Sobolev inequalities and related problems

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Abstract

In this note, we will discuss the following problem:

- Let $u \in W^{1,p}(\mathbb{R}^n)$ be a nonnegative function, and $p > 1$. Show that there exists a constant $C(n,p)$ such that the following inequality holds:

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \, \mathrm{d}x\right)^{\frac{1}{p^*}} \leq C(n,p)\left(\int_{\mathbb{R}^n} |\nabla u|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \tag{1}$$

  (where p^* is the Sobolev conjugate of p. (i.e. $\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$))
- Discuss Sobolev inequality
- Discuss Gagliardo-Nirenberg inequality
- Discuss Talenti inequality

Note

Introduction

Previous inequalities

- Triangle inequality:

$$|x + y| \leq |x| + |y| \tag{2}$$

- Cauchy inequality:

$$|xy| \leq |x||y| \tag{3}$$

- Young inequality:

$$xy \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{4}$$

  where $\frac{1}{p} + \frac{1}{q} = 1, x, y, a, b \geq 0$
- Holder inequality:

$$\int_{\mathbb{R}^n} |fg| \, \mathrm{d}x \leq \left(\int_{\mathbb{R}^n} |f|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |g|^q \, \mathrm{d}x\right)^{\frac{1}{q}} \tag{5}$$

  where $\frac{1}{p} + \frac{1}{q} = 1$
- Jesen inequality:

$$f\left(\int_{\mathbb{R}^n} g\right) \leq \int_{\mathbb{R}^n} f(g) \, \mathrm{d}x \tag{6}$$

  where f is a convex function

Sobolev space

To introduce the background of Sobolev inequality, we discuss the following problem:

$$-\Delta u = f \tag{7}$$

here f might not be continous, might just be L^2 function.

To solve this, we find the minimizer of the following functional:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x - \int_\Omega f u \, \mathrm{d}x \tag{8}$$

$(u \in \mathbb{C}_0^\infty(\mathbb{R}_n))$

We take:

$$J(u + \varepsilon\varphi) = \frac{1}{2}\int_\Omega |\nabla u + \varepsilon \nabla \varphi|^2 \, \mathrm{d}x - \int_\Omega f(u + \varepsilon\varphi) \, \mathrm{d}x \tag{9}$$

$(u, \varphi \in \mathbb{C}_0^\infty(\mathbb{R}_n))$

When u is the minimizer, we have:

$$\frac{d}{d\varepsilon} J(u + \varepsilon\varphi) = 0 \text{ at } \varepsilon = 0 \tag{10}$$

$$\frac{d}{d\varepsilon} J(u + \varepsilon\varphi) = \int_\Omega \nabla u \nabla \varphi \, \mathrm{d}x - \int_\Omega f\varphi \, \mathrm{d}x \tag{11}$$

Consider

$$\int \nabla u \nabla \varphi + \int \Delta u \varphi = 0 \tag{12}$$

we have:

$$\int_\Omega \Delta u \varphi \, \mathrm{d}x + \int_\Omega f\varphi \, \mathrm{d}x = 0 \tag{13}$$

Since φ is arbitrary, we have:

$$-\Delta u = f \tag{14}$$

Thus leading to the weak solution of the PDE.

Question: When $f \in L^2$, when does $\int u \cdot f$ make sense?

When $u \in L^2$, $f \in L^2$, we have

$$\int u \cdot f \leq \left(\int |u|^2\right)^{\frac{1}{2}} \left(\int |f|^2\right)^{\frac{1}{2}} < \infty \tag{15}$$

Thus the Sobolev space is introduced:

$$W^{1,2}(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega)\} \tag{16}$$

Measure of $W^{1,2}(\Omega)$ is defined as

$$\|f\|_{W^{1,2}(\Omega)} = \left(\int |f|^2 + \int |\nabla f|^2\right)^{\frac{1}{2}} \tag{17}$$

Trivial: $W^{1,2}(\Omega)$ defines normed space.

We can also define $W^{1,p}(\Omega)$ for $p > 2$ likewise:

$$\|f\|_{W^{1,p}(\Omega)} = \left(\int |f|^p + \int |\nabla f|^p\right)^{\frac{1}{p}} \tag{18}$$

Sobolev inequality

If Sobolev inequality holds, we have:

$$\|f\|_{L^{p^*}(\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)} \tag{19}$$

That leads to the following statement:

$$f \in W^{1,p}(\Omega) \Rightarrow f \in L^{p^*}(\Omega) \tag{20}$$

Notice $p^* > p$, thus Sobolev inequality is a statement about the integrability of the function.

We want to discuss:

When does Sobolev inequality hold?

- What's the best constant $C(n, p)$?
- What's the extremal function of the inequality?
  - Existence
  - Expression

Previous works show that:

$$u(x) = \frac{1}{\left(a + b \cdot |x|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}} \tag{21}$$

$$C(n,p) = \frac{1}{\sqrt{\pi} n^{\frac{1}{p}}} \left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma(n)}{\Gamma\left(1+n-\frac{n}{p}\right)\Gamma\left(\frac{n}{p}\right)}\right)^{\frac{1}{n}} \tag{22}$$

Question: For $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ to hold, we must have $p < n$, what happens when $p = n$?

Theorem: $\forall q \geq n$,

$$\|u\|_{L^q(\mathbb{R}^n)} \leq (\omega_{n-1})^{\frac{1}{q}-\frac{1}{n}} \left(\frac{q+1-\frac{q}{n}}{n}\right)^{\frac{1}{q}-\frac{1}{n}+1} \|u\|_{W^{1,n}(\mathbb{R}^n)} \tag{23}$$

where $\omega_{n-1} = \left|\mathbb{S}^{n-1}\right|$, volume of the unit sphere in $\mathbb{R}^n$.

Approximation of the constant:

$$\left(\frac{q+1-\frac{q}{n}}{n}\right)^{\frac{1}{q}-\frac{1}{n}+1} \sim q^{\frac{n-1}{n}} \left(\frac{1}{n}\right)^{\frac{n-1}{n}} \tag{24}$$

The $\frac{1}{n}$ here is actually $\frac{1}{n} - \frac{1}{n^2}$, I'm not sure if I've missed something or just doesn't matter.

Given by Trudinger in 1967, we have:

$$\|u\|_{L^q(\mathbb{R}^n)} \leq Cq^{\frac{n-1}{n}} \|u\|_{W^{1,n}(\mathbb{R}^n)} \tag{25}$$

Further discussion

$$\begin{aligned}\int e^u &= \sum_{j=0}^{\infty} \frac{1}{j!} \int u^j = \sum_{j=0}^{\infty} \frac{1}{j!} \|u\|_{L^j}^j \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j!} \left(C \cdot j^{\frac{n-1}{n}} \|u\|_{L^n}\right)^j \\ &= C \sum_{j=0}^{\infty} \frac{j^{j \cdot \frac{n-1}{n}}}{j!} \|u\|_{L^n}^j\end{aligned} \tag{26}$$

Using Strling's formula ($n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$) , we have:

$$\frac{j^{j \cdot \frac{n-1}{n}}}{j!} \sim \frac{1}{e^{-j+\frac{j}{n}} n^{\frac{j}{n}+\frac{1}{2}} \left(\frac{j}{n}\right)!} \leq C' \frac{1}{\left(\frac{j}{n}\right)!} \tag{27}$$

Rewrite the sum, we have:

$$\sum_{j=0}^{\infty} \frac{j^{j \cdot \frac{n-1}{n}}}{j!} \|u\|_{L^n}^j \leq C' \sum_{j=0}^{\infty} \frac{\|u\|_{L^n}^j}{\left(\frac{j}{n}\right)!} = C' e^{\|u\|_{L^n}^n} \tag{28}$$

Thus we have:

$$\int e^u \leq C e^{\|u\|_{L^n}^n} \tag{29}$$

This is called Orlicz-Sobolev inequality. Orlicz space is defined as:

$$L^{\Phi(\Omega)} = \left\{ u \in L^1(\Omega) \mid \int \Phi\left(\frac{|u|}{\lambda}\right) \mathrm{d}x \leq 1 \right\} \tag{30}$$

In 1971, Moser proved that:

$$\sup_{\|\nabla u\|_{L^n} \leq 1} \int e^{\alpha_n u^{\frac{n}{n-1}}} \leq \infty \tag{31}$$

$(u \in \mathbb{C}_0^{\infty}(\Omega), \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}, \omega_{n-1} := \left|\mathbb{S}^{n-1}\right|)$

Homework

Question 1.1

Why the power on the left hand side has to be such p^*

Hint: try replace $u(x)$ by $u(\lambda x)$, after change of variables, you will see that in order to guarantee that Sobolev inequality are independent of the scaling of u, the power on the left hand side has to be p^*

Answer

Subsitute u by $u(\lambda x)$, we have

$$\text{LHS} = \left(\int_{\mathbb{R}^n} |u(x)|^{p^*} \lambda^n \, \mathrm{d}x\right)^{\frac{1}{p^*}} = \lambda^{\frac{n}{p^*}} \left(\int_{\mathbb{R}^n} |u(x)|^{p^*} \, \mathrm{d}x\right)^{\frac{1}{p^*}} \tag{32}$$

$$\text{RHS} = C(n,p)\left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \lambda^{p-n} \, \mathrm{d}x\right)^{\frac{1}{p}} = \lambda^{\frac{n}{p}-1} C(n,p)\left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \tag{33}$$

For independence of scaling, we need $\frac{n}{p^*} = \frac{n}{p} - 1$, which gives $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Q.E.D.

Question 1.2

It has been shown during class that the extremal functions of Sobolev inequality should satisfy the following semilinear PDE:

$$-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = C|u|^{p^*-2} u \tag{34}$$

On the other hand, applying the moving plane method, we can show the positive solution of the above equation has to be radially decreasing. Combinging the two results, please try to reduce the above equation to a second order ODE and solve it:

Answer

Let $u(x) = v(r)$, where $r = |x|$, then we have

$$\begin{cases} \nabla u = \frac{x}{r} v'(r) \\ |\nabla u| = v'(r) \end{cases} \tag{35}$$

$$\begin{aligned} \text{LHS} &= -\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = -\operatorname{div}\left(v'(r)^{p-2} x v'(r)\right) = -\operatorname{div}\left(v'(r)^{p-1} x\right) \\ &= -(p-1)v'(r)^{p-2} v''(r) - (p-1) v'(r)^{p-3} v''(r) \\ &= -(p-1) v'(r)^{p-3} v''(r)(v'(r) + (p-2)) \\ \text{RHS} &= C|u|^{p^*-2} u \\ &= C v(r)^{p^*-2} v(r) \end{aligned} \tag{36}$$

Thus we have

$$-(p-1) v'(r)^{p-3} v''(r)(v'(r) + (p-2)) = C v(r)^{p-2} v(r) \tag{37}$$

Simiplify it, we have

$$-(p-1)v'(r)^{p-3}v''(r)(v'(r)+(p-2)) = Cv(r)^{p-2}v(r) \tag{38}$$

References

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aeque doleamus animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere malum nobis opinemur. Quod idem licet transferre in voluptatem, ut postea variari voluptas distinguique possit, augeri amplificarique non possit. At etiam Athenis, ut e patre audiebam facete et urbane Stoicos irridente, statua est in quo a nobis philosophia defensa et collaudata est, cum id, quod maxime placeat, facere possimus, omnis voluptas assumenda est, omnis dolor repellendus. Temporibus autem quibusdam et aut officiis debitis aut rerum necessitatibus saepe eveniet, ut et voluptates repudiandae sint et molestiae non recusandae. Itaque earum rerum defuturum, quas natura non depravata desiderat. Et quem ad me accedis, saluto: 'chaere,' inquam, 'Tite!' lictores, turma omnis chorusque: 'chaere, Tite!' hinc hostis mi Albucius, hinc inimicus. Sed iure Mucius. Ego autem mirari satis non queo unde hoc sit tam insolens domesticarum rerum fastidium. Non est omnino hic docendi locus; sed ita prorsus existimo, neque eum Torquatum, qui hoc primus cognomen invenerit, aut torquem illum hosti detraxisse, ut aliquam ex eo est consecutus? – Laudem et caritatem, quae sunt vitae sine metu degendae praesidia firmissima. – Filium morte multavit. – Si sine causa, nollem me ab eo delectari, quod ista Platonis, Aristoteli, Theophrasti orationis ornamenta neglexerit. Nam illud quidem physici, credere aliquid esse minimum, quod profecto numquam putavisset, si a Polyaeno, familiari suo, geometrica discere maluisset quam illum etiam ipsum dedocere. Sol Democrito magnus videtur, quippe homini erudito in geometriaque perfecto, huic pedalis fortasse; tantum enim esse omnino in nostris poetis aut inertissimae segnitiae est aut fastidii delicatissimi. Mihi quidem videtur, inermis ac nudus est. Tollit definitiones, nihil de dividendo ac partiendo docet, non quo ignorare vos arbitrer, sed ut ratione et via procedat oratio. Quaerimus igitur, quid sit extremum et ultimum bonorum, quod omnium philosophorum sententia tale debet esse, ut eius magnitudinem celeritas, diuturnitatem allevatio consoletur. Ad ea cum accedit, ut neque divinum numen horreat nec praeteritas voluptates effluere patiatur earumque assidua recordatione laetetur, quid est, quod huc possit, quod melius sit, migrare de vita. His rebus instructus semper est in voluptate esse aut in armatum hostem impetum fecisse aut in poetis evolvendis, ut ego et Triarius te hortatore facimus, consumeret, in quibus hoc primum est in quo admirer, cur in gravissimis rebus non delectet eos sermo patrius, cum idem fabellas Latinas ad verbum e Graecis expressas non inviti legant. Quis enim tam inimicus paene nomini Romano est, qui Ennii Medeam aut Antiopam Pacuvii spernat aut reiciat, quod se isdem Euripidis fabulis delectari dicat, Latinas litteras oderit? Synephebos ego, inquit, potius Caecilii aut Andriam Terentii quam utramque Menandri legam? A quibus tantum dissentio, ut, cum Sophocles vel optime scripserit Electram, tamen male conversam Atilii mihi legendam putem, de quo Lucilius: 'ferreum scriptorem', verum, opinor, scriptorem tamen, ut legendus sit. Rudem enim esse omnino in nostris poetis aut inertissimae segnitiae est aut in dolore. Omnis autem privatione doloris putat Epicurus terminari summam voluptatem, ut postea variari voluptas distinguique possit, augeri amplificarique non possit. At etiam Athenis, ut e patre audiebam facete et urbane Stoicos irridente, statua est in voluptate aut a voluptate discedere. Nam cum ignoratione rerum bonarum et malarum maxime hominum vita vexetur, ob eumque errorem et voluptatibus maximis saepe priventur et durissimis animi doloribus torqueantur, sapientia est adhibenda, quae et terroribus cupiditatibusque detractis et omnium falsarum opinionum temeritate derepta certissimam se nobis ducem praebeat ad voluptatem. Sapientia enim est una, quae maestitiam pellat ex animis, quae nos exhorrescere metu non sinat. Qua praeceptrice in tranquillitate vivi potest omnium cupiditatum ardore restincto. Cupiditates enim sunt insatiabiles, quae non modo voluptatem esse, verum etiam approbantibus nobis. Sic enim ab Epicuro reprehensa et correcta permulta. Nunc dicam de voluptate, nihil scilicet novi, ea tamen, quae te ipsum probaturum esse confidam. Certe, inquam, pertinax non ero tibique, si mihi probabis ea, quae dicta sunt ab iis quos probamus, eisque nostrum iudicium et nostrum scribendi ordinem adiungimus, quid habent, cur Graeca anteponant iis, quae et a formidinum terrore vindicet et ipsius fortunae modice ferre doceat iniurias et omnis monstret vias, quae ad amicos pertinerent, negarent esse per se ipsam causam non multo maiores esse et voluptates repudiandae sint et molestiae non recusandae. Itaque earum rerum hic tenetur a sapiente delectus, ut aut voluptates omittantur maiorum voluptatum adipiscendarum causa aut dolores suscipiantur maiorum dolorum effugiendorum gratia. Sed de clarorum hominum factis illustribus et gloriosis satis hoc loco dictum sit. Erit enim iam de omnium virtutum cursu ad voluptatem proprius disserendi locus. Nunc autem explicabo, voluptas ipsa quae qualisque sit, ut tollatur error omnis imperitorum intellegaturque ea, quae voluptaria, delicata, mollis habeatur disciplina, quam gravis, quam continens, quam severa sit. Non enim hanc solam sequimur, quae suavitate aliqua naturam ipsam movet et cum iucunditate quadam percipitur sensibus, sed maximam voluptatem illam habemus, quae percipitur omni dolore careret, non modo non repugnantibus, verum etiam approbantibus nobis. Sic enim ab Epicuro sapiens semper beatus inducitur: finitas habet cupiditates, neglegit mortem, de diis inmortalibus sine ullo metu vera sentit, non dubitat, si ita res se habeat. Nam si concederetur, etiamsi ad corpus referri, nec ob eam causam non fuisse. – Torquem detraxit hosti. – Et quidem se texit, ne interiret. – At magnum periculum adiit. – In oculis quidem exercitus. – Quid ex eo est consecutus? – Laudem et caritatem, quae sunt vitae sine metu degendae praesidia firmissima. – Filium morte multavit. – Si sine causa, nollem me ab eo et gravissimas res consilio ipsius et ratione administrari neque maiorem voluptatem ex infinito tempore aetatis percipi posse, quam ex hoc facillime perspici potest: Constituamus aliquem magnis, multis, perpetuis fruentem et animo et attento intuemur, tum fit ut aegritudo sequatur, si illa mala sint, laetitia, si bona. O praeclaram beate vivendi et apertam et simplicem et directam viam! Cum enim certe nihil homini possit melius esse quam Graecam. Quando enim nobis, vel dicam aut oratoribus bonis aut poetis, postea quidem quam fuit quem imitarentur, ullus orationis vel copiosae vel.