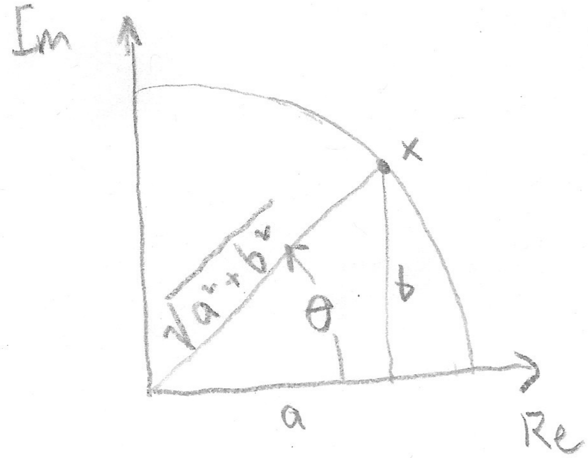
Background: on using complex notation for analytical problems that have sinusoidal and/or exponential behavior — e.g. frictional waves.

## I Identities for complex numbers




$$x = a + ib$$

$$= \sqrt{a^2+b^2}\, e^{i\theta}$$

where $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$

also $e^{i\theta} = \cos\theta + i\sin\theta$

## II A useful trig identity to memorize

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

## III Why use complex notation when solving PDE's?

It turns trigonometry into <u>simple algebra</u>, and seamlessly combines sinusoidal + exponential behavior!

## IV Complex representation of regular waves:

- A simple wave has form $\sim \cos(kx - \omega t)$ where $kx - \omega t$ is the "phase speed." It has phase speed $c = \omega / k$, and is moving to positive $x$ for $k$ positive. A good rule is to always take $\omega$ positive.

- A more useful general form for the wave is

$$\eta = \mathrm{Re}\{(A + iB) \exp iP\}$$

where $P$ = phase = $kx - \omega t$. Then by using I to rewrite the complex amplitude $A + iB$:

$$A + iB = \sqrt{A^2 + B^2} \exp i\varphi \quad \text{where } \varphi = \tan^{-1} \frac{B}{A}.$$

Then $\eta = \mathrm{Re}\{\sqrt{A^2+B^2} \exp i(P + \varphi)\}$

or $\eta = \sqrt{A^2+B^2} \cos(kx - \omega t + \varphi)$ phase lag for positive $\varphi$.

Thus by suitable choices of $A$ + $B$ we can represent a wave with arbitrary magnitude + phase.

V Using complex notation to simplify cases with incident & reflected waves. If the two have equal amplitude then it is a standing wave, but often we have to allow for more general solutions.

A full solution could be written as:

$$\eta = \mathrm{Re}\left\{ \underbrace{A^+ \exp i(kx - \omega t)}_{\text{Incident} \rightarrow} + \underbrace{A^- \exp i(-kx - \omega t)}_{\text{Reflected} \leftarrow} \right\}$$

where $A^+$ & $A^-$ are complex.

Rewrite as:

(*) $$\eta = \mathrm{Re}\left\{ \left[ A^+ \exp i(kx) + A^- \exp i(-kx) \right] \exp(-i\omega t) \right\}$$

If $A^+ = A^- = a$ (real) then this is

$$\eta = a\,\mathrm{Re}\left\{ (\cos kx + \cancel{i \sin kx} + \cos kx - \cancel{i \sin kx})(\cos \omega t - i \sin \omega t) \right\}$$

or $\eta = 2a \cos kx \cos \omega t$ a classic standing wave.

More generally we could write (*) as

$\eta = \mathrm{Re}\left\{ E(x) \exp(-i\omega t) \right\}$ where $E(x)$ is a complex function.

VI Allowing complex wave number.

say $K = K_R + i K_I$

then $\exp iKx = \exp(-K_I x) \cdot \exp i(K_R x)$

so the imaginary park of K leads to a real exponential decay of the wave field in space. Had we instead allowed a complex frequency this would have led to exponential decay (or growth!) in time.