Sobolev inequalities related problems

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1. The Sobolev inequality involved here is the following:

$$\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{\frac{1}{p^*}} \leq C(n,p)\left(\int_{\mathbb{R}^n}|\nabla u|^p\right)^{\frac{1}{p}},$$

where $u \in W^{1,p}(\mathbb{R}^n)$ and p^* satifsies $\frac{1}{p*} = \frac{1}{p} - \frac{1}{n}$. Here are several questions you can think and verify quickly.

- Why the power on the left hand side has to be such p^*? (Hint: try to replace $u(x)$ by $u(\lambda x)$, after change of variables, you will see that in order to guarantee that Sobolev inequalities are independent of λ (we call it dilation), only p^* works.)
- It has been shown during class that the extremal functions of Sobolev inequalities should satisfy the following semilinear PDE:

$$-div(|\nabla u|^{p-2}\nabla u) = C|u|^{p^*-2}u, C \text{ is some constant.}$$

  On the other hand, applying the moving plane method, we can show the the positive solution of the above equation has to be radially decreasing. Combining this two results, please try to reduce the above equation to a second order ODE and solve it.

2. Assuming the Sobolev inequalties:

$$||u||_{L^q(\Omega)} \leq Cq^{\frac{n-1}{n}}||\nabla u||_{L^n(\Omega)}$$

holds for any $q \geq 1$, try to prove the following exponential type inequality (will need to use Stirling formula):

$$\int_\Omega e^u dx \leq Ce^{\mu||u||^n_{L^n}}, C, \mu \text{ are some constants.}$$

Here are some research papers related to my lecture. They are all milestone papers and I believe you will benefit a lot if you really spend time reading and understanding them.

References

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