practice.

The next performance metric is the Max Drawdown which is compared in Figure 17. The p-values in the bottom-plot are all very close to zero so the differences between the Rebalanced and Buy&Hold portfolios are statistically significant. Looking at the distributions in the box-plot, we see that the Rebalanced portfolios generally perform worse than the Buy&Hold portfolios, that is, the Rebalanced portfolios have lower values for the Max Drawdown, so they went lower than the Buy&Hold portfolios during market crashes. The white texts are all negative, thus showing that the means for the Max Drawdowns were also lower for the Rebalanced portfolios. You may wonder why some of these data-points are very close to 0.0 so there were portfolios that hardly had any losses. This is because the start/end-dates are also chosen randomly when generating the random portfolios, so some of these will be for short time-periods where those particular stocks only decreased slightly, while other time-periods will include e.g. the market-crash in 2008-2009 and early 2020. If we look at the previous plots in Figure 7, Figure 8 and Figure 9 again, the Rebalanced portfolios often performed much worse than the Buy&Hold portfolios during the big market-crash in early 2020. The statistics in Figure 17 would seem to confirm this tendency, and perhaps we can learn a lesson here, that during a market-crash we should maybe not rebalance our portfolio so much.

But what about the effect of rebalancing when the stock-market recovers from a crash? This is considered in the next performance metric which is the Max Pullup that is compared in Figure 18. These are computed for a moving window of 1 year, because the stock-prices are generally increasing exponentially, so it would give very large numbers if considering the Max Pullup for investment periods of e.g. 10 years or more. The bottom-plot shows that the p-values are all close to zero, so the differences between the Rebalanced and Buy&Hold portfolios are all statistically significant. Because of the outliers, the y-axis of the box-plot has been limited to a max value of 2, so the plot only shows the most important data. The box-plot shows that the Rebalanced portfolios tend to have a higher Max Pullup than the Buy&Hold portfolios. This can also be seen from the white numbers which show the difference between the means of the Rebalanced and Buy&Hold portfolios, which are all positive so on average the Rebalanced portfolios performed better than the Buy&Hold portfolios in market recoveries and stock-rallies. The advantage of Rebalanced portfolios was largest at 10% (percentage points) for small portfolios with only 5 stocks, and the advantage became gradually smaller for larger portfolios, until it was only around 4% (percentage points) for portfolios with 100 stocks or more. This is interesting because it is counter-intuitive and in conflict with the simple examples in Section 3, where it seemed that the Rebalanced portfolio was worse than the Buy&Hold portfolio when some stocks were trending either up or down. So it seems that these two portfolio strategies behave in counter-intuitive ways when using real-world stock-data.

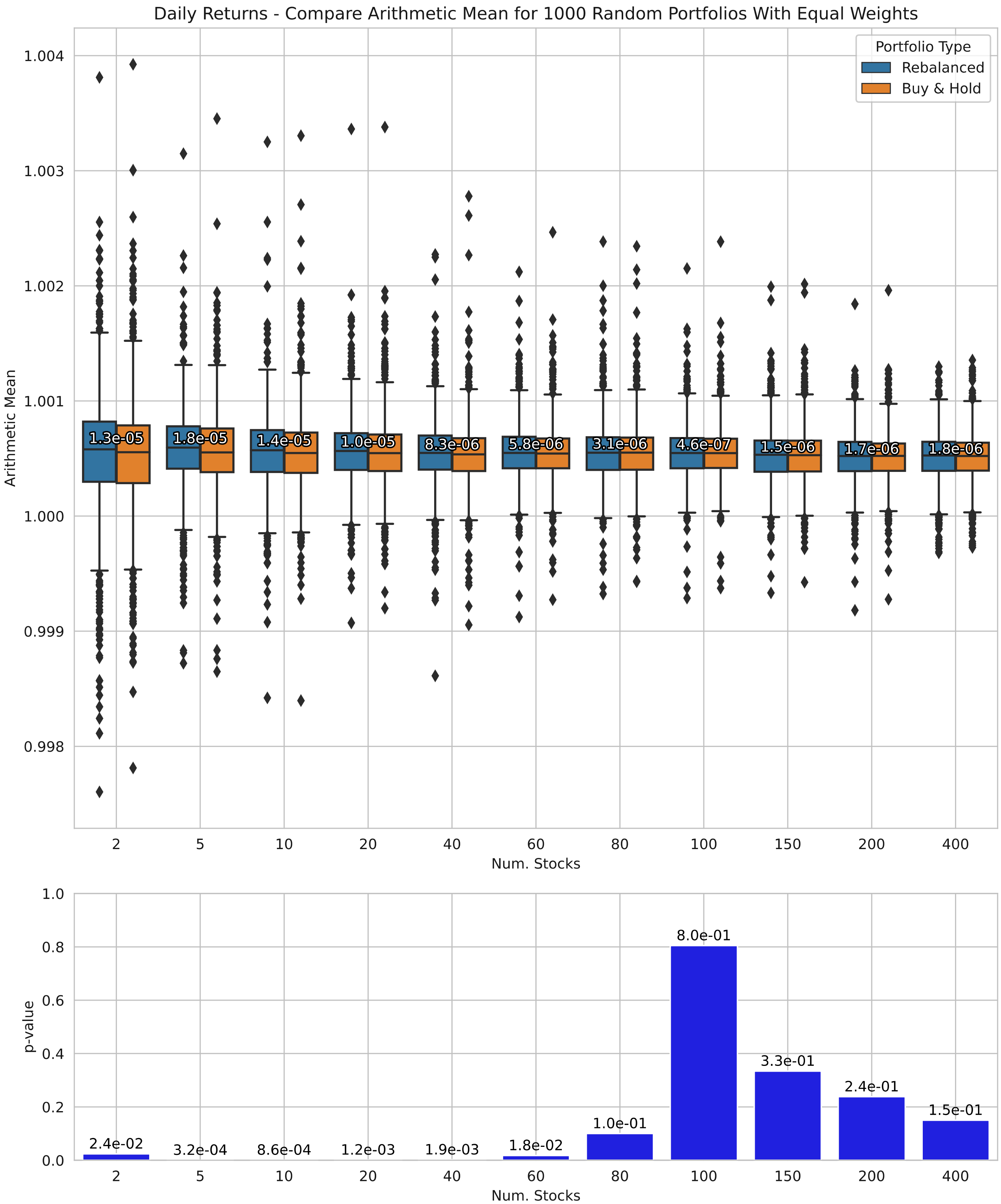

*Figure 11: Compare **arithmetic means of daily returns** for Rebalanced and Buy&Hold portfolios **between 2007 and 2021**.*

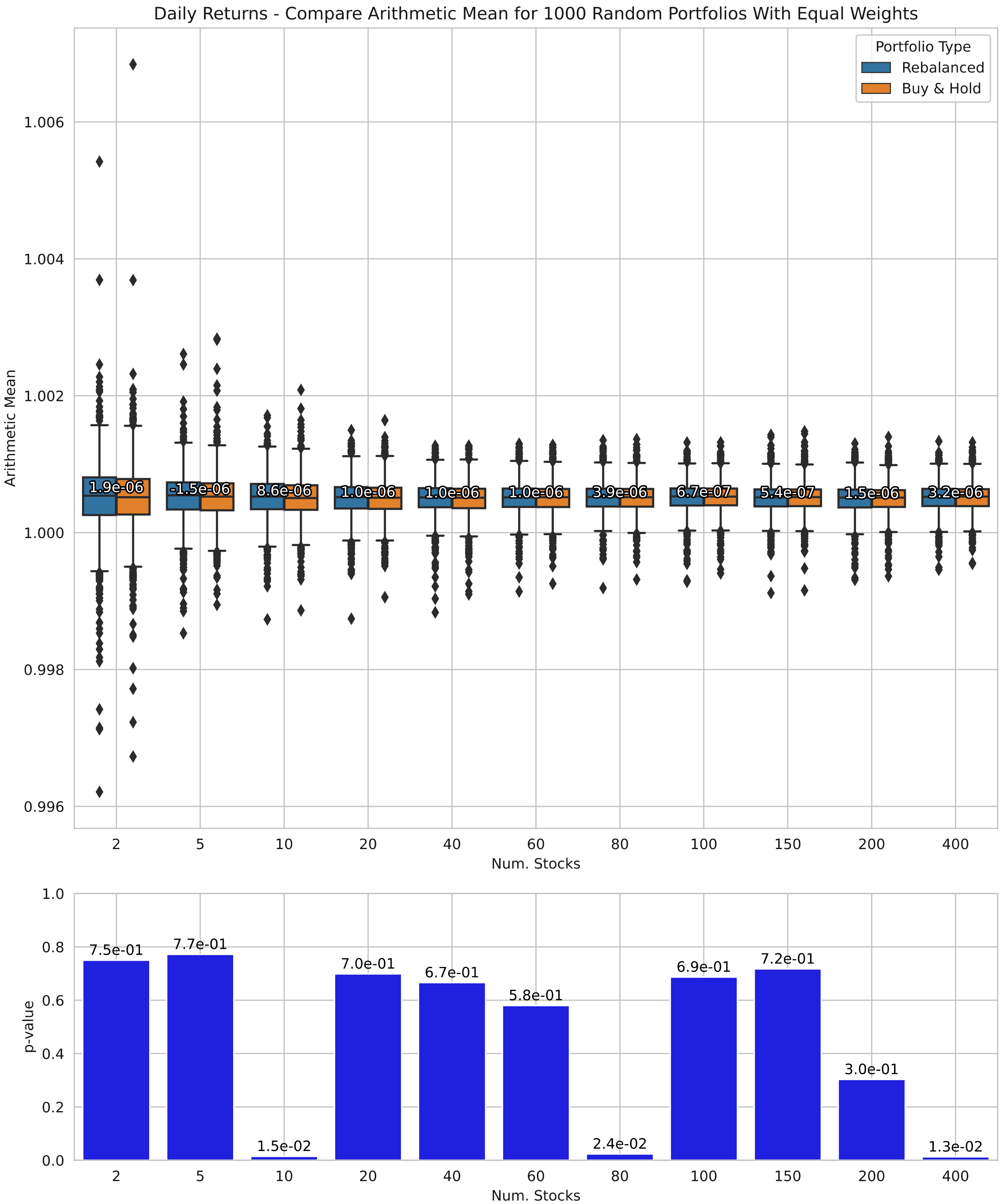

Figure 12: Compare ***arithmetic means of daily returns*** *for Rebalanced and Buy&Hold portfolios* ***between 2010 and 2021.*** *Compare this to Figure 11 for the years between 2007 and 2021.*

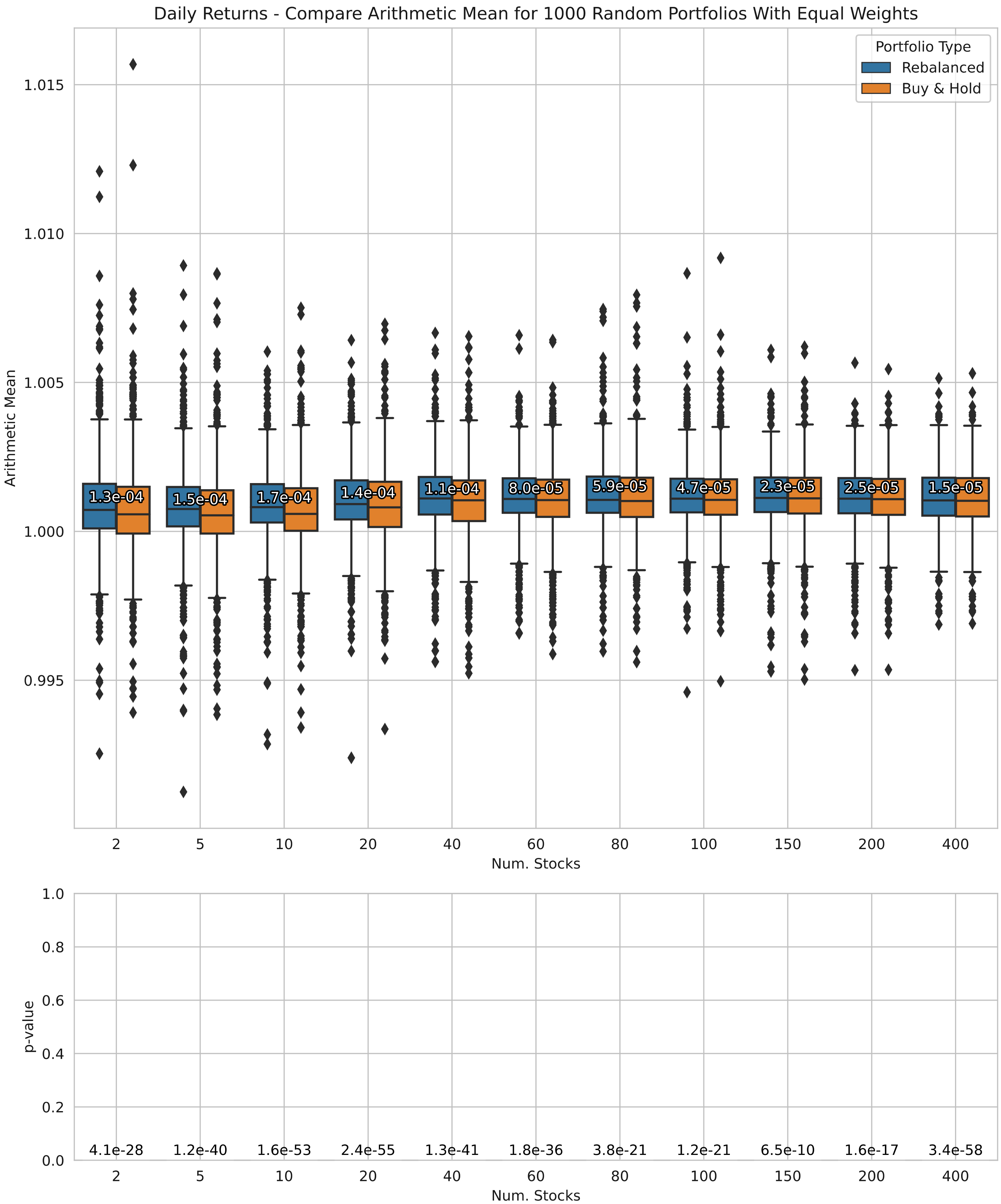

*Figure 13: Compare **arithmetic means of daily returns** for Rebalanced and Buy&Hold portfolios **between 2007 and 2010.** Compare this to Figure 11 for the years between 2007 and 2021.*

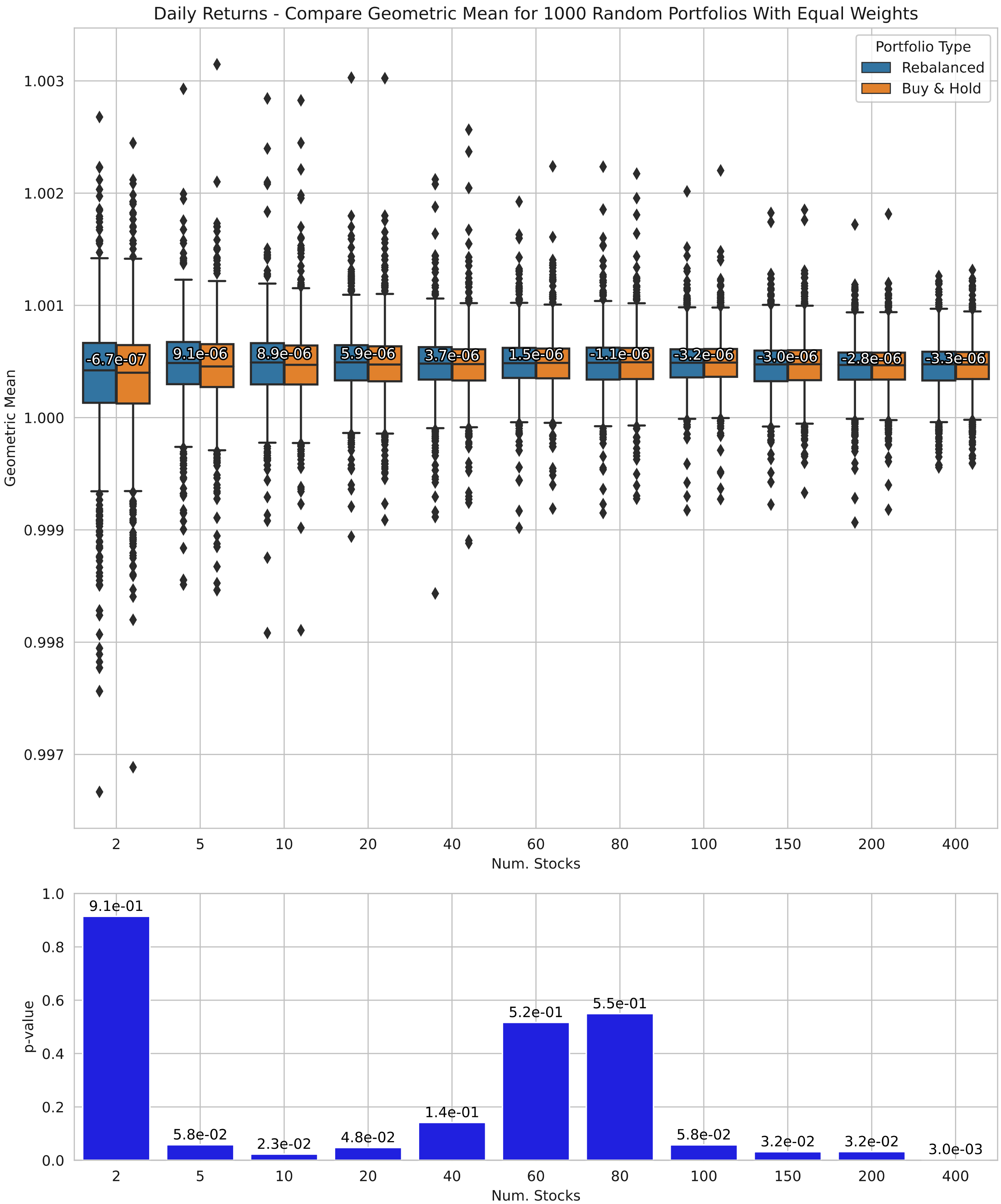

*Figure 14: Compare **geometric means of daily returns** for Rebalanced and Buy&Hold portfolios.*

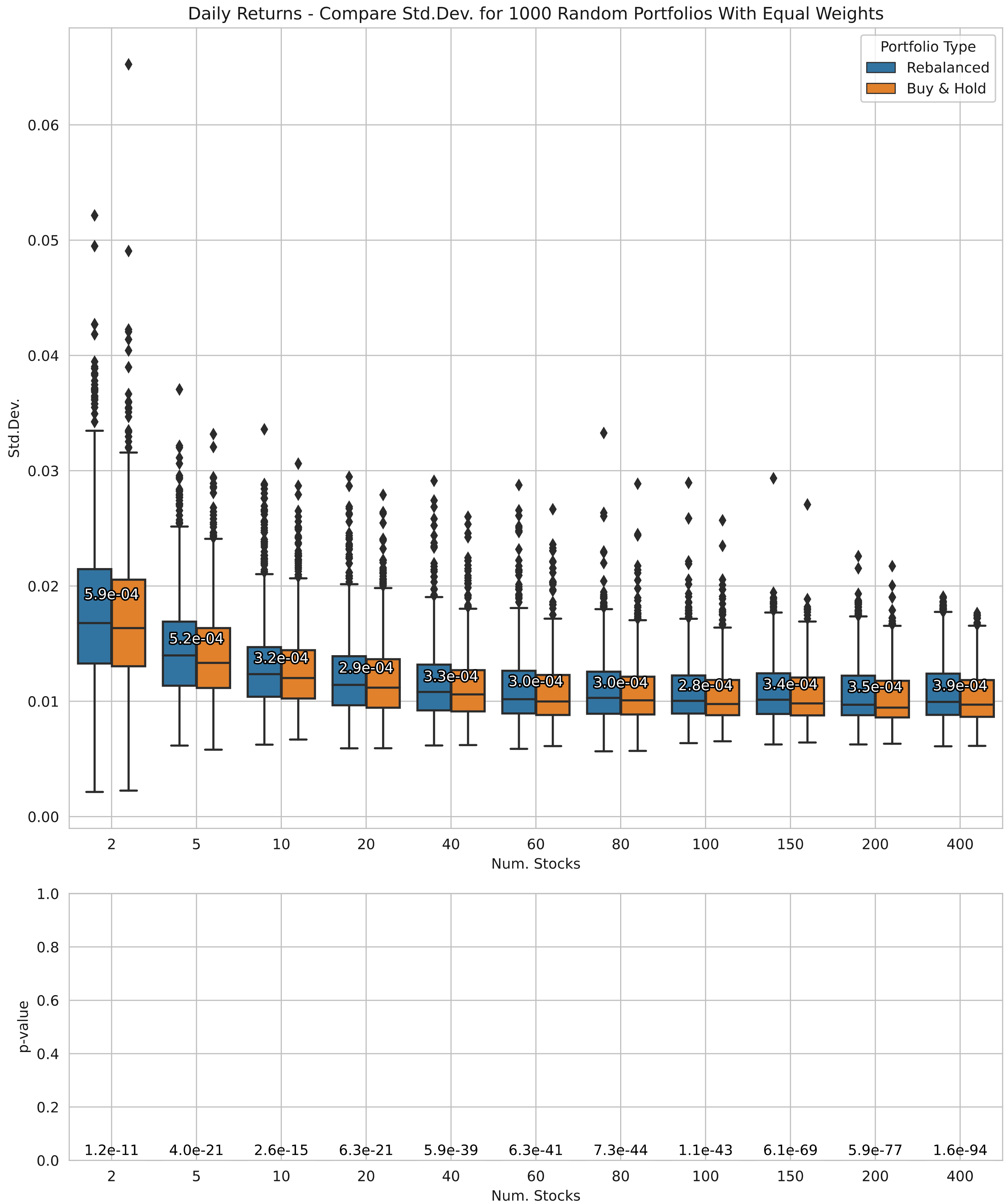

Figure 15: Compare ***standard deviations of daily returns*** *for Rebalanced and Buy&Hold portfolios.*

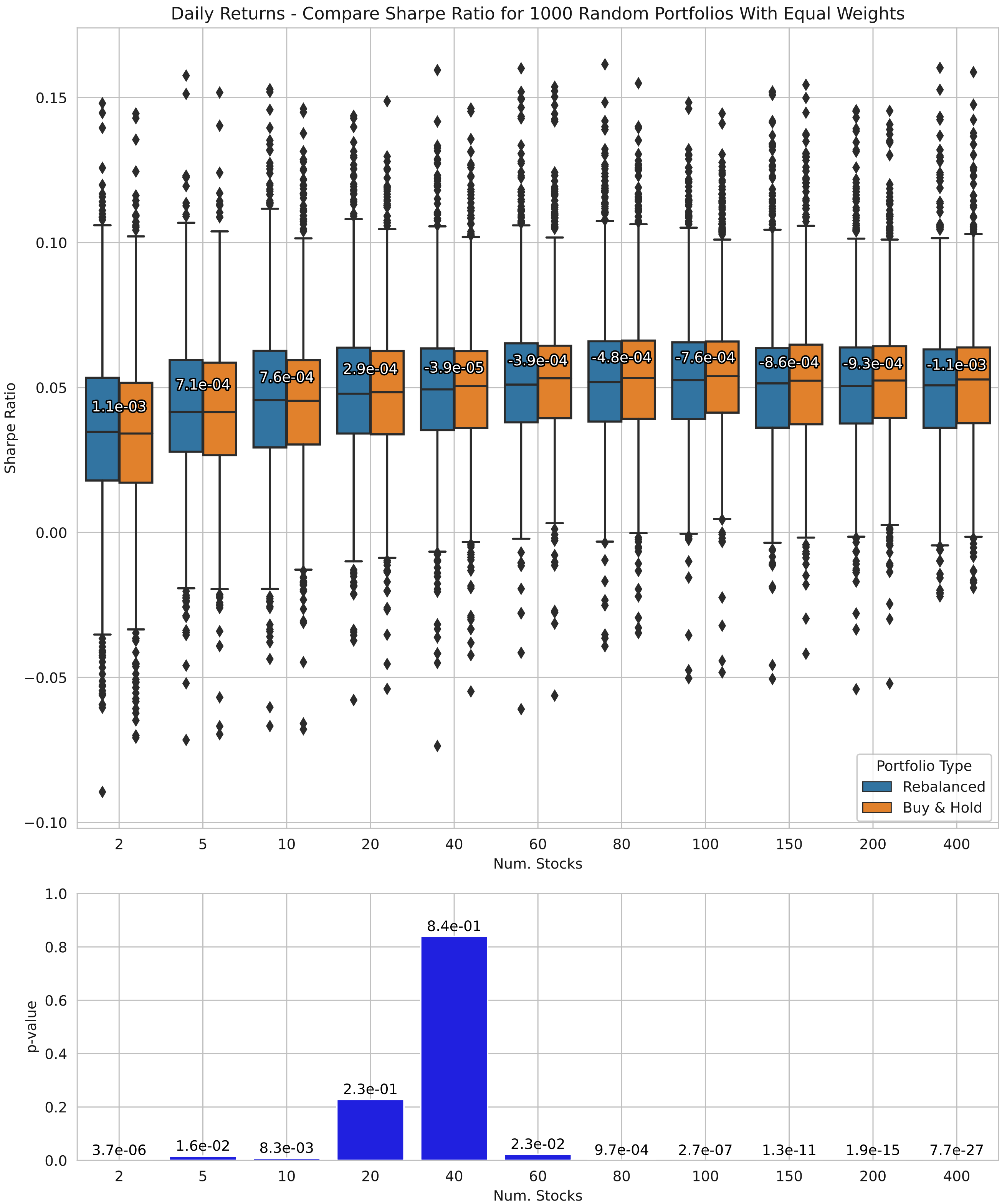

*Figure 16: Compare **Sharpe ratios of daily returns** for Rebalanced and Buy&Hold portfolios.*

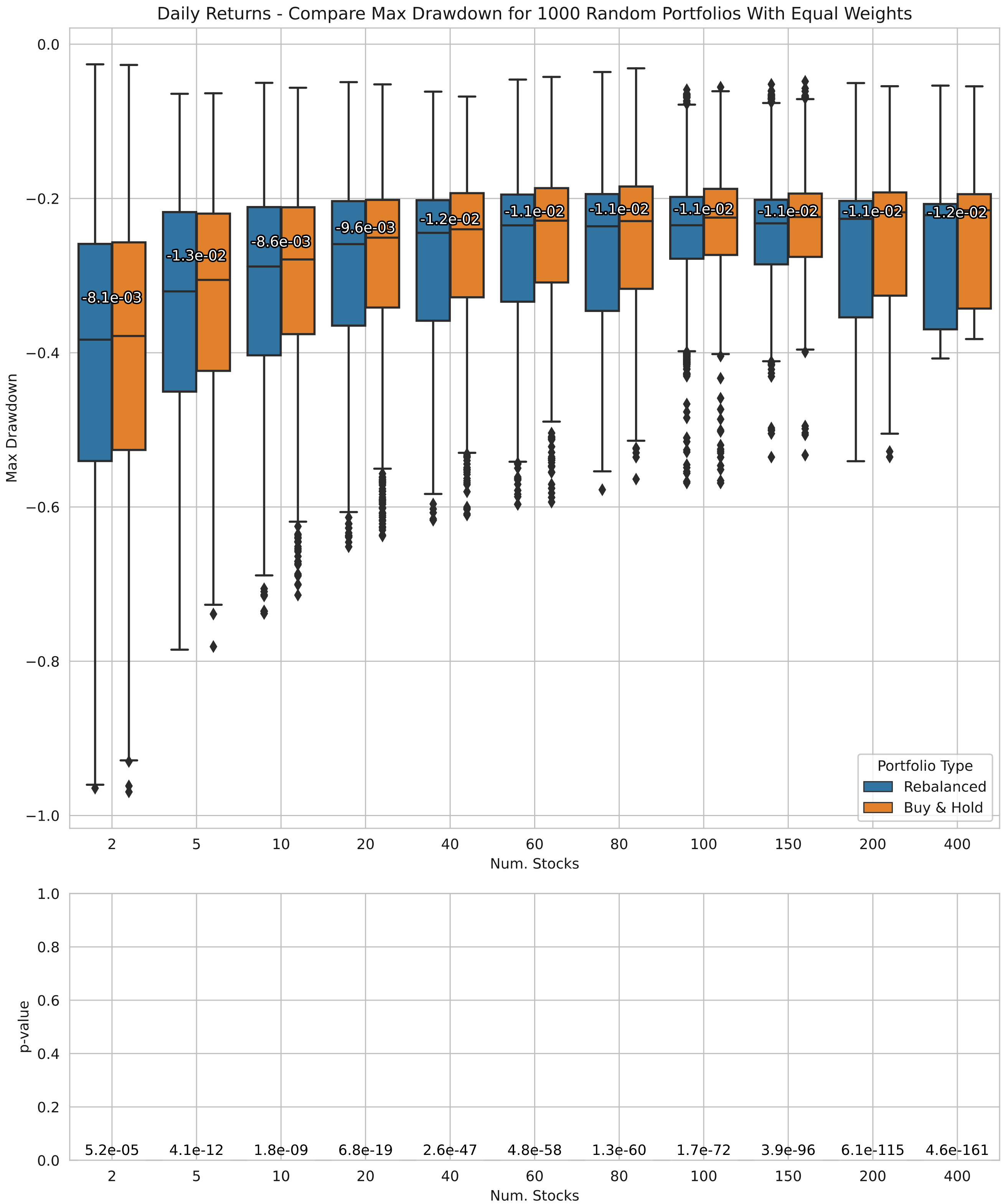

*Figure 17: Compare **Max Drawdowns of daily returns** for Rebalanced and Buy&Hold portfolios.*

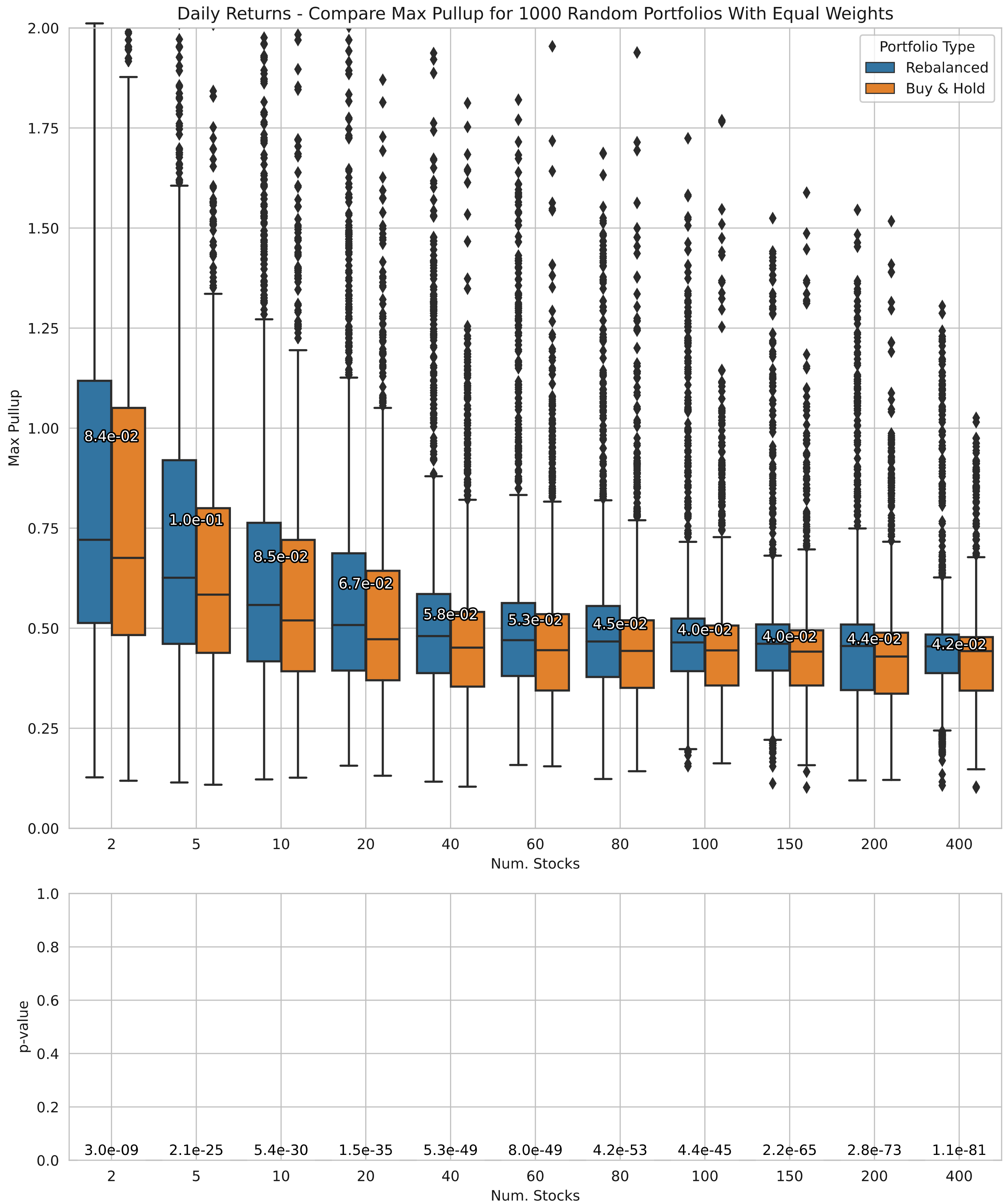

*Figure 18: Compare **Max Pullups of daily returns** for Rebalanced and Buy&Hold portfolios.*

5. Weekly Returns

In the previous section we compared Rebalanced and Buy&Hold portfolios when using daily returns, so the rebalancing back to equal stock-weights was done every single day. In this section we will study weekly rebalancing instead. The weekly stock-returns are merely resampled from the daily stock-returns. This means that we are considering weekly returns that start on a specific week-day such as Monday, and then all future rebalancing is done on Mondays as well. The computer code in Section 12 allows you to change the day of rebalancing by setting an offset in the resampling procedure. But it seems unlikely that it will make any difference on the results of this section.

5.1 Weekly Returns – Random Portfolios

Figure 19 shows the ratio between the Rebalanced and Buy&Hold portfolio for weekly stock-returns and 1000 portfolios consisting of 5 randomly selected stocks and random start/end-dates. A ratio of 1.0 means that the Rebalanced and Buy&Hold portfolios had identical returns, while a ratio above 1.0 means that the Rebalanced portfolio performed best, and vice versa, a ratio below 1.0 means that the Buy&Hold portfolio performed best. As can be seen from Figure 19, there does not seem to be a consistent advantage to either the Rebalanced or Buy&Hold portfolios.

Figure 20 shows the ratio between the two portfolio strategies for 1000 random portfolios containing 50 stocks, and Figure 21 shows it for random portfolios containing 100 stocks. The plots show that in both cases there is no consistent advantage to either portfolio strategy – however, there seems to be a few important exceptions.

Firstly, the portfolios that start near the bottom of the “Financial Crisis” in the year 2009 seem to benefit greatly from the weekly rebalancing, which seems to be consistent with the advantage of rebalancing on the Max Pullup statistic that we saw in the previous section, but there seems to be a much greater benefit to weekly instead of daily rebalancing, as can be seen by comparing Figure 20 and Figure 21 for weekly rebalancing to Figure 8 and Figure 9 for daily rebalancing.

Secondly, the Rebalanced portfolios seem to perform much worse than the Buy&Hold portfolios during market-crashes, which is especially visible in the “Corona-Virus Panic” in early 2020, and also in the mini-crash around the end of year 2015. In the big market-crash in 2008-2009 the Rebalanced portfolios do seem to slightly under-perform the Buy&Hold portfolios, but to a much smaller extent than in 2015 and 2020. It is unclear why there is such a difference.

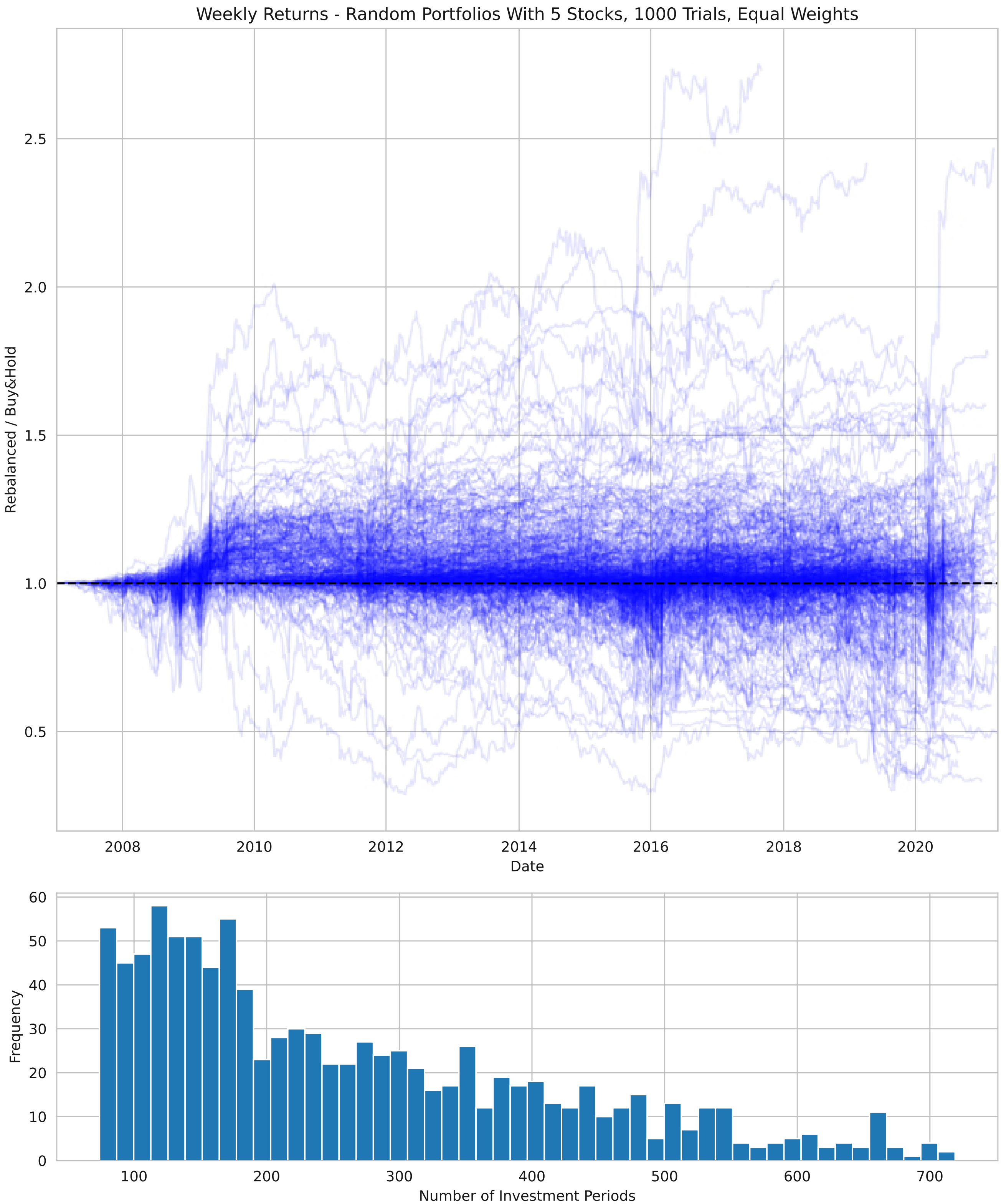

Figure 19: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using ***weekly returns****. There are* ***1000 portfolios each with 5 random stocks and random start/end-dates****. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

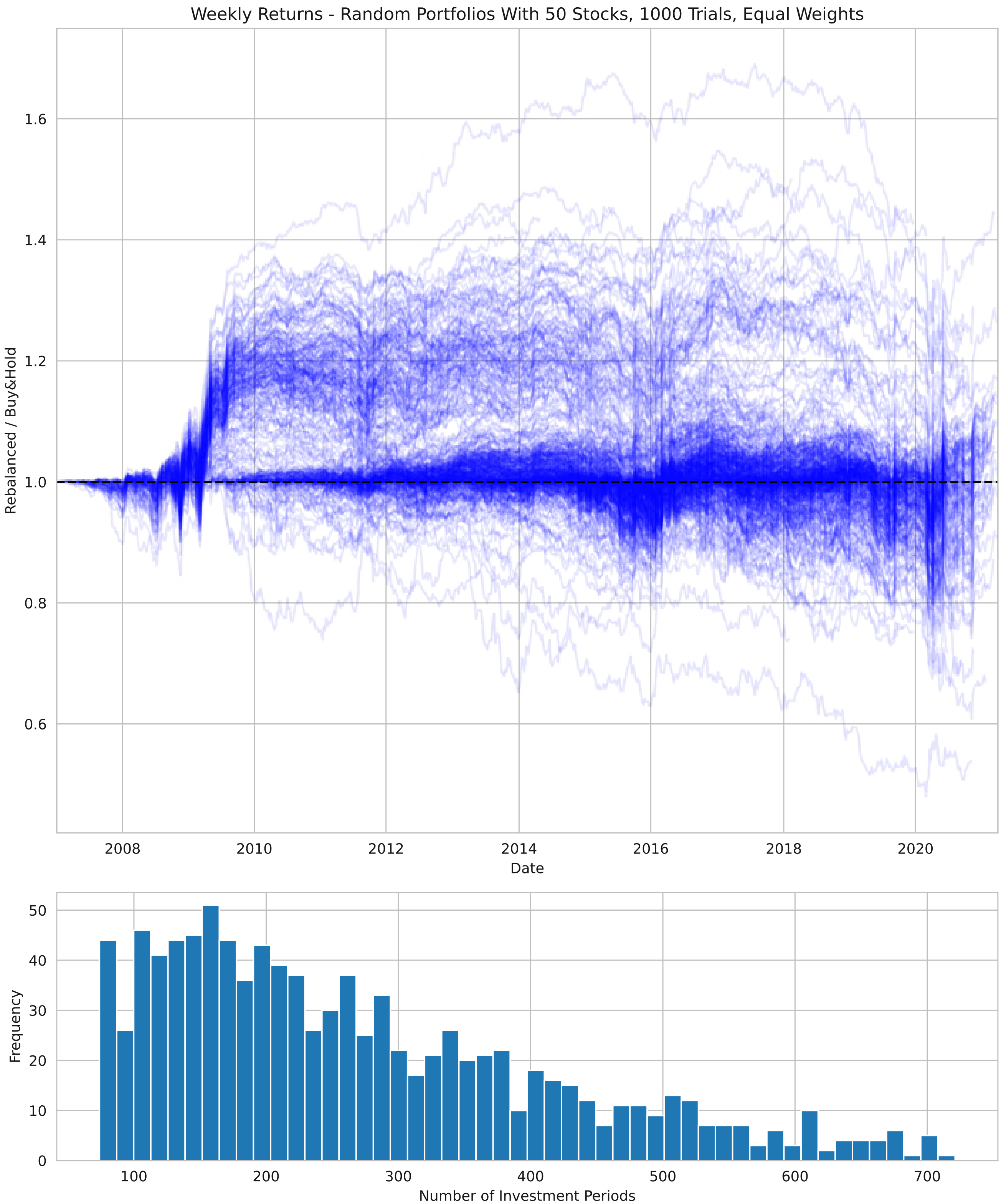

*Figure 20: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using **weekly returns**. There are **1000 portfolios each with 50 random stocks and random start/end-dates**. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

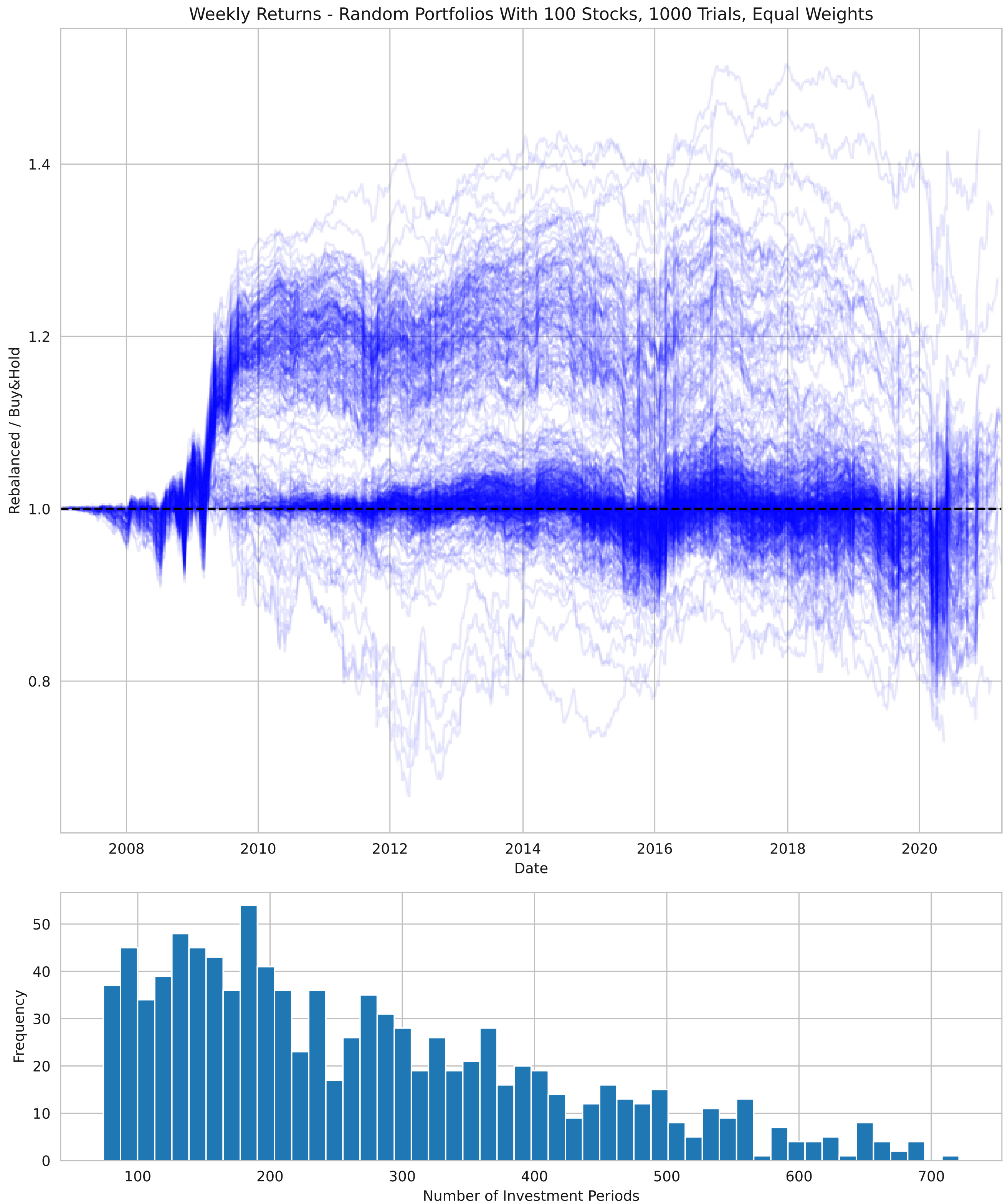

*Figure 21: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using **weekly returns**. There are **1000 portfolios each with 100 random stocks and random start/end-dates**. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

5.2 Weekly Returns – Comparing Statistics

Let us now compare the various performance statistics between the Rebalanced and Buy&Hold portfolios when using weekly stock-returns.

Figure 22 compares the arithmetic means for the weekly returns of the two portfolio strategies. The p-values in the bottom-plot are all close to zero so the average differences between the two portfolio strategies are statistically significant. The box-plot shows that the Rebalanced portfolios tend to have slightly higher arithmetic mean weekly returns than the Buy&Hold portfolios. The white texts show that depending on the portfolio size, the average difference is between 5.6e-05 and 1.1e-04, which translates into annualized excess returns for the Rebalanced portfolios between $52 \cdot 5.6\text{e-}05 \simeq 0.29\,\%$ and $52 \cdot 1.1\text{e-}04 \simeq 0.57\,\%$ because there are 52 weeks in a year. This is quite similar to the annualized excess returns for daily rebalancing as described in Section 4.2.

Figure 23 compares the geometric means for the weekly returns of the two portfolio strategies. The p-values in the bottom-plot show that for small portfolio sizes of 2, 5 and 10 stocks, the differences are so small for the two portfolio strategies, that they cannot be said to be statistically significant. For portfolio sizes of 20 stocks or more, the p-values are all close to zero so the differences are statistically significant. The white numbers in the box-plot show that the average difference between the geometric means for the two portfolio strategies is around 6e-05, which corresponds to an annualized excess return of the Rebalanced portfolios around $(1+6\text{e-}05)^{52}-1 \simeq 0.31\,\%$.

Figure 24 compares the standard deviations for the weekly returns of the two portfolio strategies. The p-values in the bottom-plot are all close to zero so the average differences between the two portfolio strategies are statistically significant. The box-plot shows that the standard deviations tend to be slightly larger for the Rebalanced portfolios compared to the Buy&Hold portfolios. The white texts show that the average difference is between 1.1e-03 and 1.7e-03 depending on portfolio size, which is a quite small difference when the standard deviations are typically between 2e-02 and 2e-03 for portfolios with 20 stocks or more.

Figure 25 compares the Sharpe ratio for the weekly returns of the two portfolio strategies. The p-values in the bottom-plot are very high for portfolios with 10, 20, 40 and 60 stocks. For portfolios with 80 stocks the p-value is around the threshold of what we consider statistically significant. For the other portfolio sizes the p-values are close to zero and the differences between the two portfolio strategies are therefore statistically significant. For small portfolio sizes of 2 and 5 stocks, the average Sharpe ratio is slightly higher for the Rebalanced portfolios, but for large portfolio sizes with 100 or more stocks, the Sharpe ratio is slightly lower for the Rebalanced portfolios. But the differences are so small that they seem to be irrelevant, even though they technically speaking are statistically significant.

Figure 26 compares the Max Drawdown for the weekly returns of the two portfolio strategies. The p-values in the bottom-plot are close to zero so the average differences between the two portfolio strategies are statistically significant. The box-plot shows that the Rebalanced portfolios tend to have lower and therefore worse Max Drawdowns than the Buy&Hold portfolios. The white texts show that the average difference is around 1% (percentage points), but the box-plot shows that the quartiles are often substantially worse for the Rebalanced portfolios. Comparing to the Max Drawdowns for daily returns in Figure 17 shows similar tendencies, but it is unclear whether daily or weekly returns tend to have worse Max Drawdowns, because the data in Figure 26 is calculated using weekly data-points so it is possible that the lowest daily share-prices occurred in-between two weekly data-points. So we cannot conclude whether the Max Drawdown was generally worse for daily or weekly rebalancing.

Figure 27 compares the Max Pullup for the weekly returns of the two portfolio strategies. The p-values in the bottom-plot are close to zero so the average differences between the two portfolio strategies are statistically significant. The box-plot shows that the weekly Rebalanced portfolios often performed substantially better than the Buy&Hold portfolios in market recoveries and stock-rallies. The white numbers show that the average difference ranged between 7.5% and 14% (percentage points) depending on the portfolio size. This is similar to our findings for daily returns in Figure 18, but the weekly rebalancing seems to be even more advantageous than daily rebalancing, although as noted above, we should probably not directly compare the daily and weekly Max Drawdown and Max Pullup because the lowest and highest stock-prices could have occurred in-between the weekly data-points.

Figure 28 is similar to Figure 27 in that it compares the Max Pullup for the weekly returns of the two portfolio strategies, but Figure 28 only uses the weekly returns between the years 2007 and 2010, which was the period of the big stock-market crash in the so-called “Financial Crisis”. The p-values are all close to zero, so the differences between the two portfolio strategies are statistically significant. For this period, the average difference in Max Pullup between the Rebalanced and Buy&Hold portfolios ranged between 19% and 24%, so there was a very large advantage to weekly rebalancing during the recovery-phase of the stock-market crash around year 2008-9.

Figure 29 also shows the Max Pullup for the weekly returns of the two portfolio strategies, but it only considers the weekly returns between the years 2010 and 2021. The p-values are close to zero, so the differences are statistically significant. But now the average difference between Rebalanced and Buy&Hold portfolios only ranges between 3.3% and 5.1%, when we are excluding the big “Financial Crisis” of 2008-9 from the data-set.

Altogether this indicates that the weekly Rebalanced portfolios often perform somewhat worse than a simple Buy&Hold strategy in the downwards-phase of a stock-market crash – but the Rebalanced portfolios often perform significantly better in the recovery-phase of a stock-market crash. This seems counter-intuitive when considering the simple examples in Section 3. It may also be very challenging to put into practice, because it may be very hard to predict when the stock-market crash has reached its bottom and starts its recovering-phase.

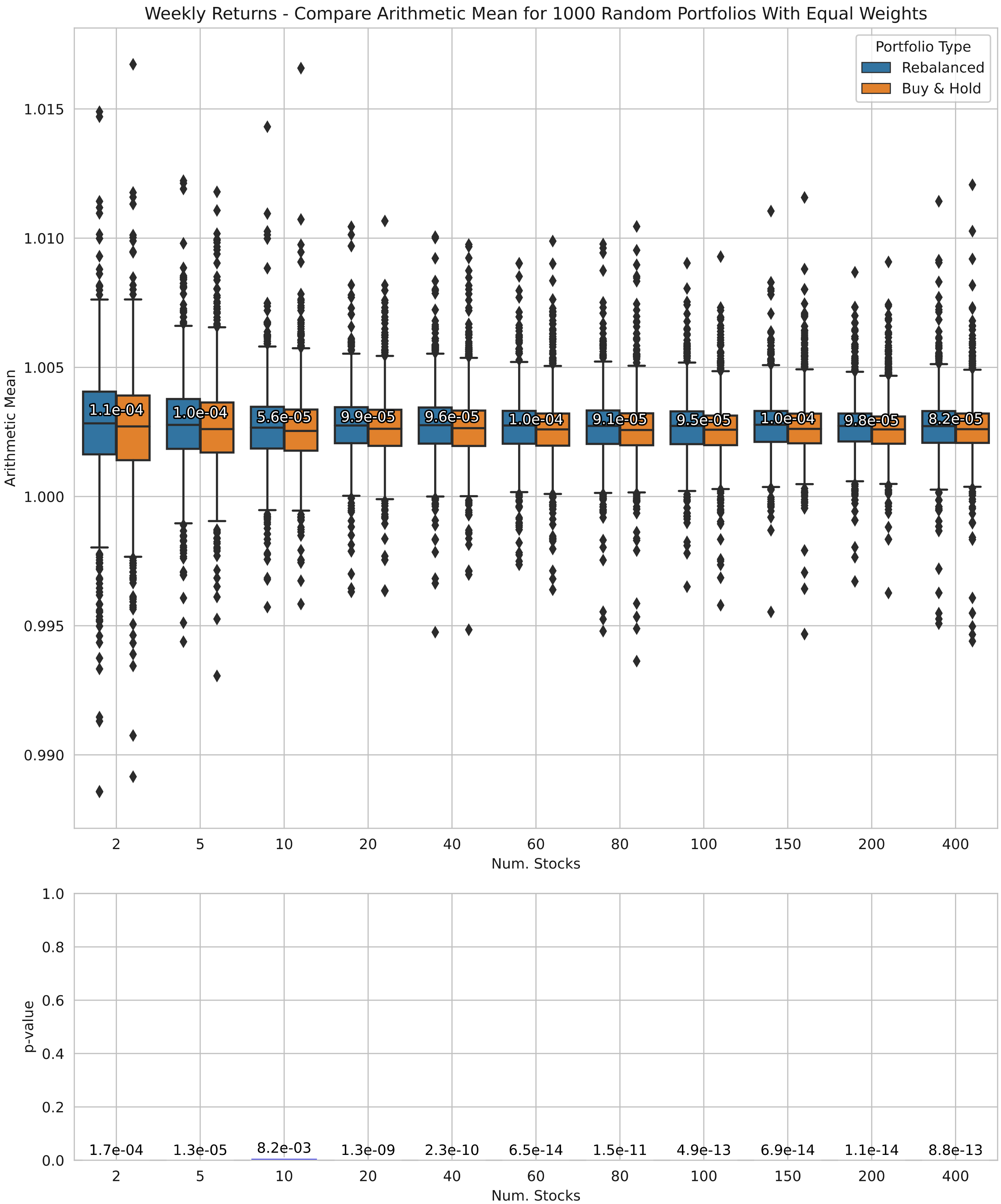

*Figure 22: Compare **arithmetic means of weekly returns** for Rebalanced and Buy&Hold portfolios.*

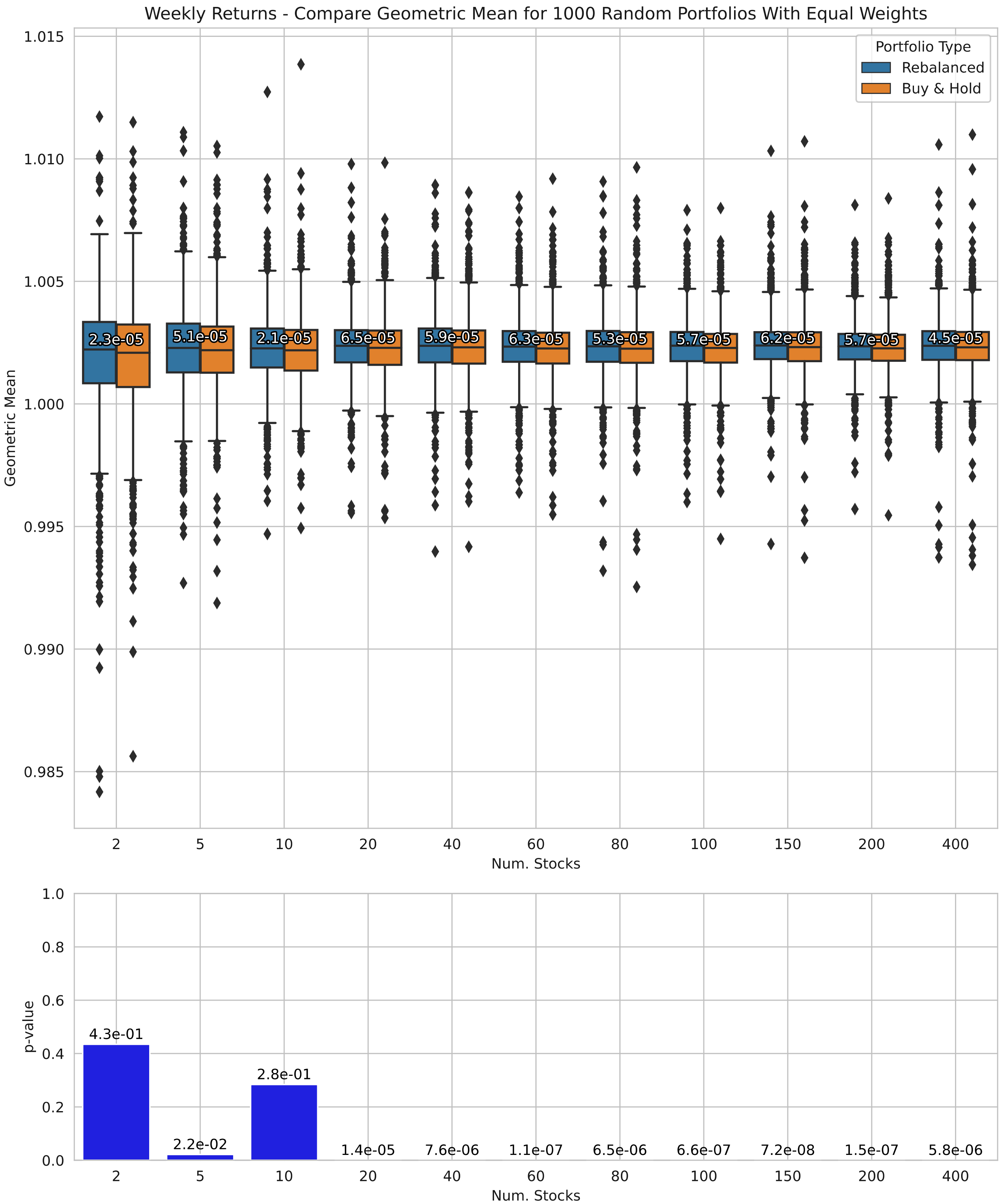

*Figure 23: Compare **geometric means of weekly returns** for Rebalanced and Buy&Hold portfolios.*

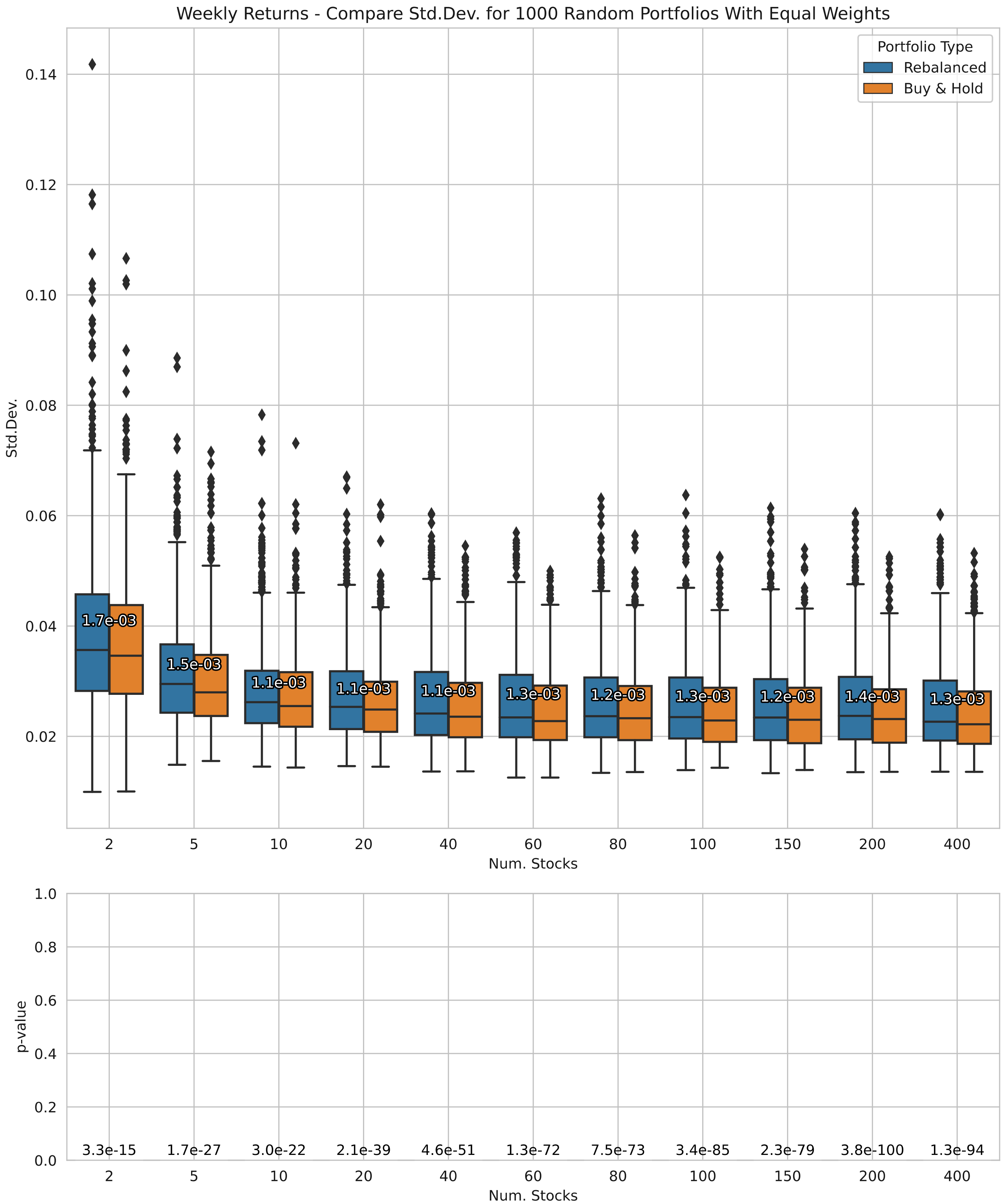

Figure 24: Compare ***standard deviations of weekly returns*** *for Rebalanced and Buy&Hold portfolios.*

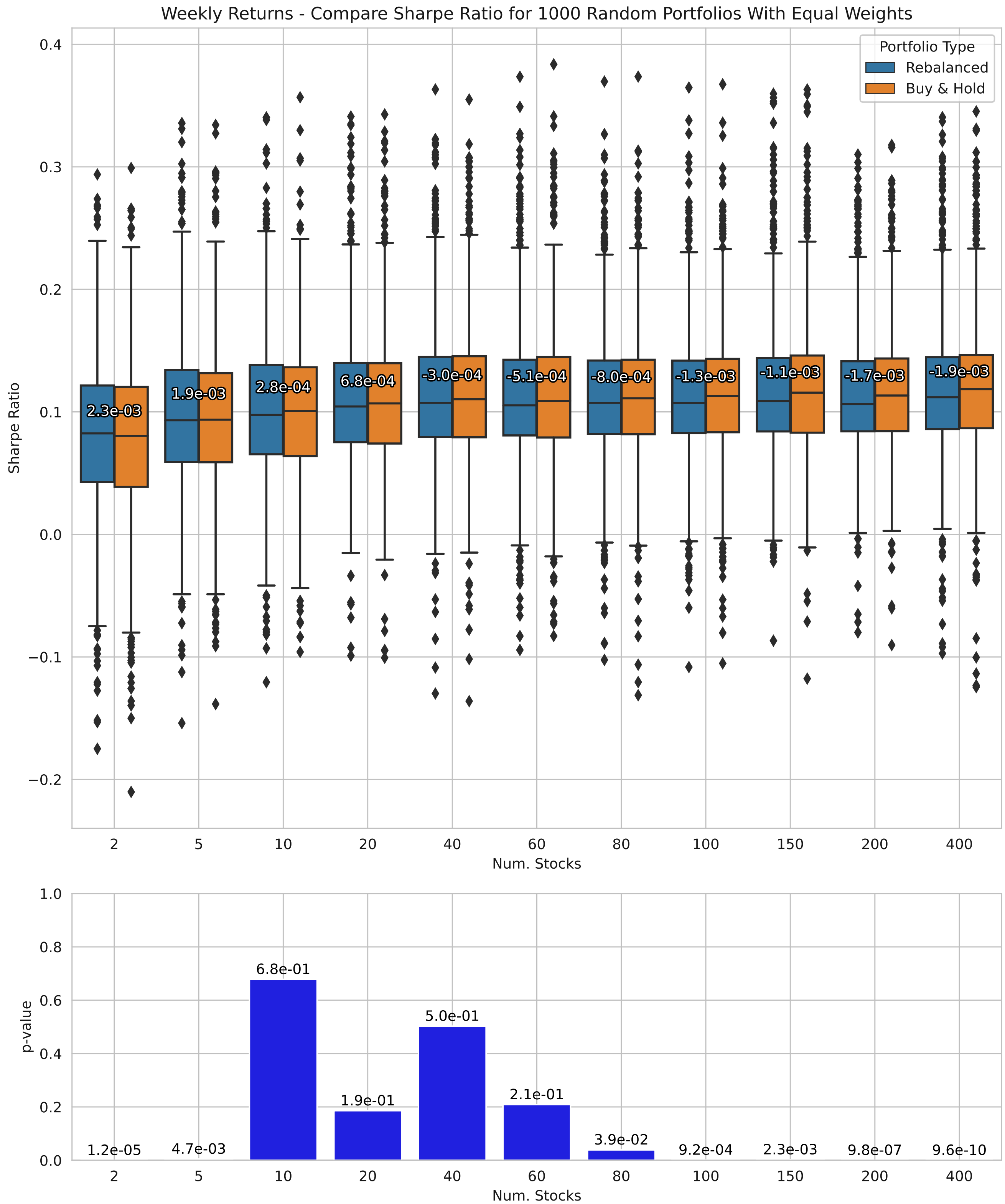

*Figure 25: Compare **Sharpe ratios of weekly returns** for Rebalanced and Buy&Hold portfolios.*

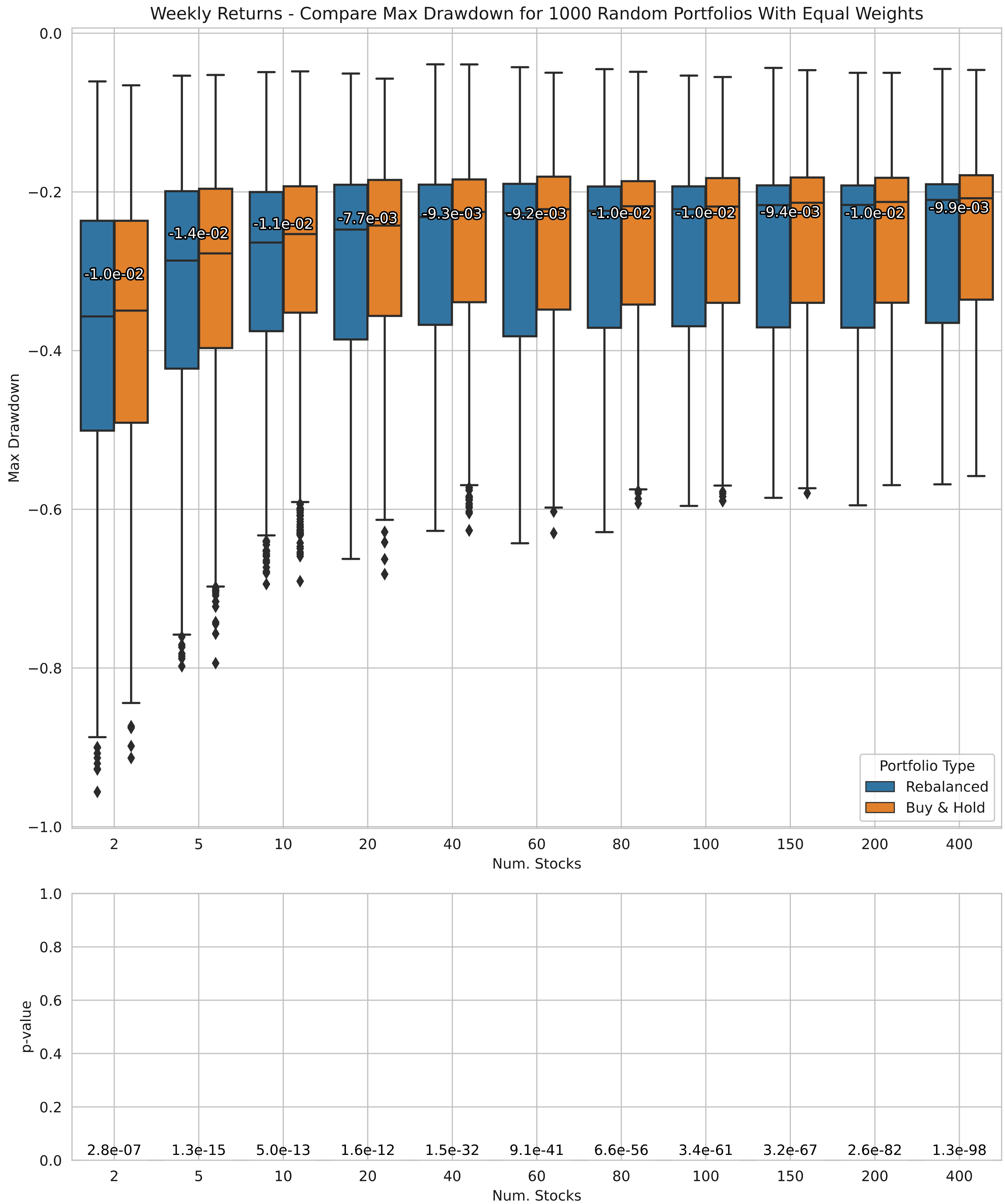

Figure 26: Compare ***Max Drawdown of weekly returns*** *for Rebalanced and Buy&Hold portfolios.*

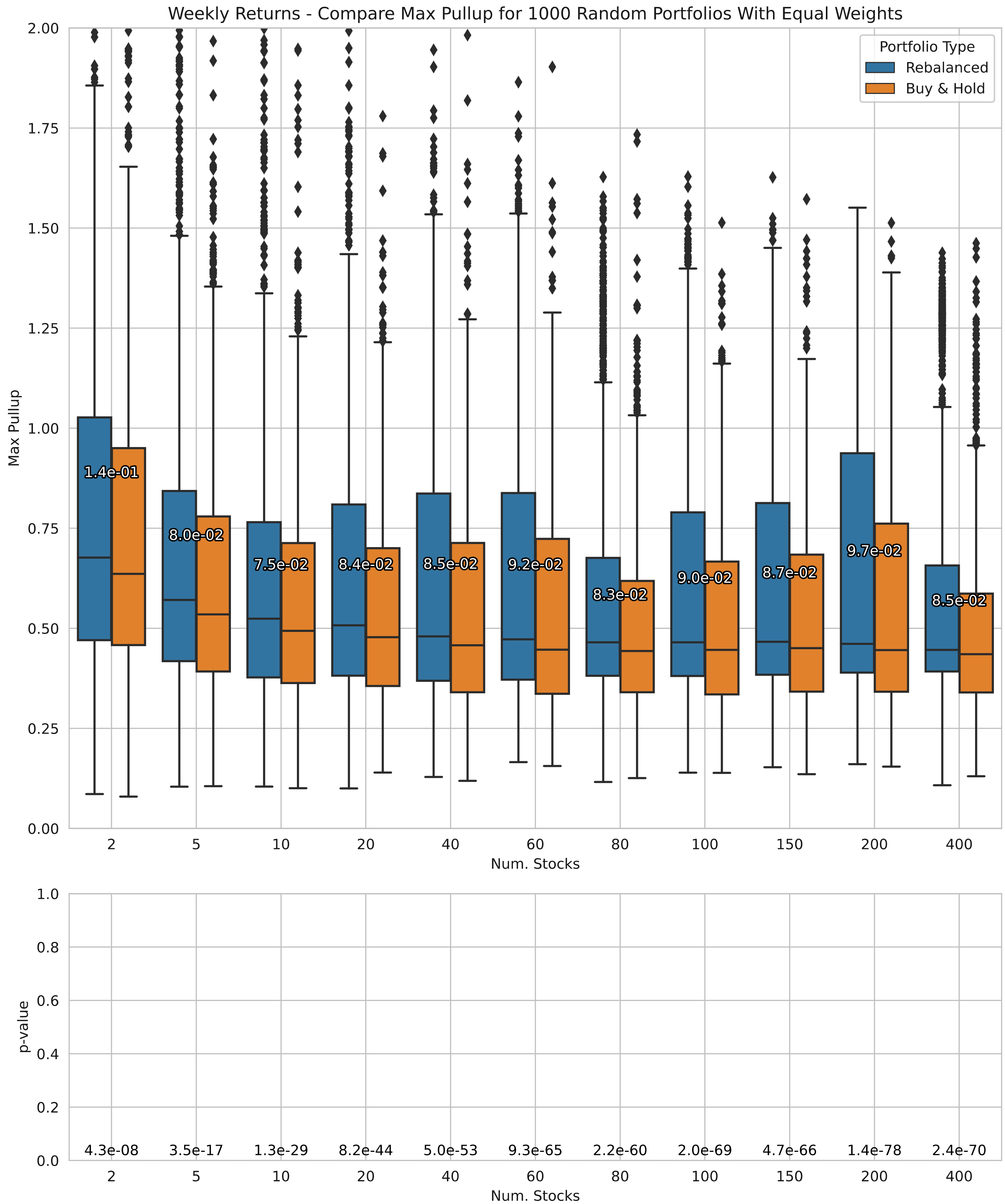

*Figure 27: Compare **Max Pullup of weekly returns** for Rebalanced and Buy&Hold portfolios.*

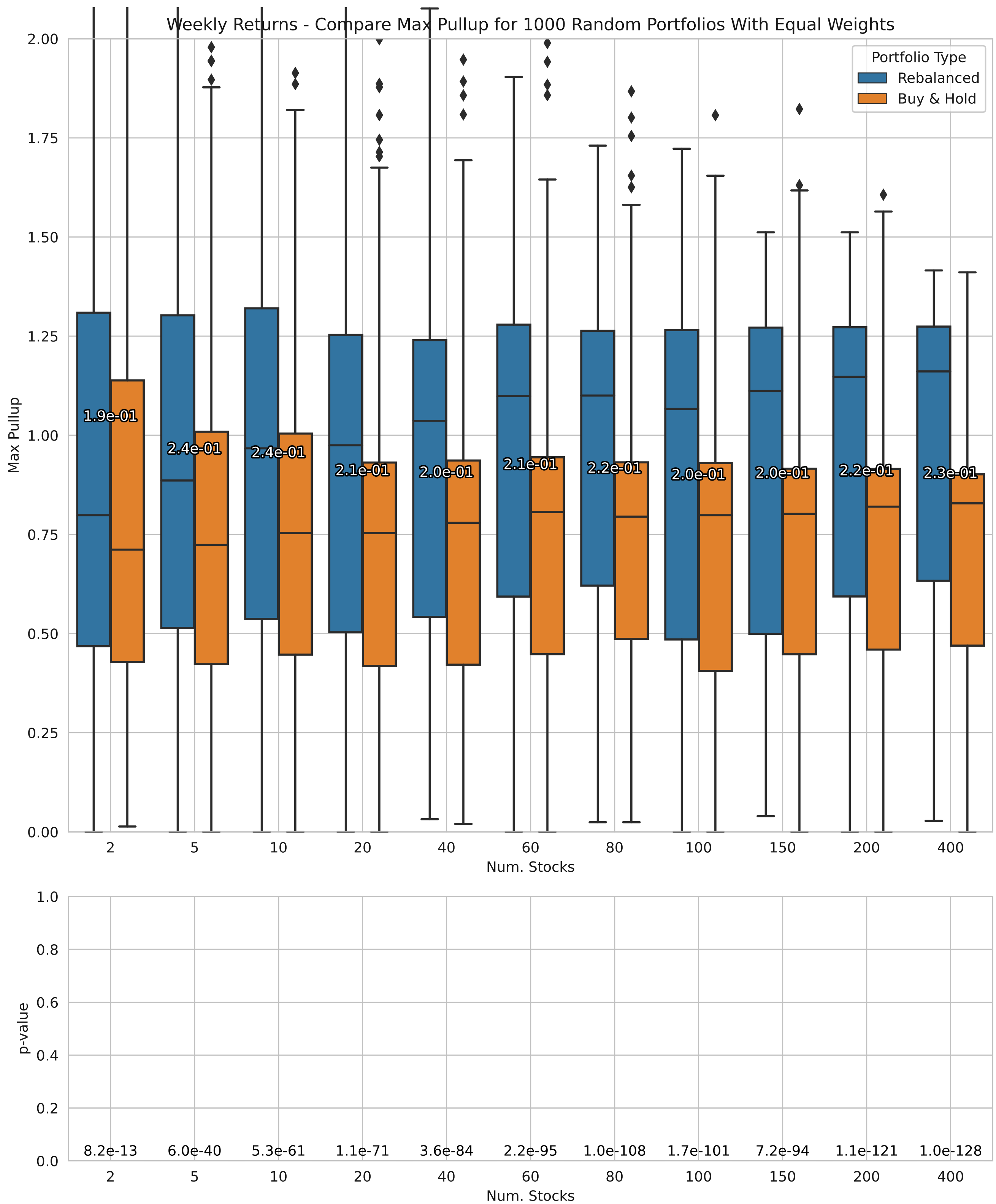

*Figure 28: Compare **Max Pullup of weekly returns** for Rebalanced and Buy&Hold portfolios **between 2007 and 2010**. Compare this to Figure 27 for the years between 2007 and 2021.*

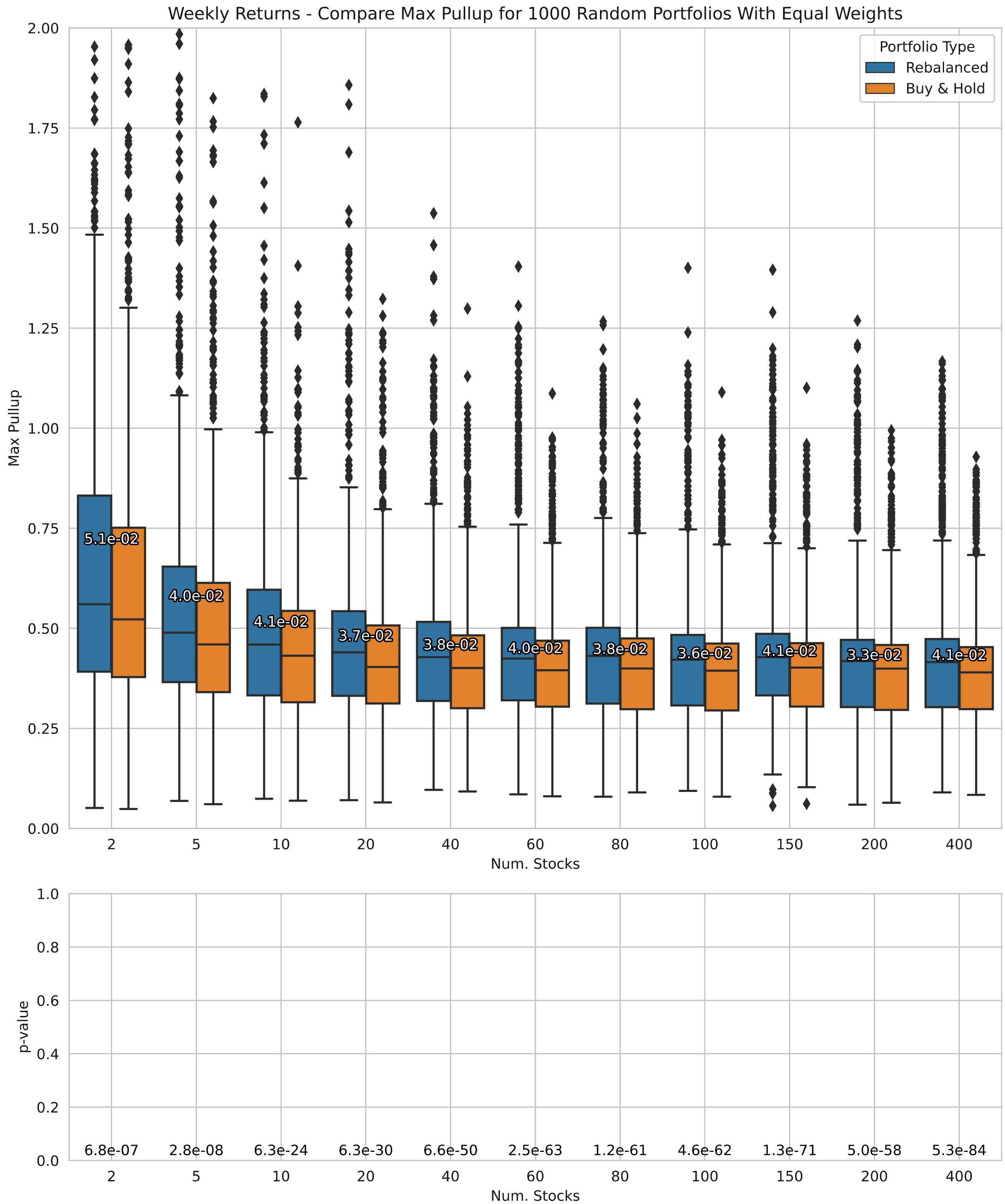

Figure 29: Compare ***Max Pullup of weekly returns*** *for Rebalanced and Buy&Hold portfolios* ***between 2010 and 2021****. Compare this to Figure 27 for the years between 2007 and 2021.*

6. Monthly Returns

Let us now consider whether the Rebalancing or Buy&Hold strategy is best for monthly stock-returns, which are merely resampled from the daily stock-returns. This means that we are considering monthly returns that start on a specific day such as the first trading-day of the month, and then all future rebalancing is done on the first trading-day of each following month as well. The computer code in Section 12 allows you to change the day of rebalancing by setting an offset in the resampling procedure. But it seems unlikely that it will make any difference on the results of this section.

6.1 Monthly Returns – Random Portfolios

Figure 30 shows the ratio between the Rebalanced and Buy&Hold portfolio for monthly stock-returns and 1000 portfolios consisting of 5 randomly selected stocks and random start/end-dates. A ratio of 1.0 means that the Rebalanced and Buy&Hold portfolios had identical returns, while a ratio above 1.0 means that the Rebalanced portfolio performed best, and vice versa, a ratio below 1.0 means that the Buy&Hold portfolio performed best. As can be seen from Figure 30, there does not seem to be a consistent advantage to either the Rebalanced or Buy&Hold portfolios.

Figure 31 shows the ratio between the two strategies when the portfolios consist of 50 stocks, and Figure 32 shows it when the portfolios consist of 100 stocks. These plots are very similar to the plots for weekly rebalancing in Figure 20 and Figure 21. We see that for the portfolios that started around year 2009, the Rebalanced portfolios were much better than the Buy&Hold portfolios. As we have discussed in the previous section for weekly returns, it seems that the Rebalanced portfolios generally perform better than the Buy&Hold portfolios in the recovery-phase of a stock-market crash, and there was a particularly large crash around year 2009. But we don’t see a similar performance difference in mid and late 2020 during the recovery-phase of the “Corona-Virus Panic”. It is possible that this is because of how the sampling function selects the random portfolios and time-periods, where it may not sample so often towards the end of the time-line. Or it is possible that there was something peculiar to the stock-market crash in 2009 that made Rebalanced portfolios perform particularly well during that recovery.

Another thing to note from both Figure 31 and especially Figure 32 is that during the downwards-phase of the stock-market crash in year 2020, the Rebalanced portfolios performed much worse than the Buy&Hold portfolios, which is consistent with our findings in the previous sections for daily and weekly rebalancing, as well as our intuition from the simple examples in Section 3.

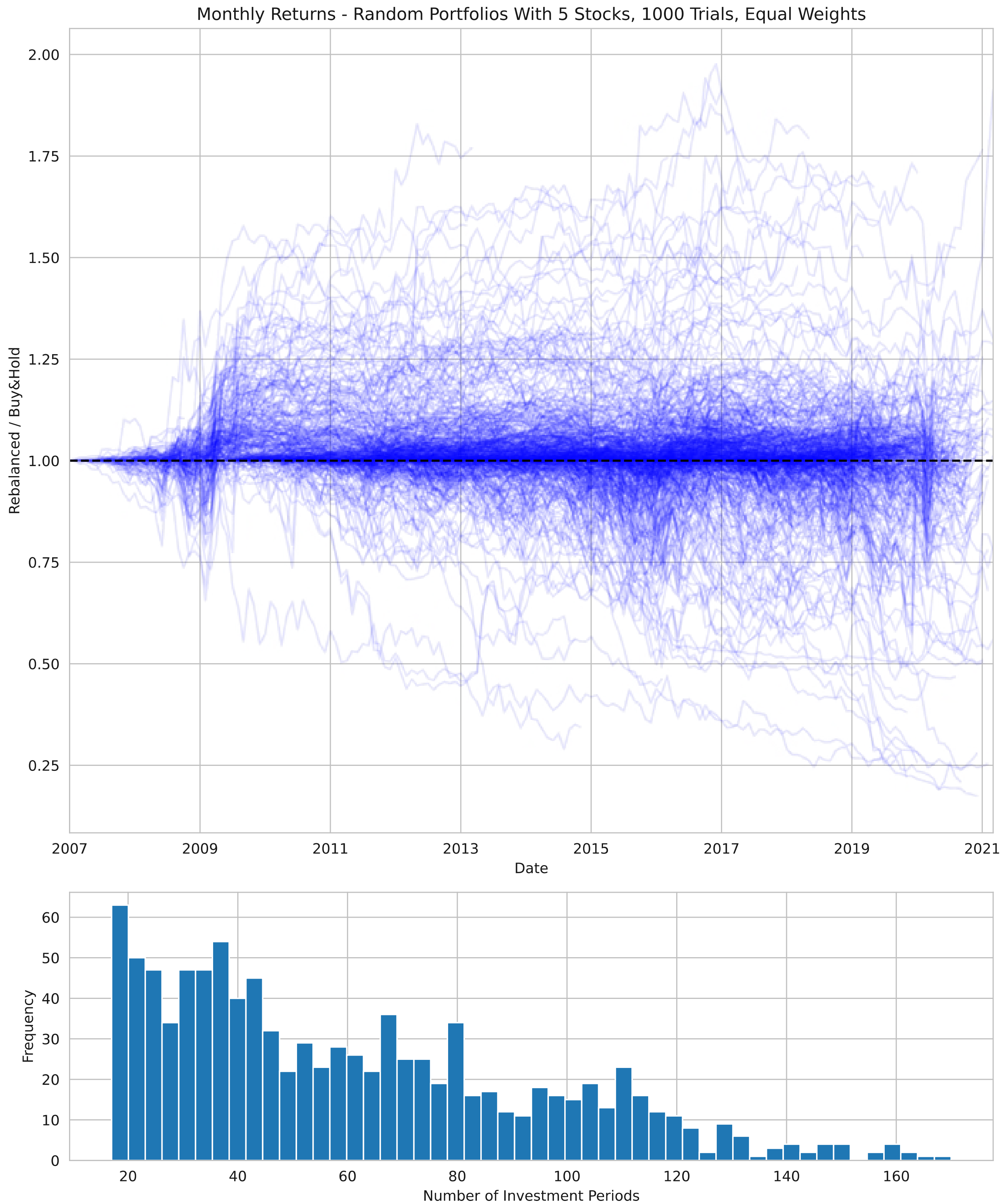

*Figure 30: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using **monthly returns**. There are **1000 portfolios each with 5 random stocks and random start/end-dates**. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

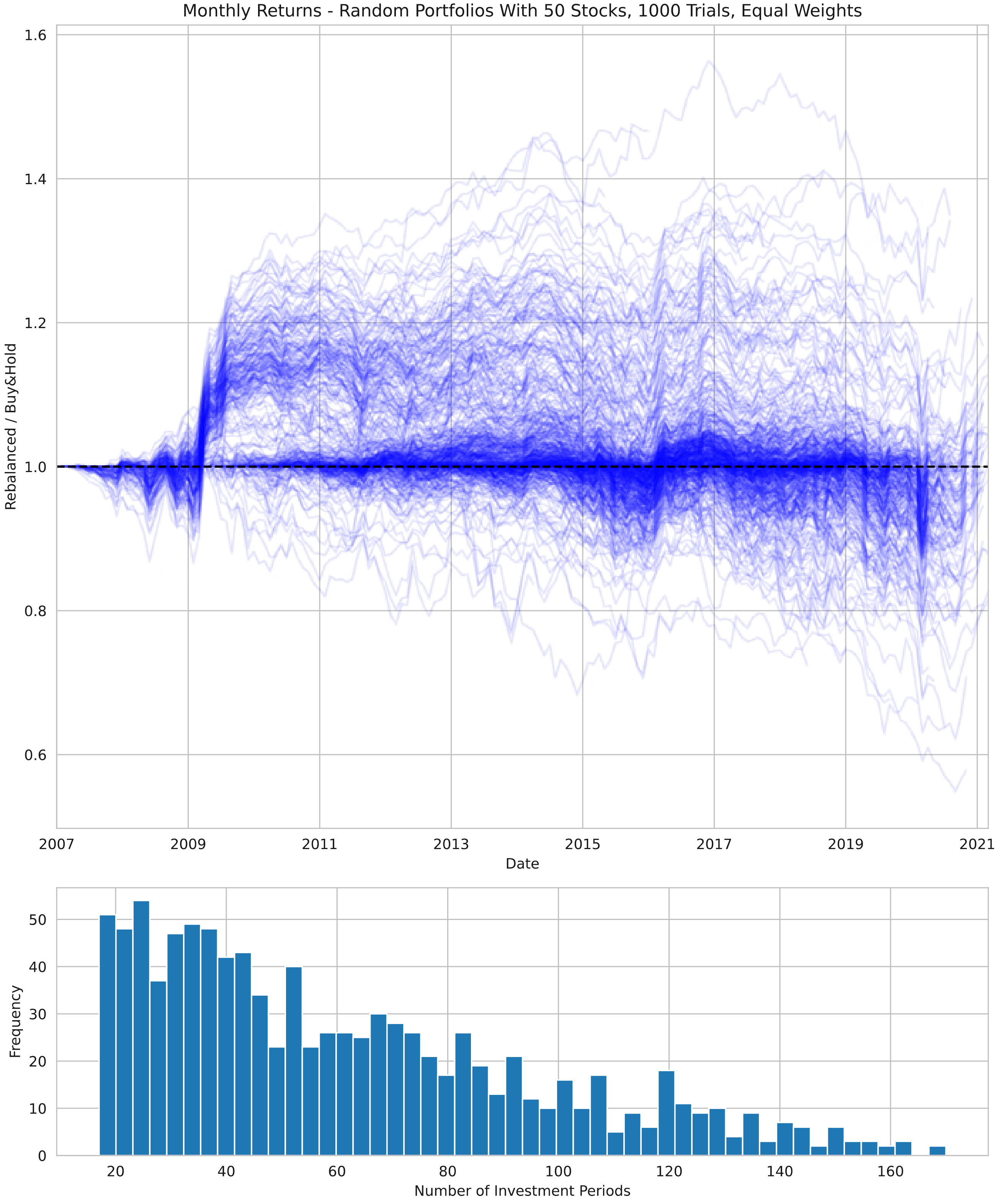

Figure 31: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using ***monthly returns****. There are* ***1000 portfolios each with 50 random stocks and random start/end-dates****. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

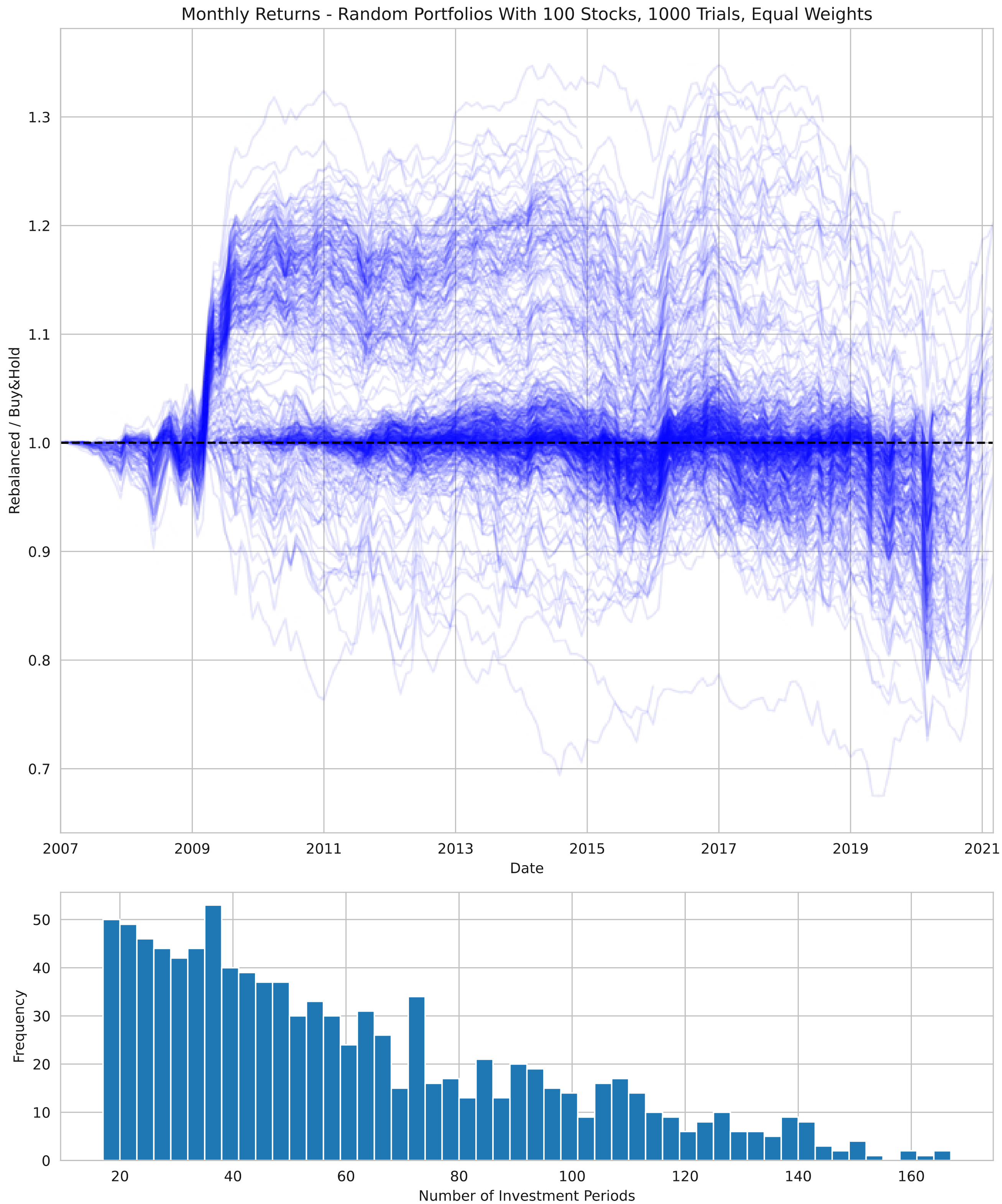

Figure 32: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using ***monthly returns****. There are* ***1000 portfolios each with 100 random stocks and random start/end-dates****. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

6.2 Monthly Returns – Comparing Statistics

Let us now compare some of the other performance statistics when using monthly stock-returns.

Figure 33 compares the arithmetic mean monthly returns for the Rebalanced and Buy&Hold portfolios. The p-values in the bottom-plot are barely statistically significant for portfolios with 2 and 5 stocks, and the p-value is very high for portfolios with 10 stocks. But portfolios with 20 stocks or more have p-values that are close to zero, so the average differences between the two portfolio strategies are statistically significant. The white text in the box-plot shows that the Rebalanced portfolios had slightly higher arithmetic mean monthly return than the Buy&Hold portfolios. On average, the monthly excess return of the Rebalanced portfolios was around 2.5e-04 which corresponds to an annualized excess return of $12 \cdot 2.5\text{e-}04 \simeq 0.3\,\%$.

But now consider Figure 34 which also compares the arithmetic mean monthly returns for the two portfolio strategies, but only for the monthly returns between years 2010 and 2021. The p-values are near zero for portfolio sizes of 10 stocks or more so the average differences are statistically significant. The white text shows that the average differences are now *negative* so the Rebalanced portfolios performed slightly *worse* than the Buy&Hold portfolios during this period. The average monthly difference was around -2.0e-04 which corresponds to an annualized difference around $12 \cdot -2.0\text{e-}04 \simeq -0.24\,\%$.

Figure 35 compares the arithmetic mean monthly returns for the period between years 2007 and 2010, where the p-values are all very close to zero so the average differences are statistically significant. The white text shows that the differences are *positive* so the Rebalanced portfolios performed *better* than the Buy&Hold portfolios. The average excess return was about 2.5e-03 per month, which corresponds to an annualized excess return of around $12 \cdot 2.5\text{e-}03 \simeq 3.0\,\%$.

So it appears that the better arithmetic mean monthly return of the Rebalanced portfolios mostly occurred between the years 2007 and 2010. As previously discussed, the reason could be that the Rebalanced portfolios appear to perform better than Buy&Hold portfolios during stock-market recoveries and there was a particularly big stock-market crash and recovery around the year 2009.

Figure 36 compares the geometric mean monthly returns for the Rebalanced and Buy&Hold portfolios for the entire data-period between the years 2007 and 2021. The p-values are very high except for the portfolios that contain 60, 80 and 150 stocks where the p-values are barely statistically significant, and for which the Rebalanced portfolios can be seen to have a slightly higher monthly return than the Buy&Hold portfolios, when measured using the geometric mean.

Figure 37 compares the standard deviations for the monthly returns of the Rebalanced and Buy&Hold portfolios, also for the entire data-period between 2007 and 2021. The p-values are close to zero so the average differences are statistically significant. As we also saw for daily and weekly stock-returns, the Rebalanced portfolios have higher standard deviations for their monthly returns than the Buy&Hold portfolios.

Figure 38 compares the Sharpe ratios for the Rebalanced and Buy&Hold portfolios, also for the entire data-period between years 2007 and 2021. The p-values are close to zero for portfolios having 20 or more stocks, so the average differences are statistically significant. The white text shows that the standard deviations tend to be slightly smaller for the Rebalanced portfolios than for the Buy&Hold portfolios.

Figure 39 compares the Max Drawdowns for the Rebalanced and Buy&Hold portfolios, again for the entire data-period between 2007 and 2021. The p-values are close to zero for all portfolio sizes so the average differences between the portfolio strategies are statistically significant. The box-plot and its white text shows that the Rebalanced portfolios tend to have slightly worse Max Drawdowns than the Buy&Hold portfolios.

Figure 40 compares the Max Pullups for the Rebalanced and Buy&Hold portfolios, also between the years 2007 and 2021. The p-values are all close to zero so the average differences for the two portfolio strategies are statistically significant. The box-plot and the white text show that the Rebalanced portfolios tend to perform significantly better than the Buy&Hold portfolios in stock-market recoveries and rallies.

Note that both Max Drawdowns and Max Pullups do not take in-between data-points into account.

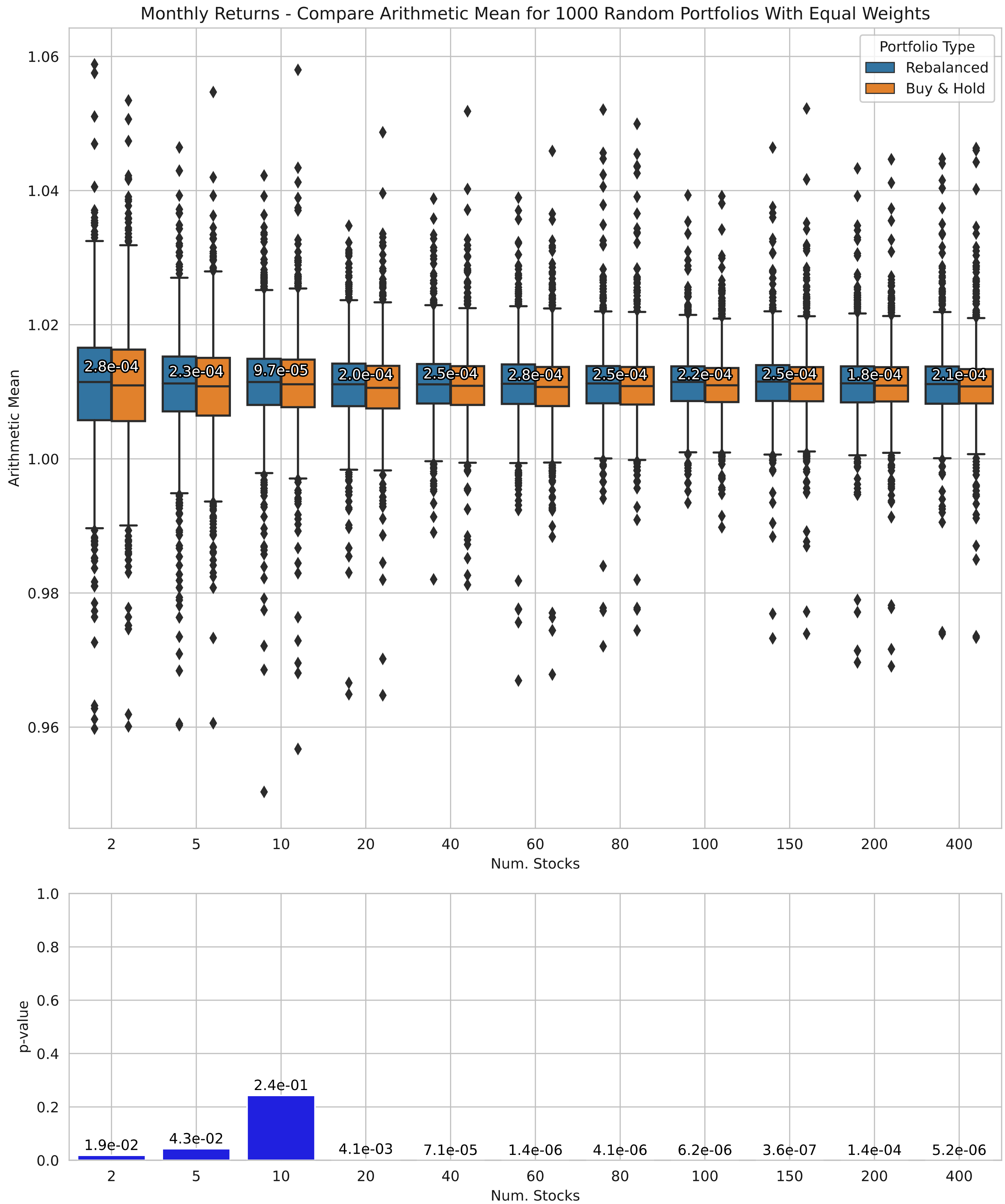

Figure 33: Compare ***arithmetic means of monthly returns*** *for Rebalanced and Buy&Hold portfolios.*

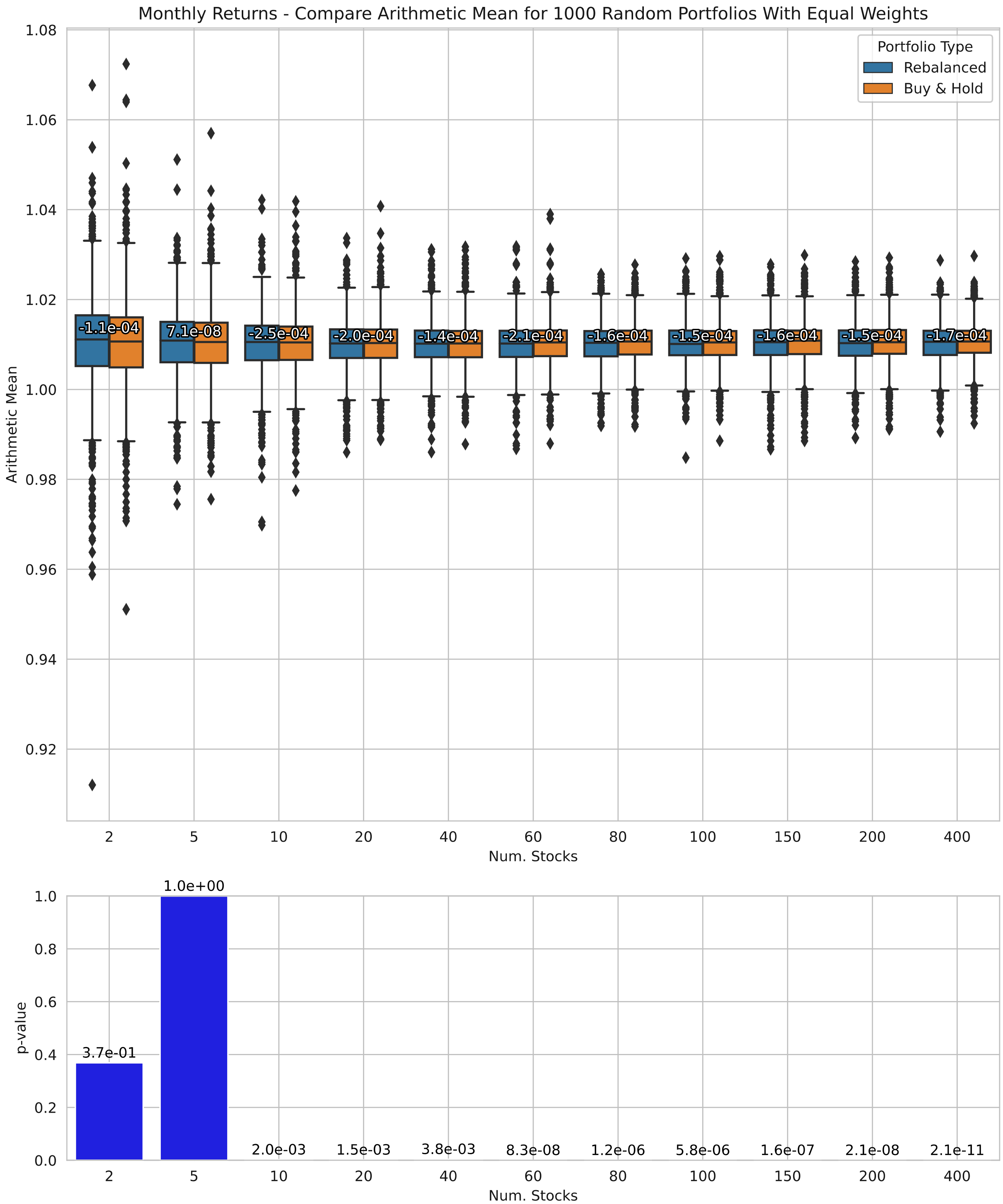

*Figure 34: Compare **arithmetic means of monthly returns** for Rebalanced and Buy&Hold portfolios **between 2010 and 2021.** Compare this to Figure 33 for the years between 2007 and 2021.*

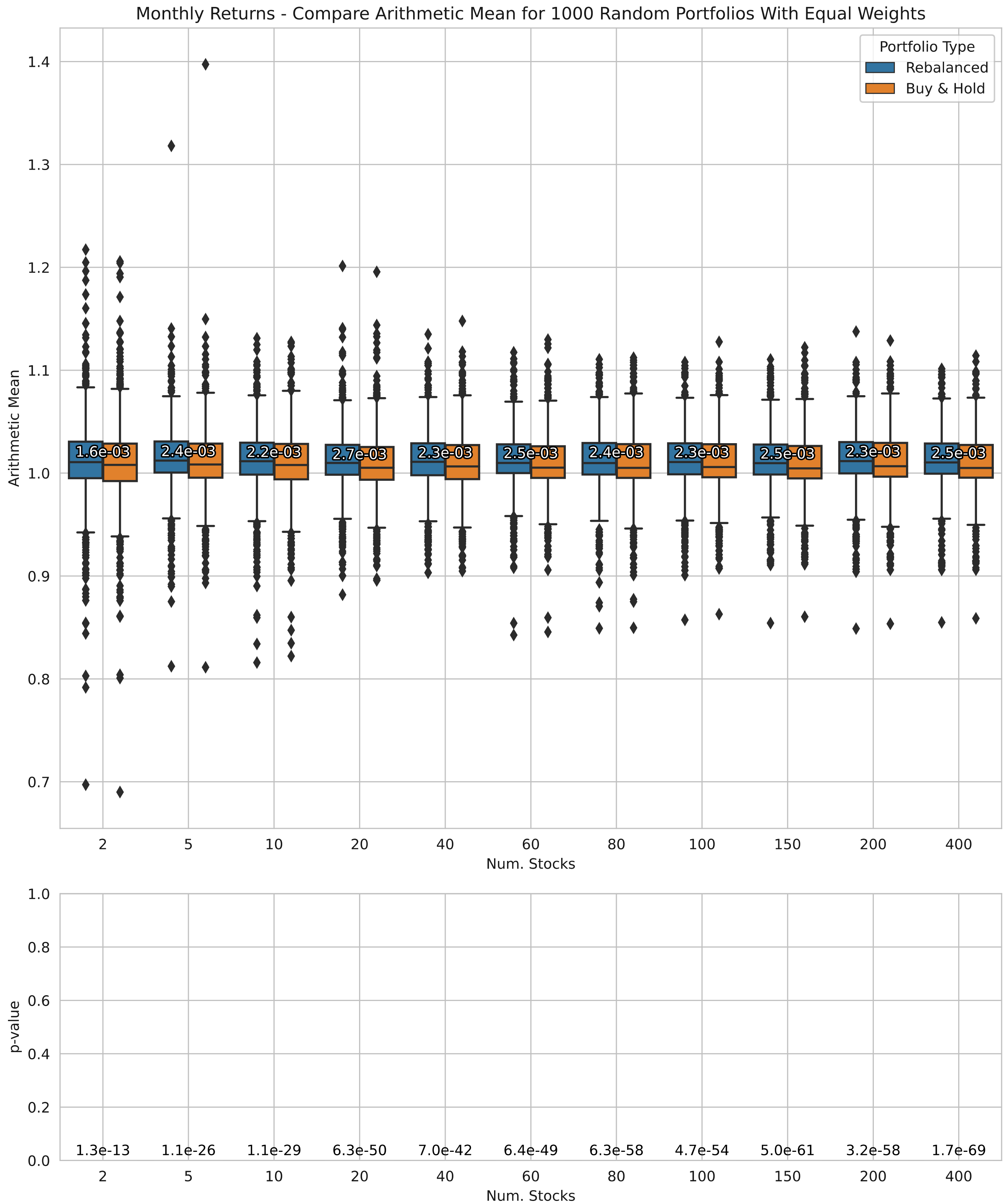

*Figure 35: Compare **arithmetic means of monthly returns** for Rebalanced and Buy&Hold portfolios **between 2007 and 2010**. Compare this to Figure 33 for the years between 2007 and 2021.*

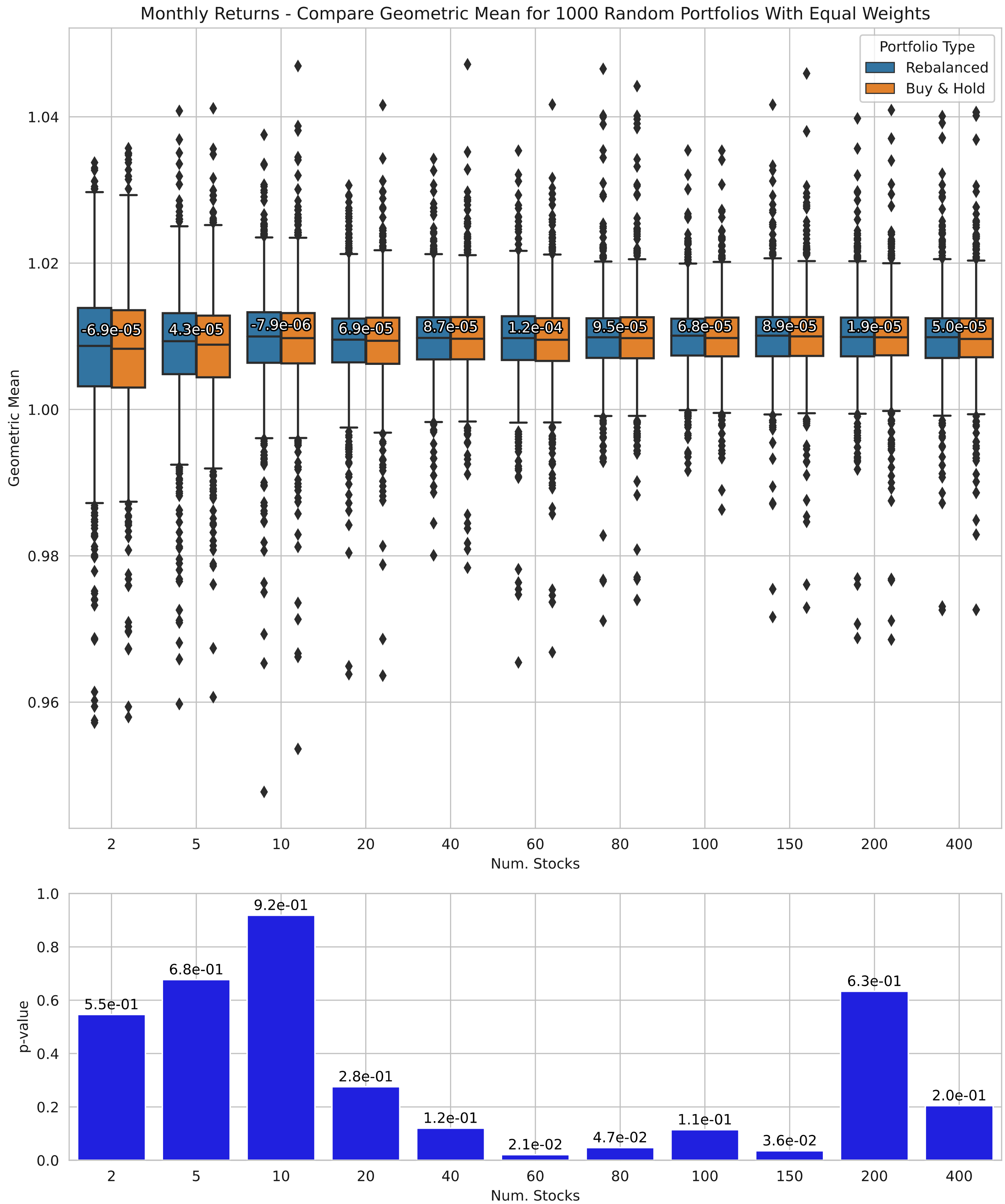

*Figure 36: Compare **geometric means of monthly returns** for Rebalanced and Buy&Hold portfolios.*

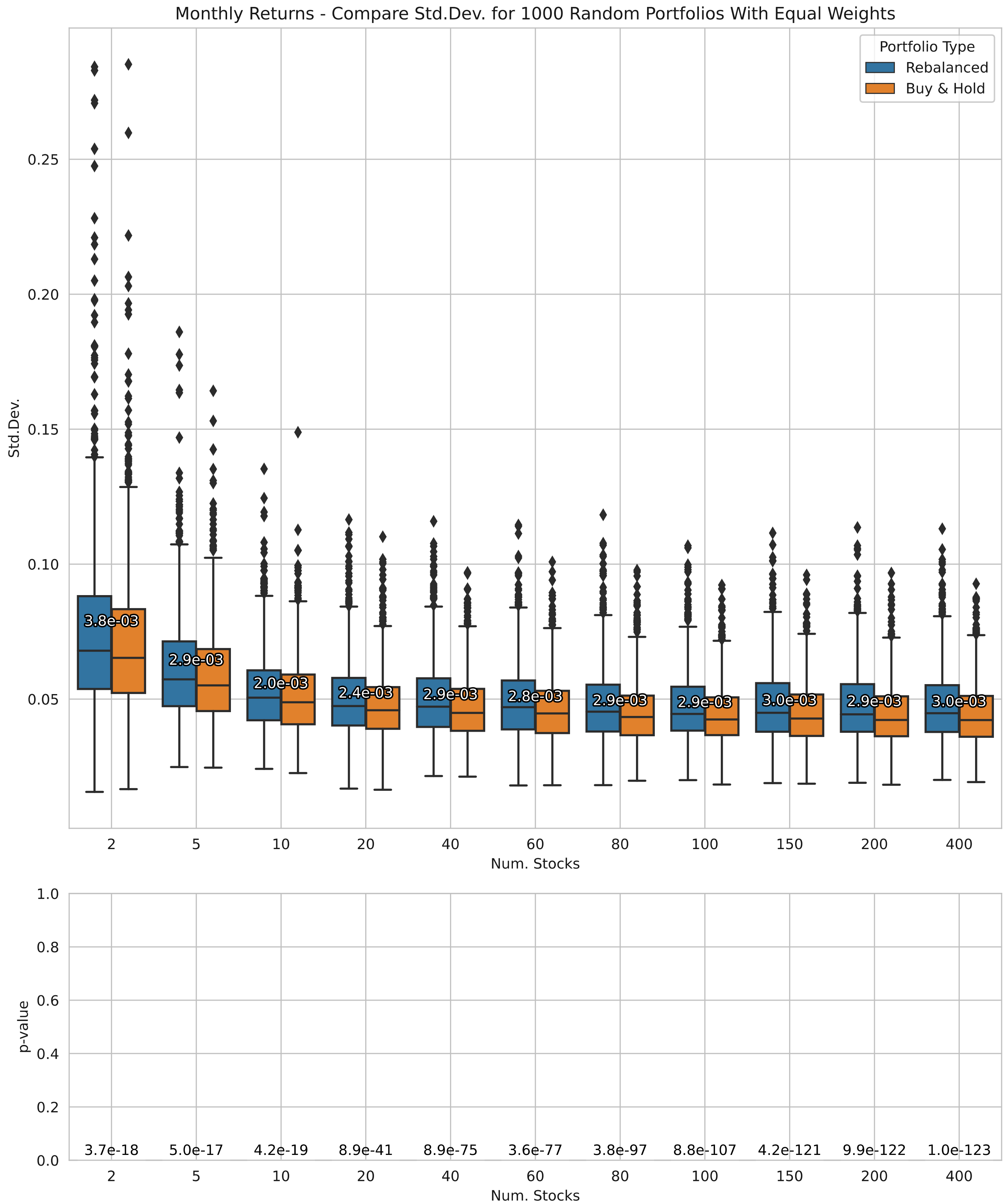

*Figure 37: Compare **standard deviations of monthly returns** for Rebalanced and Buy&Hold portfolios.*

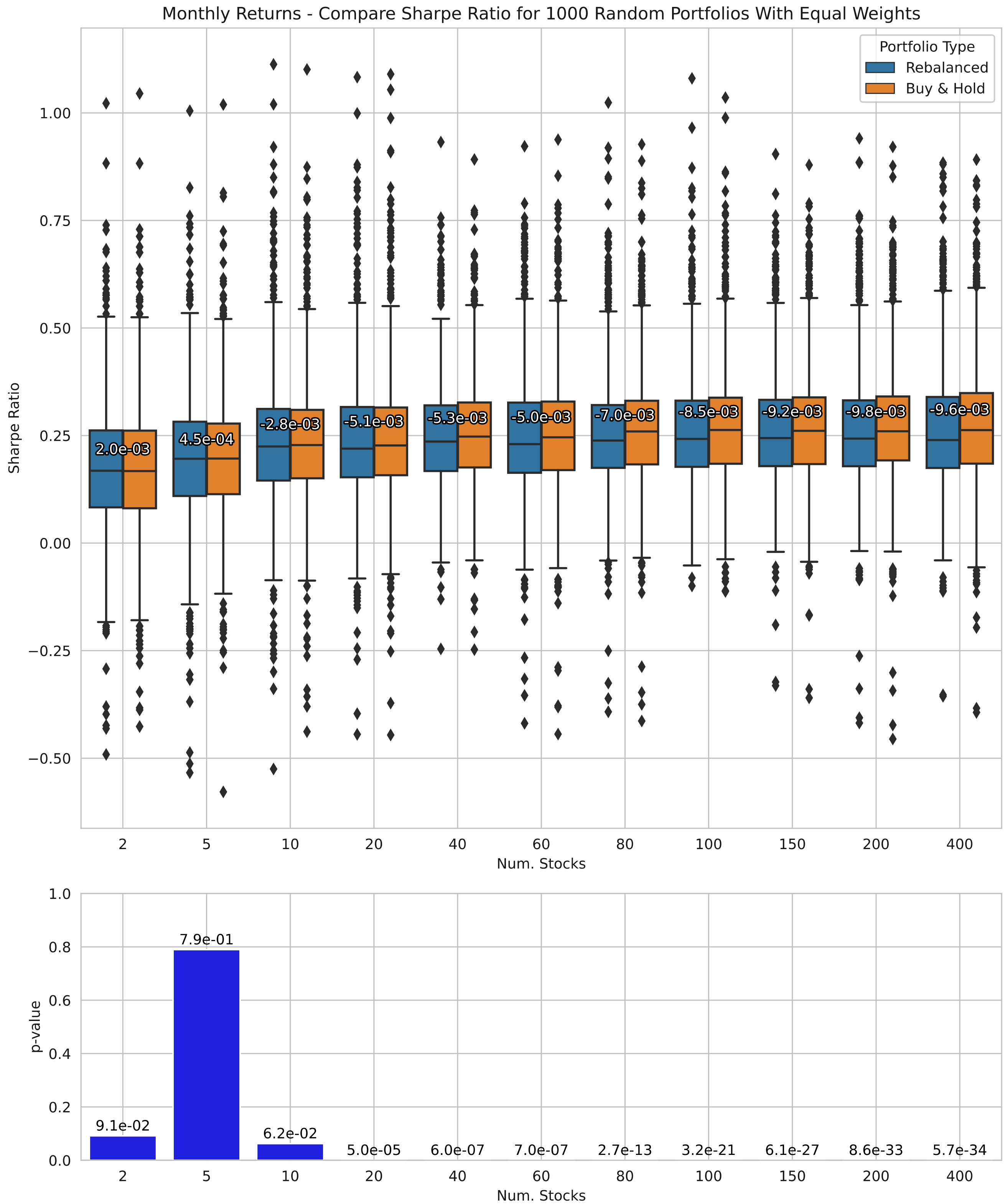

*Figure 38: Compare **Sharpe ratios of monthly returns** for Rebalanced and Buy&Hold portfolios.*

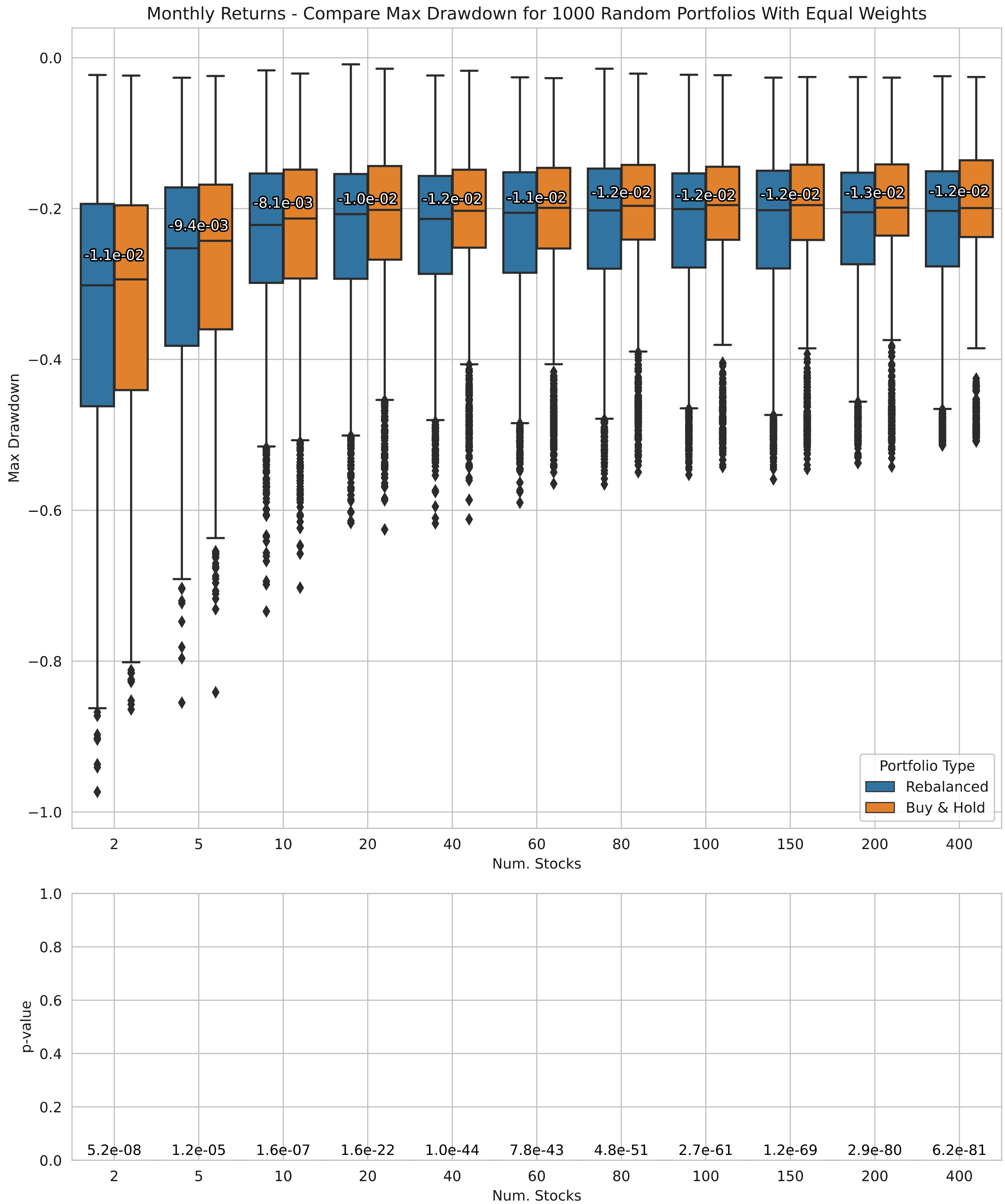

*Figure 39: Compare **Max Drawdowns of monthly returns** for Rebalanced and Buy&Hold portfolios.*

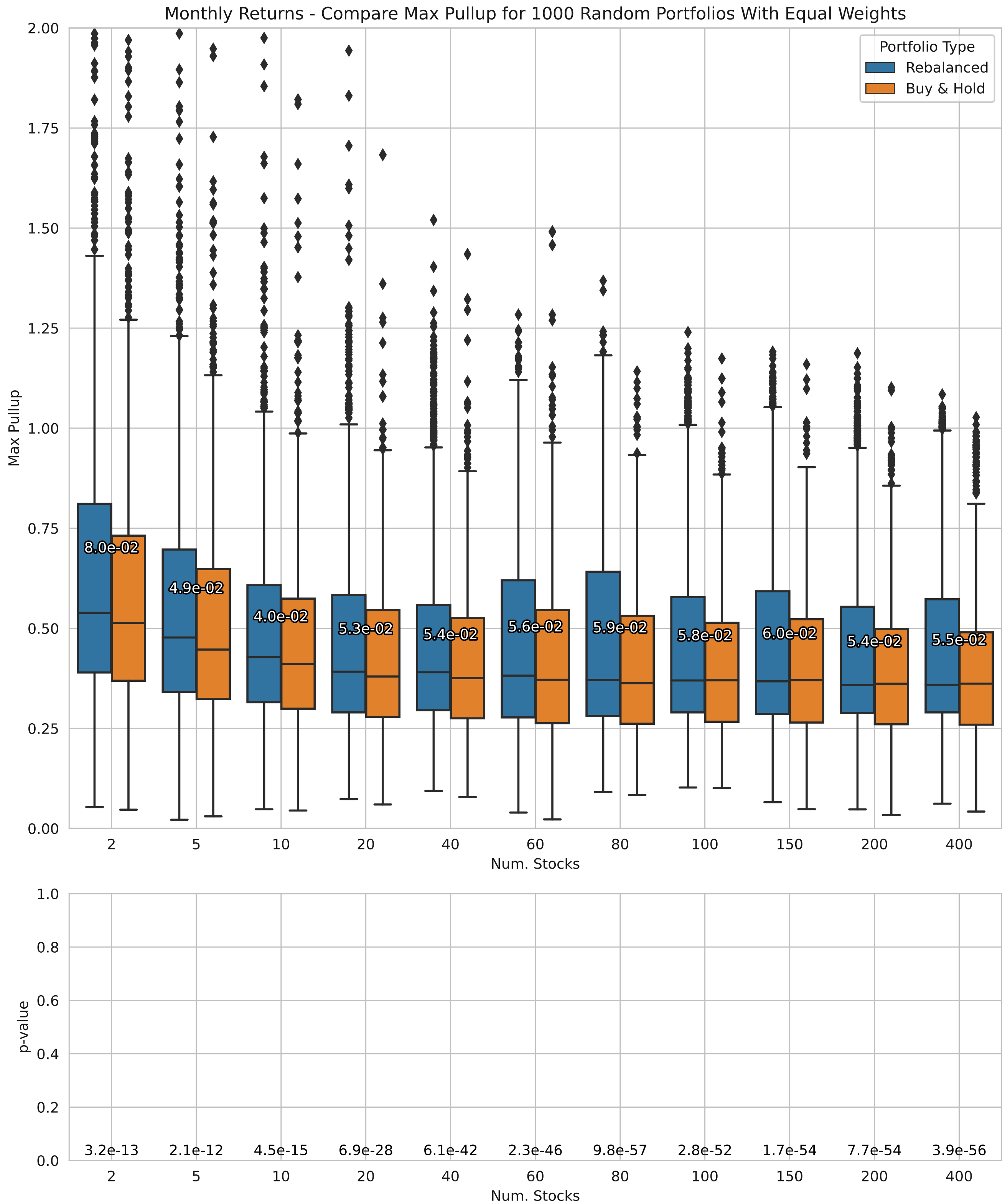

Figure 40: Compare ***Max Pullups of monthly returns*** *for Rebalanced and Buy&Hold portfolios.*

7. Annual Returns

Let us now consider whether the Rebalancing or Buy&Hold strategy is best for annual stock-returns, which are merely resampled from the daily stock-returns. This means that we are considering annual returns that start on a specific day such as the first trading-day of the year, and then all future rebalancing is done on the first trading-day of each following year as well. The computer code in Section 12 allows you to change the day of rebalancing by setting an offset in the resampling procedure. But it seems unlikely that it will make any difference on the results of this section.

In this section, we will not make box-plots comparing various performance statistics for the two portfolio strategies, because we only have 14 years of data for each stock, so we only have 14 annual data-points for each stock. It would be possible to extend the computer program in Section 12 to consider all possible rebalancing dates instead of just the first trading-day of the year, but it would require significant modifications to the computer code, and it seems that the result would most likely be similar to the previous sections for daily, weekly and monthly stock-returns, namely that there is no substantial and general advantage to either portfolio strategy, except perhaps in special circumstances such as recoveries from stock-market crashes.

7.1 Annual Returns – Random Portfolios

Figure 41 shows the ratio between the Rebalanced and Buy&Hold portfolio for annual stock-returns and 1000 portfolios consisting of 5 randomly selected stocks and random start/end-dates. A ratio of 1.0 means that the Rebalanced and Buy&Hold portfolios had identical returns, while a ratio above 1.0 means that the Rebalanced portfolio performed best, and vice versa, a ratio below 1.0 means that the Buy&Hold portfolio performed best. As can be seen from Figure 41, there does not seem to be a consistent advantage to either the Rebalanced or Buy&Hold portfolios.

Figure 42 shows it for random portfolios with 50 stocks each, and Figure 43 shows it for random portfolios with 100 stocks each. Once again there is no consistent advantage to either the Rebalanced or Buy&Hold portfolios, with exception of the portfolios that started around the years 2008-2009. As we have previously discussed, it appears that the Rebalanced portfolios perform significantly better than the Buy&Hold portfolios when the stock-market is recovering from a big crash.

Note that the dates on these plots are for the end of each year. For example, in Figure 42 and Figure 43 we see that the ratio between the Rebalanced and Buy&Hold portfolios were all significantly greater than 1.0 for the year 2009, meaning that the Rebalanced portfolios performed much greater than the Buy&Hold portfolios for the year 2009. This time-stamp actually marks the end of the year 2009, where the stock-market reached its bottom around March 2009 and then started a great recovery-rally. So the plots show us that in the year 2009, the Rebalanced portfolios performed much better than the Buy&Hold portfolios, thus suggesting that annual rebalancing may perform much better during stock-market recoveries and rallies, similar to our findings for daily, weekly and monthly rebalancing in the previous sections.

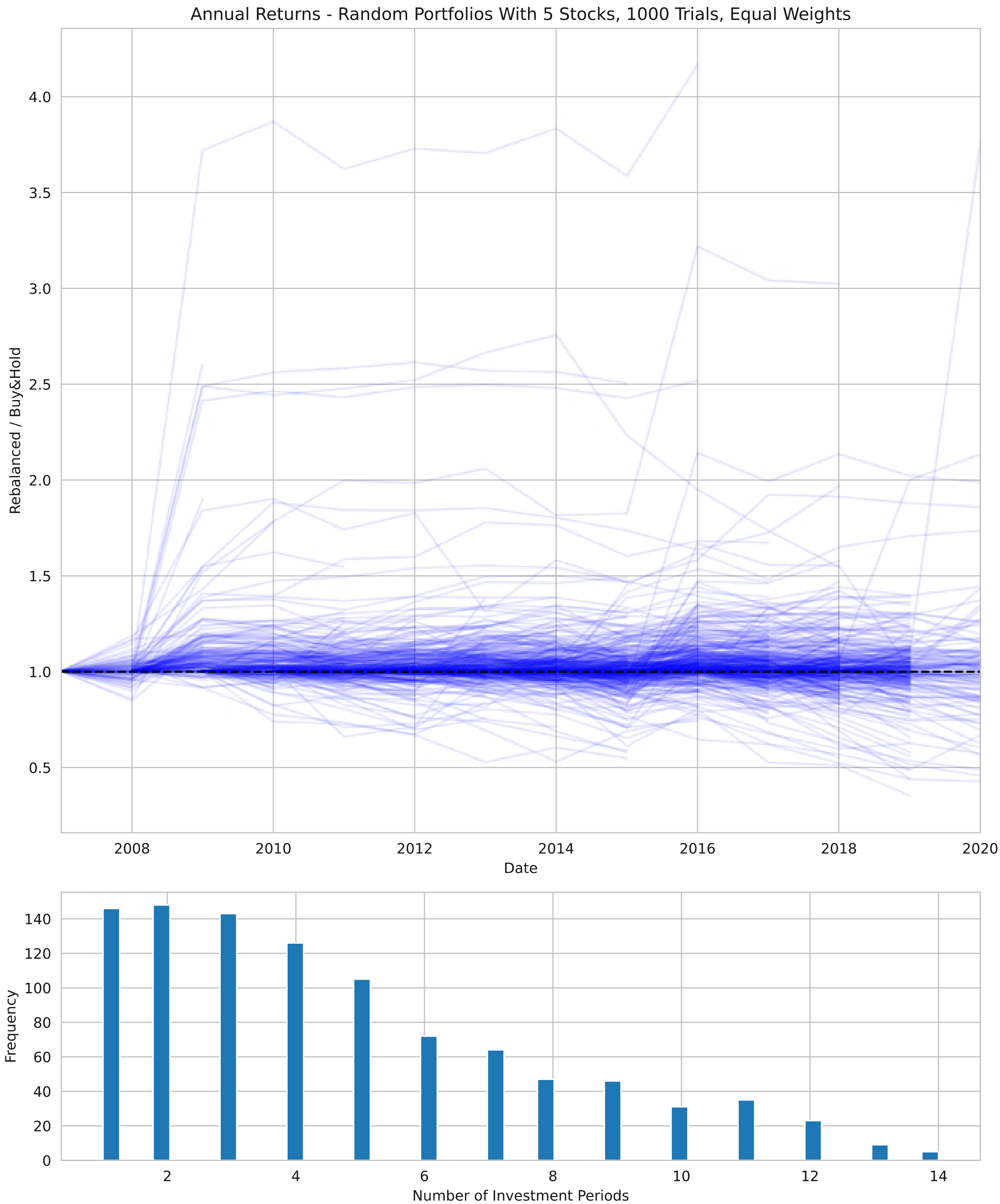

Figure 41: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using ***annual returns****. There are* ***1000 portfolios each with 5 random stocks and random start/end-dates****. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

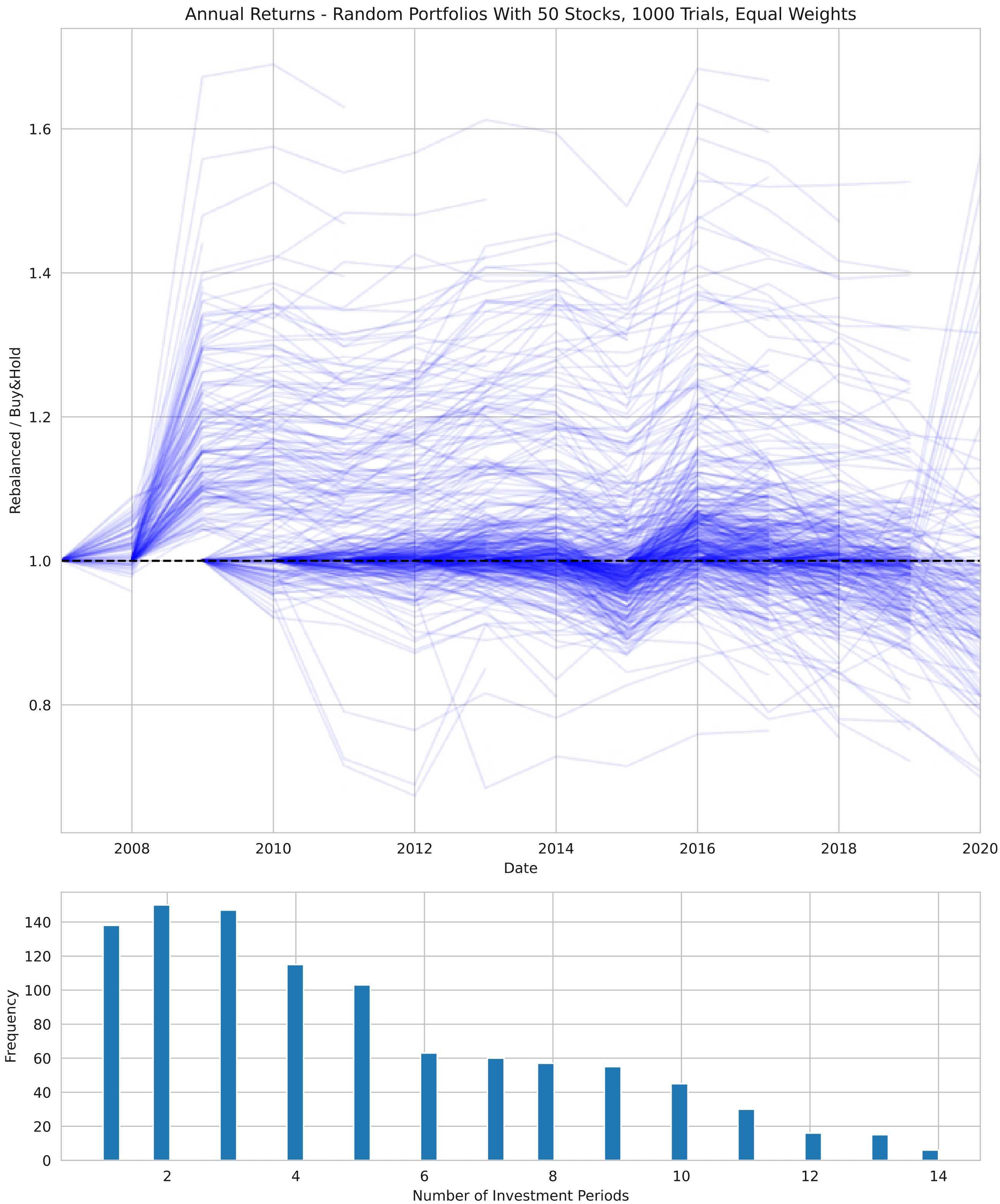

Figure 42: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using ***annual returns****. There are* ***1000 portfolios each with 50 random stocks and random start/end-dates****. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

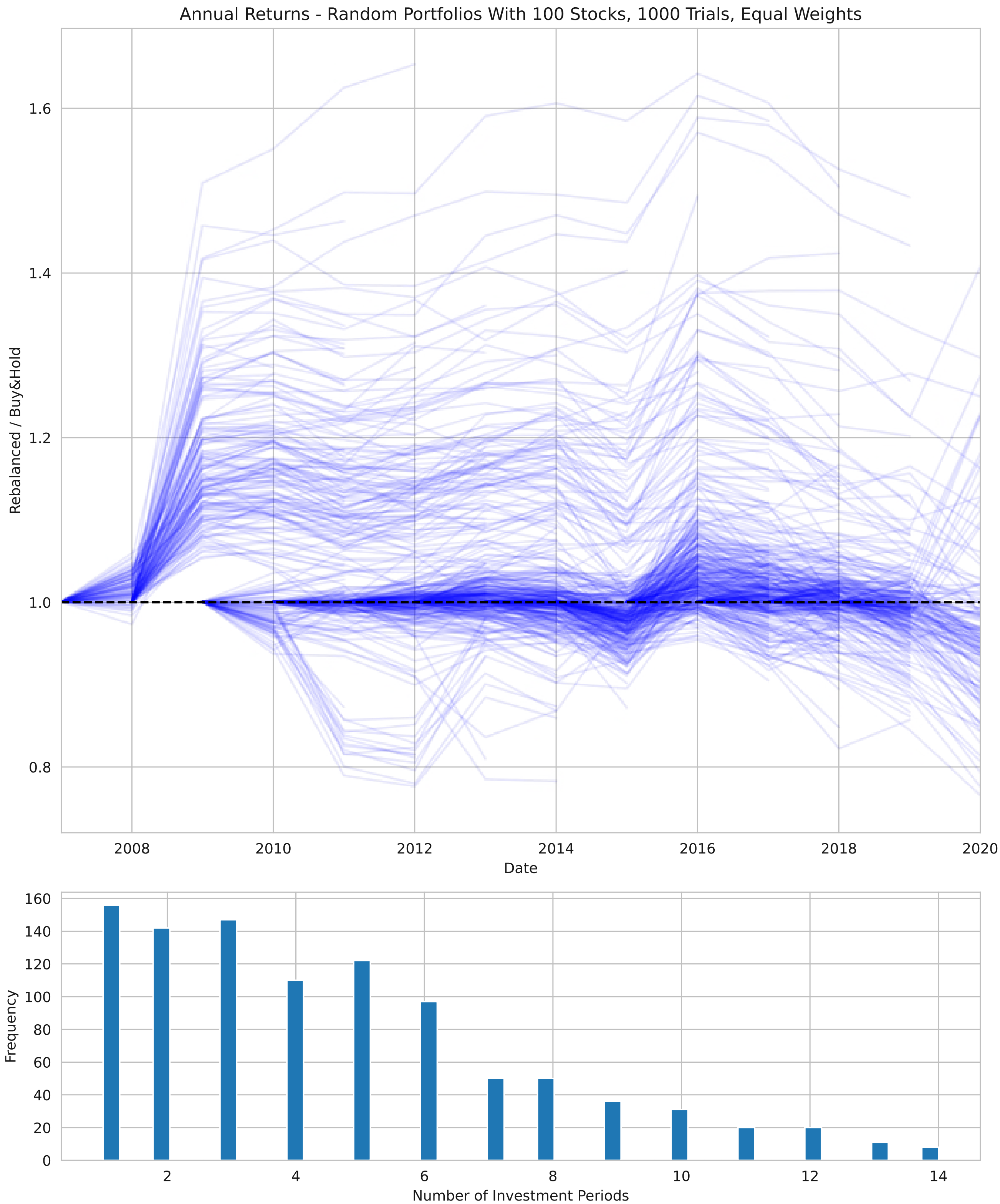

Figure 43: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using ***annual returns****. There are* ***1000 portfolios each with 100 random stocks and random start/end-dates****. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

8. Intraday Returns 1-Minute Intervals

Let us now consider intraday stock-returns. We begin with 1-minute intervals and will consider 5-minute intervals in the next section. We only have 1-minute data for the 70 stocks listed in Section 12.4 and only for a period of 9 days in April 2021, because the data was obtained from a free database on the internet that is quite slow to retrieve data from. You should therefore consider these experiments as preliminary and the results may not be representative when using intraday data for several years and hundreds or even thousands of stocks.

Figure 44 shows the cumulative returns for the 70 stocks in our data-set normalized to begin at 1.0 and the red line shows the arithmetic average at each time-step.

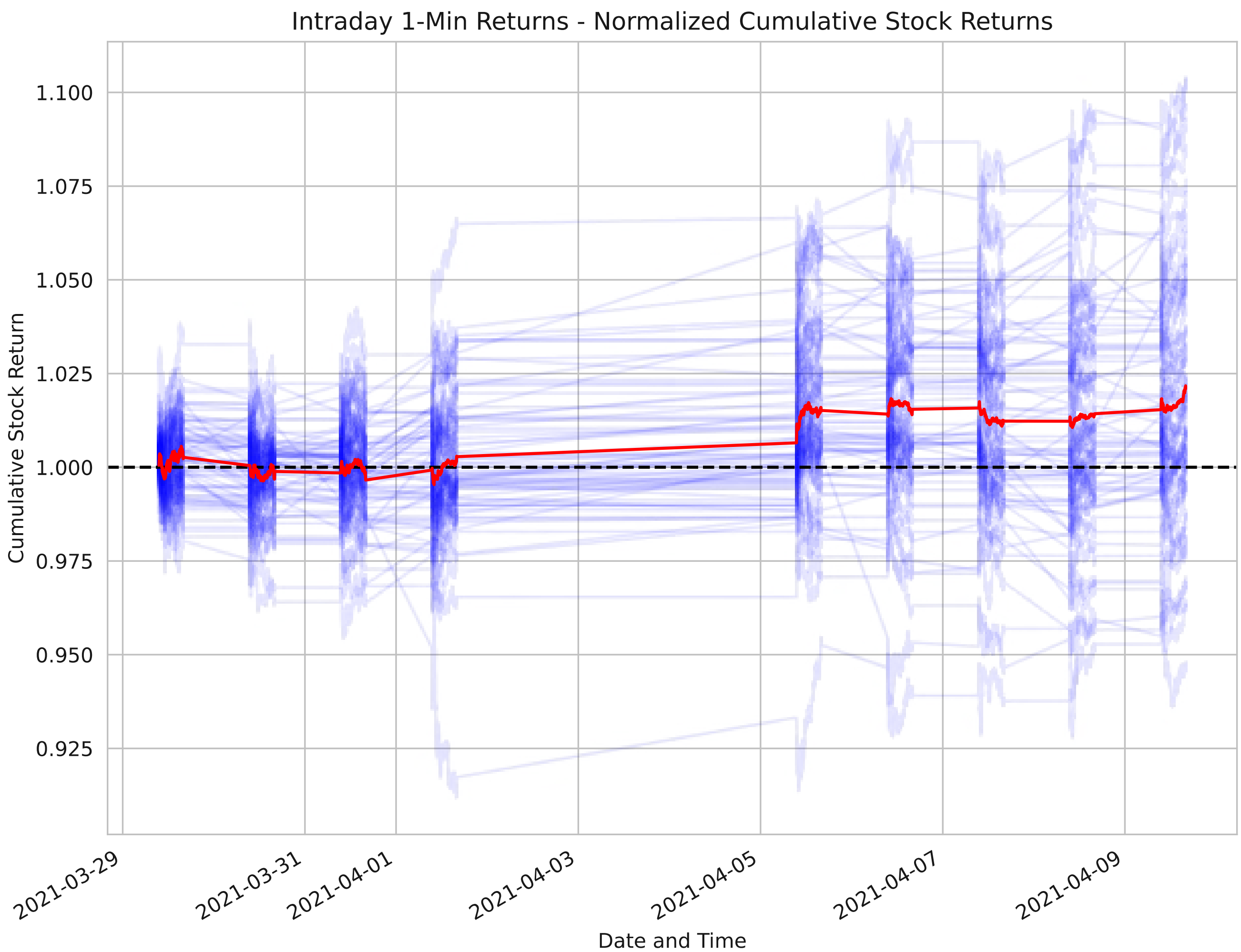

*Figure 44: **Cumulative returns** for the **70 stocks** in the data-set with **1-minute intraday returns**, which are plotted as blue lines and normalized to all start at the value 1 so they can easily be compared. The red line is the arithmetic mean for each time-step.*

8.1 Intraday Returns 1-Min Intervals – Random Portfolios

Figure 45 shows the ratio between the Rebalanced and Buy&Hold portfolios for 1-minute intraday stock-returns and 1000 portfolios consisting of 5 randomly selected stocks and random start/end-dates. A ratio of 1.0 means that the Rebalanced and Buy&Hold portfolios had identical returns, while a ratio above 1.0 means that the Rebalanced portfolio performed best, and vice versa, a ratio below 1.0 means that the Buy&Hold portfolio performed best. As can be seen from Figure 45, there does not seem to be a consistent advantage to either the Rebalanced or Buy&Hold portfolios.

Figure 46 shows it for portfolios with 10 stocks each, and Figure 47 shows it for portfolios with 40 stocks each. For portfolios with 40 stocks, most of the ratios between the Rebalanced and Buy&Hold portfolios are above 1.0 so the Rebalanced portfolios generally performed better. Although it may seem strange that we would see such a tendency for only 9 days of stock-data, there are actually more than 3500 time-steps in this data, so the rebalancing is done more than 3500 times over those 9 days. Although this result looks very interesting, we would need to conduct research on much larger data-sets with several years of 1-minute intraday data, before we conclude that Rebalanced portfolios tend to perform better than Buy&Hold portfolios. This is just a preliminary experiment and you are encouraged to make a much more thorough study of intraday rebalancing.

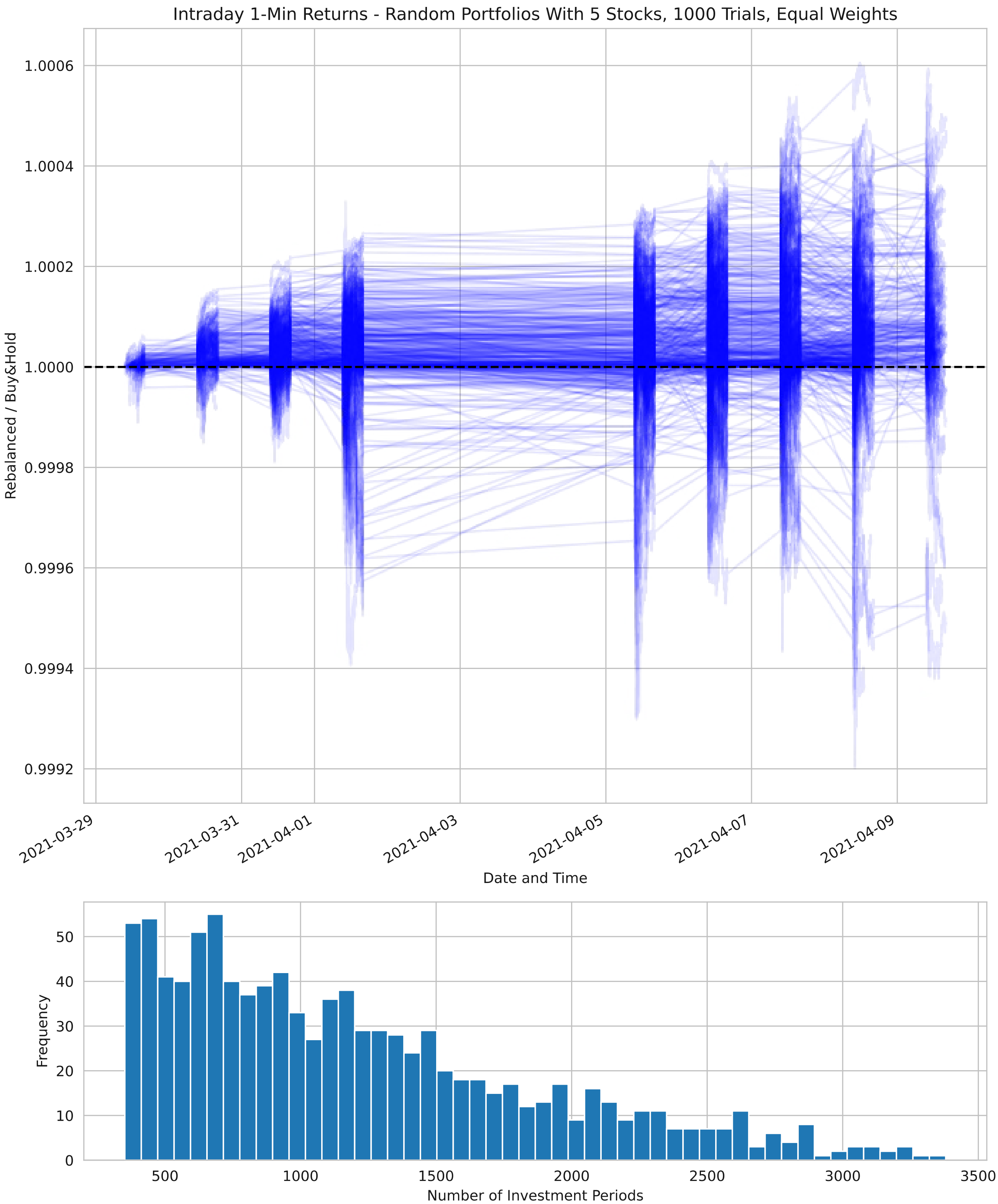

Figure 45: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using ***1-minute intraday returns****. There are* ***1000 portfolios each with 5 random stocks and random start/end-dates****. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

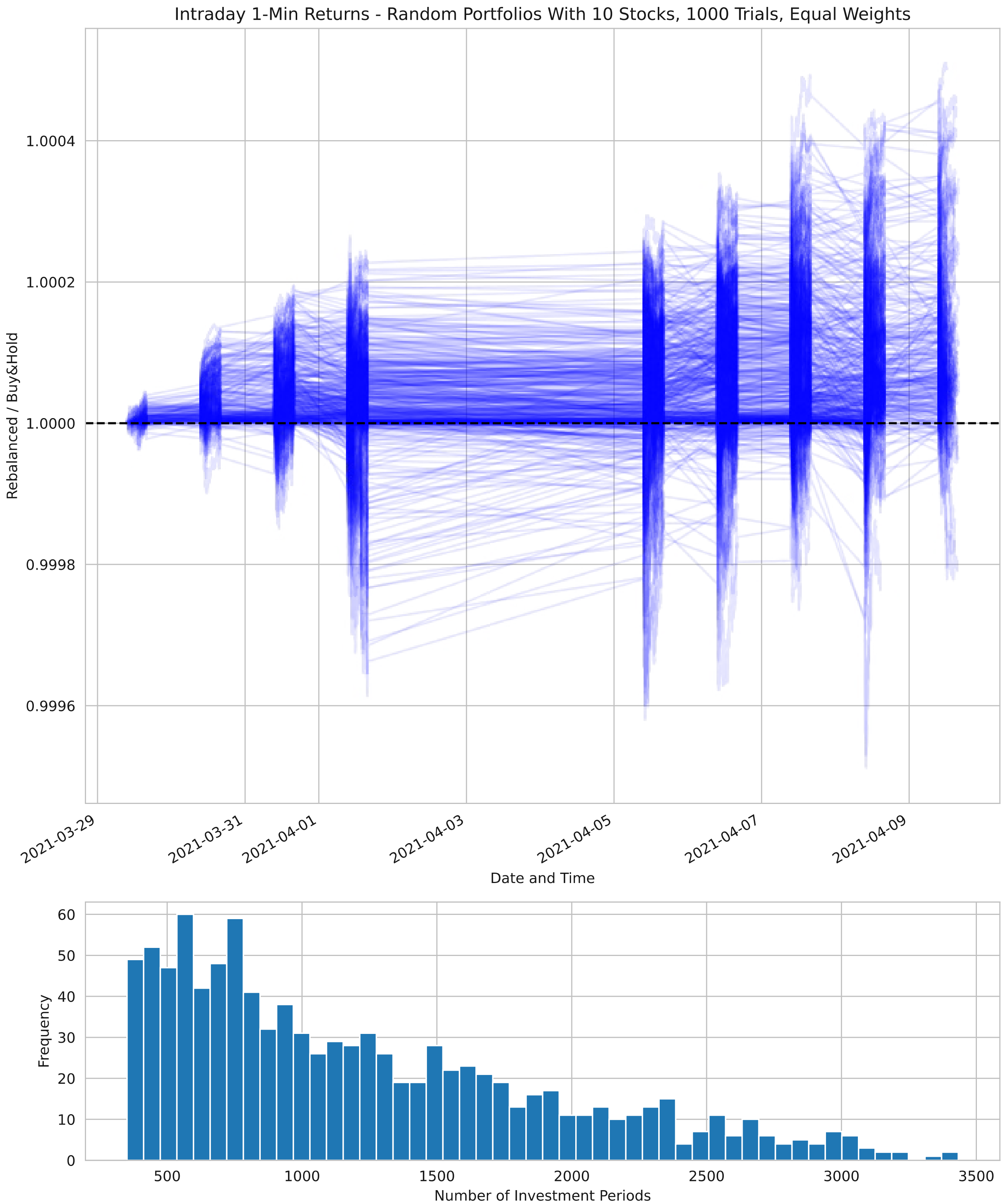

Figure 46: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using ***1-minute intraday returns****. There are* ***1000 portfolios each with 10 random stocks and random start/end-dates****. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

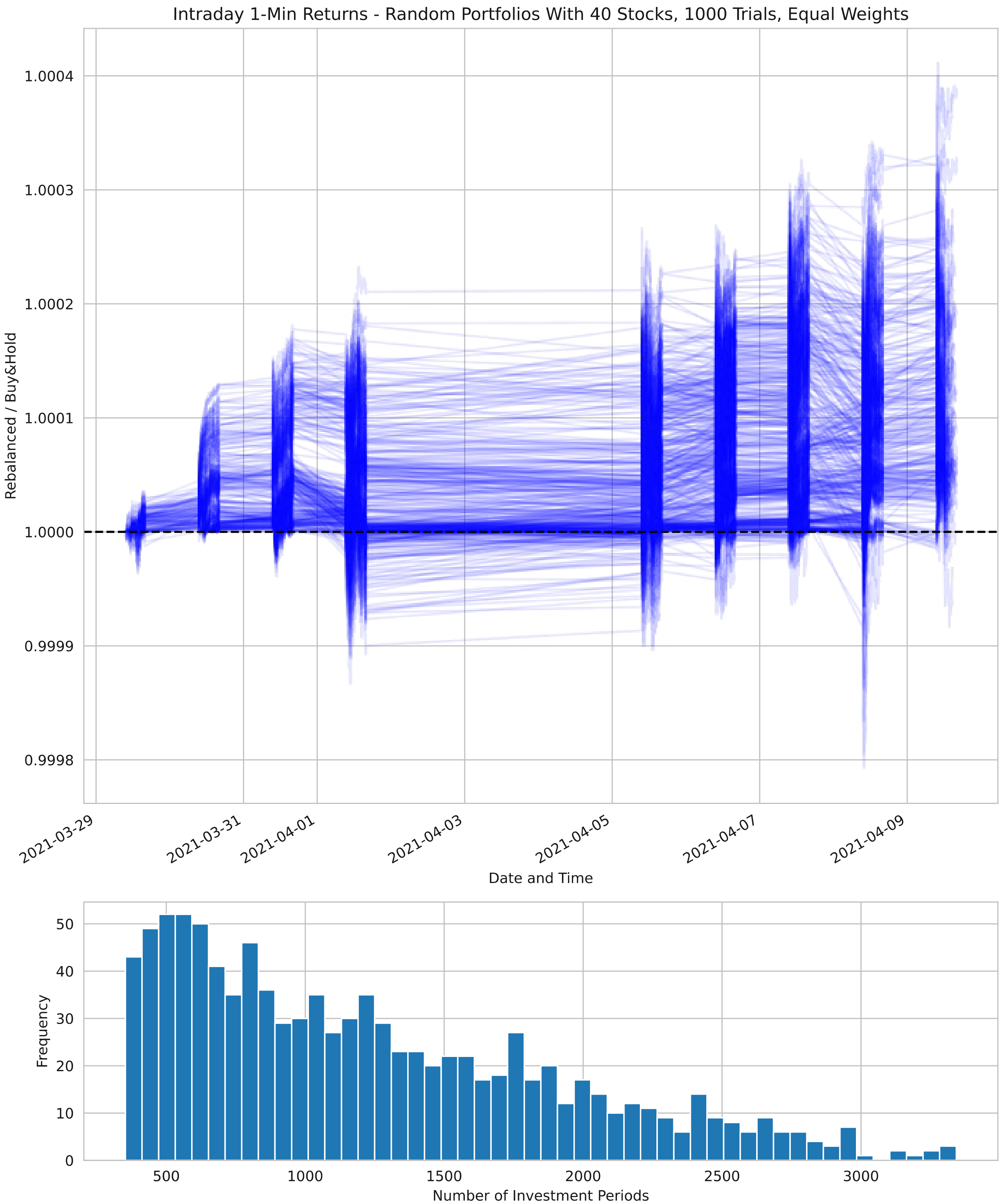

*Figure 47: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using **1-minute intraday returns**. There are **1000 portfolios each with 40 random stocks and random start/end-dates**. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

8.2 Intraday Returns 1-Min Intervals – Comparing Statistics

Figure 48 compares the arithmetic means for 1-min intraday returns of the Rebalanced and Buy&Hold portfolios. The p-values in the bottom-plot are close to zero so the differences are statistically significant. The white text on the box-plot shows the average difference between the two strategies range between 3.9e-08 and 8.0e-08 which corresponds to an annualized excess return of the Rebalanced portfolios of around $450 \cdot 250 \cdot 3.9\text{e-}08 \simeq 0.44\,\%$ and $450 \cdot 250 \cdot 8.0\text{e-}08 \simeq 0.90\,\%$, because there are 450 minutes in the 6.5 hours of a single trading-day and there are roughly 250 trading-days in a year. So rebalancing the portfolio every minute, every day, for an entire trading year, would require 112,500 trades per stock, and is only estimated to give an excess return over Buy&Hold portfolios between 0.44% and 0.90% depending on the portfolio size. It hardly seems worth the hassle.

Figure 49 compares the geometric means for 1-min intraday returns of the Rebalanced and Buy&Hold portfolios. These are nearly identical to the arithmetic means in Figure 48.

Figure 50 compares the standard deviations for 1-min intraday returns of the Rebalanced and Buy&Hold portfolios. For portfolios with only 2 stocks, the p-value is nearly 0.8 so there is probably no difference in the standard deviation for the two portfolio strategies when they only invest in 2 stocks. But for portfolios with 5, 10, 20 and 40 stocks the p-values are close to zero, so the average difference is statistically significant, although the box-plots and the white text shows that the standard deviations still appear to be nearly identical for the two portfolio strategies.

Figure 51 compares the Sharpe ratios for 1-min intraday returns of the Rebalanced and Buy&Hold portfolios. The p-values in the bottom-plot are close to zero so the average differences are statistically significant. But the box-plot shows that the Sharpe ratios are very similar for the two strategies.

Figure 52 compares the Max Drawdowns and Figure 53 compares the Max Pullups for 1-min intraday returns of the Rebalanced and Buy&Hold portfolios. The p-values in both plots are near zero so the average differences are statistically significant, but both of the box-plots show that the two portfolio strategies actually have very similar performance. But this could be because the data only covers 9 trading-days where the stocks did not really crash or rally. If we had intraday data for several years, it is possible that we would see similar tendencies as we saw in the previous sections for daily, weekly and monthly data, where the Rebalanced portfolios performed somewhat worse than Buy&Hold during stock-market crashes, but the Rebalanced portfolios also performed significantly better than Buy&Hold during the subsequent stock-market recoveries.

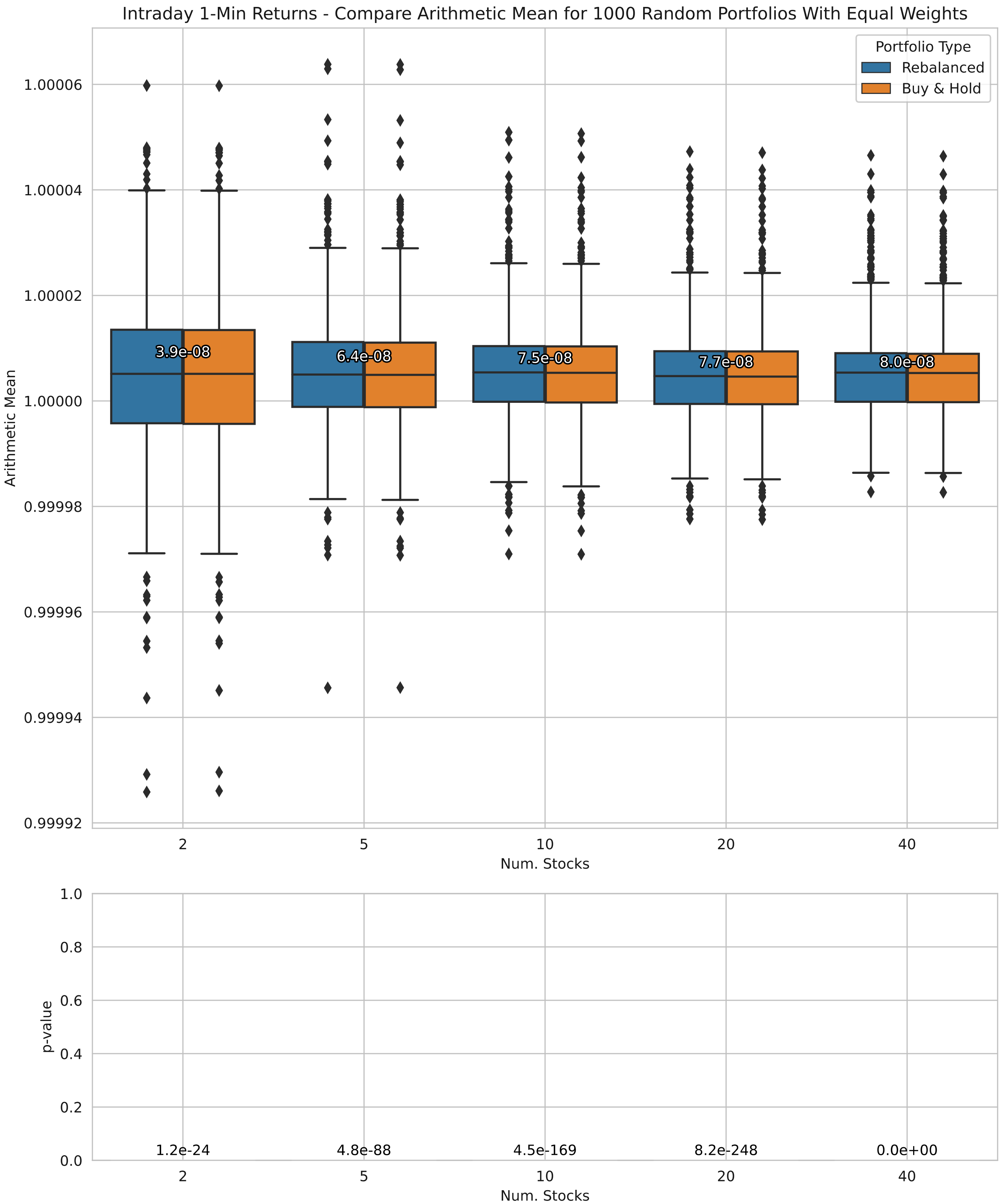

Figure 48: Compare ***arithmetic means of 1-minute intraday returns*** *for Rebalanced and Buy&Hold portfolios.*

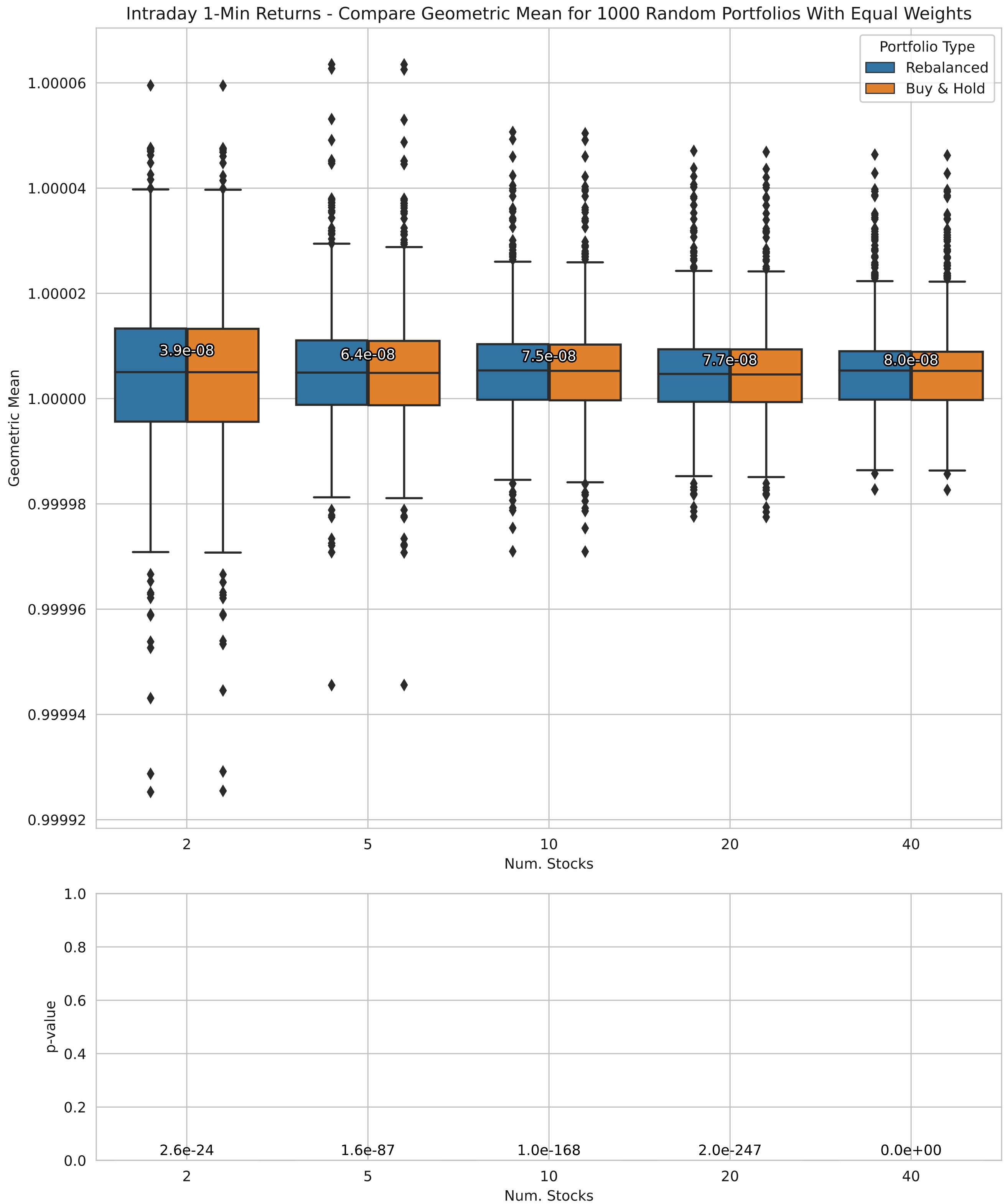

Figure 49: Compare ***geometric means of 1-minute intraday returns*** *for Rebalanced and Buy&Hold portfolios.*

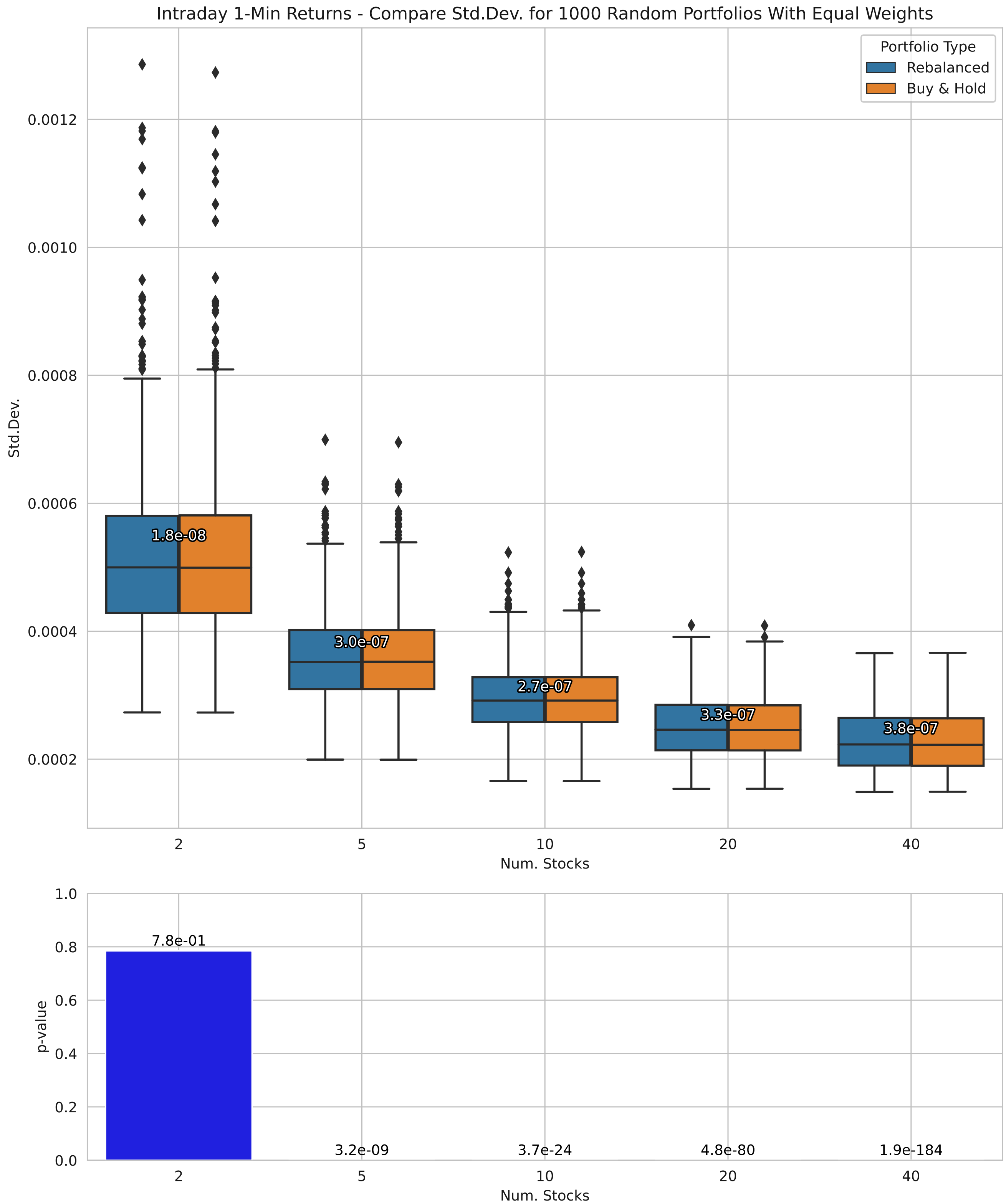

*Figure 50: Compare **standard deviations of 1-minute intraday returns** for Rebalanced and Buy&Hold portfolios.*

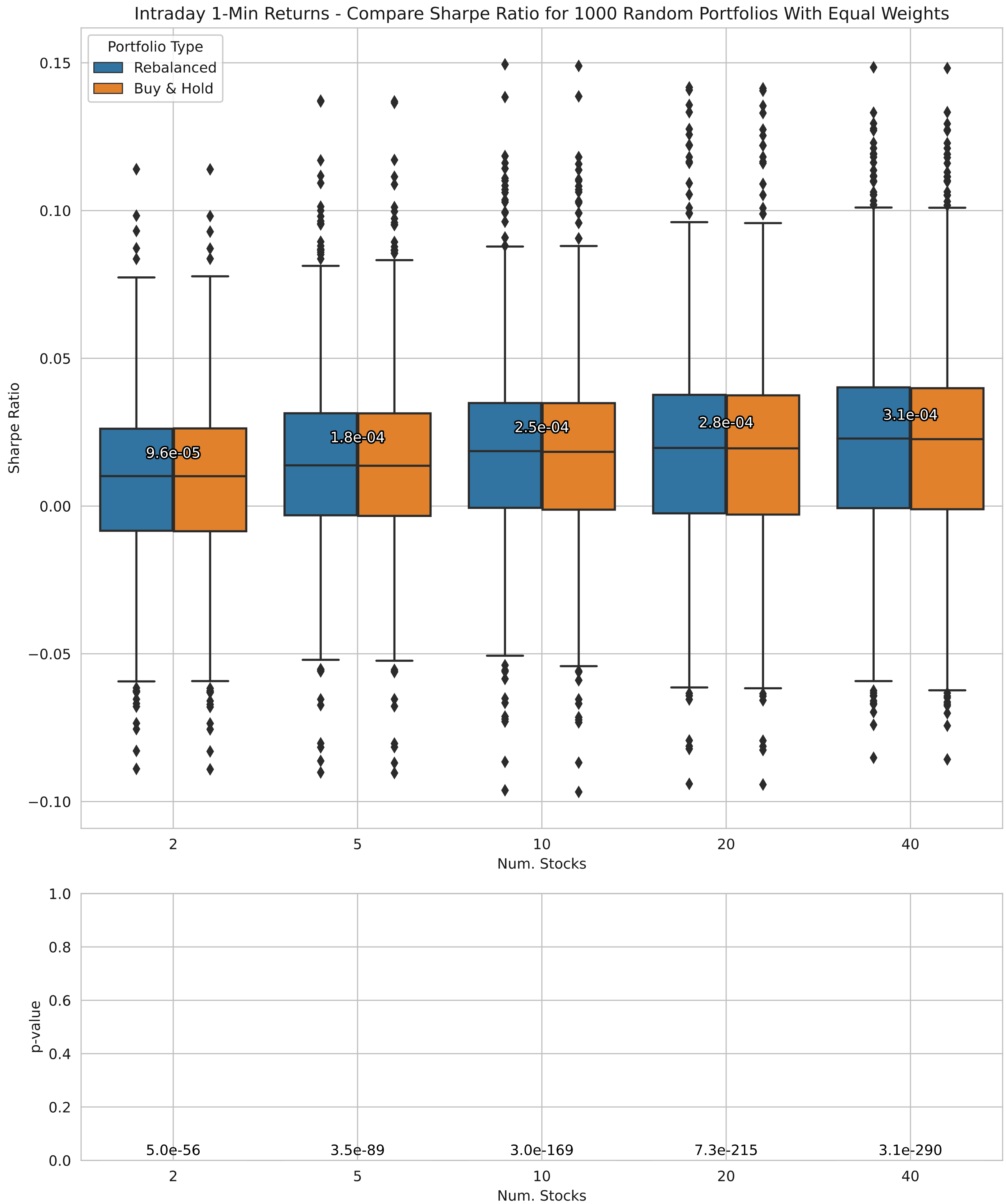

*Figure 51: Compare **Sharpe ratios of 1-minute intraday returns** for Rebalanced and Buy&Hold portfolios.*

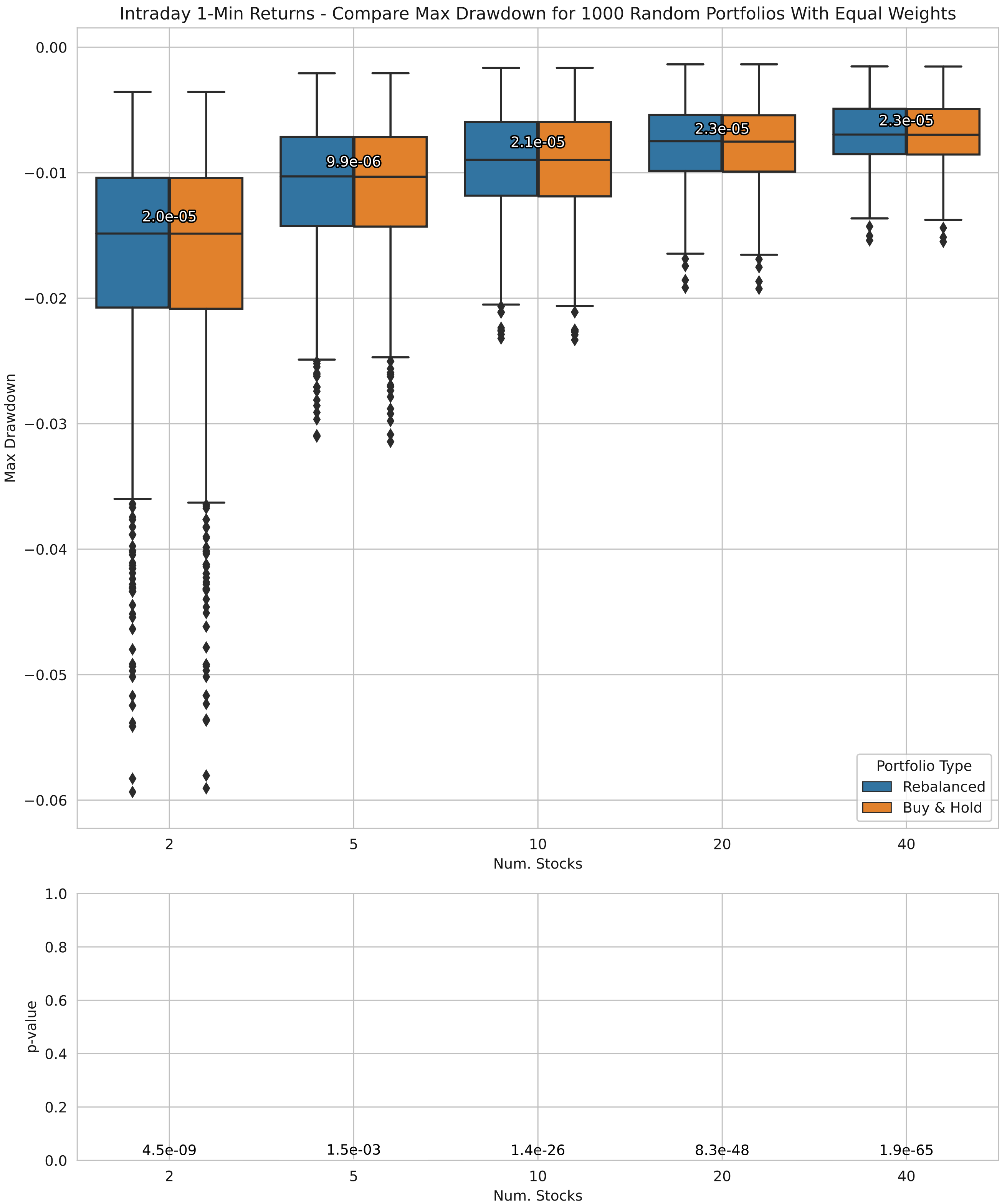

*Figure 52: Compare **Max Drawdowns of 1-minute intraday returns** for Rebalanced and Buy&Hold portfolios.*

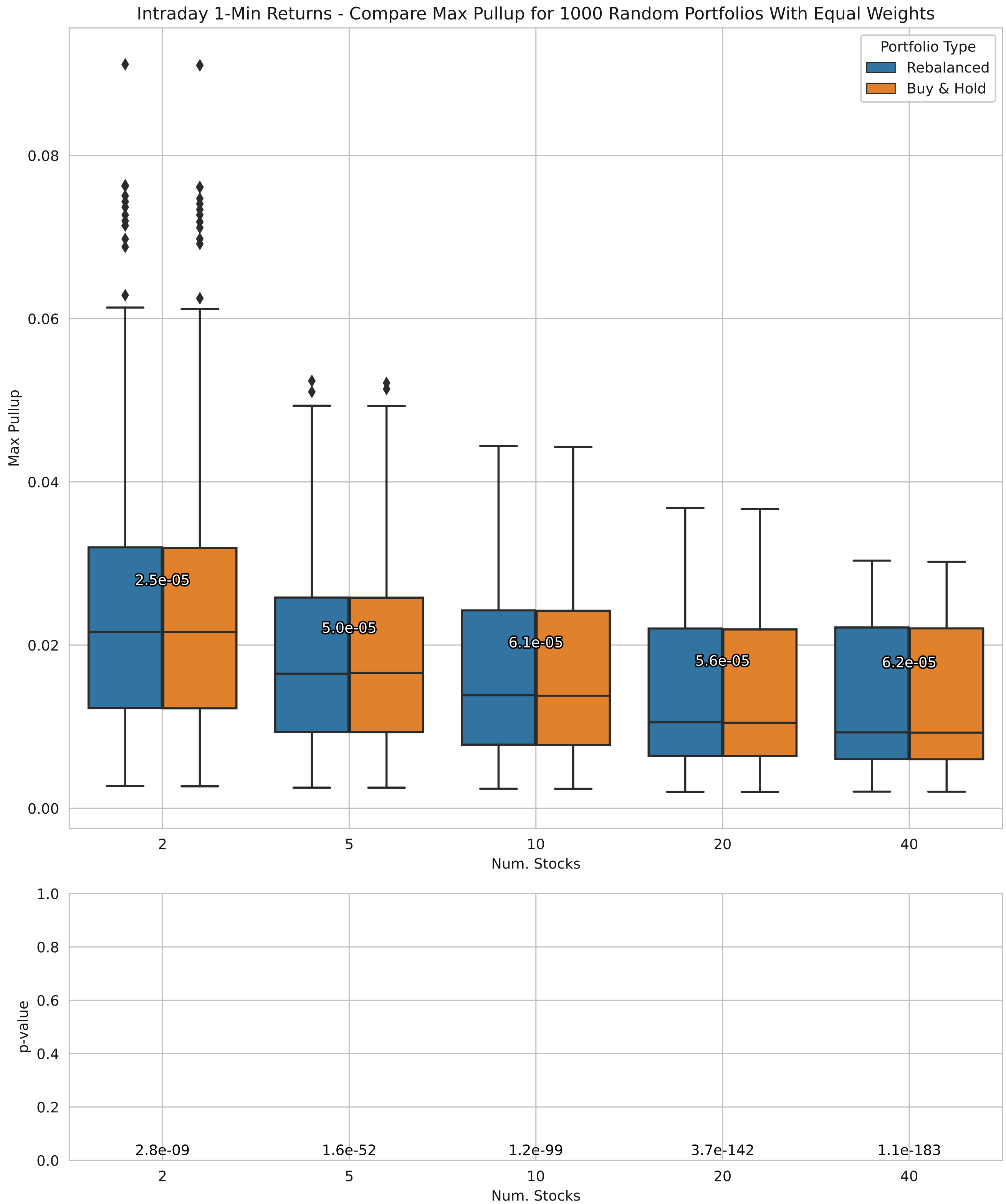

Figure 53: Compare ***Max Pullups of 1-minute intraday returns*** *for Rebalanced and Buy&Hold portfolios.*

9. Intraday Returns 5-Minute Intervals

We now consider intraday stock-returns for 5-minute intervals. We only have this data for the 70 stocks listed in Section 12.4 and for 19 days in March and April 2021. You should therefore consider these experiments as preliminary and the results may not be representative when using intraday data for several years and hundreds or even thousands of stocks. Figure 44 shows the cumulative returns for the 70 stocks in our data-set normalized to begin at 1.0 and the red line shows the arithmetic average at each time-step.

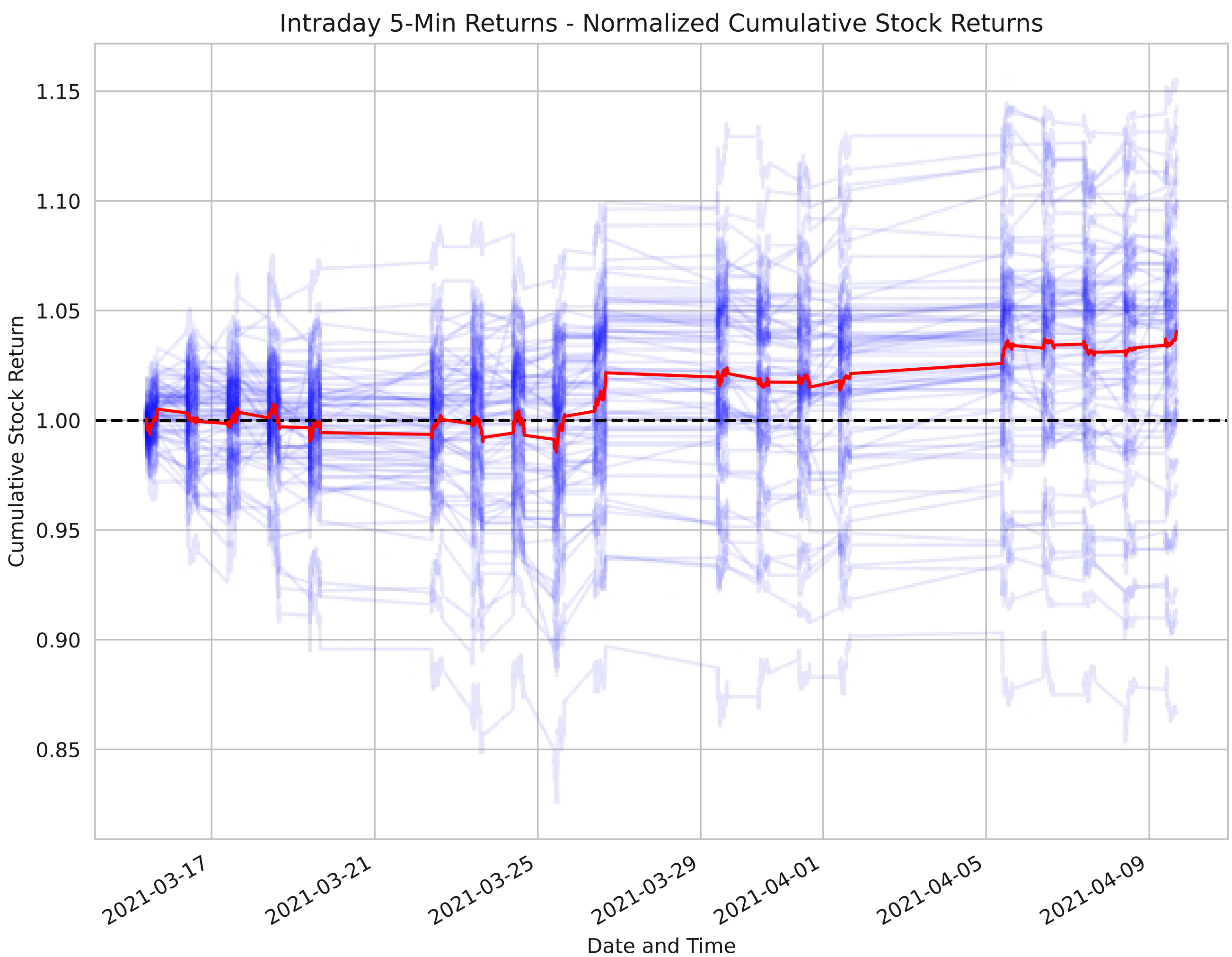

*Figure 54: **Cumulative returns** for the **70 stocks** in the data-set with **5-minute intraday returns**, which are plotted as blue lines and normalized to all start at the value 1 so they can easily be compared. The red line is the arithmetic mean for each time-step.*

9.1 Intraday Returns 5-Min Intervals – Random Portfolios

Figure 55 shows the ratio between the Rebalanced and Buy&Hold portfolios for 5-minute intraday stock-returns and 1000 portfolios consisting of 5 randomly selected stocks and random start/end-dates. A ratio of 1.0 means that the Rebalanced and Buy&Hold portfolios had identical returns, while a ratio above 1.0 means that the Rebalanced portfolio performed best, and vice versa, a ratio below 1.0 means that the Buy&Hold portfolio performed best. As can be seen from Figure 55, there does not seem to be a consistent advantage to either the Rebalanced or Buy&Hold portfolios.

Figure 56 shows it for portfolios with 10 random stocks each, and Figure 57 shows it for portfolios with 40 random stocks each, as well as random start/end-dates. For portfolios with 10 stocks, the ratios between Rebalanced and Buy&Hold portfolios are mostly above 1.0, and for portfolios with 40 stocks the ratios are nearly always above 1.0, meaning that the Rebalanced portfolios are mostly better than the Buy&Hold portfolios. Although we only have data for 19 days, there are actually almost 1500 data-points, so the portfolios are rebalanced 1500 times. So it appears that the Rebalanced portfolios are often better than Buy&Hold portfolios when the rebalancing is done every 5 minutes. This is similar to our findings in the previous section for 1-minute intervals, but once again we would need data for hundreds or even thousands of stocks over multiple years in order to investigate this properly and conclude that the Rebalanced portfolios do actually perform better than the Buy&Hold portfolios.

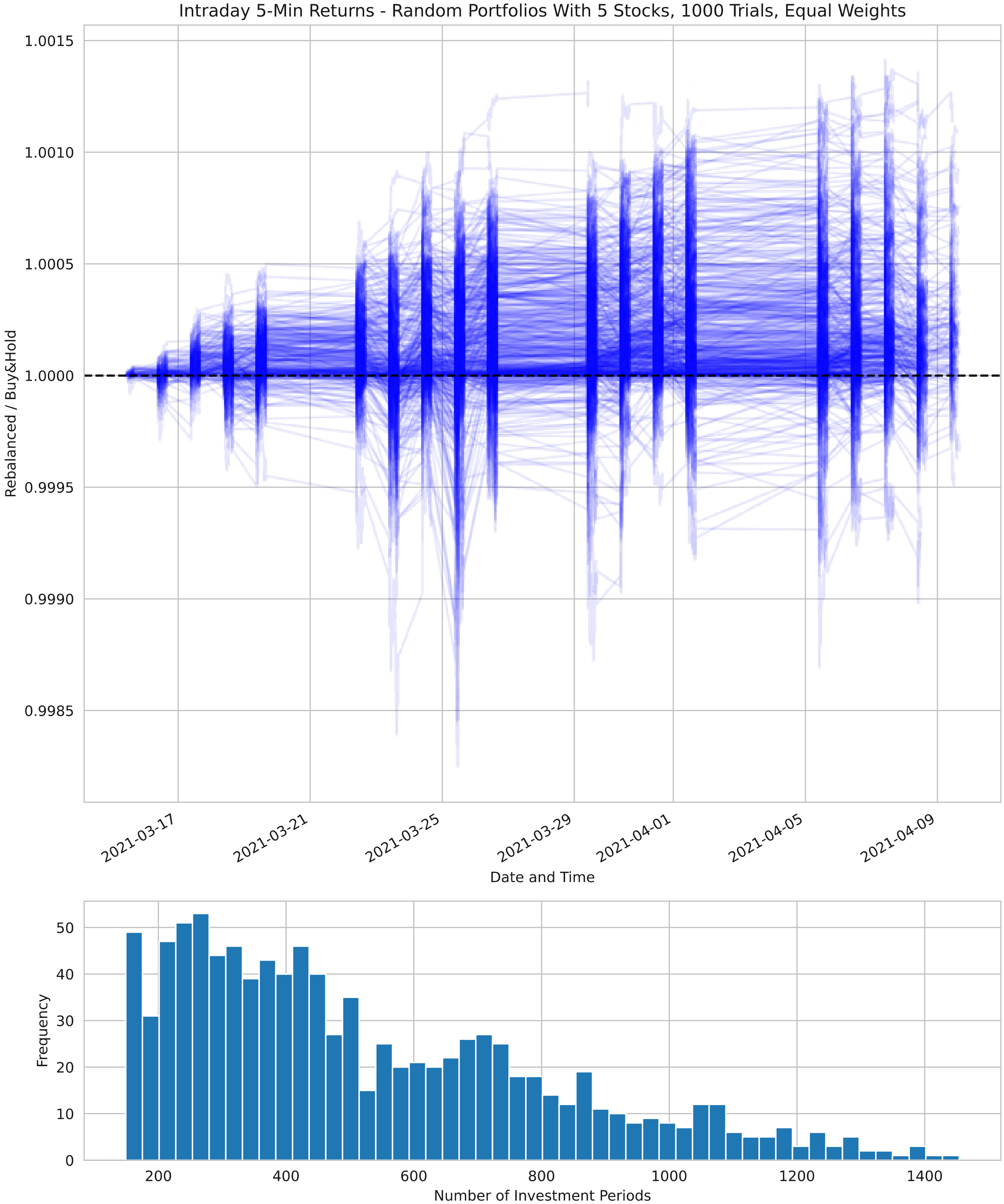

*Figure 55: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using **5-minute intraday returns**. There are **1000 portfolios each with 5 random stocks and random start/end-dates**. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

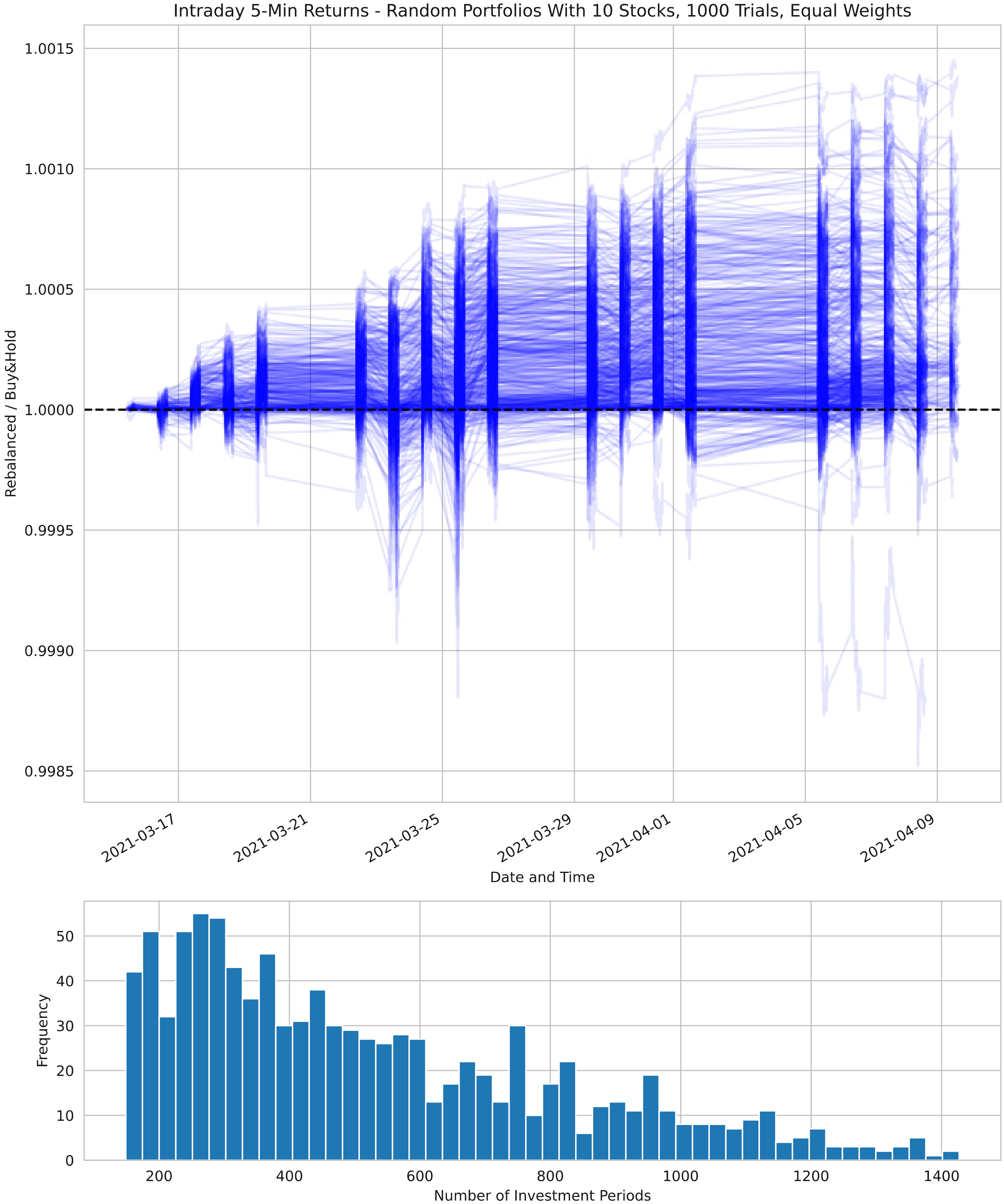

*Figure 56: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using **5-minute intraday returns**. There are **1000 portfolios each with 10 random stocks and random start/end-dates**. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

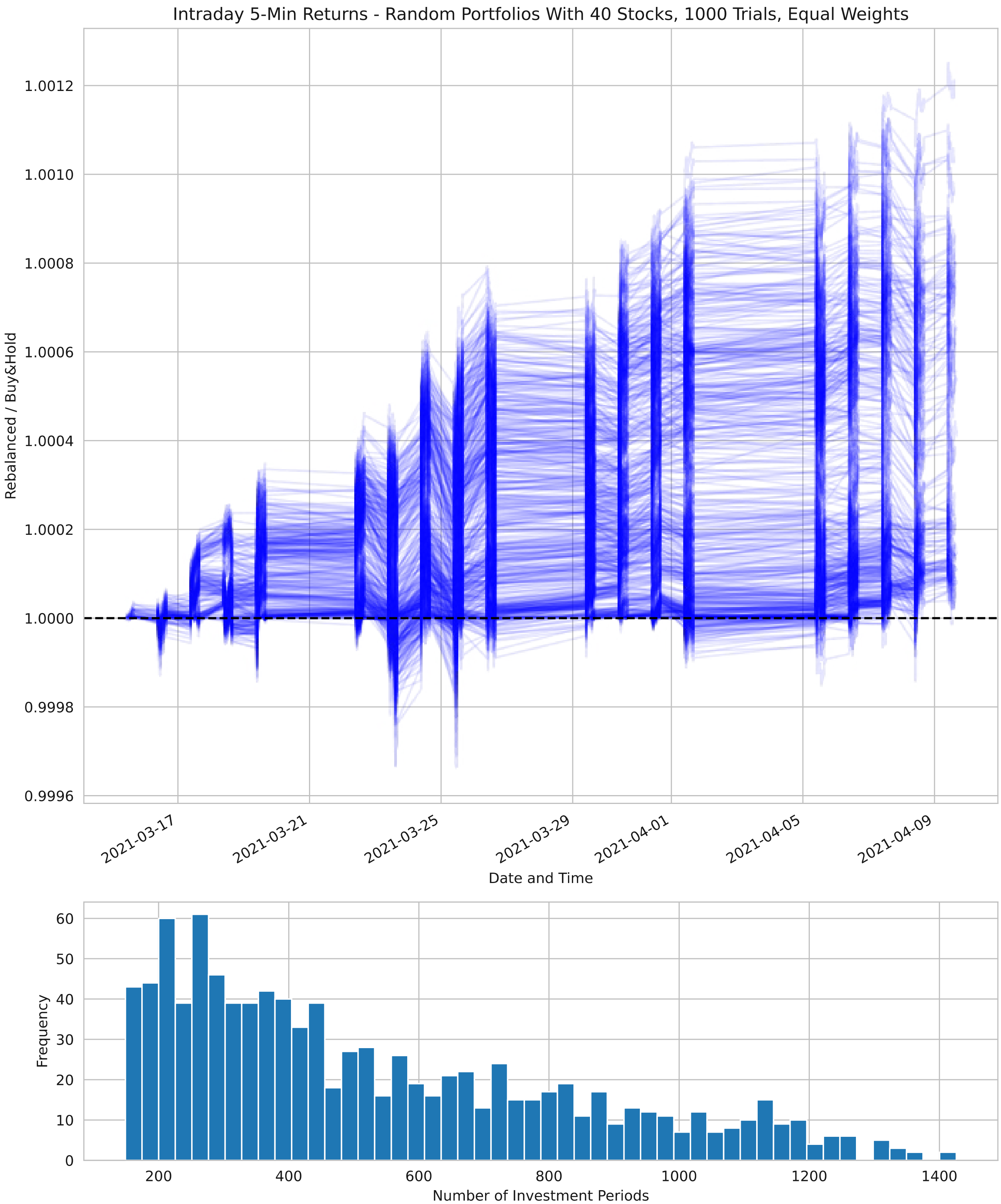

Figure 57: Top plot shows the ratio between Rebalanced and Buy&Hold portfolios using ***5-minute intraday returns****. There are* ***1000 portfolios each with 40 random stocks and random start/end-dates****. Bottom plot shows the distribution of the number of investment periods in these random portfolios.*

9.2 Intraday Returns 5-Min Intervals – Comparing Statistics

Figure 58 compares the arithmetic means for 5-min intraday returns of the Rebalanced and Buy&Hold portfolios. The p-values in the bottom-plot are very close to zero so the differences are statistically significant. The white text on the box-plot shows the average difference between the two strategies range between 2.6e-07 and 5.6e-07 which corresponds to an annualized excess return of the Rebalanced portfolios of around $90 \cdot 250 \cdot 2.6\text{e-}07 \simeq 0.59\,\%$ and $90 \cdot 250 \cdot 5.6\text{e-}07 \simeq 1.26\,\%$, because there are 90 five-minute intervals in the 6.5 hours of a single trading-day and there are roughly 250 trading-days in a year. So rebalancing the portfolio every 5 minutes, every day, for an entire trading year, would require 22,500 trades per stock, and is only estimated to give an excess return over Buy&Hold portfolios between 0.59% and 1.26% depending on the portfolio size.

Figure 59 compares the geometric means for 5-min intraday returns of the Rebalanced and Buy&Hold portfolios. These are nearly identical to the arithmetic means in Figure 58.

Figure 60 compares the standard deviations for 5-min intraday returns of the Rebalanced and Buy&Hold portfolios. Although the p-values are close to zero so the average differences are statistically significant, the box-plot and white text shows that the standard deviations are nearly identical for the Rebalanced and Buy&Hold portfolios.

Figure 61 compares the Sharpe ratios for 5-min intraday returns, and once again the p-values are close to zero so the average differences are statistically significant, but the box-plot and white text shows that the Sharpe ratios are nearly identical for the Rebalanced and Buy&Hold portfolios.

Figure 62 compares the Max Drawdowns where the p-values are close to zero for portfolios of 5 stocks or more, but the average differences between the two portfolio strategies are actually tiny, so there does not appear to be any difference in Max Drawdowns for the two portfolio strategies – at least not for this small data-set that only covers 19 days of 5-minute intervals.

Figure 63 compares the Max Pullups for 5-min intraday returns. The p-values are all close to zero so the average differences are statistically significant, and although the white text shows that the average differences are quite small, the quartiles in the box-plot do suggest that perhaps the Rebalanced portfolios perform slightly better than Buy&Hold portfolios in short intraday stock-rallies.

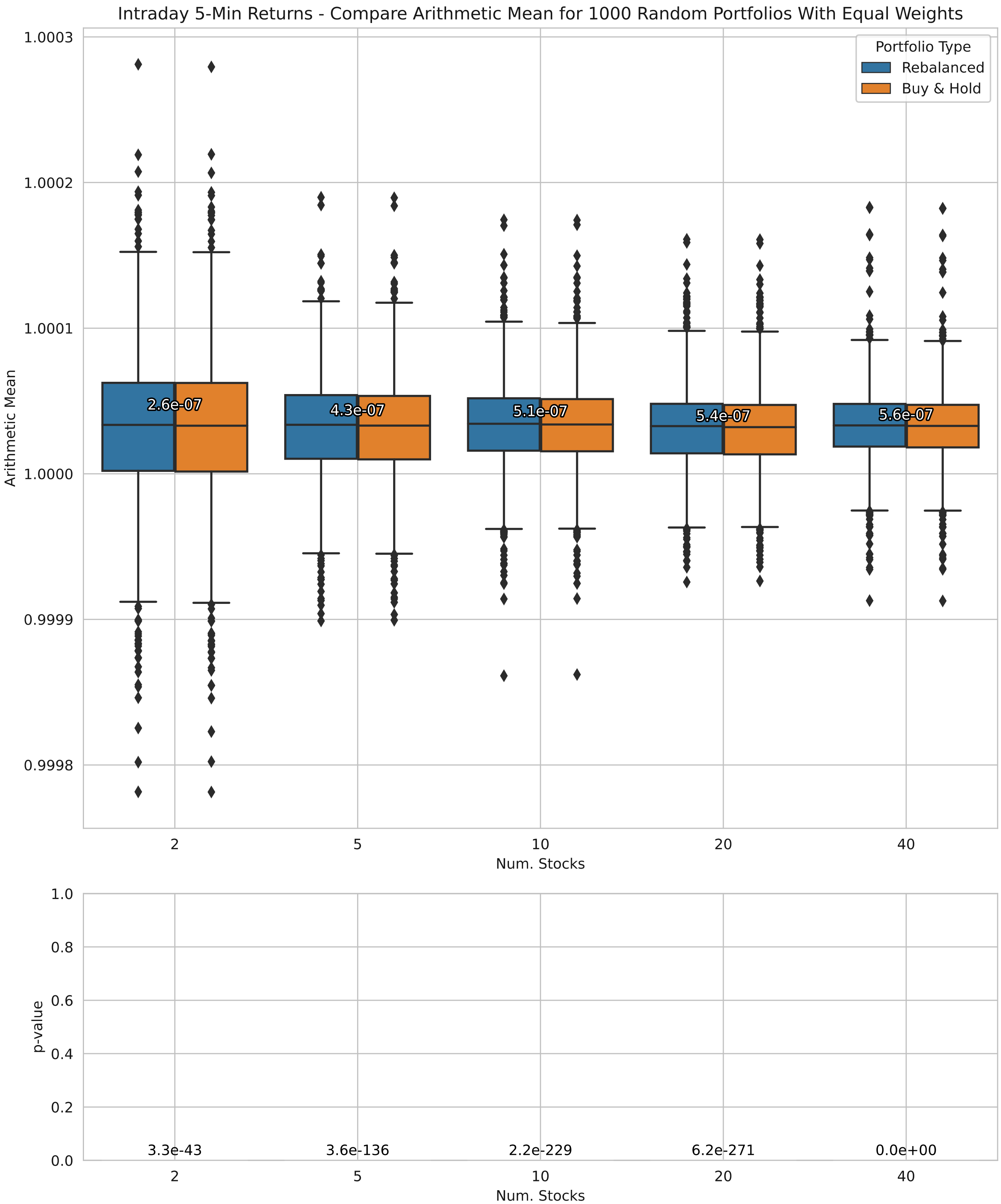

*Figure 58: Compare **arithmetic means of 5-minute intraday returns** for Rebalanced and Buy&Hold portfolios.*

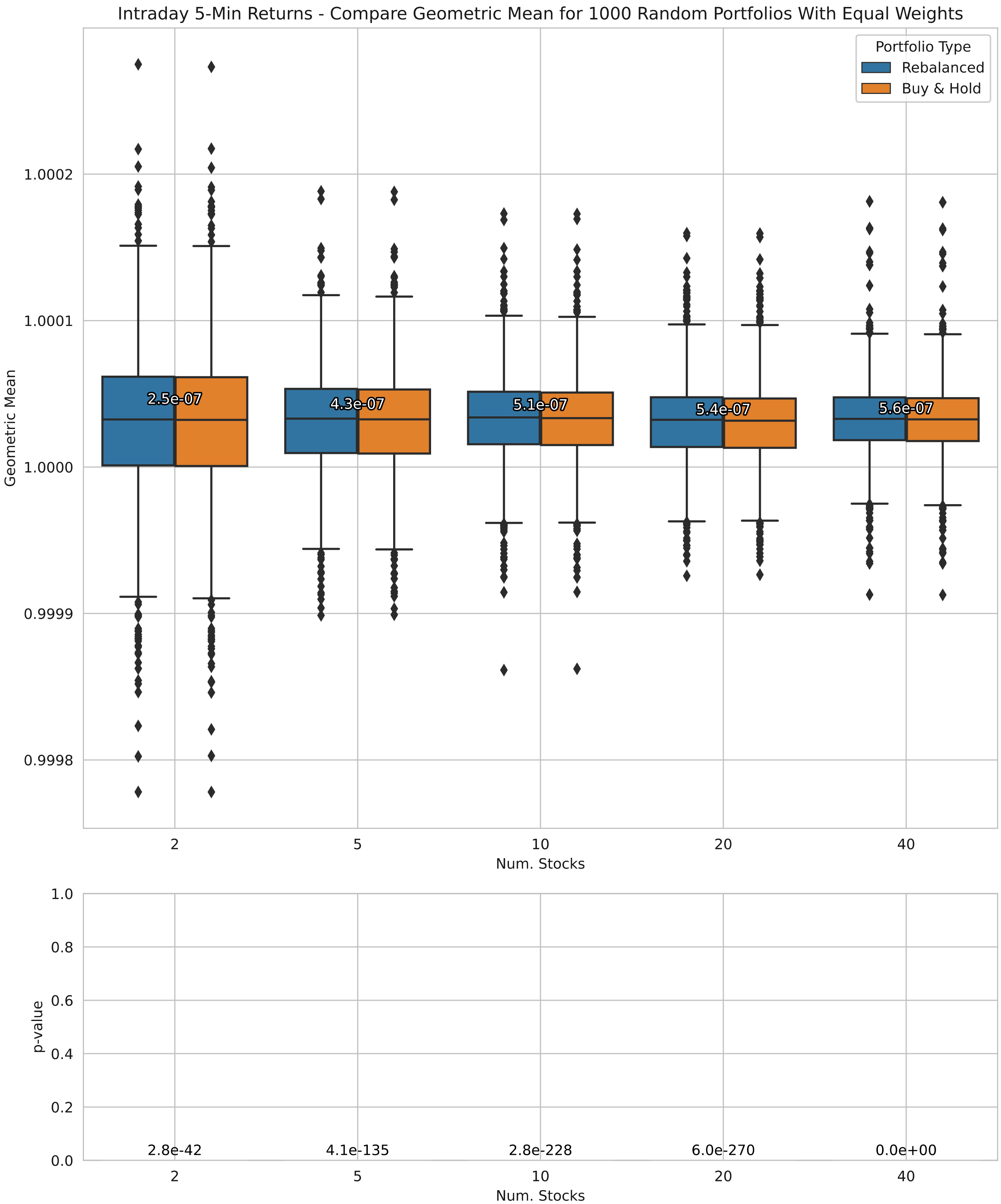

*Figure 59: Compare **geometric means of 5-minute intraday returns** for Rebalanced and Buy&Hold portfolios.*

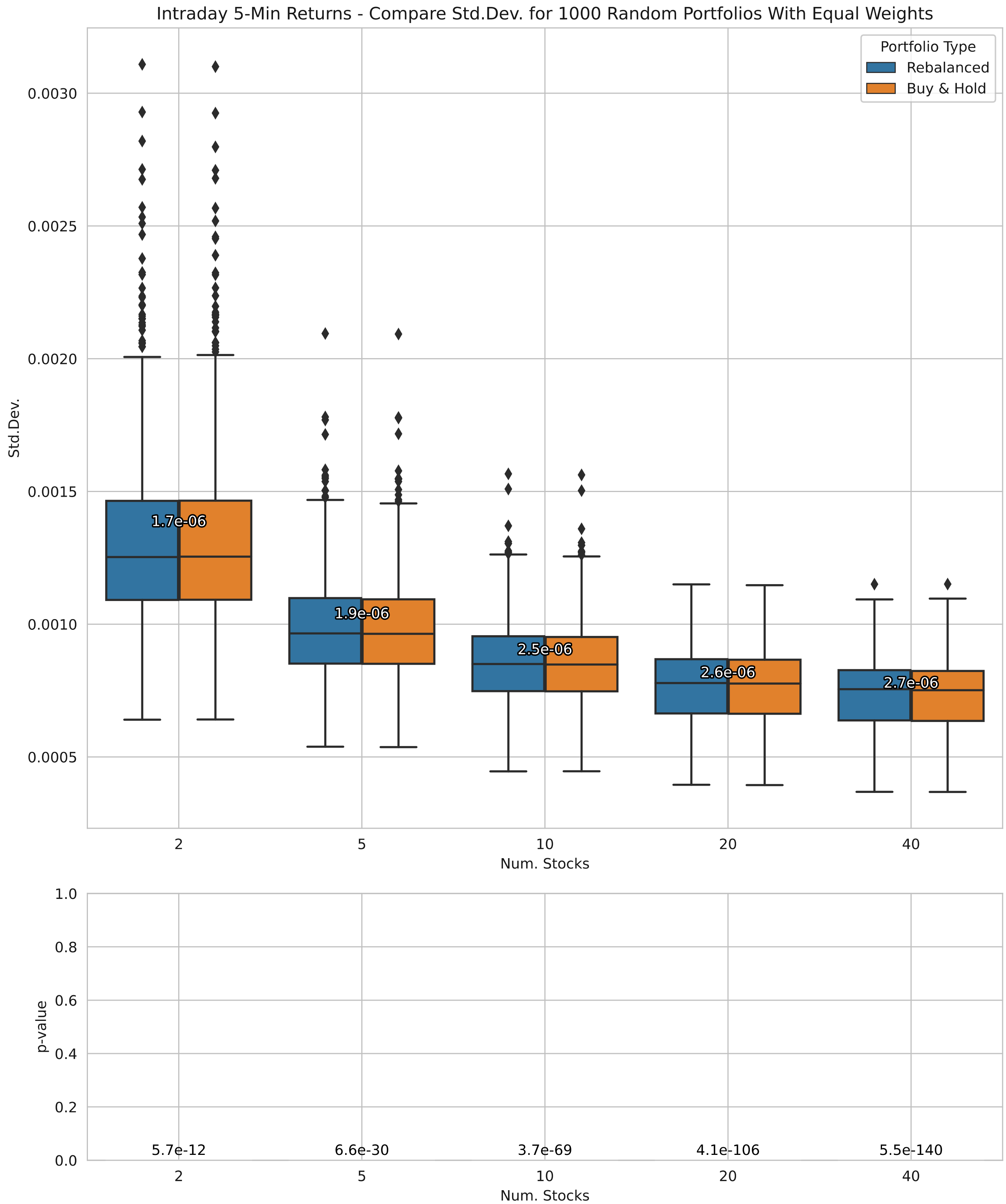

Figure 60: Compare ***standard deviations of 5-minute intraday returns*** *for Rebalanced and Buy&Hold portfolios.*

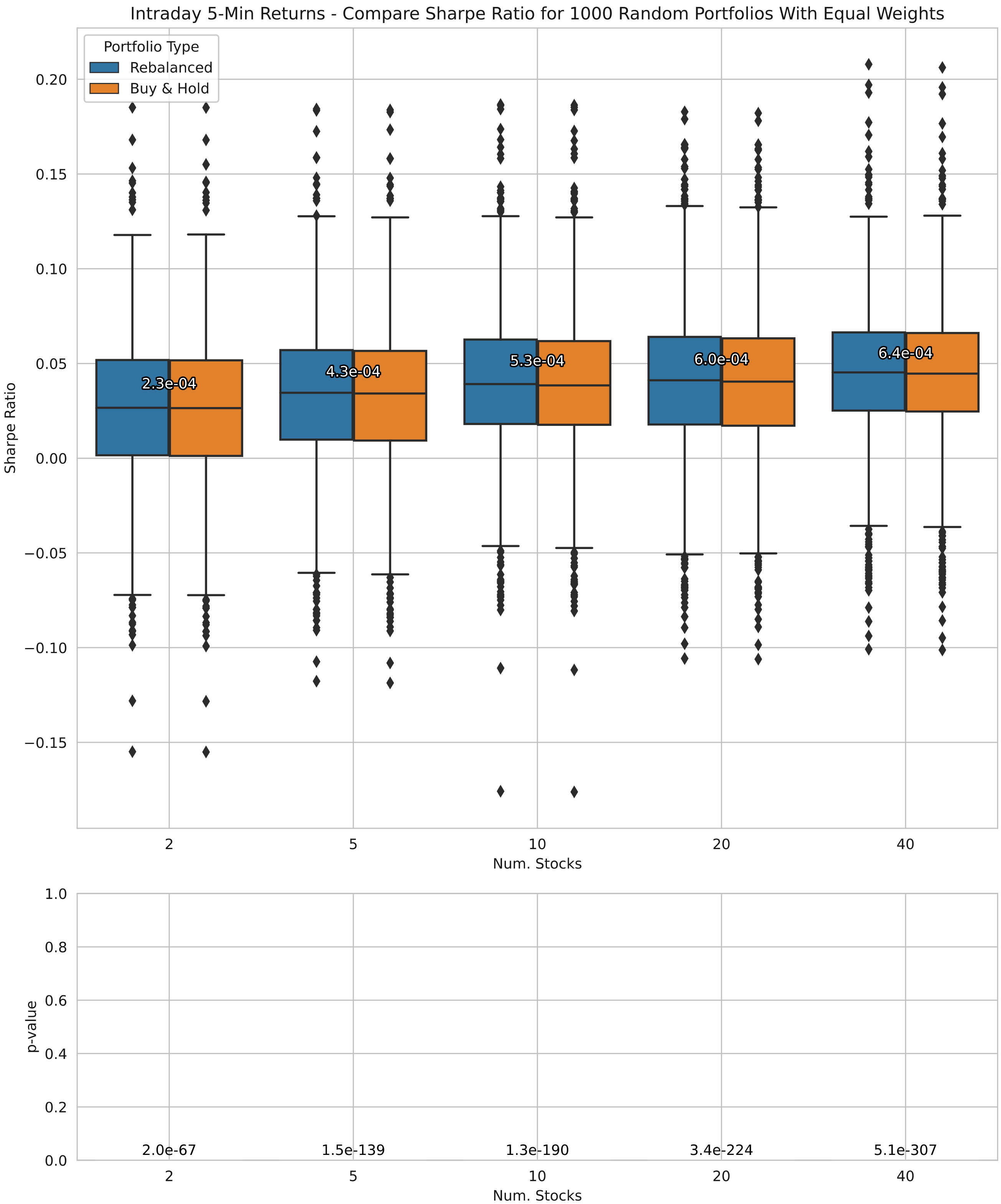

*Figure 61: Compare **Sharpe ratios of 5-minute intraday returns** for Rebalanced and Buy&Hold portfolios.*

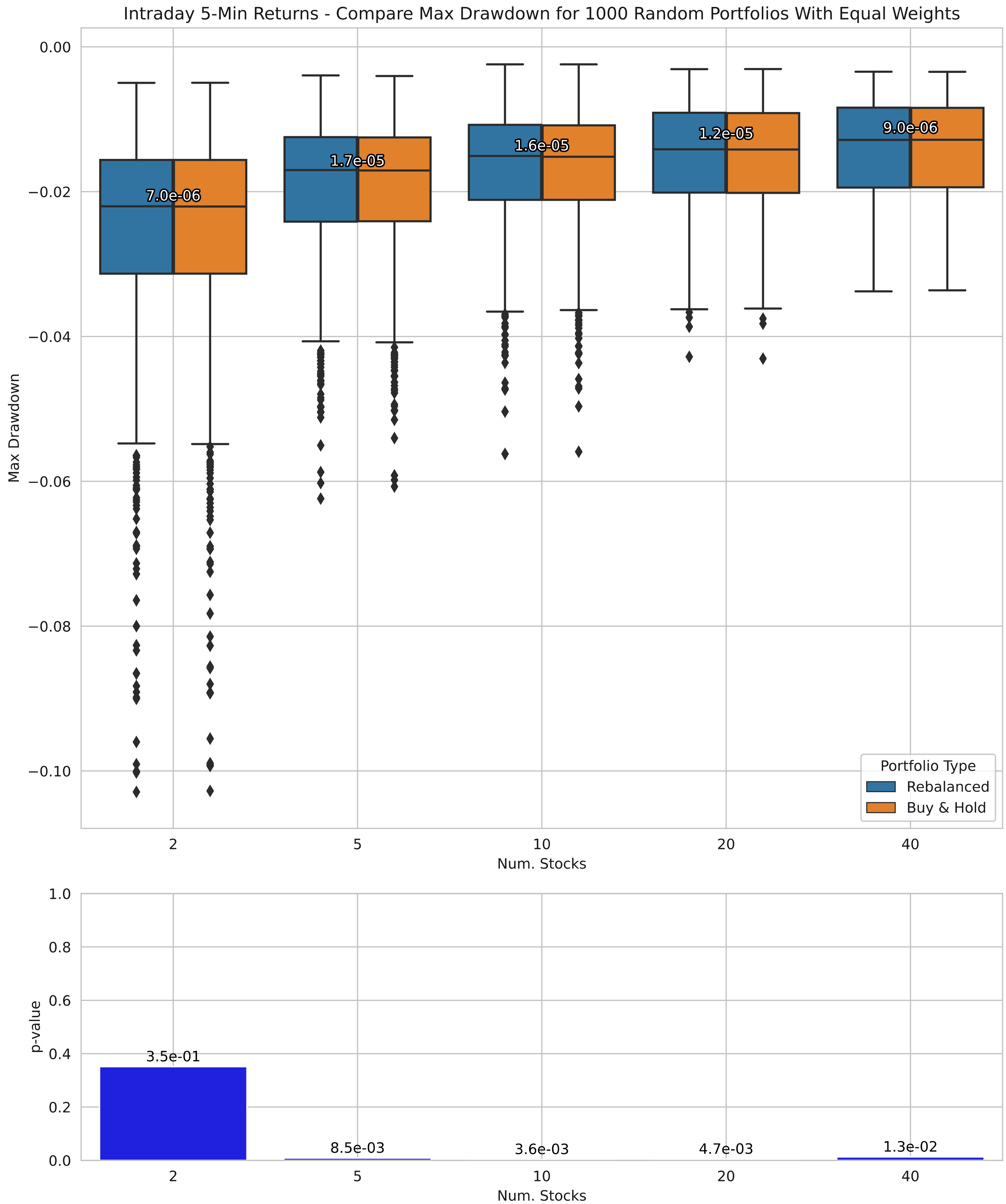

*Figure 62: Compare **Max Drawdowns of 5-minute intraday returns** for Rebalanced and Buy&Hold portfolios.*

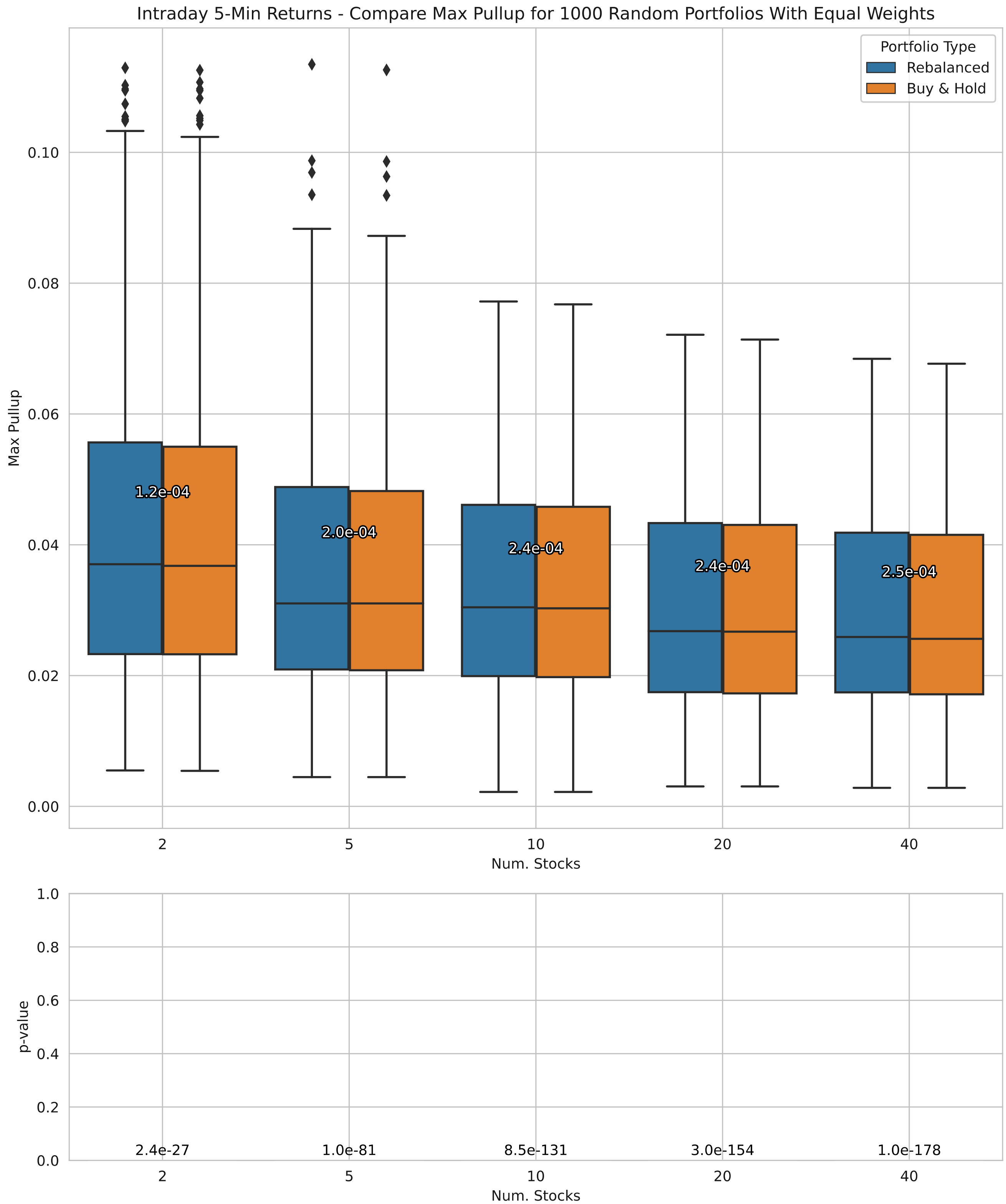

Figure 63: Compare ***Max Pullups of 5-minute intraday returns*** *for Rebalanced and Buy&Hold portfolios.*

10. Correlations

In the previous two sections, we saw that the Rebalanced portfolios with 40 random stocks usually performed better than the Buy&Hold portfolios using the same stocks, provided the rebalancing was done in 1 or 5-minute intervals. Although the data-sets were so small that we cannot conclude this holds in general, it was an interesting result that merits further research with much larger data-sets.

One possible reason why this could potentially work in general, might have to do with correlation amongst the stock-returns. Figure 64 shows histograms comparing the distributions for the correlations of stock-returns. We use the so-called Spearman correlations which are more robust to outliers than the more common Pearson correlations. The blue bars show the correlations of the 1-minute stock-returns for the 70 stocks from Section 12.4, the green bars show the correlations of the 5-minute stock-returns for the same 70 stocks, and the red bars show the correlations of daily stock-returns for those same 70 stocks again. Although these correlations are calculated for very different time-periods, there is a strong indication that the shorter-term stock-returns are much less correlated than the daily stock-returns. The 1-minute stock-returns have a correlation coefficient around 0.12 on average, while the 5-minute stock-returns have a correlation coefficient around 0.17 on average, and the daily stock-returns have a correlation coefficient around 0.41 on average. Perhaps the lower short-term correlation could help explain why the short-term stock-returns might actually benefit from frequent rebalancing.

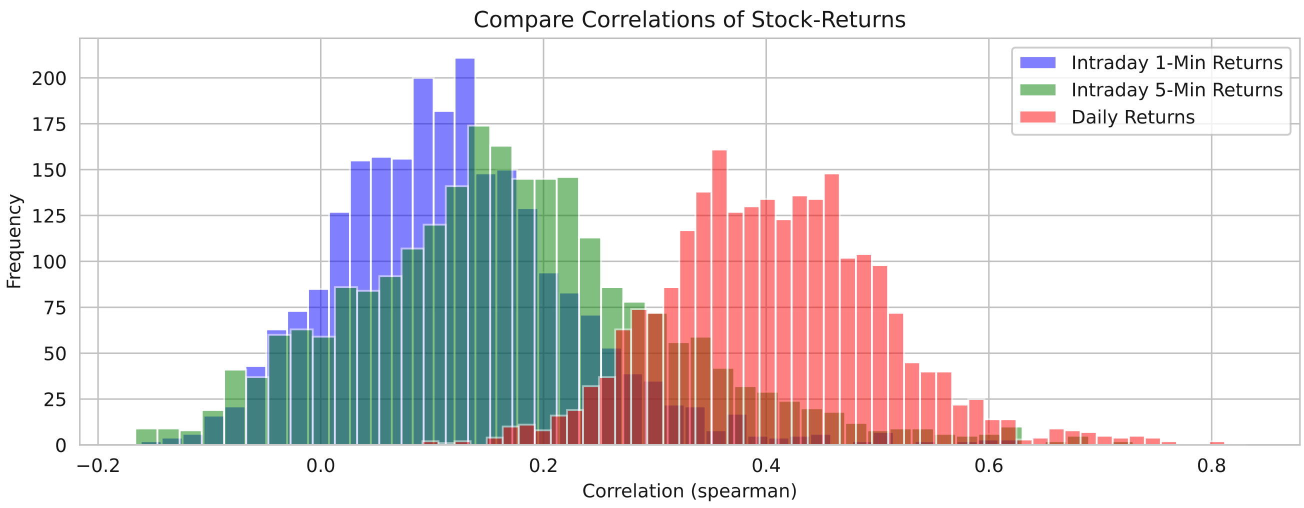

Figure 64: Compare correlation distributions for intraday and daily stock-returns.

11. Conclusion

In this paper we have conducted numerous empirical experiments with random stock-portfolios to assess whether the Rebalanced portfolios consistently performed better than the Buy&Hold portfolios, as claimed by numerous academic papers. Our many experiments did not find a consistent and substantial advantage to either portfolio strategy. For example, when considering the entire period between 2007 and 2021 for our data-set with daily stock-returns, the Rebalanced portfolios had a slightly higher arithmetic mean daily return than the Buy&Hold portfolios, but the annualized excess return was less than 0.5%, so it was hardly worth the effort to implement portfolio rebalancing for its own sake, especially when trading-costs and taxes also have to be taken into account, in which case the Rebalanced portfolios could end up significantly under-performing simple Buy&Hold portfolios.

Furthermore, the advantage of the Rebalanced portfolios seemed to occur mostly between the years 2007 and 2010, a period where the U.S. stock-markets experienced a great crash during the "Financial Crisis" of 2008-9. The many experiments in this paper suggest that the Rebalanced portfolios tend to perform somewhat worse when the stock-market is crashing, but the Rebalanced portfolios also tend to perform substantially better than the Buy&Hold portfolios when the stock-market recovers again. This seems to be even more pronounced for weekly and monthly rebalancing. This is counter-intuitive because simple experiments such as the ones in Section 3 would suggest that portfolio rebalancing is harmful whenever the stocks are trending either up or down.

Maybe this is a useful guideline we can use in our investing. If we can only choose to either keep our investments or rebalance the portfolio – that is, we cannot employ "stop-loss" tactics to move into cash when the stock-market crashes – then it appears that we should avoid rebalancing the portfolio during the stock-market crash, and then rebalance the portfolio regularly during the stock-market's recovery. And weekly rebalancing seems to be better than both daily and monthly rebalancing.

We also considered a very small data-set for intraday stock-returns of 1 and 5-minute intervals, where the Rebalanced portfolios of 40 random stocks from a data-set of only 70 stocks total, consistently performed better than the Buy&Hold portfolios. For 5-minute intervals the annualized excess return was estimated between 0.6% and 1.3% depending on the portfolio size. Because the data-sets were very small we cannot conclude that this holds in general, but it certainly merits more research with many more stocks and many more years of data. We should also investigate whether the performance difference is perhaps due to "phantom" data-points as described in Section 12.3, or perhaps due to floating-point rounding errors, which can occur when multiplying many numbers with many decimals that may require greater numerical precision.

Finally, it should be noted that our data-set probably has "survivorship bias" because it has very few stocks that became nearly worthless. Due to the counter-intuitive nature of how the Rebalanced and Buy&Hold strategies perform on real-world data, it is unclear how this "survivorship bias" could affect the conclusions of this paper.

12. Data & Computer Code

This section briefly describes the computer code and the data processing, as well as listing all the stock-tickers used in the data-sets.

12.1 Computer Code

The computer code used to generate all the experiments and plots in this paper is written in a so-called Python Notebook which is available on GitHub. You should download the entire GitHub repository so you will get all the necessary code and data-files, so you can repeat all the experiments yourself, or you can reuse the computer code with your own data for other stocks.

12.2 Data

The Python Notebook described above, automatically downloads the stock-data from freely available internet websites, although you may first need to obtain a free API-key to obtain the intraday data. This is described in more detail in the Python Notebook.

12.3 Data Processing

Both the daily and intraday stock-data is processed and cleaned before it is being used, in order to remove problematic data-points and make the analysis easier and hopefully more reliable.

The data-set for daily stock-prices actually contains more than 2600 stocks, but we remove stocks where the median daily trading-volume is less than USD 1 million, or the max daily return is greater than 100%, or more than 20% of the days have missing data. This results in only 923 stocks remaining in the data-set for daily stock-prices, which are listed in Section 12.5 below.

The data-sets with intraday stock-prices for 1 and 5-minute intervals only contain 70 stocks, which are listed in Section 12.4 below. The intraday stock-data is truncated so the pre-market and after-hours data is eliminated. This is because it may not exist for all stocks, or it may not be available for the same intervals, which may create “phantom” data-points that may distort the results when doing the portfolio rebalancing. So we only use data-points for the stock-market’s open hours between 9:30 and 16:00. These data-points are then aligned across all stocks and missing data-points are forward-filled before the single-period returns are calculated. The forward-filling creates some “phantom” data-points and it is possible that this distorts the returns of the Rebalanced portfolios which may contribute to the Rebalanced portfolios seemingly performing better than the Buy&Hold portfolios, which is something you should investigate more thoroughly if you have access to a larger intraday data-set.

12.4 Stock-Tickers for Intraday Data

The following are all 70 stock-tickers for the intraday data used in Sections 8 and 9:

AAPL, ADBE, ADM, AEP, AMZN, APD, ATVI, BLL, CL, CMG, CMI, CVX, DIS, DOV, DTE, DUK, EBAY, ED, EMR, ETR, FDX, GLW, HAL, HD, HON, IFF, INTC, IP, ITW, JCI, KMX, KO, MAR, MAS, MCD, MCO, MMM, MRK, MSFT, MSI, NFLX, NKE, NOC, NSC, PAYX, PEP, PFE, PG, PH, PM, ROK, SBUX, SCHW, SHW, SLB, SWK, T, TAP, TJX, TMO, TSN, TXN, UNP, UPS, VFC, WAT, WMB, WMT, WU, XOM

12.5 Stock-Tickers for Daily Data

The following are all 923 stock-tickers for the daily data used in Sections 4, 5, 6 and 7:

A, AAL, AAP, AAPL, AAWW, AAXN, ABC, ABG, ABMD, ABT, ACC, ACE, ACHC, ACM, ACN, ADBE, ADI, ADM, ADP, ADS, ADSK, ADTN, ADXS, AEE, AEO, AEP, AES, AET, AFG, AFL, AGCO, AGN, AGNC, AGO, AHL, AIG, AIZ, AJG, AKAM, AKS, ALB, ALE, ALGN, ALGT, ALK, ALKS, ALL, ALNY, ALR, ALV, ALXN, AMAT, AMD, AME, AMED, AMG, AMGN, AMKR, AMP, AMT, AMTD, AMZN, AN, ANDV, ANF, ANSS, ANTM, AON, AOS, APA, APC, APD, APH, ARE, ARG, ARNA, ARO, ARRS, ARW, ASH, ATHN, ATI, ATO, ATR, ATU, ATVI, ATW, AVB, AVGO, AVP, AVT, AVY, AWI, AWK, AXON, AXP, AXS, AYI, AZO, AZPN, BA, BAC, BAX, BB, BBBY, BBT, BBY, BC, BCO, BCR, BDN, BDX, BEBE, BEN, BFB, BG, BGS, BHI, BID, BIG, BIIB, BIO, BJRI, BK, BKD, BKE, BKH, BKNG, BKS, BLK, BLL, BMRN, BMS, BMY, BOH, BPL, BPOP, BR, BRCD, BRKA, BRO, BRS, BSX, BWA, BX, BXP, BYD, BZH, C, CA, CACI, CAG, CAH, CAKE, CAR, CASY, CAT, CAVM, CBB, CBI, CBRE, CBRL, CBSH, CBT, CCE, CCI, CCK, CCL, CCOI, CDE, CDNS, CDR, CE, CELG, CEQP, CERN, CF, CFR, CFX, CHD, CHE, CHRW, CI, CIEN, CINF, CL, CLC, CLF, CLGX, CLH, CLI, CLR, CLX, CMA, CMC, CMCSA, CME, CMG, CMI, CMP, CMPR, CMS, CNC, CNK, CNP, CNVR, CNX, COF, COG, COL, COLM, COO, COP, COST, CP, CPB, CPE, CPN, CPST, CPT, CR, CREE, CRI, CRL, CRM, CROX, CRR, CRS, CRUS, CRZO, CSC, CSCO, CSGP, CSL, CSX, CTAS, CTL, CTSH, CTXS, CUBE, CUZ, CVA, CVLT, CVS, CVX, CW, CXO, CXW, CY, D, DAL, DAN, DBD, DCI, DD, DDR, DDS, DE, DEI, DF, DFS, DFT, DGX, DHR, DIS, DISCA, DISH, DK, DKS, DLB, DLTR, DLX, DOV, DPZ, DRE, DRH, DRI, DRQ, DSW, DTE, DUK, DVA, DVN, DXCM, DY, EA, EAT, EBAY, ECA, ECL, ED, EEFT, EEP, EFX, EGP, EIX, EL, EMC, EME, EMN, EMR, ENDP, ENR, ENS, EOG, EPD, EPR, EQIX, EQR, EQT, ES, ESL, ESRX, ESS, ETFC, ETN, ETR, EV, EVR, EW, EWBC, EXAS, EXC, EXEL, EXP, EXPD, EXPE, EXR, F, FAST, FCN, FCX, FDS, FDX, FE, FFIV, FICO, FII, FIS, FISV, FITB, FL, FLEX, FLIR, FLO, FLR, FLS, FMC, FNB, FNSR, FOSL, FR, FRT, FSLR, FTI, FUL, G, GBX, GCO, GD, GE, GEO, GHC, GILD, GIS, GLW, GNTX, GNW, GOV, GPC, GPN, GPS, GRA, GRMN, GS, GT, GWR, GWW, GXP, H, HAL, HAS, HBAN, HBHC, HBI, HCP, HD, HE, HES, HFC, HIBB, HIW, HK, HL, HLF, HOG, HOLX, HON, HP, HPQ, HR, HRB, HRC, HRL, HRS, HSIC, HST, HSY, HUM, HUN, HXL, IAC, IBKC, IBKR, IBM, ICE, IDA, IDCC, IDXX, IEX, IFF, IGT, ILMN, INCY, INFN, INGR, INT, INTC,

INTU, IO, IONS, IP, IPG, IPGP, IPI, IPXL, IRBT, IRET, IRM, ISIL, ISRG, IT, ITRI, ITT, ITW, IVR, IVZ, JACK, JAKK, JBHT, JBL, JBLU, JCI, JCOM, JEC, JEF, JKHY, JLL, JNJ, JNPR, JOY, JPM, JWN, K, KATE, KBH, KBR, KDP, KEX, KEY, KIM, KLAC, KMB, KMX, KO, KRC, KSS, KSU, L, LAMR, LAZ, LB, LBTYA, LDOS, LECO, LEG, LEN, LH, LII, LKQ, LL, LLL, LLY, LMT, LNC, LNG, LNT, LOGI, LOGM, LOPE, LOW, LPX, LRCX, LSTR, LULU, LUV, LVLT, LVS, LXP, LYV, M, MA, MAA, MAC, MAN, MAR, MAS, MASI, MAT, MCD, MCHP, MCK, MCO, MD, MDLZ, MDP, MDR, MDRX, MDSO, MDT, MDU, MELI, MET, MGLN, MGM, MHK, MHP, MIC, MIDD, MJN, MKC, MKL, MKSI, MKTX, MLM, MMC, MMM, MMS, MNKD, MNRO, MNST, MO, MOS, MPW, MPWR, MRK, MRO, MRVL, MS, MSCC, MSCI, MSFT, MSI, MSM, MTB, MTD, MTG, MTN, MTOR, MTZ, MU, MUR, MWW, MXIM, MYGN, MYL, NAV, NBIX, NBL, NCR, NDAQ, NDSN, NE, NEE, NEM, NEU, NFG, NFLX, NFX, NI, NKE, NKTR, NNN, NOC, NOV, NRG, NS, NSC, NTAP, NTGR, NTRS, NUAN, NUE, NURO, NUVA, NVAX, NVDA, NVR, NWE, NWL, NXST, NYT, O, OA, OC, OCN, ODFL, ODP, OFC, OGE, OHI, OI, OII, OKE, OLED, OLN, OMC, OMI, ON, ORCL, ORI, ORLY, OSK, OVV, OXY, PAA, PACW, PAYX, PBCT, PBI, PCAR, PCG, PCH, PDCO, PDLI, PENN, PEP, PETM, PFE, PFG, PG, PGR, PH, PHH, PHM, PII, PKG, PKI, PLCE, PLD, PM, PNC, PNM, PNR, PNRA, PNW, PODD, POM, POOL, PPC, PPG, PPL, PRGO, PRU, PRXL, PSA, PSEG, PTEN, PVH, PX, PXD, PZZA, QCOM, QRTEA, R, RAD, RAI, RAX, RCII, RCL, RDC, RDN, RE, REGN, REN, RES, RF, RGA, RGC, RGLD, RHI, RHT, RIG, RJF, RL, RMD, RNR, ROK, ROP, ROST, RPM, RRC, RS, RSG, RTN, RYN, S, SAFM, SAM, SANM, SBAC, SBGI, SBH, SBUX, SCG, SCHW, SCI, SCSS, SEE, SEIC, SF, SFLY, SGEN, SGMS, SGY, SHO, SHW, SIG, SIRI, SIVB, SJM, SKT, SKX, SLAB, SLB, SLG, SLM, SM, SMG, SMTC, SNA, SNDK, SNH, SNI, SNPS, SO, SOHU, SON, SONC, SPG, SPLS, SPR, SPWR, SPXC, SPY, SRCL, SRE, SSYS, STE, STI, STJ, STLD, STRA, STT, STX, STZ, SVU, SWK, SWKS, SWN, SWY, SXT, SYK, SYMC, SYNA, SYY, T, TAP, TCBI, TCO, TDC, TDG, TDW, TECD, TECH, TEL, TEN, TER, TEVA, TFX, TGI, TGNA, TGT, THC, THO, THS, TIF, TIVO, TJX, TKR, TLRD, TMK, TMO, TMUS, TOL, TPR, TPX, TRN, TROW, TRUE, TRV, TSCO, TSN, TSS, TTEK, TTWO, TUP, TWX, TXN, TXRH, TXT, TYL, UAL, UDR, UFS, UGI, UHS, UIS, ULTA, ULTI, UMPQ, UNFI, UNH, UNM, UNP, UPS, URBN, URI, USB, USG, UTHR, UTX, V, VAL, VAR, VFC, VIA, VIAC, VLO, VMC, VMI, VMW, VNO, VR, VRSK, VRSN, VRTX, VRX, VSAT, VTR, VZ, WAB, WAT, WBA, WBC, WBMD, WCC, WCG, WDC, WEC, WELL, WEN, WERN, WEX, WFC, WFM, WFT, WGL, WHR, WLK, WM, WMB, WMT, WOR, WPC, WR, WRE, WRI, WSM, WSO, WTR, WU, WWD, WWW, WY, WYND, WYNN, X, XCO, XEC, XEL, XLNX, XOM, XPER, XPO, XRAY, XRX, Y, YELL, YHOO, YUM, ZION, ZMH

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