Lecture Notes: Sobolev inequalities

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Text colored in blue were added after class.

Preknowledge

Weak derivative

From now on, we'll refer to weak derivative as just derivative, and classical derivative as classical derivative.

Weak derivative is introduced to generalize the concept of derivative to functions that are not differentiable in the classical sense.

Using *integration by parts theorem*, we can extend the definition of derivative to a larger class of functions (e.g. $u \in L^1(\Omega)$).

Definition: $u \in L^1_{\text{loc}}(\Omega)$ is said to have a weak derivative $v \in L^1_{\text{loc}}(\Omega)$ if:

$$\int_\Omega u(x)\varphi'(x)\mathrm{d}x \equiv -\int_\Omega v(x)\varphi(x)\mathrm{d}x \quad \forall \varphi \in \mathbb{C}_c^\infty(\Omega) \tag{1}$$

In this case, we say v is the weak derivative of u and write $v = u'$.

Generalization: In n dimensons, and multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, if we have:

$$\int_\Omega u(x)D^\alpha\varphi(x)\mathrm{d}x \equiv (-1)^{|\alpha|}\int_\Omega v(x)\varphi(x)\mathrm{d}x \quad \forall \varphi \in \mathbb{C}_c^\infty(\Omega) \tag{2}$$

here

$$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_n^{\alpha_n}} \tag{3}$$

We say v is the weak derivative of u and write $v = D^\alpha u$.

Properties:

Weak derivatives are well-defined in the sense that:

- if u has a weak derivative v, then v is unique.

$$\begin{cases} v_1 = D^\alpha u \\ v_2 = D^\alpha u \end{cases} \Rightarrow v_1 \equiv v_2 \quad a.e. \tag{4}$$

- If $u \in C^{k(\Omega)}$, then $D^\alpha u = D^\alpha u$ (classical derivative)

Holder continuity

Definition: u is said to be Holder continuous with exponent γ if there exists a constant C such that:

$$|u(x) - u(y)| \leq C|x - y|^{\gamma} \quad \forall x, y \in \Omega \tag{5}$$

We write $u \in C^{0,\gamma}(\Omega)$

Norm on Holder space is defined as:

$$\|u\|_{C^{0,\gamma}(\Omega)} = \|u\|_{C^{0}(\Omega)} + [u]_{\gamma} = \sup u(x) + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \tag{6}$$

Properties:

- A function on an interval satisfying the condition with $\gamma > 1$ is constant.

Proof:

$$\begin{gathered} f : I \to Y \\ f \in C^{0,\gamma}(I) \Rightarrow f \in C^{0}(I) \end{gathered} \tag{7}$$

If f isn't constant, say $x \neq y$, $f(x) \neq f(y)$, then let:

$$\begin{aligned} x_{i(n)} &= x + \frac{i}{n}(y - x) \quad i = 0, 1, ..., n, \forall n \in \mathbb{N} \\ d_Y(f(y), f(x)) &\leq \sum_{i=1}^{n} d_Y\left(f\left(x_{i(n)}\right), f(x_{i-1}(n))\right) \\ &\leq C \sum_{i=1}^{n} \left|x_{i(n)} - x_{i-1}(n)\right|^{\gamma} = C \sum_{i=1}^{n} \frac{|y - x|^{\gamma}}{n^{\gamma}} \end{aligned} \tag{8}$$

So we have:

$$d_Y(f(y), f(x)) \leq C|y - x|^{\gamma} n^{1-\gamma} \quad \forall n \in \mathbb{N} \tag{9}$$

Since $d_n \to 0 (n \to \infty), 1 - \gamma < 0$, we have that, for $\varepsilon = d_Y(f(y), f(x)) > 0, \exists n_0$ that $d_{n_0} < \varepsilon$, which is a contradiction.

So $\forall x, y \in I, f(x) = f(y)$, which means f is constant.

- If $\gamma = 1$, then the condition is Lipschitz continuity.
- The condition implies uniform continuity.

Note

Introduction

Previous inequalities

- Triangle inequality:

$$|x+y| \le |x| + |y| \tag{10}$$

- Cauchy inequality:

$$|xy| \le |x||y| \tag{11}$$

- Young inequality:

$$xy \le \frac{x^p}{p} + \frac{y^q}{q} \tag{12}$$

  where $\frac{1}{p} + \frac{1}{q} = 1, x, y \ge 0$
- Holder inequality:

$$\int_{\mathbb{R}^n} |fg| \mathrm{d}x \le \left(\int_{\mathbb{R}^n} |f|^p \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |g|^q \mathrm{d}x \right)^{\frac{1}{q}} \tag{13}$$

  where $\frac{1}{p} + \frac{1}{q} = 1$
- Jesen inequality:

$$f\left(\int_{\mathbb{R}^n} g \right) \le \int_{\mathbb{R}^n} f(g) \mathrm{d}x \tag{14}$$

  where f is a convex function

Background

To introduce the background of Sobolev inequality, we start by discussisng the following problem:

$$\Delta u + f = 0 \tag{15}$$

Here f might not be continous, might just be L^2 function.

To solve this, we first find the minimizer of the following functional:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega fu\mathrm{d}x \tag{16}$$

($u \in \mathbb{C}_c^\infty(\mathbb{R}_n)$, meaning that $|u \neq 0| < \infty$)

We take:

$$J(u + \varepsilon\varphi) = \frac{1}{2}\int_\Omega |\nabla u + \varepsilon\nabla\varphi|^2 \mathrm{d}x - \int_\Omega f(u + \varepsilon\varphi)\mathrm{d}x \tag{17}$$

($u, \varphi \in \mathbb{C}_c^\infty(\mathbb{R}_n)$)

When u is the minimizer, we have:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u + \varepsilon\varphi) = 0 \text{ at } \varepsilon = 0 \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u + \varepsilon\varphi) = \int_\Omega \nabla u \nabla\varphi \mathrm{d}x - \int_\Omega f\varphi\mathrm{d}x \tag{19}$$

Consider

$$\int \nabla u \nabla \varphi + \int \Delta u \varphi = \int_{\partial\Omega} \nabla u \cdot \varphi \equiv 0 \tag{20}$$

we have:

$$\begin{gathered} \int_\Omega \Delta u \varphi \mathrm{d}x + \int_\Omega f\varphi \mathrm{d}x \equiv 0, \forall \varphi \in \mathbb{C}_0^\infty(\mathbb{R}^n) \\ \Rightarrow \Delta u + f = 0 \end{gathered} \tag{21}$$

Thus leading to the weak solution of the PDE.

Question: When $f \in L^2$, when does $\int u \cdot f$ make sense?

When $u \in L^2$, $f \in L^2$, using Holder, we have

$$\int u \cdot f \leq \left(\int |u|^2\right)^{\frac{1}{2}} \left(\int |f|^2\right)^{\frac{1}{2}} < \infty \tag{22}$$

Sobolev space

Sobolev space is introduced:

$$
\begin{aligned}
W^{1,2}(\Omega) &= \{u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega)\} \\
\|u\|_{W^{1,2}(\Omega)} &= \left(\int |u|^2 + \int |\nabla u|^2\right)^{\frac{1}{2}}
\end{aligned}
\tag{23}
$$

Trivial: $W^{1,2}(\Omega)$ defines normed space.

We can also define $W^{1,p}(\Omega)$ for $p > 2$ likewise:

$$
\begin{aligned}
W^{1,p}(\Omega) &= \left\{u \in L^p(\Omega) \mid \nabla u \in L^{p(\Omega)}\right\} \\
\|u\|_{W^{1,p}(\Omega)} &= \left(\int |u|^p + \int |\nabla u|^p\right)^{\frac{1}{p}}
\end{aligned}
\tag{24}
$$

More generally, we can define $W^{k,p}(\Omega)$ for $k \in \mathbb{N}$:

$$
\begin{aligned}
W^{k,p}(\Omega) &= \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), \forall 0 \leq |\alpha| \leq k\} \\
\|u\|_{W^{k,p}(\Omega)} &= \left(\sum_{0 \leq |\alpha| \leq k} \int |D^\alpha u|^p\right)^{\frac{1}{p}} \\
\|u\|_{W^{k,\infty}} &= \operatorname{esssup}_{x \in \Omega} \sum_{0 \leq |\alpha| \leq k} |D^\alpha u(x)|
\end{aligned}
\tag{25}
$$

Note: Sobolev space is a Hilbert space, and it's a Banach space for $p > 1$.

Sobolev inequality

If Sobolev inequality holds, we have:

$$\|f\|_{L^{p^*}(\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)} \tag{26}$$

That leads to the following statement:

$$f \in W^{1,p}(\Omega) \Rightarrow f \in L^{p^*}(\Omega) \tag{27}$$

Notice $p^* > p$, thus Sobolev inequality is a statement about the integrability of the function.

Discussions:

- If and when does Sobolev inequality hold?
- What's the best constant $C(n, p)$?
- What's the extremal function of the inequality?
  - Existence
  - Expression

Previous works show that:

$$u(x) = \frac{1}{\left(a + b \cdot |x|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}} \tag{28}$$

$$C(n,p) = \frac{1}{\sqrt{\pi} n^{\frac{1}{p}}} \left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma(n)}{\Gamma\left(1+n-\frac{n}{p}\right)\Gamma\left(\frac{n}{p}\right)}\right)^{\frac{1}{n}} \tag{29}$$

Question: For $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ to hold, we must have $p < n$, what happens when $p = n$?

Theorem: $\forall q \geq n$,

$$\|u\|_{L^q(\mathbb{R}^n)} \leq (\omega_{n-1})^{\frac{1}{q}-\frac{1}{n}} \left(\frac{q+1-\frac{q}{n}}{n}\right)^{\frac{1}{q}-\frac{1}{n}+1} \|u\|_{W^{1,n}(\mathbb{R}^n)} \tag{30}$$

where $\omega_{n-1} = \left|\mathbb{S}^{n-1}\right|$, volume of the unit sphere in $\mathbb{R}^n$.

Approximation of the constant:

$$\left(\frac{q+1-\frac{q}{n}}{n}\right)^{\frac{1}{q}-\frac{1}{n}+1} \sim q^{\frac{n-1}{n}} \left(\frac{1}{n}\right)^{\frac{n-1}{n}} \tag{31}$$

The $\frac{1}{n}$ here is actually $\frac{1}{n} - \frac{1}{n^2}$, I'm not sure if I've missed something or just doesn't matter.

Given by Trudinger in 1967, we have:

$$\|u\|_{L^q(\mathbb{R}^n)} \leq Cq^{\frac{n-1}{n}} \|u\|_{W^{1,n}(\mathbb{R}^n)} \tag{32}$$

We can show that:

$$
\begin{aligned}
\int e^u &= \sum_{j=0}^{\infty} \frac{1}{j!} \int u^j = \sum_{j=0}^{\infty} \frac{1}{j!} \|u\|_{L^j}^j \\
&\le \sum_{j=0}^{\infty} \frac{1}{j!} \left(C \cdot j^{\frac{n-1}{n}} \|u\|_{L^n}\right)^j \\
&= C \sum_{j=0}^{\infty} \frac{j^{j \cdot \frac{n-1}{n}}}{j!} \|u\|_{L^n}^j
\end{aligned}
\tag{33}
$$

Using Strling's formula ($n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$) , we have:

$$
\frac{j^{j \cdot \frac{n-1}{n}}}{j!} \sim \frac{1}{e^{-j+\frac{j}{n}} n^{\frac{j}{n}+\frac{1}{2}} \left(\frac{j}{n}\right)!} \le C' \frac{1}{\left(\frac{j}{n}\right)!}
\tag{34}
$$

Rewrite the sum, we have:

$$
\sum_{j=0}^{\infty} \frac{j^{j \cdot \frac{n-1}{n}}}{j!} \|u\|_{L^n}^j \le C' \sum_{j=0}^{\infty} \frac{\|u\|_{L^n}^j}{\left(\frac{j}{n}\right)!} = C' e^{\|u\|_{L^n}^n}
\tag{35}
$$

Thus we have:

$$
\int e^u \le C e^{\|u\|_{L^n}^n}
\tag{36}
$$

This is called Orlicz-Sobolev inequality. Orlicz space is defined as:

$$
L^{\Phi}(\Omega) := \left\{ u \in L^1(\Omega) \mid \int \Phi\left(\frac{|u|}{\lambda}\right) \mathrm{d}x \le 1 \right\}
\tag{37}
$$

In 1971, Moser proved that:

$$
\sup_{\|\nabla u\|_{L^n} \le 1} \int e^{\alpha_n u^{\frac{n}{n-1}}} \le \infty
\tag{38}
$$

$(u \in \mathbb{C}_0^{\infty}(\Omega), \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}, \omega_{n-1} := \left|\mathbb{S}^{n-1}\right|)$

Before we continue, let's prove the following inequalities:

Gagliardo-Nirenberg-Sobolev inequality

Goal: Say $1 \leq p < n$, there exisits $C(n,p)$ such that:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n) \tag{39}$$

Proof:

Step 1: $p = 1, p^* = \frac{n}{n-1}$

$$|u(x)| = \left|\int_{-\infty}^{x_i} \partial_i u(x_1, ..., x_i, ..., x_n) \mathrm{d}x_i\right| \leq \int_{-\infty}^{\infty} |\nabla u| \mathrm{d}x_i \tag{40}$$

$$\begin{aligned} |u(x)|^{\frac{n}{n-1}} &\leq \left(\prod_{i=1}^{n} \int_{-\infty}^{x_i} |\nabla u| \mathrm{d}x_i\right)^{\frac{1}{n-1}} \\ \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} \mathrm{d}x_1 &\leq \int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} \int_{-\infty}^{x_i} |\nabla u| \mathrm{d}x_i\right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\ &= \left(\int_{-\infty}^{\infty} |\nabla u| \mathrm{d}x_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} |\nabla u| \mathrm{d}y_i\right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\ &\leq \left(\int_{-\infty}^{\infty} |\nabla u| \mathrm{d}x_1\right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u| \mathrm{d}x_1 \mathrm{d}y_i\right)^{\frac{1}{n-1}} \end{aligned} \tag{41}$$

Repeating this process, it's trivial to show that:

$$\begin{aligned} &\int_{\mathbb{R}^k} |u|^{\frac{n}{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 ... \mathrm{d}x_k \\ \leq &\prod_{i=1}^{k} \left(\int_{\mathbb{R}^k} |\nabla u| \mathrm{d}x_1 \mathrm{d}x_2 ... \mathrm{d}x_k\right)^{\frac{1}{n-1}} \prod_{i=k+1}^{n} \left(\int_{\mathbb{R}^{k+1}} |\nabla u| \mathrm{d}x_1 \mathrm{d}x_2 ... \mathrm{d}x_k \mathrm{d}y_i\right)^{\frac{1}{n-1}} \end{aligned} \tag{42}$$

Subsitute $k = n$, we have:

$$\begin{aligned} \int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \mathrm{d}x &\leq \left(\int_{\mathbb{R}^n} |\nabla u| \mathrm{d}x\right)^{\frac{n}{n-1}} \\ \left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \mathrm{d}x\right)^{\frac{n-1}{n}} &\leq \int_{\mathbb{R}^n} |\nabla u| \mathrm{d}x \end{aligned} \tag{43}$$

Step 2: $1 < p < n$

Subsitute u with u^γ, we have:

$$
\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} \mathrm{d}x\right)^{\frac{n-1}{n}} &\leq \int_{\mathbb{R}^n} \left|\nabla |u|^{\gamma}\right| \mathrm{d}x \\
&= \gamma \int |u|^{\gamma-1} |\nabla u| \mathrm{d}x \\
&\leq \gamma \left(\int |u|^{\frac{(\gamma-1)p}{p-1}} \mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int |\nabla u|^{p} \mathrm{d}x\right)^{\frac{1}{p}}
\end{aligned}
\tag{44}
$$

Last inequality uses Holder inequality.

Letting $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}$, we have $\gamma = \frac{p(n-1)}{n-p}$, thus: $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} = p^{*}$, leading to the final result.

Morrey inequality

Goal: Assuming $n < p \leq \infty$. Then there exists a constant $C = C(n,p)$ (depending only on n and p, not with u), for every $u \in C^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, we have:

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \gamma = 1 - \frac{n}{p} \tag{45}$$

Thus: if $u \in W^{1,p}(\mathbb{R}^n)$, then u is Holder continuous with exponent $\gamma = 1 - \frac{n}{p}$.(after a possible redefinition on a set of measure zero)

Proof:

$\forall x \in \mathbb{R}^n, \rho > 0, z \in \partial B_1, 0 < t < \rho,$

$$|u(x+tz) - u(x)| = \left|\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} u(x+sz)\mathrm{d}s\right| \leq \int_0^t |\nabla u(x+sz)|\mathrm{d}s \tag{46}$$

$$\begin{aligned}\int_{\partial B(0,1)} |u(x+tz) - u(x)|\mathrm{d}z &\leq \int_0^t \int_{\partial B(0,1)} |\nabla u(x+sz)|\mathrm{d}z\mathrm{d}s \\ &= \int_{B(0,1)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\mathrm{d}y\end{aligned} \tag{47}$$

$$\begin{aligned}\int_{B(x,\rho)} |u(y) - u(x)|\mathrm{d}y &= \int_0^\rho \mathrm{d}s \int_{\partial B(x,s)} |u(y) - u(x)|\mathrm{d}y \\ &= \int_0^\rho s^{n-1}\mathrm{d}s \int_{\partial B(0,1)} |u(x+sz) - u(x)|\mathrm{d}s \\ &\leq \int_0^\rho s^{n-1}\mathrm{d}s \int_{B(x,\rho)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\mathrm{d}s \\ &= \frac{\rho^n}{n} \int_{B(x,\rho)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\mathrm{d}s\end{aligned} \tag{48}$$

Setting $\rho = 1$, we have:

$$\begin{aligned}|u(x)| &\leq \frac{1}{|B(0,1)|}\left(\int_{B(0,1)} |u(x) - u(y)| + |u(y)|\mathrm{d}y\right) \\ &\leq \frac{1}{n}\int_{B(0,1)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\mathrm{d}y + \frac{1}{|B(0,1)|}\int_{B(0,1)} |u(y)|\mathrm{d}y \\ &\leq C_1\|\nabla u\|_{L^p} + C_2\|u\|_{L^p(B(0,1))} \\ &\leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}\end{aligned} \tag{49}$$

Equation holds for every x, thus we have:

$$\sup_{x\in\mathbb{R}^n} |u(x)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{50}$$

Estamating $|u(x) - u(y)|$, take $\rho = |x-y|, O = B(x,\rho) \cap B(y,\rho)$,

$$
\begin{aligned}
|u(x)-u(y)| \leq \frac{1}{|O|}\int_O |u(x)-u(z)|\mathrm{d}z + \frac{1}{|O|}\int_O |u(y)-u(z)|\mathrm{d}z \\
\frac{1}{|O|}\int_O |u(x)-u(z)|\mathrm{d}z &\leq C\frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} |u(x)-u(z)|\mathrm{d}z \\
&\leq C\frac{1}{|B(x,\rho)|}\frac{\rho^n}{n}\int_{B(x,\rho)} \frac{|\nabla u(z)|}{|x-z|^n}\mathrm{d}z \\
&\leq C\rho^{1-\frac{n}{p}}\|\nabla u\|_{L^p(B(x,\rho))}
\end{aligned} \tag{51}
$$

Given all that,

$$
\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \gamma = 1-\frac{n}{p} \tag{52}
$$

Homework

Question 1.1

Why the power on the left hand side has to be such p^*

Hint: try replace $u(x)$ by $u(\lambda x)$, after change of variables, you will see that in order to guarantee that Sobolev inequality are independent of the scaling of u, the power on the left hand side has to be p^*

Answer

Subsitute u with $u(\lambda x)$, we have

$$\text{LHS} = \left(\int_{\mathbb{R}^n} |u(x)|^{p^*} \lambda^n \mathrm{d}x\right)^{\frac{1}{p^*}} = \lambda^{\frac{n}{p^*}} \left(\int_{\mathbb{R}^n} |u(x)|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \tag{53}$$

$$\text{RHS} = C(n,p)\left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \lambda^{p-n} \mathrm{d}x\right)^{\frac{1}{p}} = \lambda^{\frac{n}{p}-1} C(n,p)\left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \tag{54}$$

Since $C(n,p) \not\sim u(x)/u(\lambda x)$, we need $\lambda^{\frac{n}{p^*}} \equiv \lambda^{\frac{n}{p}-1} \Rightarrow \frac{n}{p^*} = \frac{n}{p} - 1$, which gives $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Question 1.2

It has been shown during class that the extremal functions of Sobolev inequality should satisfy the following semilinear PDE:

$$-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = C|u|^{p^*-2} u \tag{55}$$

On the other hand, applying the moving plane method, we can show the positive solution of the above equation has to be radially decreasing. Combinging the two results, please try to reduce the above equation to a second order ODE and solve it:

Answer

$$\begin{aligned} -\text{LHS} = \nabla\left(|\nabla u|^{p-2} \nabla u\right) &= (p-2)|\nabla u|^{p-2} \Delta u + |\nabla u|^{p-2} \Delta u = (p-1)|\nabla u|^{p-2} \Delta u \\ &= -C|u|^{p^*-2} u \end{aligned} \tag{56}$$

Deducing to ODE: (and taking $p = 2, u > 0$)

$$\frac{\partial^2 u}{\partial x^2} = -Cu^{2^*-1} \tag{57}$$

General solution for this ODE is:

$$u = \tag{58}$$

~~Haven't come up with a solution yet, will update later~~

Question 2

Assuming the Sobolev inequalties:

$$\|u\|_{L^q(\Omega)} \leq Cq^{\frac{n-1}{n}} \|u\|_{L^n(\Omega)} \tag{59}$$

holds for any $q \geq 1$, try to prove the following exponential type inequality (will need to use Stirling formula):

$$\int_\Omega e^u \mathrm{d}x \leq Ce^{\mu \|u\|_{L^n}^n} \quad C, \mu \text{ are some constants} \tag{60}$$

Answer

This is already proven in previous sections, copying the proof here for completeness:

We can show that:

$$\begin{aligned} \int e^u &= \sum_{j=0}^{\infty} \frac{1}{j!} \int u^j = \sum_{j=0}^{\infty} \frac{1}{j!} \|u\|_{L^j}^j \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j!} \left(C \cdot j^{\frac{n-1}{n}} \|u\|_{L^n}\right)^j \\ &= C \sum_{j=0}^{\infty} \frac{j^{j \cdot \frac{n-1}{n}}}{j!} \|u\|_{L^n}^j \end{aligned} \tag{61}$$

Using Strling's formula ($n! \sim \frac{n^n}{e^n} \sqrt{2\pi n}$) , we have:

$$\frac{j^{j \cdot \frac{n-1}{n}}}{j!} \sim \frac{1}{e^{-j+\frac{j}{n}} n^{\frac{j}{n}+\frac{1}{2}} \left(\frac{j}{n}\right)!} \leq C' \frac{1}{\left(\frac{j}{n}\right)!} \tag{62}$$

Rewrite the sum, we have:

$$\sum_{j=0}^{\infty} \frac{j^{j \cdot \frac{n-1}{n}}}{j!} \|u\|_{L^n}^j \leq C' \sum_{j=0}^{\infty} \frac{\|u\|_{L^n}^j}{\left(\frac{j}{n}\right)!} = C' e^{\|u\|_{L^n}^n} \tag{63}$$

Thus we have:

$$\int e^u \leq Ce^{\|u\|_{L^n}^n} \tag{64}$$

Notice you could just take $C' = Ce^\mu$ for a cleaner formula.