For the jupyter notebook we use this octave kernel: https://github.com/Calysto/octave_kernel

```
pip install octave_kernel
cd <root of the project>
jupyter notebook
```

In [1]:
```
format longE

# setup the path to include the 'utils' directory
directory = pwd
addpath(genpath(directory))
```

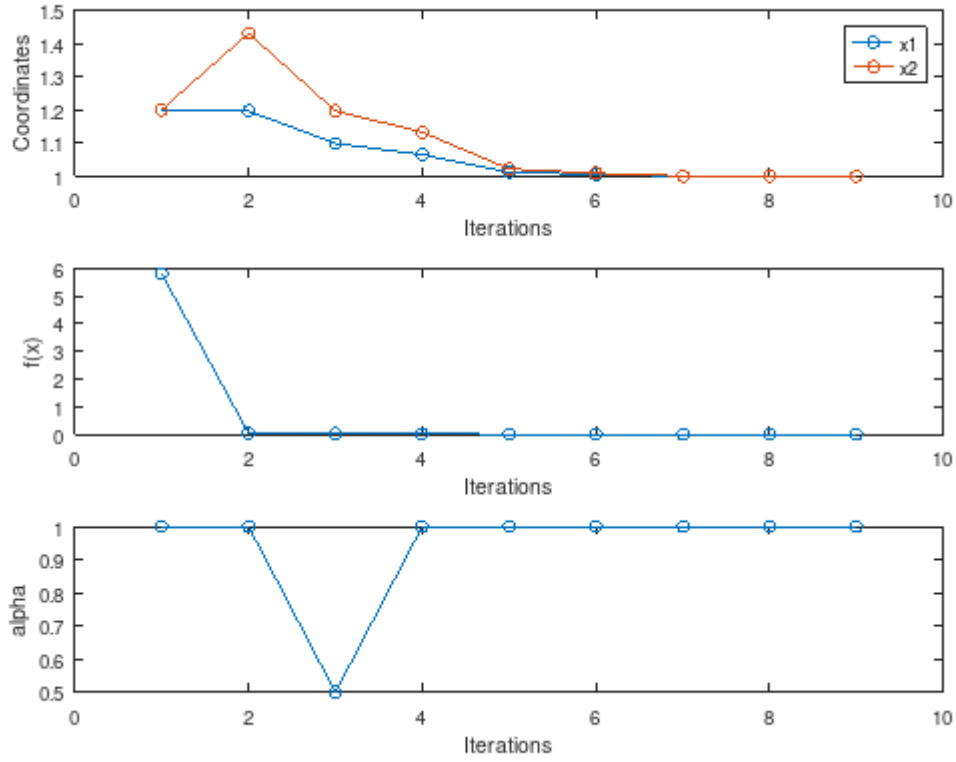
```
directory = /home/thodkatz/repos/personal/math-optimization
```

In [5]:
```
# define rosenbrock function

function f = rosen_sym()
    syms x y
    f = 100*(y-x^2)^2 + (1-x)^2;
end
```

In [2]:
```
is_backtracking_wolfe_weak = 1;
is_wolfe_strong = 2;
is_backtracking_armijo = 3;
is_none = 4;
```

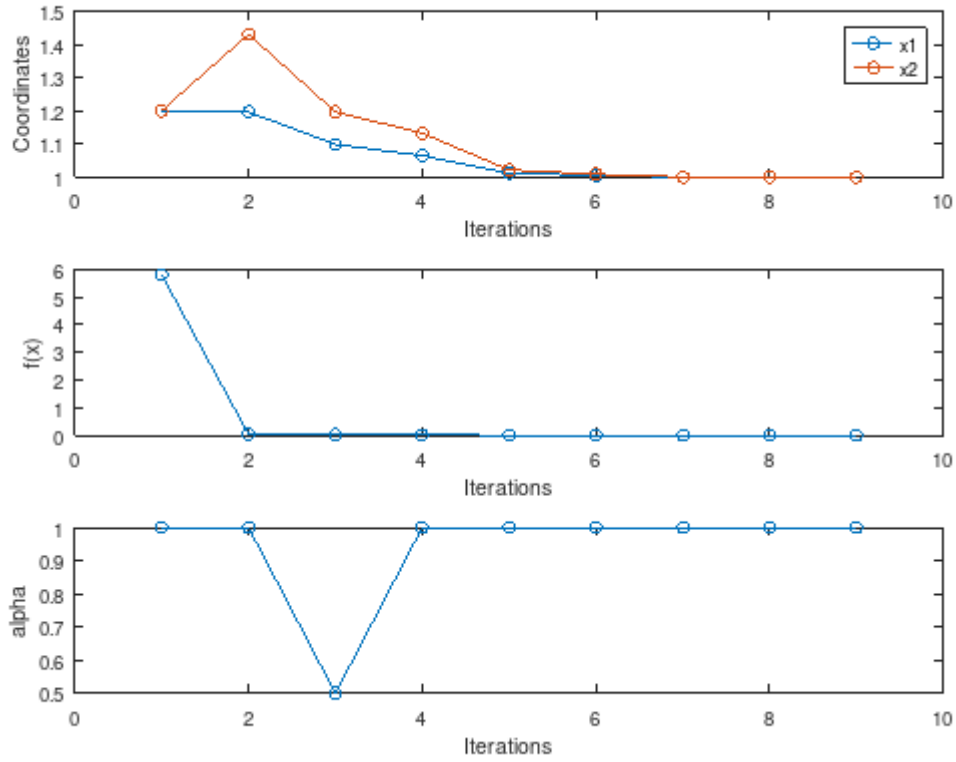
In [8]:
```
[xmin, ymin] = newton(rosen_sym, [1.2,1.2]', is_backtracking_wolfe_weak)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   1.000000000000089e+00
   1.000000000000147e+00

ymin = 1.074235775027810e-25
```



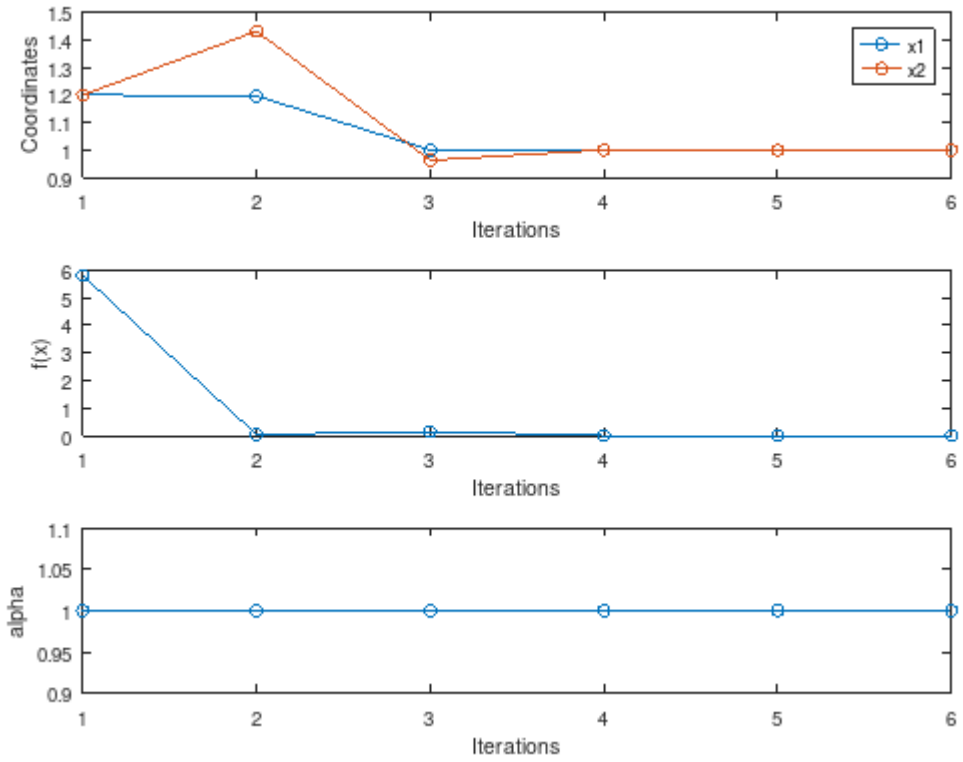
In [31]:
```
[xmin, ymin] = newton(rosen_sym, [1.2,1.2]', is_backtracking_armijo)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   1.000000000000089e+00
   1.000000000000147e+00

ymin = 1.074235775027810e-25
```



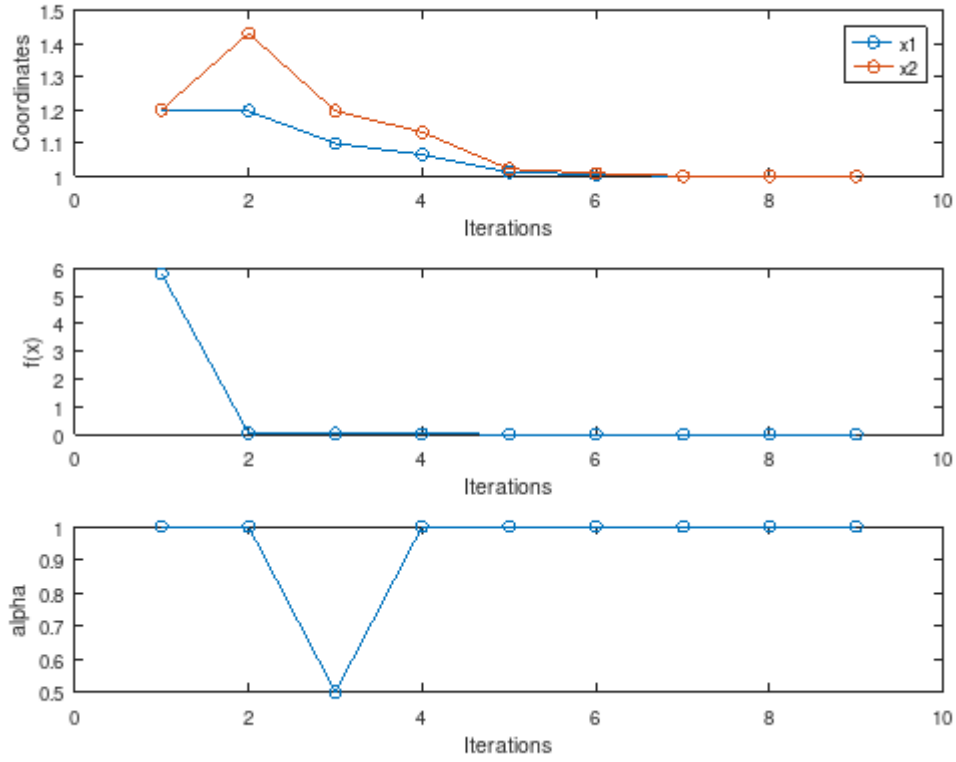
In [32]:
```
[xmin, ymin] = newton(rosen_sym, [1.2,1.2]', is_none)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   1.000000000000043e+00
   1.000000000000087e+00

ymin = 1.874777245064311e-27
```



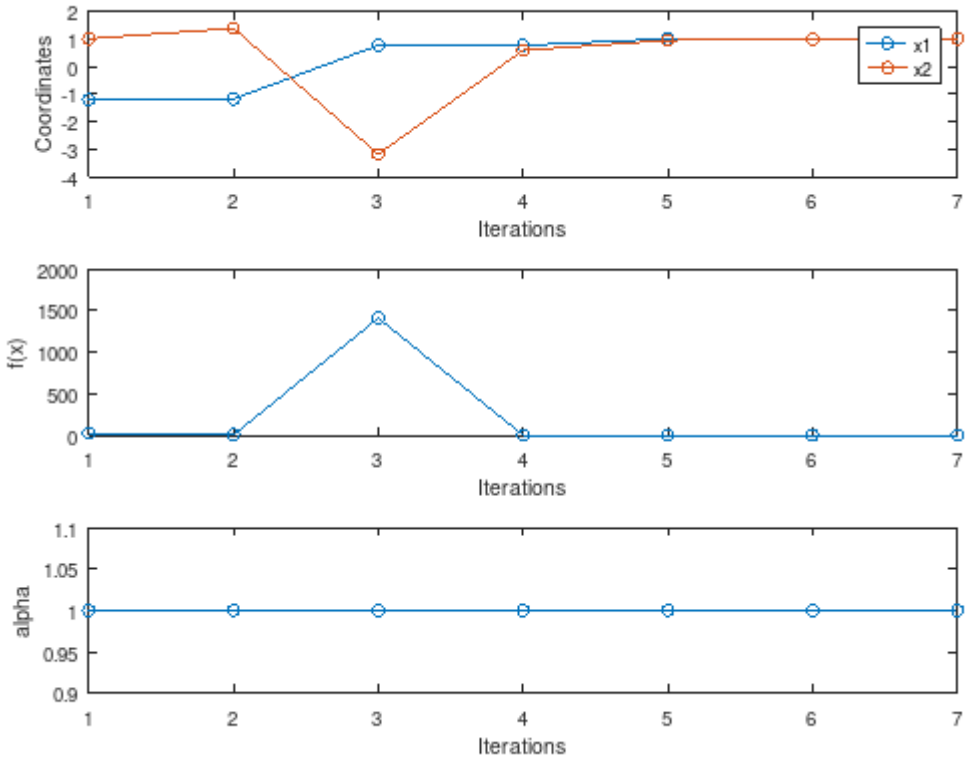
In [34]:
```
c = [1e-4 0.9];
rho = 2;
[xmin, ymin] = newton(rosen_sym, [1.2,1.2]', is_wolfe_strong, c, rho)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   1.000000000000086e+00
   1.000000000000141e+00

ymin = 9.853839060823770e-26
```



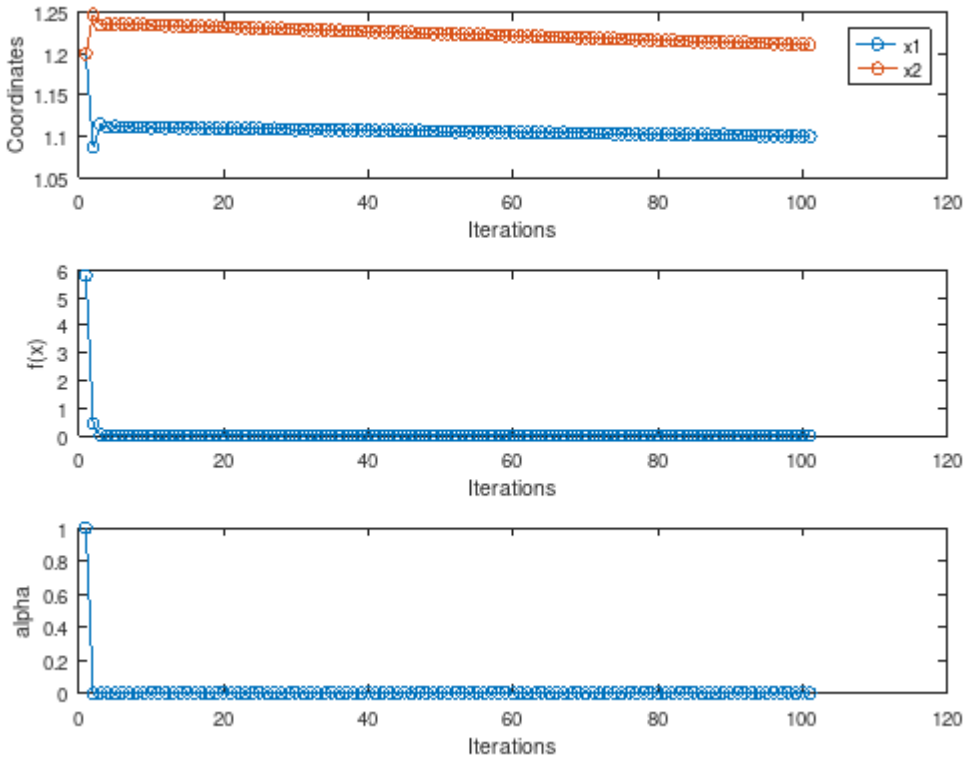
In [35]:
```
[xmin, ymin] = newton(rosen_sym, [-1.2,1]', is_none)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   9.999999999999999e-01
   9.999999999814724e-01

ymin = 3.432646187536322e-20
```



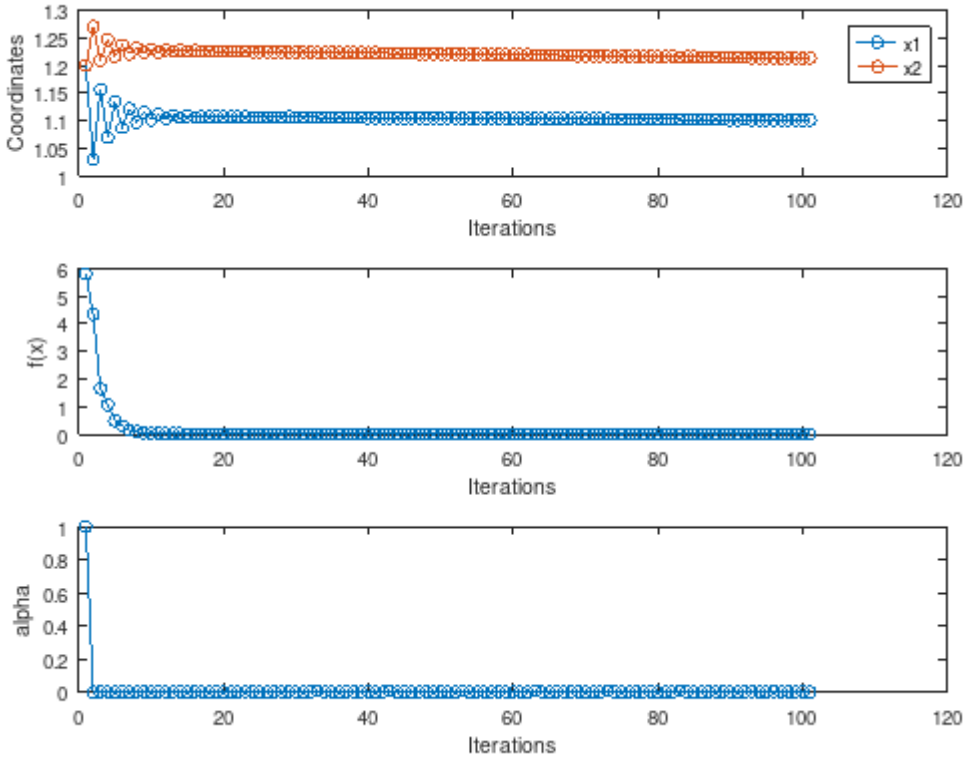
In [9]:
```
[xmin, ymin] = steepest_descent(rosen_sym, [1.2,1.2]', is_backtracking_wolfe_weak)
```

```
STARTED Line search using steepest descent
ENDED Line search using steepest descent
xmin =

   1.100114182959402e+00
   1.210524486569367e+00

ymin = 1.003031733471683e-02
```



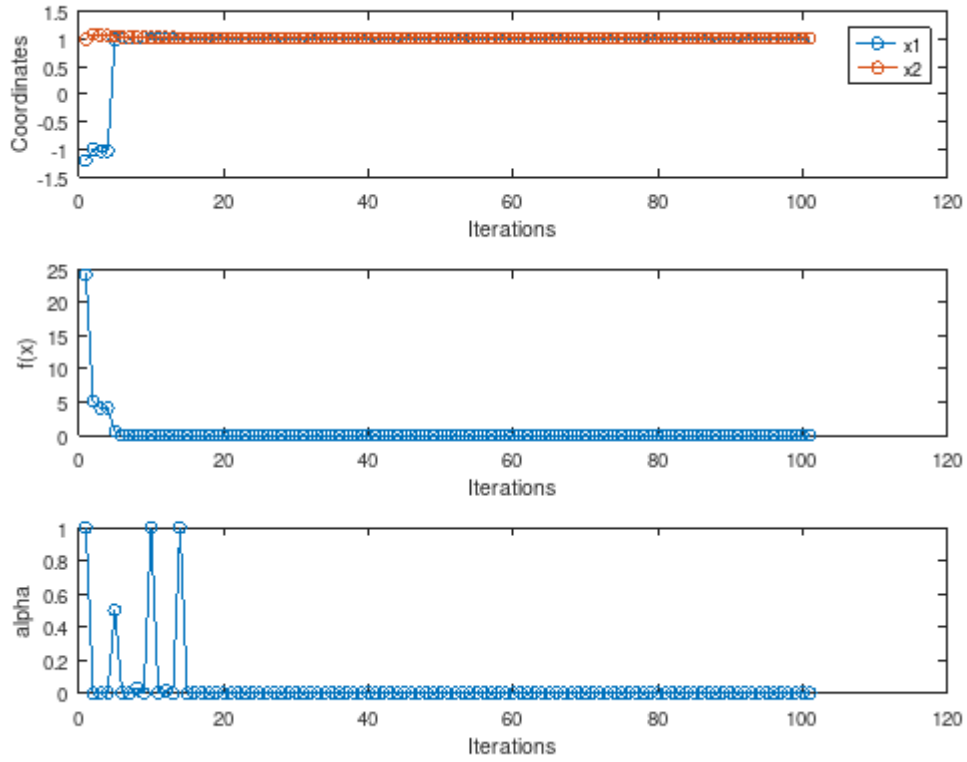
In [9]:
```
c = [1e-4 0.9];
rho = 2;
[xmin, ymin] = steepest_descent(rosen_sym, [1.2,1.2]', is_wolfe_strong, c, rho)
```

```
STARTED Line search using steepest descent
ENDED Line search using steepest descent
xmin =

   1.101165615095637e+00
   1.213154431748026e+00

ymin = 1.026914078728015e-02
```



In [10]:
```
[xmin, ymin] = steepest_descent(rosen_sym, [-1.2,1]', is_backtracking_armijo)
```

```
STARTED Line search using steepest descent
ENDED Line search using steepest descent
xmin =

   1.005567246468059e+00
   1.011207521826797e+00

ymin = 3.117092447874594e-05
```



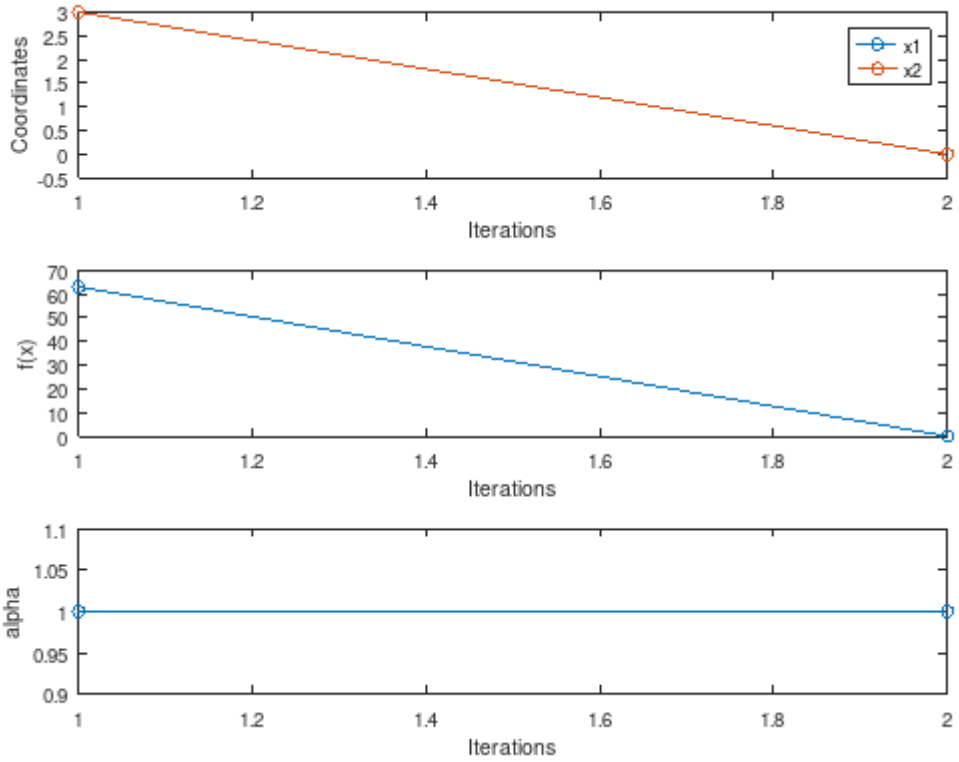
In [11]:
```
[xmin, ymin] = steepest_descent(rosen_sym, [-1.2,1]', is_none)
```

```
STARTED Line search using steepest descent
error: Failed to converge. Inf value reached
error: called from
    steepest_descent at line 11 column 13
```

In [11]:
```
% f(x,y) = x^2 + 4y^2 + 2xy -- fmin = 0 at x = 0, y = 0 -- Search domain: -3 ≤ x, y ≤

function f = f1_sym()
    syms x y
    f = x^2 + 4*y^2 + 2*x*y;
end
```

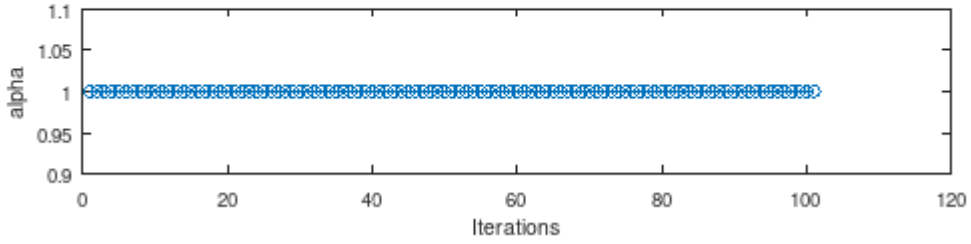
In [16]:
```
[xmin, ymin] = newton(f1_sym, [3,3]', is_none)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   1.332267629550188e-15
  -8.881784197001252e-16

ymin = 2.563797941968288e-30
```



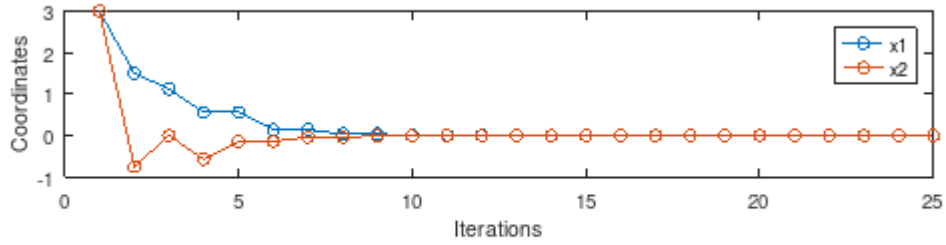
In [17]:
```
[xmin, ymin] = steepest_descent(f1_sym, [3,3]', is_none)
```

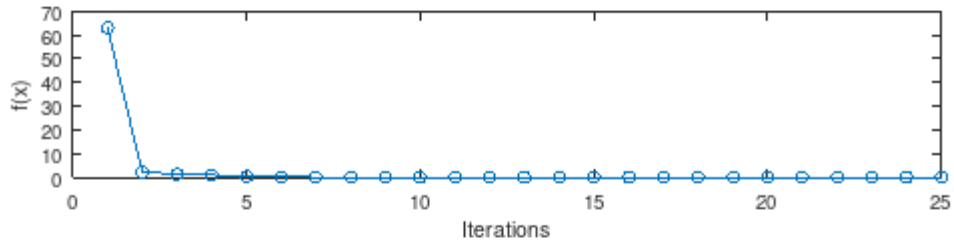
```
STARTED Line search using steepest descent
ENDED Line search using steepest descent
xmin =

   1.406335921346234e+88
   4.644812019489721e+88

ymin = 1.013392274910718e+178
```

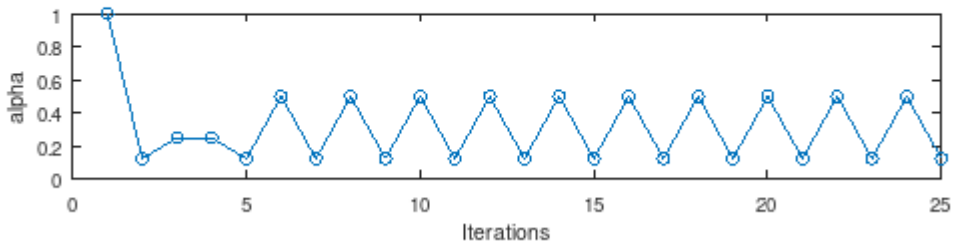


In [15]:
```
[xmin, ymin] = steepest_descent(f1_sym, [3,3]', is_backtracking_armijo)
```

```
STARTED Line search using steepest descent
ENDED Line search using steepest descent
xmin =

   5.364418029785156e-07
  -1.341104507446289e-07

ymin = 2.158273559871304e-13
```





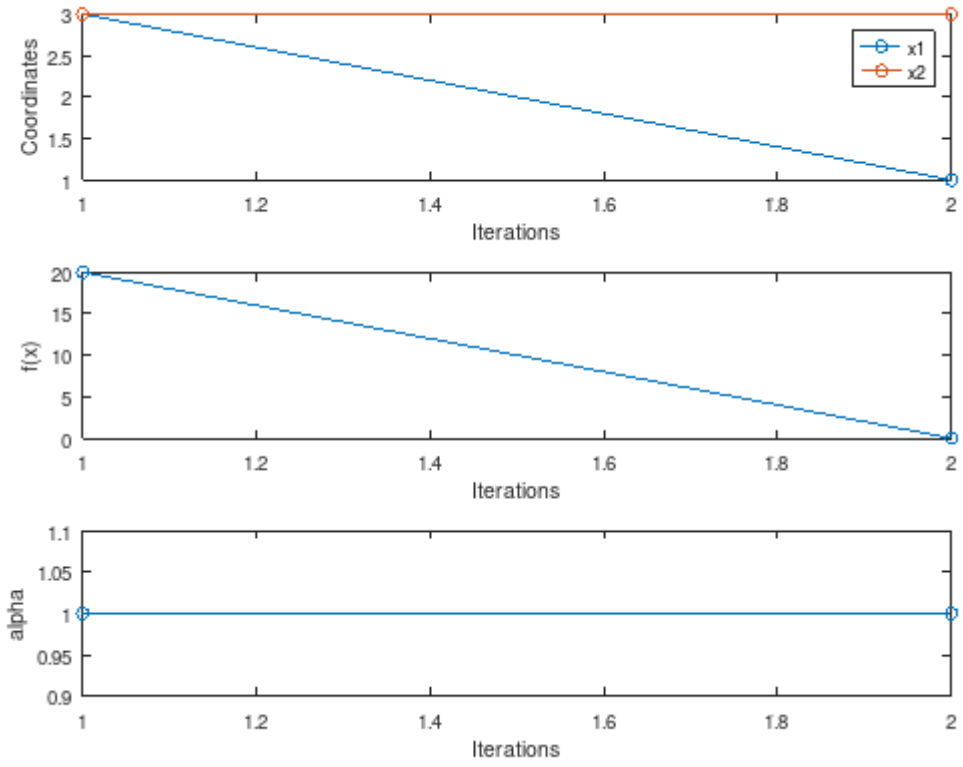


In [21]:
```
% f(x,y) = (x + 2y - 7)^2 + (2x + y - 5)^2 -- fmin = 0 at x = 1, y = 3 -- Search doma

function f = f2_sym()
    syms x y
```

```
    f = (x+2*y-7)^2 + (2*x + y - 5)^2;
end
```

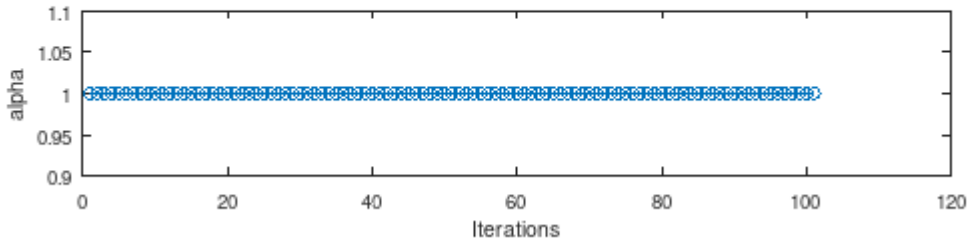
In [22]:
```
[xmin, ymin] = newton(f2_sym, [3,3]', is_none)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   1.000000000000000e+00
   3.000000000000000e+00

ymin =                    0
```



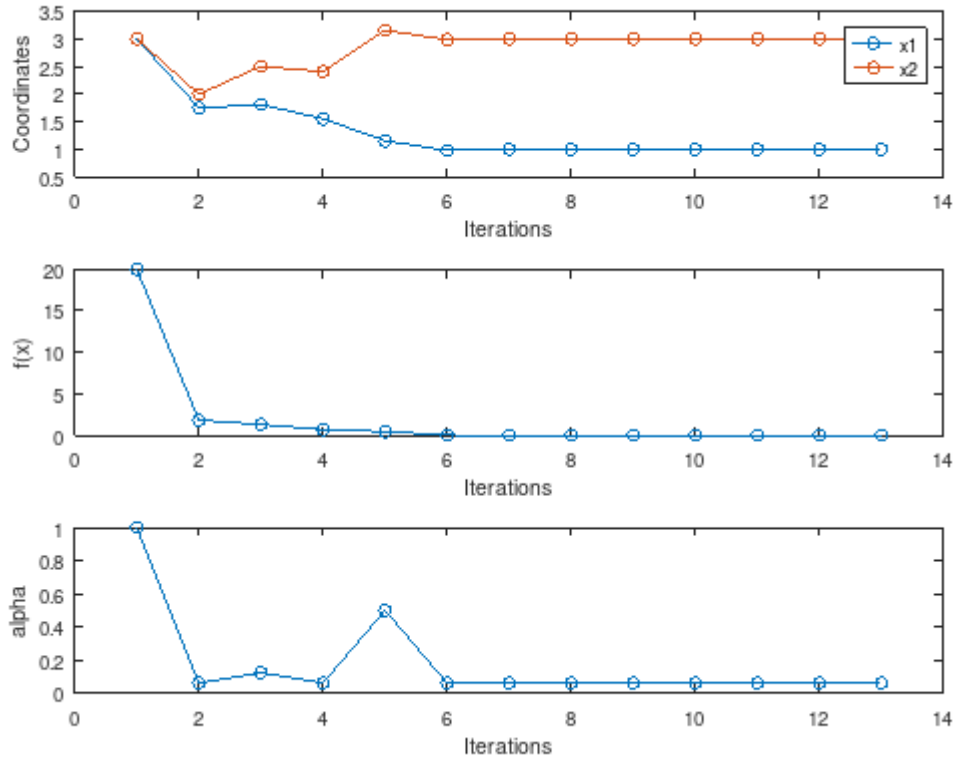
In [23]:
```
[xmin, ymin] = steepest_descent(f2_sym, [3,3]', is_none)
```

```
STARTED Line search using steepest descent
ENDED Line search using steepest descent
xmin =

   1.108899372780782e+123
   1.108899372780782e+123

ymin = 2.213384074116501e+247
```



In [24]:
```
[xmin, ymin] = steepest_descent(f2_sym, [3,3]', is_backtracking_armijo)
```

```
STARTED Line search using steepest descent
ENDED Line search using steepest descent
xmin =

   1.000000009313226e+00
   3.000000009313226e+00

ymin = 1.561251128379126e-15
```

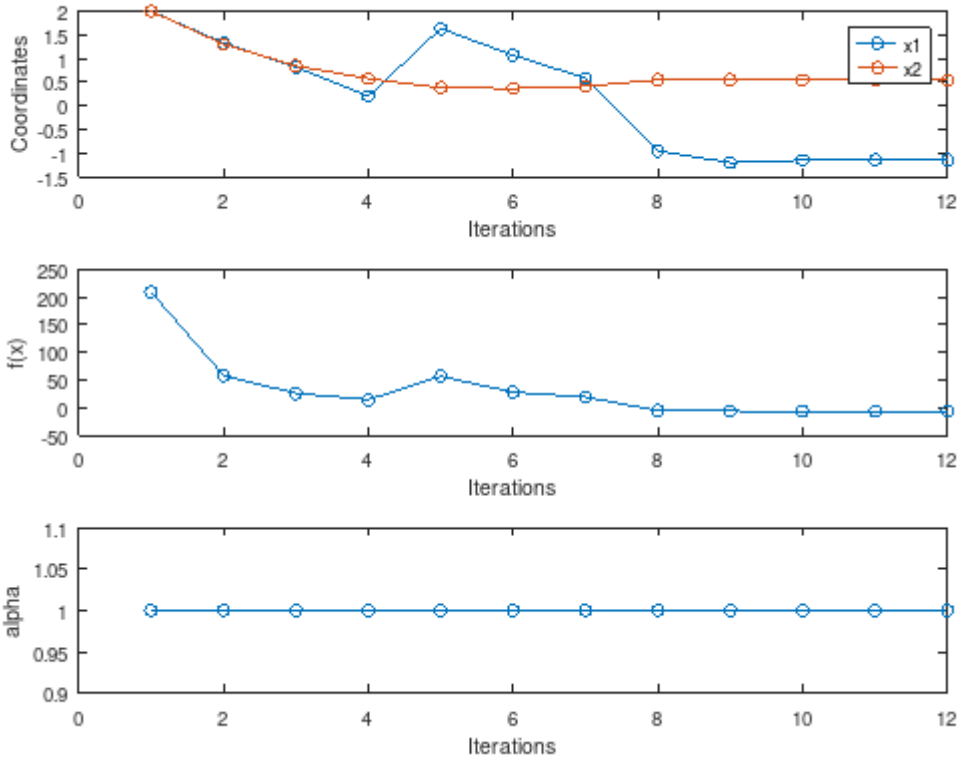


In [25]:
```
% f(x,y) = 5x^4 + 6y^4 - 6x^2 + 2xy + 5y^2 +15x - 7y + 13 -- fmin = -6.4931 at x = -1

function f = f3_sym()
    syms x y
```

```
    f = 5*x^4 + 6*y^4 - 6*x^2 + 2*x*y + 5*y^2 + 15*x - 7*y + 13;
end
```

In [26]:
```
[xmin, ymin] = newton(f3_sym, [2,2]', is_none)
```

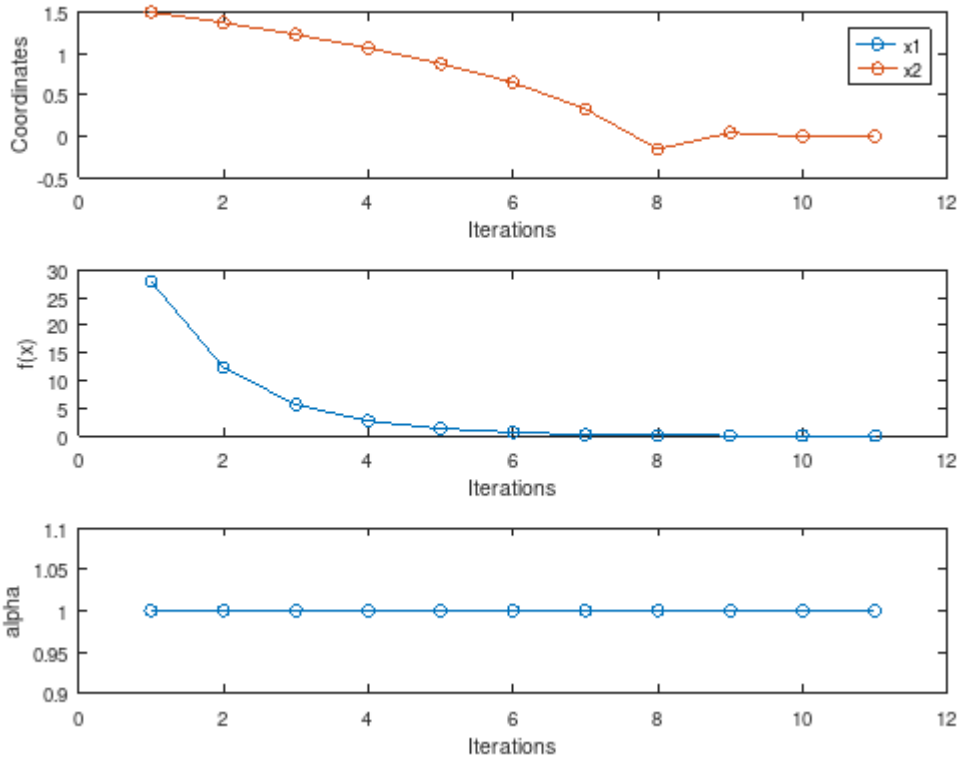
```
STARTED Line search using newton
ENDED Line search using newton
xmin =

  -1.142054928448438e+00
   5.433724812106187e-01

ymin = -6.496118935491067e+00
```



In [30]:
```
% f(x,y) = (x^2)^(y^2 + 1) + (y^2)^(x^2+1) -- fmin = 0 at x = 0, y = 0 Search domain:

function f = f4_sym()
    syms x y
    f = (x^2)^(y^2 + 1) + (y^2)^(x^2 + 1);
end
```

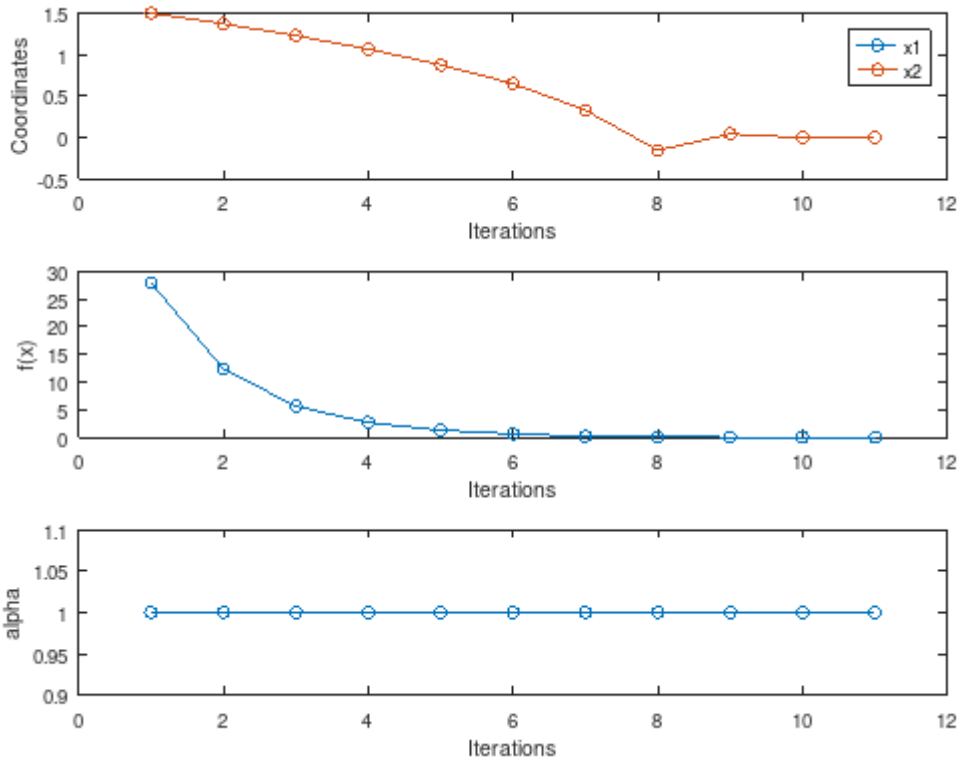
In [31]:
```
[xmin, ymin] = newton(f4_sym, [1.5,1.5]', is_none)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   2.112513120457876e-07
   2.112513120460045e-07

ymin = 8.925423368210264e-14
```



In [32]:
```
[xmin, ymin] = newton(f4_sym, [1.5,1.5]', is_backtracking_armijo)
```

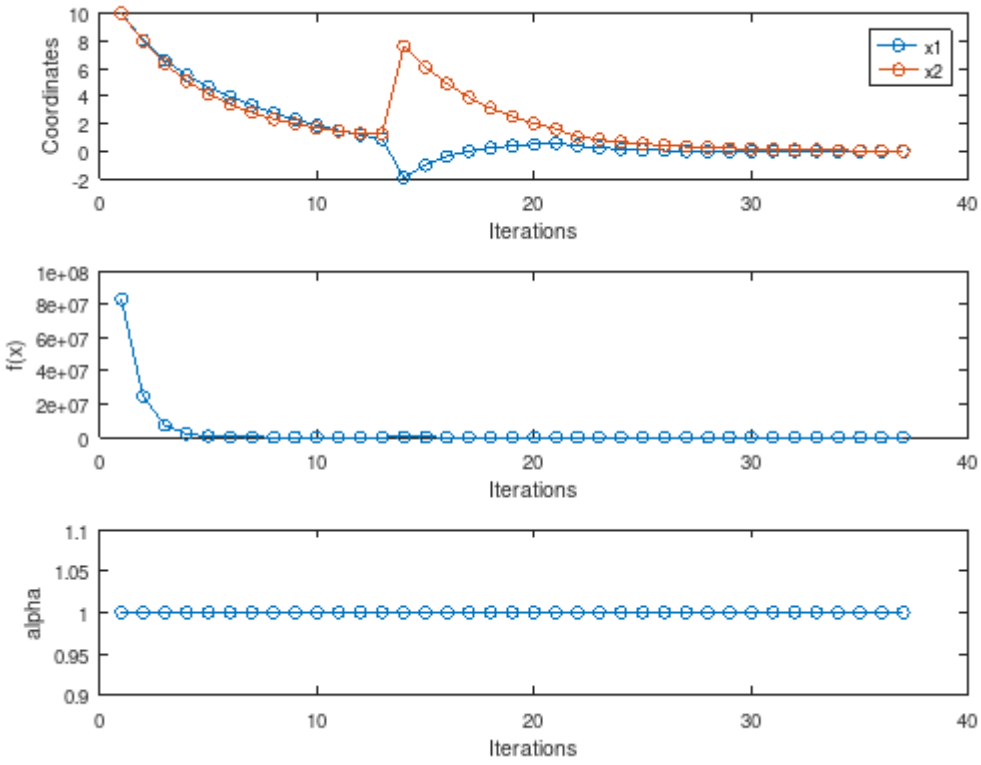
```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   2.112513120457876e-07
   2.112513120460045e-07

ymin = 8.925423368210264e-14
```



In [35]:
```
% f(x,y,z)=(x^2+y^3-z^4)^2+(2*x*y*z)^2+(2*x*y-3*y*z+x*z)^2 -init x=y=z=10-obj =0.0000

function f = f4_sym()
    syms x y z
    f = (x^2 + y^3 - z^4)^2 + (2*x*y*z)^2 + (2*x*y-3*y*z+x*z)^2;
end
```

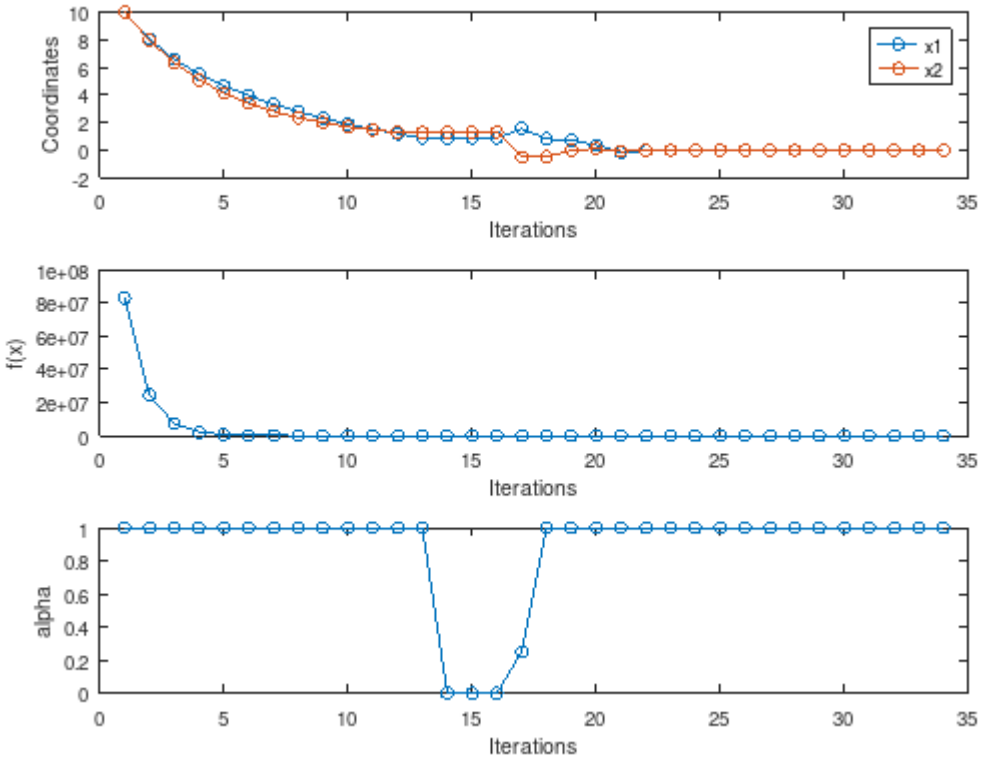
In [37]:
```
[xmin, ymin] = newton(f4_sym, [10,10,10]', is_none)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

   2.525601052449970e-04
   3.792558884054644e-02
   1.684963639464687e-04

ymin = 2.982697279570769e-09
```



In [38]:
```
[xmin, ymin] = newton(f4_sym, [10,10,10]', is_backtracking_armijo)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

  -2.126482159228893e-05
  -7.097574255828430e-06
   9.006911588552714e-02

ymin = 4.331130112795951e-09
```

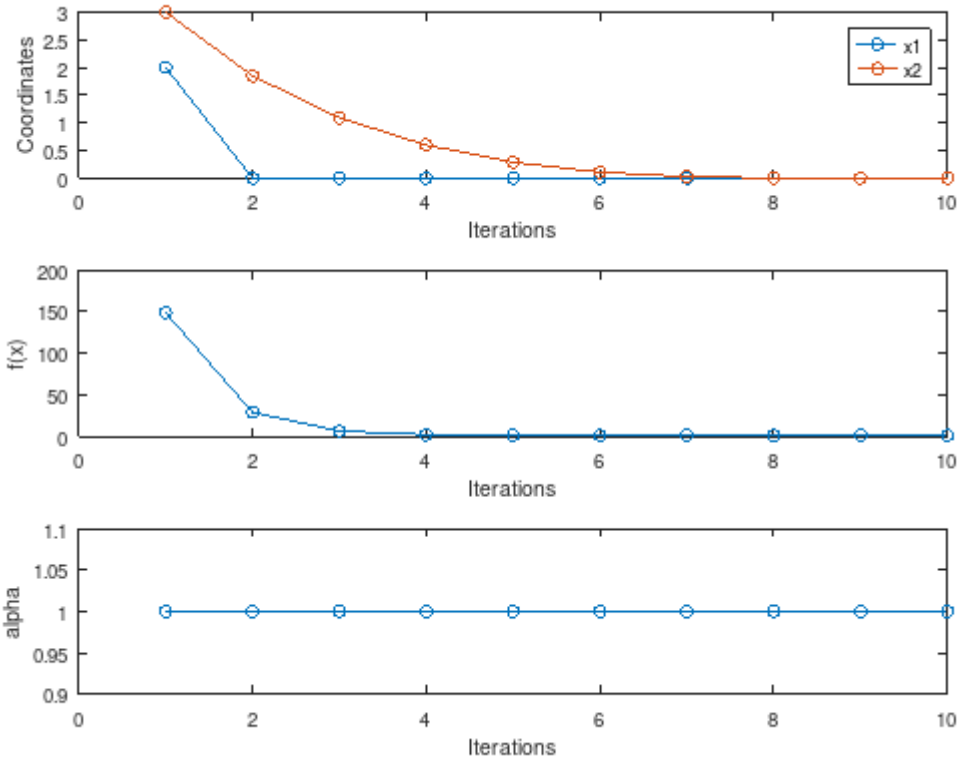


In [39]:
```
% f(x,y,z)=x^2+(y+y^2)^2+(-1+exp(z))^2-init x=2,y=3,z=-8 -obj =1.58063e-15 at x=3.024

function f = f5_sym()
```

```
    syms x y z
    f = x^2 + (y+y^2)^2 + (-1+exp(z))^2;
end
```

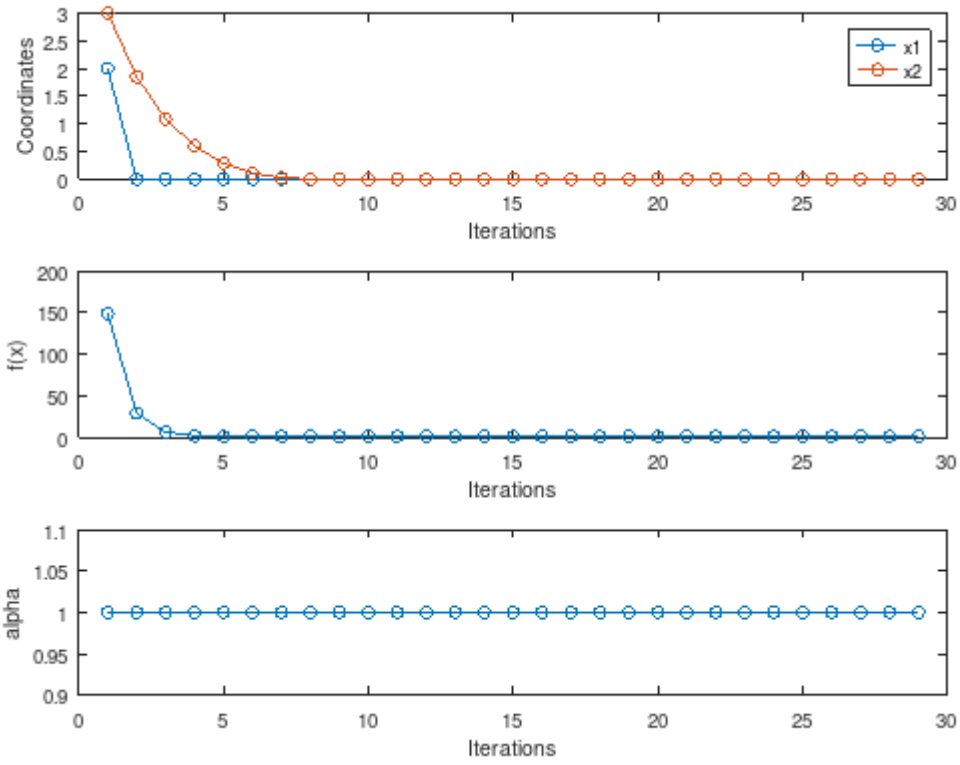
In [41]:
```
[xmin, ymin] = newton(f5_sym, [2,3,-8]', is_none)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

                         0
   9.037342696635868e-11
  -1.700053081326613e+01

ymin = 9.999999172451862e-01
```



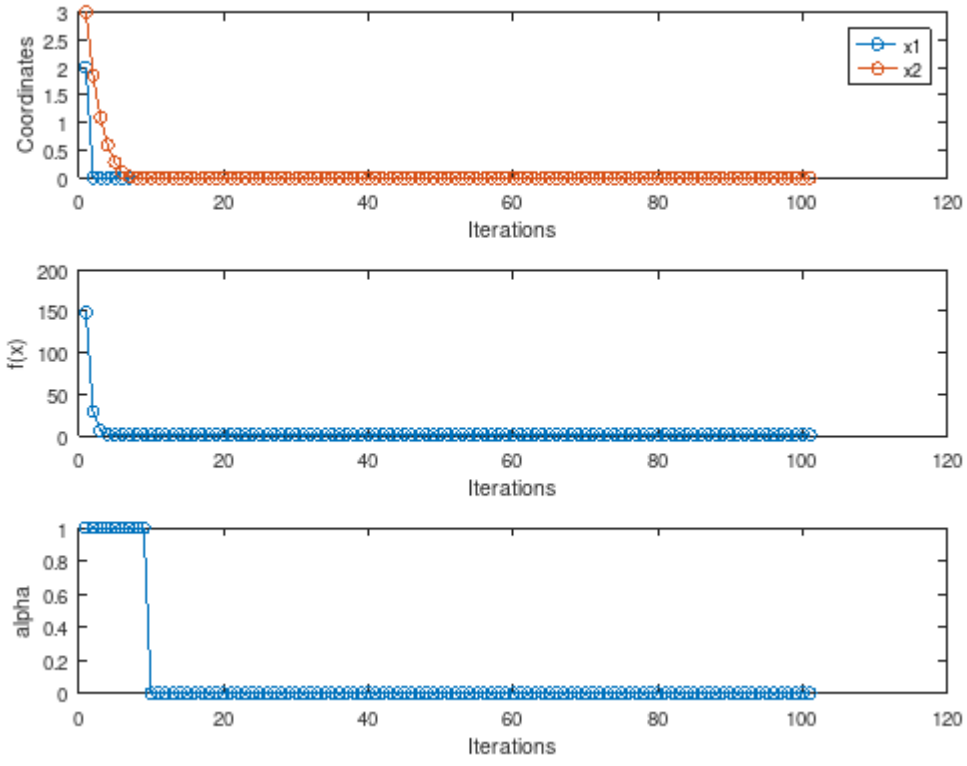
In [48]:
```
c = 0.1;
a = 1;
rho = 0.5;
eps = 1e-15;
[xmin, ymin] = newton(f5_sym, [2,3,-8]', is_none, c, rho, a, eps)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

                         0
                         0
  -3.600053087872423e+01

ymin = 9.999999999999996e-01
```



In [42]:
```
[xmin, ymin] = newton(f5_sym, [2,3,-8]', is_backtracking_armijo)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

                         0
   5.016816498800536e-06
  -1.609037446045994e+01

ymin = 9.999997944033796e-01
```



In [4]:
```
% f(x,y,z,k)=(x-1)^2+ (x-y)^2 + (y-z)^2 + (z-k)^2 -init x=y=z=k=0.1 -obj =1.13719e-10

function f = f6_sym()
```

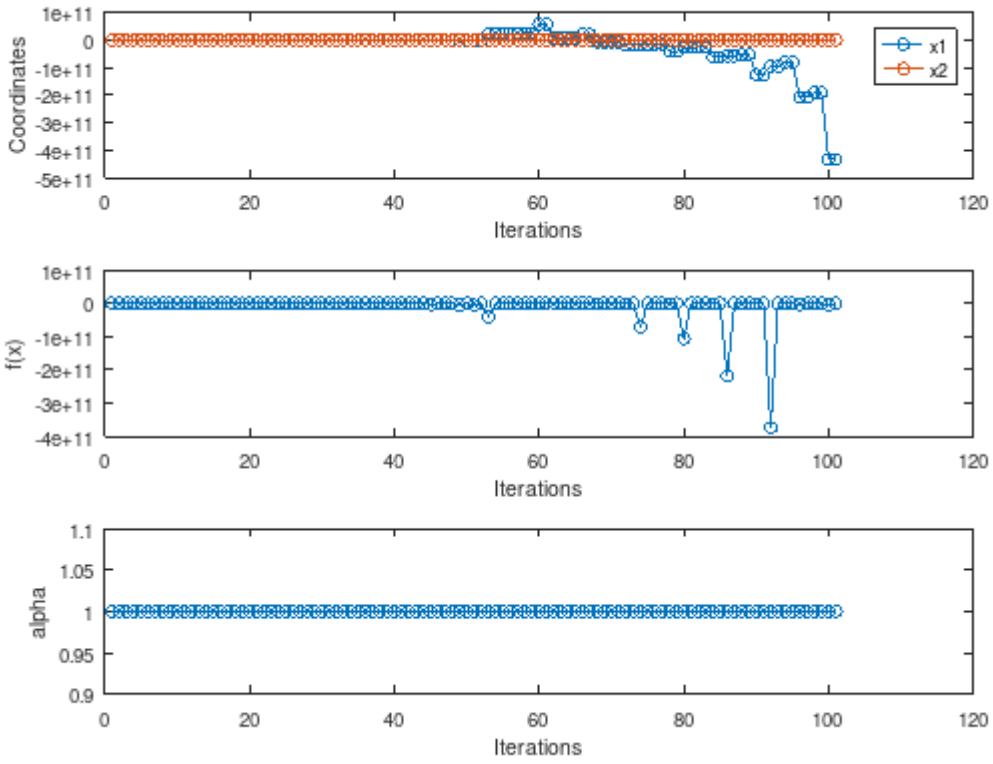
```
    syms x y z k
    f = (x-1)^2 + (x-sqrt(y))^2 + (y-sqrt(z))^2 + (z-sqrt(k))^2;
end
```

In [55]:
```
[xmin, ymin] = newton(f6_sym, [0.1,0.1,0.1, 0.1]', is_none)
```

```
STARTED Line search using newton
warning: matrix singular to machine precision, rcond = 3.84378e-17
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 1.05562e-17
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 9.22692e-17
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 1.09759e-17
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 2.51442e-18
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 4.85082e-17
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 1.38253e-19
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 1.39665e-20
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 5.11049e-21
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 5.79529e-19
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 6.55107e-20
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 1.68254e-20
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 3.50783e-18
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12
```

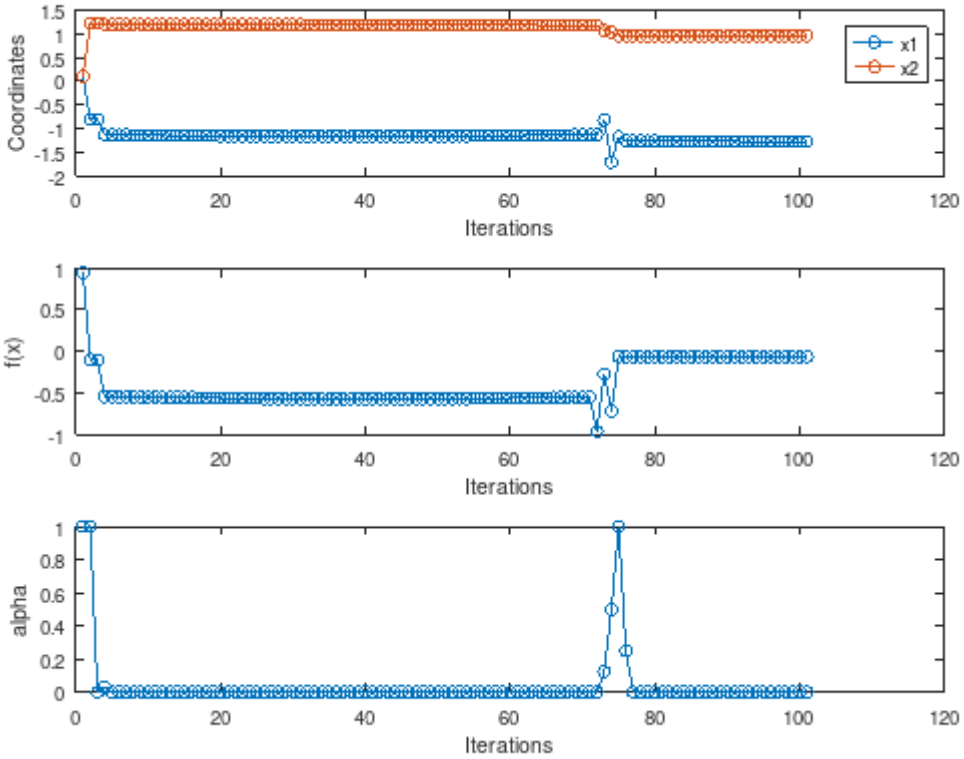
```
warning: matrix singular to machine precision, rcond = 7.31484e-20
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 3.24632e-20
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 1.11058e-17
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 2.44158e-20
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 9.92171e-21
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 4.61337e-18
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 8.36128e-21
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 3.97562e-21
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 2.15291e-18
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 3.62003e-21
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 2.14543e-21
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 8.59473e-19
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 1.27106e-21
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 1.28246e-21
```

```
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

warning: matrix singular to machine precision, rcond = 4.15522e-22
warning: called from
    linsolve at line 113 column 7
    newton at line 17 column 12

ENDED Line search using newton
xmin =

  -4.322698464918635e+11 - 8.000537338984439e+10i
   1.381466687851960e+01 - 5.163031407545673e+00i
   6.017627427824312e+02 - 5.492879505389458e+02i
   6.058645234507984e+04 - 6.602579532046878e+05i

ymin =  2.746315255066210e+02 - 2.645557009114806e+02i
```



In [56]:
```
[xmin, ymin] = newton(f6_sym, [0.1,0.1,0.1, 0.1]', is_backtracking_armijo)
```

```
STARTED Line search using newton
ENDED Line search using newton
xmin =

  -1.275648809119521e+00 + 1.948144597017925e-01i
   9.616769665903564e-01 + 2.034956822374415e-01i
   7.897232587945412e-01 + 7.155710473173073e-01i
  -1.999978995807808e-02 + 1.211620234828866e+00i

ymin = -6.761901558860435e-02 - 3.428204759301077e-02i
```



In [11]:
```
point = [0.1, 0.1, 0.1, 0.1]';
[sym_vars, sym_point] = point2sym(f6_sym, point);
gradient(f6_sym, sym_vars)

% that is a very interesting use case where the gradient of the function contains the
% during the algorithm the coordinates occur as negative values, the gradient will ha
% causing the algorithm to fail. How to resolve this?
```

```
ans = (sym 4×1 matrix)

⎡     -(-√k + z)      ⎤
⎢     ──────────      ⎥
⎢         √k          ⎥
⎢                     ⎥
⎢   4·x - 2·√y - 2    ⎥
⎢                     ⎥
⎢             x - √y  ⎥
⎢2·y - 2·√z - ──────  ⎥
⎢               √y    ⎥
⎢                     ⎥
⎢             y - √z  ⎥
⎢-2·√k + 2·z - ──────  ⎥
⎣               √z    ⎦
```

In [12]:
```
[xmin, ymin] = newton(f6_sym, [0.1,0.1,0.1, 0.1]', is_none)
```

```
STARTED Line search using newton
grad_x =

   6.837722339831621e-01
  -2.232455532033676e+00
   2.513167019494862e-01
   2.513167019494862e-01

hess_x =

 Columns 1 through 3:

   1.581138830084190e+00                         0                         0
                       0   4.000000000000000e+00  -3.162277660168379e+00
                       0  -3.162277660168379e+00   3.581138830084189e+00
  -3.162277660168379e+00                         0  -3.162277660168379e+00

 Column 4:

  -3.162277660168379e+00
                       0
  -3.162277660168379e+00
   3.581138830084189e+00

pk =

  -9.007863458665037e-01
   1.123432485182670e+00
   7.150777546132995e-01
  -2.341654069164140e-01

x =

  -8.007863458665038e-01
   1.223432485182670e+00
   8.150777546132995e-01
  -1.341654069164140e-01

grad_x =

   1.000000000000000e+00 - 1.499278186945096e-01i
   1.088096805663315e+00 +                       0i
   1.275026982371085e+00 - 7.325719266158484e-01i
   7.316691861671722e-01 + 4.355164398194780e-01i

hess_x =

 Column 1:

                       0 - 9.361287156580934e-02i
                       0 +                       0i
                       0 +                       0i
                       0 + 1.117484917613046e+00i

 Column 2:

                       0 +                       0i
   4.000000000000000e+00 +                       0i
  -1.107644715395292e+00 +                       0i
                       0 +                       0i

 Column 3:

                       0 +                       0i
  -1.107644715395292e+00 +                       0i
   2.831287885840065e+00 +                       0i
```

```
                        0 + 2.730107348283325e+00i

 Column 4:

                        0 + 1.117484917613046e+00i
                        0 +                     0i
                        0 + 2.730107348283325e+00i
   2.000000000000001e+00 + 8.292934141653923e+00i

pk =

  -1.423330724650509e+00 - 4.087683087212133e+00i
  -2.794960774433231e-01 + 7.588194104774014e-02i
  -2.698293387271827e-02 + 2.740298942180558e-01i
   1.493151460851215e-02 + 5.524372082566643e-01i

x =

  -2.224117070517013e+00 - 4.087683087212133e+00i
   9.439364077393468e-01 + 7.588194104774014e-02i
   7.880948207405812e-01 + 2.740298942180558e-01i
  -1.192338923079018e-01 + 5.524372082566643e-01i

grad_x =

   1.248380745508082e+00 - 8.332489963205604e-02i
  -2.562736254846421e-02 - 7.178253742919916e-04i
   5.990255760003247e-01 - 5.317054821071064e-01i
  -2.384857627204484e+00 + 5.400479866441846e+00i

hess_x =

 Column 1:

   4.890643299680655e-03 - 2.772059552754935e-02i
                        0 +                     0i
                        0 +                     0i
  -2.368388469234151e-01 - 3.984914930592609e-01i

 Column 2:

                        0 +                     0i
   4.000000000000000e+00 +                     0i
  -1.079471205566668e+00 + 1.823189059347235e-01i
                        0 +                     0i

 Column 3:

                        0 +                     0i
  -1.079471205566668e+00 + 1.823189059347235e-01i
   2.566817473579127e+00 - 2.543062946552845e-01i
  -8.354983418330865e-01 + 1.035064802391020e+00i

 Column 4:

  -2.368388469234151e-01 - 3.984914930592609e-01i
                        0 +                     0i
  -8.354983418330865e-01 + 1.035064802391020e+00i
   1.316715635084135e+00 - 7.051926389414603e-01i

pk =

   1.915059037547561e+00 + 2.324692368266934e+00i
  -2.954795222489507e-01 - 3.183807006802650e-01i
  -8.937788368499017e-01 - 1.331386516255413e+00i
   1.225369458227050e+00 - 2.589699909428512e+00i
```

```
x =

  -3.090580329694521e-01 - 1.762990718945199e+00i
   6.484568854903962e-01 - 2.424987596325249e-01i
  -1.056840161093204e-01 - 1.057356622037357e+00i
   1.106135565919149e+00 - 2.037262701171848e+00i

grad_x =

  -6.977939206456022e-01 + 3.463032778908568e-01i
  -7.896037718514604e-01 + 5.586050610629136e-01i
  -2.424890077630105e+00 - 8.663146722498631e-01i
   1.195910610250543e+00 - 1.393342508405850e+00i

hess_x =

 Column 1:

   1.339293168358686e-02 + 4.838574271990483e-01i
                       0 +                      0i
                       0 +                      0i
  -4.807430054295933e-01 - 5.723498415802536e-01i

 Column 2:

                       0 +                      0i
   4.000000000000000e+00 +                      0i
  -6.509498471357734e-01 - 7.192565263101082e-01i
                       0 +                      0i

 Column 3:

                       0 +                      0i
  -6.509498471357734e-01 - 7.192565263101082e-01i
   1.827619859439025e+00 + 2.648567489882229e-01i
  -5.644484585529632e-01 - 3.358126785478284e-01i

 Column 4:

  -4.807430054295933e-01 - 5.723498415802536e-01i
                       0 +                      0i
  -5.644484585529632e-01 - 3.358126785478284e-01i
   2.150257821061297e+00 - 9.078470931976668e-03i

pk =

  -3.724124902899846e-01 + 1.284389614949988e-01i
   2.602355438505135e-01 + 1.725215253476313e-01i
   1.128229426561456e+00 + 6.716412733603565e-01i
  -4.862676046514791e-01 + 9.280303284046234e-01i

x =

  -6.814705232594367e-01 - 1.634551757450200e+00i
   9.086924293409095e-01 - 6.997723428489361e-02i
   1.022545410452136e+00 - 3.857153486770002e-01i
   6.198679612676694e-01 - 1.109232372767225e+00i

grad_x =

   4.807106666930232e-02 + 7.469089736936252e-02i
  -4.221309922062040e-01 + 9.513624309925633e-02i
   2.334532080362790e-01 + 2.793912012512642e-01i
  -1.919253231601457e-01 - 1.677122476255448e-01i
```

```
hess_x =

 Column 1:

  -8.396017983642418e-02 + 2.561852673496773e-01i
                       0 +                       0i
                       0 +                       0i
  -4.167633847832241e-01 - 6.252888760754363e-01i

 Column 2:

                       0 +                       0i
   4.000000000000000e+00 +                       0i
  -9.410504331416073e-01 - 1.715865173604968e-01i
                       0 +                       0i

 Column 3:

                       0 +                       0i
  -9.410504331416073e-01 - 1.715865173604968e-01i
   2.356647522464461e+00 + 1.785720741103630e-01i
  -7.651422850368081e-01 - 4.489278831735196e-01i

 Column 4:

  -4.167633847832241e-01 - 6.252888760754363e-01i
                       0 +                       0i
  -7.651422850368081e-01 - 4.489278831735196e-01i
   2.127114943824844e+00 + 3.596911541424528e-01i

pk =

  -4.164380267997263e-01 - 4.339496269943910e-01i
   1.198778242551963e-01 - 6.449975978584818e-02i
   2.847234149593647e-02 - 1.782564037535282e-01i
   1.482519850303639e-01 - 2.117753657269954e-01i

x =

  -1.097908550059163e+00 - 2.068501384444591e+00i
   1.028570253596106e+00 - 1.344769940707418e-01i
   1.051017751948072e+00 - 5.639717524305283e-01i
   7.681199462980333e-01 - 1.321007738494220e+00i

grad_x =

   1.549824658177898e-03 + 1.471529778789778e-02i
  -4.109254474507829e-03 - 5.454857779631950e-03i
   1.564245036237955e-02 - 5.241914615521814e-03i
  -5.518658370334162e-03 - 4.766783184997569e-02i

hess_x =

 Column 1:

  -9.716891966763973e-02 + 1.897714465746892e-01i
                       0 +                       0i
                       0 +                       0i
  -3.367644735855744e-01 - 5.600078216191479e-01i

 Column 2:

                       0 +                       0i
   4.000000000000000e+00 +                       0i
  -8.880122094763274e-01 - 2.232000765653980e-01i
                       0 +                       0i
```

```
 Column 3:

                       0 +                       0i
  -8.880122094763274e-01 - 2.232000765653980e-01i
   2.326632349645496e+00 + 2.276757852567817e-01i
  -7.011988652992344e-01 - 4.033985320791783e-01i

 Column 4:

  -3.367644735855744e-01 - 5.600078216191479e-01i
                       0 +                       0i
  -7.011988652992344e-01 - 4.033985320791783e-01i
   2.150564242197679e+00 + 2.775059583349199e-01i

pk =

  -2.199783737392530e+00 - 6.645602976944607e+00i
   1.944645528974954e-01 - 2.930567397836594e-02i
   7.868911076294586e-01 - 3.359319905528552e-01i
   1.483070587341600e+00 - 1.744619630057580e+00i

x =

  -3.297692287451693e+00 - 8.714104361389198e+00i
   1.223034806493601e+00 - 1.637826680491077e-01i
   1.837908859577531e+00 - 8.999037429833836e-01i
   2.251190533639633e+00 - 3.065627368551801e+00i

grad_x =

  -2.455151467317919e-01 - 3.599064386409499e-02i
   1.049167203356220e-01 - 9.395542573443660e-03i
   3.372531881690408e-01 - 1.192910519071926e-01i
   9.834357008007149e-01 - 1.122364217475504e+00i

hess_x =

 Column 1:

  -2.546326516015265e-02 + 6.182936741281234e-02i
                       0 +                       0i
                       0 +                       0i
  -1.862001273626291e-01 - 2.695510847559112e-01i

 Column 2:

                       0 +                       0i
   4.000000000000000e+00 +                       0i
  -6.810076822501465e-01 - 1.577737633369408e-01i
                       0 +                       0i

 Column 3:

                       0 +                       0i
  -6.810076822501465e-01 - 1.577737633369408e-01i
   2.179691931966965e+00 + 1.101350676786895e-01i
  -4.574615333517071e-01 - 2.316265981955757e-01i

 Column 4:

  -1.862001273626291e-01 - 2.695510847559112e-01i
                       0 +                       0i
  -4.574615333517071e-01 - 2.316265981955757e-01i
   2.080152063031179e+00 + 1.122672513495903e-01i
```

```
pk =

  -9.393341569668878e-02 + 4.249209065112457e-01i
  -9.365212367554744e-02 + 9.806611495215486e-03i
  -3.662140604107971e-01 + 1.286474091097710e-01i
  -5.995147580999218e-01 + 5.852922533754946e-01i

x =

  -3.391625703148382e+00 - 8.289183454877953e+00i
   1.129382682818054e+00 - 1.539760565538922e-01i
   1.471694799166734e+00 - 7.712563338736126e-01i
   1.651675775539711e+00 - 2.480335115176306e+00i

grad_x =

   4.265241617377475e-03 + 3.714701720356824e-03i
   1.423816816363876e-02 + 2.892984564893952e-04i
   2.051093523236514e-02 - 6.049840721455757e-03i
   4.835699203965316e-03 + 1.895551193999978e-04i

hess_x =

 Column 1:

  -2.085908197443034e-02 + 5.152753379724095e-02i
                       0 +                       0i
                       0 +                       0i
  -1.862417751460103e-01 - 2.774317574145744e-01i

 Column 2:

                       0 +                       0i
   4.000000000000000e+00 +                       0i
  -7.533040149693520e-01 - 1.854282088215690e-01i
                       0 +                       0i

 Column 3:

                       0 +                       0i
  -7.533040149693520e-01 - 1.854282088215690e-01i
   2.221324948531338e+00 + 1.477293944988259e-01i
  -5.106729682468836e-01 - 2.734764143917104e-01i

 Column 4:

  -1.862417751460103e-01 - 2.774317574145744e-01i
                       0 +                       0i
  -5.106729682468836e-01 - 2.734764143917104e-01i
   2.088305989348810e+00 + 1.352175029131003e-01i

pk =

   4.752870649266758e+00 - 1.793629178346973e+01i
   1.504113376190952e-01 + 5.880194581558014e-02i
   8.434242043935003e-01 + 1.050073285811234e-01i
   2.930729329114571e+00 - 1.021921117163000e+00i

x =

   1.361244946118376e+00 - 2.622547523834768e+01i
   1.279794020437149e+00 - 9.517411073831210e-02i
   2.315119003560234e+00 - 6.662490052924892e-01i
   4.582405104654282e+00 - 3.502256232339306e+00i

grad_x =
```

```
  -1.190494464010005e-01 - 1.200713815735787e-01i
   4.535161006047206e-02 + 5.280196362726108e-02i
   2.555013521428515e-01 + 1.526266795436748e-01i
   1.730112095115280e+00 + 6.125306547913212e-03i

hess_x =

 Column 1:

  -1.178626280909248e-03 + 2.139633761329329e-02i
                       0 +                       0i
                       0 +                       0i
  -1.415158687489903e-01 - 1.343609307368498e-01i

 Column 2:

                       0 +                       0i
   4.000000000000000e+00 +                       0i
  -6.379666073770314e-01 - 8.997179581878760e-02i
                       0 +                       0i

 Column 3:

                       0 +                       0i
  -6.379666073770314e-01 - 8.997179581878760e-02i
   2.161430587372499e+00 + 5.821156447378208e-02i
  -3.944252776688009e-01 - 1.334675344051713e-01i

 Column 4:

  -1.415158687489903e-01 - 1.343609307368498e-01i
                       0 +                       0i
  -3.944252776688009e-01 - 1.334675344051713e-01i
   2.066588514546721e+00 + 5.593436686522500e-02i

pk =

   1.106549692447293e-01 + 1.174805145343404e+00i
  -5.895363697637501e-02 - 3.996556339043853e-02i
  -3.159300521961557e-01 - 1.232595795391041e-01i
  -9.565197915329547e-01 + 6.663890650336449e-02i

x =

   1.471899915363105e+00 - 2.505067009300428e+01i
   1.220840383460774e+00 - 1.351396741287506e-01i
   1.999188951364078e+00 - 7.895085848315934e-01i
   3.625885313121327e+00 - 3.435617325835941e+00i

grad_x =

   2.863538721230229e-03 + 2.399940600670498e-03i
   2.866385605106103e-03 + 7.616908165004377e-03i
   1.077874274165236e-02 + 1.064646457473861e-02i
   9.022806487291035e-04 - 8.075002306924262e-04i

hess_x =

 Column 1:

   1.213116546347549e-03 + 1.983111200830605e-02i
                       0 +                       0i
                       0 +                       0i
  -1.452374169510450e-01 - 1.369542063669188e-01i
```

```
 Column 2:

                       0 +                       0i
   4.000000000000000e+00 +                       0i
  -6.700579925575230e-01 - 1.275160784303269e-01i
                       0 +                       0i

 Column 3:

                       0 +                       0i
  -6.700579925575230e-01 - 1.275160784303269e-01i
   2.175152606628344e+00 + 8.545823851992063e-02i
  -4.156448853893855e-01 - 1.656429315850384e-01i

 Column 4:

  -1.452374169510450e-01 - 1.369542063669188e-01i
                       0 +                       0i
  -4.156448853893855e-01 - 1.656429315850384e-01i
   2.069674435908872e+00 + 6.643146020773125e-02i

pk =

   1.804117395614923e+01 - 4.324531885246701e+01i
   1.699350212507124e-01 + 1.869677436412409e-02i
   1.005708246175848e+00 - 6.841193819794623e-02i
   4.273321831695671e+00 - 1.910902407948085e+00i

x =

   1.951307387151234e+01 - 6.829598894547129e+01i
   1.390775404711487e+00 - 1.164428997646265e-01i
   3.004897197539926e+00 - 8.579205230295396e-01i
   7.899207144816998e+00 - 5.346519733784026e+00i
```